Weierstrass Approximation Theorem

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Polynomials are "dense" in the set of all continuous functions over the interval $[0,1]$ in the following sense:

Theorem 1 (Weierstrass Approximation–Polynomial Version)**.** Given a continuous function f on $[0,1]$ and any number $n > 0$, one can always find a polynomial p_n such that

$$\max_{x\in[0,1]} |f(x) - p_n(x)| < \frac{1}{n}.$$

Loosely speaking, any continuous function on $[0,1]$ can be approximated by a sequence of polynomials. We illustrate an application of this theorem by an example which is of importance in probability and statistics.

Example (Hausdorff Moment Problem)**.** Suppose f is a continuous real-valued function on $[0,1]$ such that

$$\int_0^1 f(x)x^k\,dx = 0 \qquad \text{for all } k = 0, 1, 2, \ldots$$

Show that $f(x) = 0$ for all $x \in [0,1]$.

Solution. We claim that $\int_0^1 (f(x))^2\,dx = 0$. Let us see why after proving this claim we can immediately proceed to conclude that $f(x) = 0$ everywhere.

Since f^2 is continuous and $f^2(x) \geq 0$, the integral $\int_0^1 (f(x))^2\,dx$ can equal zero only if the integrand $f^2(x)$ is zero for all x. Of course, this is equivalent to $f(x) = 0$ everywhere. So it suffices to prove our claim.

Let $n > 0$ be arbitrary and let p_n be a polynomial as in Theorem 1. Then observe the trivial identity that

$$f^2(x) = f(x)p_n(x) + f(x)(f(x) - p_n(x)).$$

Integrating both sides of this equality from 0 to 1 yields

$$\begin{aligned}\int_0^1 f^2(x)\,dx &= \int_0^1 f(x)p_n(x)\,dx + \int_0^1 f(x)(f(x) - p_n(x))\,dx \\ &= 0 + \int_0^1 f(x)(f(x) - p_n(x))\,dx.\end{aligned}$$

The integral $\int_0^1 f(x)p_n(x)\,dx$ is zero because, if $p_n(x) = a_0 + \cdots + a_m x^m$, then we can use a term by term integration to obtain $\int_0^1 f(x)p_n(x)\,dx = \sum_{k=0}^{m} a_k \int_0^1 f(x)x^k\,dx = 0$ by assumption. Thus so far we have

$$\int_0^1 f^2(x)\,dx = \int_0^1 f(x)(f(x) - p_n(x))\,dx.$$

Next observe that by triangle inequality for integration,

$$\begin{aligned}\int_0^1 f^2(x)\,dx = \left|\int_0^1 f^2(x)\,dx\right| &= \left|\int_0^1 f(x)(f(x) - p_n(x))\,dx\right| \\ &\leq \int_0^1 |f(x)||f(x) - p_n(x)|\,dx.\end{aligned}$$

Finally note that since $|f|$ is continuous on $[0, 1]$, it achieves a maximum M, therefore

$$|f(x)||f(x) - p_n(x)| \leq M\left(\frac{1}{n}\right),$$

and

$$\int_0^1 |f(x)||f(x) - p_n(x)|\,dx \leq \int_0^1 M\left(\frac{1}{n}\right)\,dx = M\left(\frac{1}{n}\right).$$

Thus

$$0 \leq \int_0^1 f^2(x)\,dx \leq M\left(\frac{1}{n}\right) \text{ for any } n > 0.$$

Letting $n \to \infty$, we see that $\int_0^1 f^2(x)\,dx = 0$ as claimed.

Exercise. Revisit this example after you take a course in analysis and try to give an alternate one-line solution for it.

Weierstrass Approximation Theorem has the following counterpart for continuous **periodic** functions. Recall that a **trigonometric polynomial** (of period 1) is a function of the form $\sum_{k=-m}^{m} a_k e^{2\pi ikx}$.

Theorem 2 (Weierstrass Approximation Theorem–Trigonometric Version)**.** Given a continuous function f on $[0,1]$ with $f(0) = f(1)$ and any number $n > 0$, one can always find a trigonometric polynomial p_n such that

$$\max_{x\in[0,1]} |f(x) - p_n(x)| < \frac{1}{n}.$$

After the break, we will briefly discuss this theorem in the context of Fourier series.