Windows and Kernels

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Definition 1 (Window)**.** A *window* (sometimes called a *summation procedure*) is a sequence of functions $\alpha_N \colon \mathbb{Z} \to \mathbb{R}$ such that

(a) $0 \leq \alpha_N(n) \leq 1$;

(b) $\alpha_N(n) = 0$ for all except finitely many n;

(c) For all n, $\alpha_N(n) \to 1$ as $N \to \infty$.

Example 1 (Dirichlet Window)**.** The *Dirichlet window* is given by

$$\delta_N(n) = \begin{cases} 1 & \text{if } |n| \leq N \\ 0 & \text{if } |n| > N. \end{cases}$$

Example 2 (Fejér Window)**.** The *Fejér window* is defined by

$$\alpha_N(n) = \begin{cases} 1 - \frac{|n|}{N} & \text{if } |n| \leq N \\ 0 & \text{if } |n| > N. \end{cases}$$

The first two pages in Chapter 4 motivate this definition.

Every window corresponds to a *kernel*.

Definition 2. The *kernel* of a window α_N is

$$\underbrace{K_{\alpha_N}(x)}_{\text{kernel}} = \sum_{n=-\infty}^{\infty} \underbrace{\alpha_N(n)}_{\text{window}} e^{2\pi i n x}$$

Note for a fixed N the above summation is in fact finite because $\alpha_N(n) = 0$ for all except finitely many n.

Example 3 (Dirichlet Kernel)**.** The first example above implies that the Dirichlet kernel D_N is the function

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi inx}.$$

Example 4 (Fejér Kernel)**.** We can use the second example to write down the Fejér Kernel Δ_N as

$$\Delta_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi inx}.$$

Since the Nth and $-N$th terms of the Fejér kernel vanish, Δ_N is a trigonometric polynomial of degree $N - 1$.

The problems with the Dirichlet summation $S_N(f) = D_N * f$ stem from the "high frequency noise" coming from the abrupt discontinuity of Dirichlet window. Signal processing teaches us that we must take tapered windows, which fade out around the edges.

In fact, for any continuous function f of period 1, $\sigma_N(f) = \Delta_N * f$ converges to f "uniformly," that is, for each $n > 0$, the inequality

$$\max_x |\sigma_N(f)(x) - f(x)| < \frac{1}{n}$$

holds for all sufficiently large N.

Moreover, the Fejér summation procedure avoids the Gibbs phenomenon for piecewise continuous functions. This is because unlike D_N that takes both positive and negative values, $\Delta_N(x) \geq 0$ for all x. One can check this positivity property, say, by finding a closed form for Δ_N. This computation is carried out at the beginning of Chapter 4.