Matrix Lie Groups–Homework*5

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Please provide the details of your work for each problem. All problems are partial credit.

1. (*5 points*) Recall that $A \in M_n(\mathbb{R})$ is called **skew-symmetric** if $A^T = -A$.

(a) Show that if $A \in M_n(\mathbb{R})$ is skew-symmetric and n is odd, then A is not invertible. *Hint:* Compute $\det A$. Note that this property does not hold for even n as you can easily check for $n = 2$.

(b) Show that the eigenvalues of any skew-symmetric matrix are purely imaginary, that is, the real part of each eigenvalue is zero. *Hint:* Mimic the proof of the fact that the eigenvalues of any symmetric matrix are real.

(c) Show that the set of all $n \times n$ skew-symmetric matrices V with the bracket operation $[A, B] = AB - BA$ is an abstract Lie algebra. *Hint:* Show that V is a real vector space that is closed under the bracket operation. You can take the Jacobi identity for granted since it is satisfied for M_n.

2. (*3 points*) Let X be a 2×2 matrix with $\mathrm{tr}(X) = 0$.

(a) Show that $X^2 = -\det(X)I$. (*Hint:* See Exercises 1 and 6 from the previous Homework.)

(b) Conclude that

$$\exp X = \cos(\sqrt{\det X})I + \frac{\sin(\sqrt{\det X})}{\sqrt{\det X}}X.$$

Hint: Use the power series expansion for the exp function.
Educational Note: If $\det X = 0$, then the coefficient of X in the formula above should be interpreted as 1 in accordance with $\lim_{x\to 0} \frac{\sin x}{x} = 1$. In a future exercise you will learn how to apply the above formula to matrices with non-zero trace by taking advantage of another property of the exponential function.

*Submit on Wednesday, March 22 in class.

(c) Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Use part (b) to calculate $\exp(tA)$ for arbitrary $t \in \mathbb{C}$. (This gives an alternative solution to Exercise 3 from the previous Homework.)

3. (*2 points*) (Stillwell, Exercises 4.2.2 and 4.2.3) Recall that when $AB = BA$, $\exp A \exp B = \exp(A + B)$. Since we can have $uv \neq vu$ for quaternions u and v, it can be expected that we can have $e^u e^v \neq e^{u+v}$.

(a) Explain why $i = e^{i\pi/2}$ and $j = e^{j\pi/2}$. (*Hint:* Use the formula for computing $e^{u\theta}$ where u is a unit purely imaginary quaternion and θ is a real number.)

(b) Deduce from the previous part that at least one of $e^{i\pi/2}e^{j\pi/2}$, $e^{j\pi/2}e^{i\pi/2}$ is not equal to $e^{i\pi/2+j\pi/2}$

4. (*2 points*) (Optional Bonus Problem) Show that for $A \in M_n(\mathbb{C})$ and any *orthonormal* basis $\{u_1, \ldots, u_n\}$ of $\mathbb{C}^n$, $\|A\|^2 = \sum_{j,k=1}^n |\langle Au_j, u_k\rangle|^2$. Conclude that if λ is any eigenvalue of A, then $|\lambda| \leq \|A\|$.

The following **Practice Problems** will not be collected/graded.

5. (*0 points*) (Practice Problem) Let V be an inner product space. Recall that by the Cauchy-Schwarz inequality, for any $u, v \in V$, we have

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Show that

$$|\langle u, v\rangle| = \|u\|\|v\|$$

holds if and only if $\{u, v\}$ is linearly dependent. *Hint:* Examine the proof of the Cauchy-Schwarz inequality.

6. (*0 points*) (Practice Problem) Show, by a direct calculation, that the bracket operation $[A, B] = AB - BA$ defined on M_n satisfies the Jacobi identity.

7. (*0 points*) (Practice Problem) Consider the sequence of matrices $M_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ and the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then one can *define* the convergence $M_n \to M$ in the following two ways:

(a) **Entrywise** We say that $M_n \to M$ if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lim_{n\to\infty} a_n & \lim_{n\to\infty} b_n \\ \lim_{n\to\infty} c_n & \lim_{n\to\infty} d_n \end{bmatrix}.$$

(b) **Convergence in the norm** We say that $M_n \to M$, if $\|M_n - M\| \to 0$ as a sequence of real numbers.

Show that the two definitions above are equivalent. That is, $M_n \to M$ in the sense of (a) if and only if $M_n \to M$ in the sense of (b).

8. (*0 points*) (Practice Problem) Solve Exercises 4.1.1–4.1.4 in Stillwell to prove that $e^{A+B} = e^A e^B$ when $AB = BA$.

9. (*0 points*) (Practice Problem) We know that if $AB = BA$, then $e^A e^B = e^{A+B}$. However, in general, $e^A e^B = e^{A+B}$ does not imply that $AB = BA$. Prove that nevertheless, the following statement holds: $AB = BA$ if and only if $e^{tA} e^{tB} = e^{t(A+B)}$ for all real t.
Hint: Calculate the second derivative of both sides of the desired equality at $t = 0$.

10. (*0 points*) (Practice Problem) Show that for arbitrary matrices $A, B \in M_n(\mathbb{C})$ we have

$$e^{AB} A = A e^{BA}.$$

Hint: Use the power series expansion of the exponential function and explain why $(AB)^n A = A(BA)^n$ for arbitrary matrices A and B.