MAT472–Adjoint Maps

Ehssan Khanmohammadi

Let G be a matrix group and fix $g \in G$. Recall that the *conjugation* map by g is an invertible homomorphism of G denoted by $C_g\colon G \to G$ where

$$C_g(x) = gxg^{-1}.$$

We saw earlier that $(D_I C_g)\colon \mathfrak{g} \to \mathfrak{g}$ is an invertible Lie algebra homomorphism denoted by Ad_g, where

$$(D_I C_g)(X) = \mathrm{Ad}_g(X) = gXg^{-1}.$$

From the definition above, it is easy to see that $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$.

> $\mathrm{Ad}_g(X)$ measures the failure of g to commute with elements of G near I in the direction of X.

If we allow $g \in G$ to vary, this process produces a map

$$\begin{aligned} \mathrm{Ad}\colon G &\to GL(\mathfrak{g}) = \{\text{all invertible linear maps from } \mathfrak{g} \text{ to itself}\} \\ g &\mapsto \mathrm{Ad}_g \end{aligned}$$

that is a group homomorphism.

Exercise 1. In this exercise, we show that the directional derivative

$$(D_I\,\mathrm{Ad})\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \{\text{all linear maps from } \mathfrak{g} \text{ to itself}\}$$

is given by

$$(D_I\,\mathrm{Ad})(X) = \mathrm{ad}_X$$

where $\mathrm{ad}_X\colon \mathfrak{g} \to \mathfrak{g}$ is the linear map $\mathrm{ad}_X(Y) = [X, Y]$.

Let $X, Y \in \mathfrak{g}$ and choose a smooth curve $\gamma\colon (-\epsilon, \epsilon) \to G$ such that $\gamma(0) = I$ and $\gamma'(0) = X$. Show that

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\gamma(t)}(Y) = [X, Y].$$

Derivations

Let A be a (Lie) algebra. A **derivation** of A is a linear map $D\colon A \to A$ such that

$$D(ab) = aD(b) + D(a)b \quad \text{for all } a, b \in A.$$

This property is reminiscent of the product rule in calculus. The set of all derivations of A, denoted by $\mathrm{Der}(A)$ is not empty. (why?)

Exercise 2. Let L be a Lie algebra. Verify that ad_X is a derivation for all $X \in L$:

$$(\mathrm{ad}_X)[Y, Z] = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)].$$

Exercise 3. Let A be a (Lie) algebra.

(a) Show that $\mathrm{Der}(A)$ is a vector space.

(b) Let $D, E \in \mathrm{Der}(A)$. Show that

$$[D, E] = D \circ E - E \circ D$$

is also a derivation, but $D \circ E$ need not be a derivation.