Orthogonal Projections and the Gram-Schmidt Process

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Review of the Theory

Recall that when $\{w_1, \dots, w_p\}$ is an orthogonal basis for a subspace W of an inner product space V, we can find the *orthogonal projection* of any vector v in V onto W by the formula

$$\operatorname{Proj}_W v = \frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \cdots + \frac{\langle v, w_p\rangle}{\langle w_p, w_p\rangle} w_p. \tag{1}$$

Remark. There are several pieces of notation and terminology for the orthogonal projection of v onto W. (You just need to keep in mind that they refer to the same thing!)

$$\begin{aligned}\operatorname{Proj}_W v &= \text{the orthogonal projection of } v \text{ onto } W\\ &= \text{the best approximation to } v \text{ in } W\\ &= \text{the closest point to } v \text{ in } W\end{aligned}$$

Theorem 1 (Orthogonal Projection)**.** Let V be an inner product space and $B = \{w_1, \dots, w_p\}$ an orthogonal basis for a finite-dimensional subspace $W \subseteq V$.

(a) Any vector $v \in V$ can be uniquely written in the form $v = v_1 + v_2$ where $v_1 \in W$ and $v_2 \in W^\perp$. The vector v_1 equals $\operatorname{Proj}_W v$.

(b) The lengths satisfy $\|v\|^2 = \|v_1\|^2 + \|v_2\|^2$.

(c) The vector v_1 is the best approximation to v in W, in the sense that

$$\|v - v_1\| < \|v - w\| \quad \text{for any } w \in W \text{ with } w \neq v_1.$$

The Gram-Schmidt Process (GS for short) is an algorithm named after two mathematicians who discovered how one can transform any basis for a given vector space to an *orthogonal* basis. The **idea** is to use orthogonal projections.

Theorem 2 (Gram-Schmidt Process)**.** Let $\{w_1, \dots, w_p\}$ be a basis for W. Then $\{v_1, \dots, v_p\}$ is an orthogonal basis for W where $v_1, \dots, v_p$ are defined recursively as follows

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \\ v_3 &= w_3 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \\ &\vdots \end{aligned}$$

In other words,

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= w_2 - \operatorname{Proj}_{v_1} w_2 \\ v_3 &= w_3 - \operatorname{Proj}_{v_1, v_2} w_3 \\ &\vdots \end{aligned}$$

Review Examples

Example 1. Let $V = C[-1, 1]$ be equipped with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$ and let $W = \text{Span}\{p_0, p_1\}$ where $p_0(x) = 1, p_1(x) = x$. Find the best approximation to $p_2(x) = x^2$ in W.

Solution. We are asked to find $\operatorname{Proj}_W p_2$. Note that $p_0 \perp p_1$ with the given inner product because $\int_{-1}^{1} (1)x\, dx = 0$. Thus we can use the orthogonal projection formula (1).

$$\begin{aligned} \operatorname{Proj}_W p_2 &= \frac{\langle p_2, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle} p_1 \\ &= \underline{\hspace{4cm}}\ \mathbf{1} + \underline{\hspace{4cm}}\ x \end{aligned}$$

I will let you compute the coefficients of 1 and x by carrying out the necessary integrations.

Example 2. Let $V = C[0, 1]$ be equipped with the inner product $\langle f, g\rangle = \int_{0}^{1} f(x)g(x)\, dx$ and let $W = \text{Span}\{p_0, p_1\}$ where $p_0(x) = 1, p_1(x) = x$. Find the best approximation to $p_2(x) = x^2$ in W.

Solution. Again we need to find $\operatorname{Proj}_W p_2$. Note that with the new inner product $\int_{0}^{1} (1)x\, dx \neq 0$. Thus we cannot use the orthogonal projection formula (1) directly.

- **Step I** Use the Gram-Schmidt process to find an orthogonal basis for W, say $\{q_0, q_1\}$. Indeed, the short calculation

$$\begin{aligned} q_0 &= p_0 \\ q_1 &= p_1 - \operatorname{Proj}_{q_0} p_1 \end{aligned}$$

gives $q_0(x) = 1, q_1(x) = x - \frac{1}{2}$.

- **Step II** Solve the problem as in the previous Example.

$$\begin{aligned}\operatorname{Proj}_W p_2 &= \frac{\langle p_2, q_0\rangle}{\langle q_0, q_0\rangle} q_0 + \frac{\langle p_2, q_1\rangle}{\langle q_1, q_1\rangle} q_1 \\ &= \underline{\hspace{6cm}}\ q_0 + \underline{\hspace{6cm}}\ q_1\end{aligned}$$

I will let you compute the coefficients of q_0 and q_1 by carrying out the necessary integrations.