Matrix Lie Groups–Homework*8

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Please provide the details of your work for each problem. All problems are partial credit.

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra.

- A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a **Lie subalgebra** of $\mathfrak{g}$ if for any $X \in \mathfrak{h}, Y \in \mathfrak{h}$, we have $[X, Y] \in \mathfrak{h}$, that is, $\mathfrak{h}$ is closed under the Lie bracket. The latter condition can be written as $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.
- A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is said to be an **ideal** of $\mathfrak{g}$ if for any $X \in \mathfrak{g}, Y \in \mathfrak{h}$, we have $[X, Y] \in \mathfrak{h}$. The latter condition can be written as $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

1. (*2 points*) Let $\phi \colon L_1 \to L_2$ be a Lie algebra homomorphism. Show that

$$\ker \phi = \{X \in L_1 \mid \phi(X) = 0\}$$

is an ideal of L_1.

Educational Note: With a suitable definition for the quotient of Lie algebras by their ideals, one can show the **First Isomorphism Theorem** which gives the following isomorphism of Lie algebras:

$$L_1/\ker \phi \simeq \operatorname{im} \phi.$$

2. (*3 points*) (Constructing New Ideals) Suppose A and B are ideals of an abstract Lie algebra L.

(a) Show that $A + B$ is an ideal of L where

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Educational Note: The construction $A + B$ is used in the **Second Isomorphism Theorem**. Can you make a guess about the statement of this theorem by way of analogy with the results from group theory?

*Submit on Friday, April 21 in class.

(b) Prove that $[A, B]$ is an ideal of L where

$$[A, B] = \text{Span}\{[a, b] \mid a \in A, b \in B\}.$$

3. (*2 points*) Let N be a subgroup of a matrix group G. We know that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Show that if N is a normal subgroup, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Hint: Recall from class that

$$[X, Y] = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} a(t)b(s)a(t)^{-1}$$

for suitably chosen curves a and b. Alternatively, you can use the definition/properties of adjoint maps to solve this problem.

4. (*3 points*)

(a) Show that $SL_n(\mathbb{C})$ is closed in $M_n(\mathbb{C})$.

(b) Show that $SO(n)$ is both open and closed in $O(n)$. (You can freely use any results from class.)

5. (*2 points*) (Optional Bonus Problem, Tapp Exercise 7.7) Let G be a matrix group. Define the **identity component**, G_0, of G as:

$$\{g \in G \mid \exists \text{ continuous } \gamma\colon [0, 1] \to G \text{ with } \gamma(0) = I \text{ and } \gamma(1) = g\}.$$

(a) Prove that G_0 is a normal subgroup of G.

(b) Is G_0 necessarily closed in G?

The following **Practice Problems** will not be collected/graded.

6. (*0 points*) (Practice Problem) Show that $SL_n(\mathbb{C})$ is path-connected.
Hint: Modify our proof for the fact that $GL_n(\mathbb{C})$ is path-connected.

7. (*0 points*) (Practice Problem) Show that $SU(2)$ is path-connected.
Hint: Recall that each $A \in SU(2)$ has the form

$$A = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}, |\alpha|^2 + |\beta|^2 = 1.$$

Prove that

$$\alpha = u \cos\theta, \beta = v \sin\theta, |u| = |v| = 1$$

for some real θ and complex numbers u and v. Then write u and v in the exponential form to find

$$\alpha = e^{i\phi}\cos\theta, \beta = e^{i\psi}\sin\theta.$$

Use this to define a path from I to A in $SU(2)$.

8. (*0 points*) (Practice Problem) Show that $SO(3)$ is path-connected. *Hint:* In a previous worksheet we proved that any $A \in SO(3)$ represents a space rotation and in particular A is similar to a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

You can find an inductive proof for path-connectedness of $SO(n)$ on page 53 of Stillwell.

9. (*0 points*) (Practice Problem) For a Lie algebra L, we write L' for $[L, L]$ and call L' the **derived** (or commutator) algebra of L. Show that

$$\mathfrak{sl}_2(\mathbb{C})' = \mathfrak{sl}_2(\mathbb{C}).$$

10. (*0 points*) (Practice Problem)

(a) (Warm-up) Show that the intersection of any two subgroups of a group is also a subgroup.

(b) Let H and K be two closed subgroups of a matrix Lie group G. Show that $H \cap K$ is a closed subgroup of G and that

$$T_I(H \cap K) = T_I(H) \cap T_I(K).$$