Matrix Lie Groups–Homework*3

Ehssan Khanmohammadi

Please provide the details of your work for each problem. All problems are partial credit.

1. (*3 points*) The *trace* of an $n \times n$ real (or complex) matrix $M = [m_{ij}]$ is defined to be the sum of the diagonal entries of M:

$$\operatorname{tr}(M) = m_{11} + m_{22} + \cdots + m_{nn}.$$

Observe that trace defines a linear function from the set of $n \times n$ matrices to the real (or complex) numbers, that is, $\operatorname{tr}\colon \operatorname{Mat}(n \times n, \mathbb{R}) \to \mathbb{R}$.

(a) Show that for any two $n \times n$ matrices A and B, we have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(b) Use (a) to prove that any two similar matrices have the same trace. See also the bonus problems below.

Educational Note: Trace is an example of a *class function* since it is constant on each similarity class of matrices.

2. (*4 points*)

(a) Conclude from (a) of the previous problem that there are no $n \times n$ matrices A and B satisfying $AB - BA = I$. (Equivalently, it is not possible to find two linear maps of finite-dimensional spaces $S, T\colon \mathbb{R}^n \to \mathbb{R}^n$ with $S \circ T - T \circ S = \operatorname{Id}$.)

(b) The set of all polynomials

$$\mathcal{P} = \{p(x) = a_m x^m + \cdots + a_1 x + a_0 \mid m \geq 0, a_j \in \mathbb{R}\}$$

with the usual addition and scalar multiplication of polynomials is a real vector space. Show that $\mathcal{P}$ is infinite-dimensional by finding an infinite basis for $\mathcal{P}$.

(c) Define functions $S, T\colon \mathcal{P} \to \mathcal{P}$ by $T(p(x)) = xp(x)$ and $S(p(x)) = p'(x)$. Show that S and T are linear transformations. (You can assume facts from calculus.)

*Submit on Monday, Feb 20 in class.

(d) Continuing the notation of the previous part show that $S \circ T - T \circ S = \mathrm{Id}$.

Educational Note: The above equation is the *Canonical Commutation Relation* (CCR) used in quantum mechanics. This exercise points out one of the reasons why quantum observables should be studied as operators over infinite-dimensional vector spaces.

3. (*3 points*) Let S be the set of all $n \times n$ real matrices which have trace equal to zero.

(a) Show that S is a subspace of the vector space of all $n \times n$ matrices.

(b) Find the dimension of S.

Educational Note: The set of all such matrices forms the Lie algebra of $SL(n, \mathbb{R})$ as we shall see later.

4. (*2 points*) (Optional Bonus Problem) Let $M \in \mathrm{Mat}(n \times n, \mathbb{R})$ be a matrix with zero trace. Show that M can be written as a sum of matrices of the form $AB - BA$.

Hint: One can solve this problem based on the construction of an explicit basis for the space of trace zero matrices. This is done in problem 3 (b), so let us sketch the solution to that problem first.
Let E_{ij} be the matrix that is zero everywhere except for the (i, j)-entry where it equals 1. Then a possible solution to problem 3 (b) is to describe a basis for S as follows:

$$B = \{E_{ij}, E_{11} - E_{kk} \mid i \neq j, k = 2, \ldots, n\},$$

for instance if $n = 2$, then

$$B = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Now to finish the solution to this problem, it remains to show that each of the basis vectors (matrices) above can be written in the form $AB - BA$. This is not too hard; note that $E_{1n} = E_{1k}E_{kn} - E_{kn}E_{1n}$ for $n \neq 1$.

Educational Note: In fact it can be shown that if $\mathrm{tr}(M) = 0$, then $M = AB - BA$ for some matrices A and B, but the proof of this statement is more difficult than the exercise above.

The following **Practice Problem** will not be collected/graded.

5. (*0 points*) (Practice Problem) Suppose $f\colon \mathrm{Mat}(n \times n, \mathbb{R}) \to \mathbb{R}$ is a linear function such that for any two $n \times n$ matrices, $f(AB) = f(BA)$. Prove that f is a scalar multiple of the trace function.