MAT472–Exercises in Lie Algebras

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1 Definition of Lie Algebras

Exercise 1. Let L be a Lie algebra.

(a) Show that $[v, 0] = 0 = [0, v]$ for all $v \in L$.

(b) Suppose that $x, y \in L$ satisfy $[x, y] \neq 0$. Show that x and y are linearly independent.

2 Lie Algebra Homomorphisms

Recall that a function $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ between (real) Lie algebras $(\mathfrak{g}_1, [\ ,\ ]_1)$ and $(\mathfrak{g}_2, [\ ,\ ]_2)$ is called a **Lie algebra homomorphism** if it respects both the vector space structure and the bracket multiplication:

The function $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism if

- f is a linear map:

$$f(cX + Y) = cf(X) + f(Y)$$

  for $X, Y \in \mathfrak{g}_1$ and $c \in \mathbb{R}$

- $f([X, Y]_1) = [f(X), f(Y)]_2$ for $X, Y \in \mathfrak{g}_1$.

Exercise 2. (Composition of Lie algebra homomorphims)

(a) Suppose $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\psi\colon \mathfrak{g}_2 \to \mathfrak{g}_3$ are Lie algebra homomorphisms. Show that $\psi \circ \phi$ is also a Lie algebra homomorphism.

(b) Show that if $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a bijective Lie algebra homomorphism, then $\phi^{-1}\colon \mathfrak{g}_2 \to \mathfrak{g}_1$ is also a Lie algebra homomorphism. In this case, ϕ is called an isomorphism of Lie algebras. This suggests the following definition.

Two Lie algebras are **isomorphic** if there exists a bijective Lie algebra homomorphism between them.

(c) Let L_1 and L_2 be two abelian Lie algebras. Show that L_1 and L_2 are isomorphic if and only if they have the same dimension.

Exercise 3. Suppose $f\colon G_1 \to G_2$ is a homomorphism of matrix groups.

(a) Show that if γ_1 is a smooth curve in G_1 with initial position I_{G_1}, then γ_2 is a smooth curve in G_2 with initial position I_{G_2}.

(b) Conclude from the previous part that $D_{I_{G_1}} f$ is map from $\mathfrak{g}_1$ to $\mathfrak{g}_2$.
Hint: What is the relation between $\gamma_1'(0)$ and $\gamma_2'(0)$?

3 Lie Algebras of Classical Groups

Exercise 4. Recall that

$$U(n) = \{A \in M_n(\mathbb{C}) \mid AA^* = I\}.$$

(a) Suppose $X \in \mathfrak{u}(n)$, so that $X = \gamma'(0)$ for some smooth curve γ in $U(n)$ with initial position I. Show that $X + X^* = 0$, that is, X is a skew-Hermitian matrix.

Hint: Differentiate $\gamma(t)\gamma(t)^* = I$.

(b) Suppose X is skew-Hermitian matrix. Prove that there exists a curve γ in $U(n)$ with initial position I whose initial velocity is X.

Hint: Use the exponential map.