Matrix Lie Groups–Homework[*]4

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Please provide the details of your work for each problem. All problems are partial credit.

1. (*2 points*) Recall that the *Cayley–Hamilton Theorem* states that **any square matrix satisfies its characteristic equation.** That is, if

$$\chi_A(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0,$$

then

$$\chi_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_0 I = \mathbf{0}.$$

You can verify this in the 2×2 case directly (see the Practice Problems below.)

(a) Verify the Cayley–Hamilton Theorem for diagonal matrices.

Hint: Let A be an $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Use the following factorization of $\chi_A(x)$ to evaluate $\chi_A(A)$:

$$\chi_A(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0 = (x - \lambda_1)\ldots(x - \lambda_n).$$

(b) Use the previous part to verify the Cayley–Hamilton Theorem for diagonalizable matrices.

2. (*2 points*) Let $A \in M_n(\mathbb{C})$ be a matrix whose only eigenvalue is 0. Prove that $A^n = \mathbf{0}$. (*Hint:* What is the characteristic polynomial of A?)

3. (*3 points*) (Tapp, Exercise 6.3) For a matrix $A \in M_n(\mathbb{C})$, define $\exp(A)$ as follows

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

(a) Let $A \in M_n(\mathbb{C})$ be the diagonal matrix

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}.$$

Calculate $\exp(A)$.

[*]Submit on Monday, March 6 in class.

(b) Use the previous part to prove that $\det \exp(A) = e^{\mathrm{tr}(A)}$ for a diagonal matrix A.

(c) Prove that $\det \exp(A) = e^{\mathrm{tr}(A)}$ when A is a diagonalizable matrix. (*Hint:* Observe that $\exp(XYX^{-1}) = X \exp(Y) X^{-1}$.)

4. (*3 points*) (Tapp, Exercise 6.4) Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Calculate $\exp(tA)$ for arbitrary $t \in \mathbb{C}$. (*Hint*: A is diagonalizable.)

5. (*2 points*) (Optional Bonus Problem) Let $A_1, A_2 \in M_n(\mathbb{C})$ be diagonalizable matrices. That is, assume that we can find matrices P_i and diagonal matrices D_i such that $A_i = P_i D_i P_i^{-1}$ for $i = 1, 2$. Show that if $A_1 A_2 = A_2 A_1$, then there exists a matrix P such that

$$A_1 = PD_1P^{-1} \quad \text{and} \quad A_2 = PD_2P^{-1}.$$

The following **Practice Problems** will not be collected/graded.

6. (*0 points*) (Practice Problem) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary two-by-two matrix. Recall that $\mathrm{tr}(A) = a + d$ and $\det A = ad - bc$. Check, by an explicit computation, that

$$A^2 - \mathrm{tr}(A)A + \det(A)I_2 = \mathbf{0}$$

where I_2 is the two-by-two identity matrix. In other words, A satisfies a polynomial equation of the form $x^2 + c_1 x + c_0 = 0$.

7. (*0 points*) (Practice Problem) Let A be an invertible matrix. Check that $(A^{-1})^T = (A^T)^{-1}$. (*Hint:* $AA^{-1} = I$. Now take transpose from both sides.)

8. (*0 points*) (Practice Problem) Let A be a skew-symmetric matrix, that is, with the property that $A^T = -A$. Show that $R = (I + A)(I - A)^{-1}$ is orthogonal. (You can take the invertibility of $(I - A)$ for granted, or use Problem 10 (a) below to prove it.)

9. (*0 points*) (Practice Problem) Let J_n be the $n \times n$ matrix that is one everywhere. For instance,

$$J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find all eigenvalues of J_n. (*Hint:* J_n is a symmetric matrix. The Spectral Theorem implies that the algebraic multiplicity of eigenvalue 0 equals the geometric multiplicity of eigenvalue 0, namely nullity of J_n.)

10. (*0 points*) (Practice Problem)

(a) Let $A \in M_n(\mathbb{C})$ and $c \in \mathbb{C}$. Show that λ is an eigenvalue of A if and only if $\lambda + c$ is an eigenvalue of $B = A + cI$.

(b) Let J_n be the matrix described above. Find the eigenvalue of $J_n - I$.

Educational Note: $J_n - I$ is the *adjacency matrix* of a "complete graph" on n vertices. This matrix is zero everywhere except for the diagonal entries where it is zero. This indicates that all the nodes of the graph are connected to each other and no vertex is connected to itself.

11. (*0 points*) (Practice Problem)

(a) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that there is no matrix B such that $A = B^2$. (That is, A does not have a "square root.")

(b) Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Show that there is a matrix B (possibly with complex entries) such that $A = B^2$. (*Hint:* A is diagonalizable by the Spectral Theorem.)