Matrix Lie Groups–Homework*2

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Please provide the details of your work for each problem. All problems are partial credit.

1. (*2 points*) Recall that a subset $\{v_1, \ldots, v_k\}$ of $\mathbb{R}^n$ is called *orthonormal* if

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } i, j = 1, \ldots, k.$$

Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear function. Show that if $\{v_1, \ldots, v_n\}$ and $\{T(v_1), \ldots, T(v_n)\}$ are orthonormal bases for $\mathbb{R}^n$, then T is an isometry.

2. (*2 points*) Recall that for a complex matrix A, its *Hermitian transpose* A^* is defined by $A^* = \overline{A}^T$. Let A be a two-by-two complex matrix such that $AA^* = I$ and $\det A = 1$. Prove that A is of the form

$$A = \begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix}$$

for two complex numbers u, v such that $|u|^2 + |v|^2 = 1$. Also verify the converse, namely that any such matrix satisfies $AA^* = I$ and $\det A = 1$.

Educational Note: The set of all such matrices forms a group called the **special unitary group** of size two and is denoted by $SU(2)$. This exercise proves that $SU(2)$ is isomorphic with the group of unit quaternions and S^3.

3. (*3 points*) (Exercise 3.9, Tapp) Recall that two groups G_1 and G_2 are said to be isomorphic if there exists an *isomorphism* $f\colon G_1 \to G_2$ between them. That is, a bijection f with $f(ab) = f(a)f(b)$ for all $a, b \in G_1$.

Define a map $f\colon O(n) \to SO(n) \times \{1, -1\}$, for n and odd number, as follows

$$f(A) = (\det A \cdot A, \det A).$$

(a) Prove that if n is odd, then f is an isomorphism.

*Submit on Monday, Feb 13 in class.

(b) Prove that $O(2)$ is not isomorphic to $SO(2) \times \{1, -1\}$. (Note: You need to show that there is *no isomorphism* $g\colon O(2) \to SO(2) \times \{1, -1\}$. The function f defined above certainly cannot be an isomorphism because for $n = 2$ it is not well-defined.)

4. (*3 points*) Suppose the linear transformations $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ have the standard matrices M_1 and M_2 with

$$M_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}.$$

(a) Show that M_1 and M_2 are in $SO(3)$.

(b) Describe T_1 and T_2 geometrically.

(c) Suppose (a, b, c) is a point at distance 1 from the origin in $\mathbb{R}^3$. Describe geometrically how you would choose the angles θ_1 and θ_2 so that $T_2(T_1(a, b, c)) = (0, 0, 1)$.

(d) Continuing (c), assume (a, b, c) is in the first octant and write expressions for $\sin\theta_i$ and $\cos\theta_i$ $(i = 1, 2)$ in terms of a, b, and c.

5. (*2 points*) (Optional Bonus Problem) Suppose a is an irrational real number. Show that the set E_a of numbers of the form $e^{2\pi i n a}, n \in \mathbb{Z}$, is dense in the unit circle S^1.
Hint: Show that if we divide S^1 into N equally sized "bins" of length $2\pi/N$, there is at least one bin that contains infinitely many elements of E_a. Then use the fact that E_a is a subgroup of S^1. Your solution should show *explicitly* where the irrationality of a is used.

Educational Note: This problem (whose statement is borrowed from Hall's Lie groups, Lie algebras, and Representations) is related to some deeper results in number theory, including Weyl's equidistribution theorem. We briefly discussed the latter in *Fourier Series* last Spring.

6. (*2 points*) (Optional Bonus Problem) A matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is said to be the *limit* of sequence of matrices $M_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$, written $M_n \to M$, if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lim\limits_{n\to\infty} a_n & \lim\limits_{n\to\infty} b_n \\ \lim\limits_{n\to\infty} c_n & \lim\limits_{n\to\infty} d_n \end{bmatrix}.$$

In other words, we define the convergence of a sequence of matrices entrywise.

Let a be an irrational real number and let G be the group

$$G = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

Prove that any matrix of the form

$$\begin{bmatrix} e^{it} & 0 \\ 0 & e^{is} \end{bmatrix} \quad s, t \in \mathbb{R}$$

is the limit of a sequence of matrices in G.

Educational Note: A subgroup $G \subset GL(n, \mathbb{C})$ is said to be *closed* in $GL(n, \mathbb{C})$, if for any sequence $M_n \in G$ convergent to some matrix $M \in GL(n, \mathbb{C})$, we have $M \in G$. This exercise shows that G is not a closed subgroup of $GL(n, \mathbb{C})$ under the limit operation in $GL(2, \mathbb{C})$. Another example, in lower dimensions, is given by $GL(1, \mathbb{Q})$ that is not closed in $GL(1, \mathbb{C})$. We will go back to these examples at the end of this course when we learn about the topology of matrix groups.

The following **Practice Problems** will not be collected/graded.

7. (*0 points*) Exercises 1.5.1–1.5.5 from Stillwell.

8. (*0 points*) Exercises 2.2.1–2.2.3 from Stillwell.