MAT472–The Geometry of Space Rotations

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Our goal in this worksheet is a geometric description of the transformations in $SO(3)$. But as a warm-up, let us start in two dimensions first.

Exercise 1. Answer these questions by reasoning geometrically.

(a) Start in two dimensions and suppose T is a rotation of $\mathbb{R}^2$ about the origin. Recall that the nonzero vector v is an *eigenvector* for the linear transformation T with *eigenvalue* $\lambda \in \mathbb{R}$ if $T(v) = \lambda v$. What are the eigenvector(s) and eigenvalue(s) of T (if any)? Does the size of the angle of rotation matter?

(b) Still in two dimensions, suppose T is a reflection of $\mathbb{R}^2$ across a line through the origin spanned by the nonzero vector u. What are the eigenvector(s) and eigenvalue(s) of T (if any)? Does the choice of the mirror line matter?

(c) Now return to three dimensions and suppose T is a rotation of $\mathbb{R}^3$ around an axis spanned by the nonzero vector u. What are the eigenvector(s) and eigenvalue(s) of T (if any)? Does the size of the angle of rotation matter? Does the choice of the axis matter?

Exercise 2. Let M be a matrix in $SO(3)$.

(a) Justify each step of the following argument.

$$\begin{aligned}\det(M - I) &= \det(M - M^T M)\\ &= \det((I - M^T)M)\\ &= \det(I - M^T)\\ &= \det(I - M)\end{aligned}$$

(b) Explain why (a) tells you that $\det(M - I) = 0$.

(c) Explain why (b) tells you that M has an eigenvector with eigenvalue $+1$.

Exercise 3. Suppose M is in $O(3)$ and $\det M = -1$. Use a variant of the above argument to show that M has an eigenvector with eigenvalue -1.

Exercise 4. Let M be a matrix in $SO(3)$.

(a) Let u be an eigenvector for M with eigenvalue $+1$. Explain why multiplication by M preserves the space $u^\perp$ defined by

$$u^\perp = \{v \in \mathbb{R}^3 \mid u \cdot v = 0\}.$$

(b) Consider the special case of $u = [0, 0, 1]$. What is $u^\perp$ in this case? Since multiplication by M preserves $u^\perp$ we can think of it as if it were $T\colon \mathbb{R}^2 \to \mathbb{R}^2$. What is the general form of the standard matrix of T?

(c) Continuing (b), describe the geometric effect of multiplying by M on $u^\perp$, and then describe the geometric effect of M on $\mathbb{R}^3$.

Exercise 5.

(a) Find a diagonal matrix D in $O(3)$ but not in $SO(3)$.

(b) Show that $O(3) = SO(3) \cup SO(3)D$ where

$$SO(3)D = \{MD \mid M \in SO(3)\}.$$