Matrix Lie Groups–Homework*7

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Please provide the details of your work for each problem. All problems are partial credit.

1. (*2 points*) Let G_1 and G_2 be two matrix groups with Lie algebras $(\mathfrak{g}_1, [\ ,\ ]_1)$ and $(\mathfrak{g}_2, [\ ,\ ]_2)$, respectively.

We know that the Cartesian product $G_1 \times G_2$ is a group under componentwise multiplication with the identity element $I = (I_1, I_2)$ where I_1 and I_2 are the identity elements of G_1 and G_2, respectively.

(a) Show that

$$T_I(G_1 \times G_2) = \mathfrak{g}_1 \times \mathfrak{g}_2.$$

In other words, the tangent space at the identity of $G_1 \times G_2$ consists of pairs of matrices (X, Y) with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_2$.

(b) (Practice Problem. Part b will not be graded.) Show that $T_I(G_1 \times G_2)$ is a Lie algebra with the bracket operation

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_1, [Y_1, Y_2]_2).$$

2. (*3 points*)

(a) Give an example of a Lie algebra of dimension one. Make sure to explicitly define the Lie bracket.

(b) Prove that any two Lie algebras of dimension one are isomorphic.

Parts (a) and (b) together prove that there exists a unique one-dimensional Lie algebra, up to isomorphism.

3. (*2 points*) In class we showed that

$$(D_I \det)(X) = \operatorname{tr}(X)$$

*Submit on Friday, April 7 in class.

where $(D_I \det)(X)$ is the directional derivative of the det function in the direction of X at the identity. Verify that

$$(D_A \det)(X) = \det(A)\,\mathrm{tr}(A^{-1}X)$$

for any invertible matrix A.

If L is a Lie algebra with basis $B = \{X_1, \ldots, X_n\}$, because of bilinearity of $[\ ,\ ]$, the bracket is completely determined for any two vectors by knowing the values of $[X_i, X_j]$ only. Note that each $[X_i, X_j] \in L$ can itself be expressed in terms of the basis vectors $\{X_1, \ldots, X_n\}$:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

The c_{ij}^k are called the **structure constants** of L with respect to B. We emphasize that c_{ij}^k depend on the choice of basis in general.

4. (*3 points*) Let L_1 and L_2 be Lie algebras. Show that L_1 is isomorphic to L_2 if and only if there is a basis B_1 of L_1 and a basis B_2 of L_2 such that the structure constants of L_1 with respect to B_1 are equal to the structure constants of L_2 with respect to B_2.

5. (*2 points*) (Optional Bonus Problem) Let A be a 2×2 matrix with distinct eigenvalues α and β. Show that

$$\exp(A) = \left(\frac{e^\alpha - e^\beta}{\alpha - \beta}\right) A + \left(\frac{\alpha e^\beta - \beta e^\alpha}{\alpha - \beta}\right) I_2.$$

Hint: Let χ_A be the characteristic polynomial of A. By the division algorithm for polynomials (long division), if m is a positive integer, then

$$x^m = \chi_A(x) q_m(x) + r_m(x),$$

where $q_m(x)$ and $r_m(x)$ are polynomials, and $\deg r_m(x) < \deg \chi_A(x) = 2$. Thus $r_m(x) = a_m x + b_m$ for real numbers a_m and b_m. Show that

$$a_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \text{ and } b_m = \frac{\beta^m \alpha - \alpha^m \beta}{\alpha - \beta}.$$

The following **Practice Problem** will not be collected/graded.

6. (*0 points*) (Practice Problem) Find the structure constants of $\mathfrak{sl}_2(\mathbb{R})$ with respect to the basis given by the matrices

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$