Matrix Lie Groups–Homework*6

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Please provide the details of your work for each problem. All problems are partial credit.

1. (*2 points*) (Tapp, Exercise 5.10) Let G be a matrix group with the Lie algebra $\mathfrak{g}$. Prove that the tangent space to G at any $A \in G$ is

$$T_A G = A\mathfrak{g} = \{AB \mid B \in \mathfrak{g}\}$$

Hint: By definition, the tangent space $T_A G$ consists of the initial velocity of curves in G with initial position A.

Educational Note: This exercise shows that the vector spaces $T_I G$ and $T_A G$ are isomorphic under the linear map $X \mapsto AX$.

2. (*4 points*) Let $G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a > 0, b \in \mathbb{R} \right\}$.

(a) Show that G is subgroup of GL_2 under matrix multiplication.

(b) Find the Lie algebra of G. *Hint:* The Lie algebra of G is defined to be the tangent space to G at the identity, that is, $\mathfrak{g} = T_I G$. Find the initial velocity of all curves in G with initial position I.

(c) Use the previous part to find the dimension of G.

Educational Note: The group G is isomorphic with the group of *affine transformations* of $\mathbb{R}$, namely $\{f_{a,b} \mid f_{a,b}(x) = ax + b, a > 0, b \in \mathbb{R}\}$ where the group operation is the composition of functions.

3. (*3 points*) In class we proved that $\frac{d}{dt} e^{tX} = X e^{tX}$. Show that e^{tX} is the *only* differentiable function f satisfying the initial value differential equation

$$f'(t) = Xf(t) \text{ and } f(0) = I$$

as follows.

*Submit on Wednesday, March 29 in class.

(a) Prove that for any solution f of the equation above, the product $e^{-tX}f(t)$ is a constant, that is, $e^{-tX}f(t) = c$ for some c. *Hint:* A differentiable function $g\colon \mathbb{R} \to \mathbb{R}^N$ is constant if and only if $g'(t) = 0$ for all t.

(b) Show that $c = I$ to conclude that $f(t) = e^{tX}$.

4. (*1 point*) Give an example of a matrix group G and a matrix X such that $e^X \in G$ but $X \notin \mathfrak{g}$.

Educational Note: It turns out that if $e^{tX} \in G$ for all real t, then $X \in \mathfrak{g}$. In fact, later we shall see that

$$\mathfrak{g} = \{X \mid e^{tX} \in G \text{ for all real } t\}.$$

5. (*2 points*) (Optional Bonus Problem) Consider the matrix $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ in $SL_2(\mathbb{C})$. Show that there is no matrix $X \in \mathfrak{sl}_2(\mathbb{C})$ with $A = e^X$.

Educational Note: In class we showed that there is no matrix $X \in \mathfrak{sl}_2(\mathbb{R})$ such that $A = e^X$ by resorting to the fact that the equation $-1 = e^\lambda$ has no real solution. This argument is no longer valid in the complex case.

The following **Practice Problems** will not be collected/graded.

6. (*0 points*) Solve the following system of differential equations:

$$\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = 2y_2 \end{cases} \quad \text{with } y_1(0) = 1, y_2(0) = 3.$$

Hint: Let A be an $n \times n$ matrix, and let $\mathbf{y}$ be an $n \times 1$ column vector whose entries depend smoothly on a parameter t. Then the system of differential equations $\mathbf{y}' = A\mathbf{y}$ with the initial condition $\mathbf{y}(0)$ has the unique solution $\mathbf{y}(t) = e^{tA}\mathbf{y}(0)$. Compare with problem 3 above.

7. (*0 points*) (Practice Problem) Let A be an $n \times n$ matrix with all eigenvalues equal to λ. Show that e^A is given by the following *finite* sum

$$e^A = e^{\lambda I} \sum_{k=0}^{n-1} \frac{(A - \lambda I)^k}{k!}.$$

Hint: $e^A = e^{\lambda I} e^{A - \lambda I}$ (why?)