Some Properties of Fourier Coefficients

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The best way to learn is to do; the worst way to teach is to talk.
-*Paul Halmos*

Question 1. Suppose f and g are integrable periodic functions of period 1.

(a) Define $h(x) = af(x) + bg(x)$ for complex numbers a and b. Explain why $\widehat{h}(n) = a\widehat{f}(n) + b\widehat{g}(n)$.

(b) Let $h(x) = f(x)g(x)$. Is it necessarily true that $\widehat{h}(n) = \widehat{f}(n)\widehat{g}(n)$? Either prove the given equation, or disprove it by an finding a counterexample.

Exercise 1. Let f be an integrable periodic function of period 1.

(a) Define the function g by $g(x) = f(x+a)$ for a real number a. Find a relation between the Fourier coefficients of g, namely $\widehat{g}(n)$, and the Fourier coefficients of f, namely $\widehat{f}(n)$.

(b) Define the function g by $g(x) = f(x)e^{2\pi ikx}$ for an integer k. Find a relation between the Fourier coefficients of f and g.

(c) Compute the Fourier coefficients of g, and h in terms of Fourier coefficients of f, where $g(x) = f(-x)$ and $h(x) = \overline{f(x)}$.

Exercise 2. Let f, g, and h be continuous periodic functions of period 1. Define the *convolution* of f and g to be the function $f * g$ defined by

$$(f * g)(x) = \int_0^1 f(t)g(x-t)\,dt. \tag{1}$$

Loosely speaking, convolutions correspond to "weighted averages."

(a) Explain why $f * (g + h) = f * g + f * h$, and $(cf) * g = c(f * g)$ for any complex number c.

(b) Is $f * g$ a periodic function? What about integrable?

(c) Equation (1) certainly looks asymmetric in f and g. Nevertheless $f * g = g * f$. Check this!

(d) Compute $\widehat{f * g}(n)$ in terms of the Fourier coefficients of f and g.

Educational Note: One can define the convolution operation for larger classes of functions. For instance, if f and g are two *integrable* functions of period 1 as in Equation 1, then we still get all of the above results. The idea is that any such integrable function can be approximated by a continuous function in the sense of Theorem 0.22 in your textbook, and indeed all your calculations above for continuous functions carry over to integrable functions without much difficulty. Also, a rule of thumb to keep in mind is that, the convolution $f * g$ is smoother than either f or g. For instance, the convolution of two integrable functions is *continuous*.

Question 2. Let f, g, and h be continuous functions of period 1.

(a) Does the **associative law** hold for the convolution of these functions? In other words, is it necessarily true that $f * (g * h) = (f * g) * h$. Prove or disprove this equality!

(b) Does there exist an **identity element** for the convolution operation? In other words, does there exist a continuous function of period 1 such that $f * g = g$ for all continuous functions g of period 1? One of your future homework problems will ask you to revisit this question!