Quaternions

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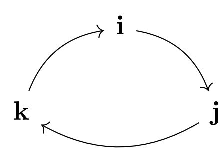
1 Introduction

In 1843 Hamilton discovered a system of arithmetic for quadruples called the **quaternions**. A quaternion is a vector in the four-dimensional vector space $\mathbb{H}$ over real numbers with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The vector space $\mathbb{H}$ is equipped with a noncommutative vector multiplication given by

$$\begin{aligned} 1\mathbf{i} = \mathbf{i}1 = \mathbf{i}, \quad 1\mathbf{j} = \mathbf{j}1 = \mathbf{j}, \quad 1\mathbf{k} = \mathbf{k}1 = \mathbf{k}, \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \end{aligned}$$

and extending by linearity. The multiplication rules can be remembered easily by the following diagram.



Note that since $\mathbb{C}$ is spanned by $\{1, \mathbf{i}\}$ with the same multiplication rules, we can view $\mathbb{C}$ as a subset of $\mathbb{H}$.

We shall frequently identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{H}$ with $\mathbb{R}^4$ below.

Warning: Due to the non-commutativity of $\mathbf{H}$, to ensure that the matrix multiplication map T defined by $v \mapsto Av$ is linear for $v \in \mathbf{H}^n$ we should do the scalar multiplication from *right* so that

$$T(vc) = A(vc) = (Av)c = (T(v))c.$$

But if we were scaling from left, then in general the T would not be linear because $Ac \neq cA$ and

$$T(cv) = A(cv) \neq c(Av) = c(T(v)).$$

From now on we will assume that all the vector spaces involved are right vector spaces.

2 Properties

(a) For $q = a+\mathbf{i}b+\mathbf{j}c+\mathbf{k}d$ in $\mathbb{H}$, we define its **conjugate** to be $\bar{q} = a-\mathbf{i}b-\mathbf{j}c-\mathbf{k}d$. It is easy to check that

$$q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2.$$

The quantity $q\bar{q}$ is usually denoted by $|q|^2$. Therefore, if $q \neq 0$, then $\bar{q}/|q|^2$ gives the standard representation of q^{-1}.

(b) A quaternion is called **pure imaginary** if its real part is zero, that is, if it is of the form $\mathbf{i}b+\mathbf{j}c+\mathbf{k}d$. We can view any such pure imaginary quaternion as a vector in $\mathbb{R}^3$ using the identification

$$\mathbf{i}b + \mathbf{j}c + \mathbf{k}d \leftrightarrow [b\ c\ d] \in \mathbb{R}^3.$$

Then a straightforward calculation shows that for pure imaginary quaternions p_1 and p_2,

$$\underbrace{p_1p_2}_{\text{prod. in } \mathbb{H}} = \underbrace{-p_1 \cdot p_2}_{\text{dot prod. in } \mathbb{R}^3} + \underbrace{p_1 \times p_2}_{\text{cross prod. in } \mathbb{R}^3} \tag{1}$$

and thus we conclude that

- p_1p_2 is real if and only if p_1 and p_2 are collinear in $\mathbb{R}^3$.
- p_1p_2 is pure imaginary if and only if p_1 and p_2 are orthogonal in $\mathbb{R}^3$.

Exercise 1.

(i) Verify Equation (1).

(ii) Conclude that if p is a pure imaginary quaternion of length one, then $p^2 = -1$, thus the second degree equation $x^2 = -1$ has infinitely many solutions over $\mathbb{H}$.

(c) For any two quaternions q_1 and q_2 we have $\overline{q_1q_2} = \overline{q_2}\ \overline{q_1}$. (Note the reversal of order.) You are invited to check this property of conjugation by a direct calculation. Below we shall learn a second method of its verification using matrices.

3 Quaternions as Complex Matrices

Recall that any complex number $a + \mathbf{i}b$ has a matrix representation

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

We would like to find an analogous representation for any quaternion $q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ such that the quaternionic multiplication corresponds to the matrix multiplication. Note that q can be written in the form $q = a + \mathbf{i}b + \mathbf{j}(c - \mathbf{i}d)$. We claim that the matrix

$$M_q = \begin{bmatrix} a + \mathbf{i}b & -c - \mathbf{i}d \\ c - \mathbf{i}d & a - \mathbf{i}b \end{bmatrix}$$

gives a representation of q. In other words, if $q = z + \mathbf{j}w$ for $z, w \in \mathbb{C}$, then

$$q \leftrightarrow M_q = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}.$$

Note that $\det \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} = |q|^2$.

Exercise 2. Check that for a complex number z, we have $\mathbf{j}z\mathbf{j}^{-1} = \bar{z}$.

Exercise 3. Consider $q_1 = 2 + \mathbf{i}$ and $q_2 = \mathbf{j} + \mathbf{k}$. Compute $q_1 q_2$ in two ways:

(a) By a direct calculation using the definition of $\mathbb{H}$.

(b) By representing q_1 and q_2 as matrices.

(c) Verify that your answers in (a) and (b) are the same.

Exercise 4. Let $q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$.

(a) Find the matrix representation of $\bar{q}$.

(b) Verify that $M_{\overline{q}} = \overline{M_q}^T$ where $\overline{M_q}$ is the matrix obtained from M_q by taking the (complex) conjugate of each (complex) entry of M and T denotes the transpose.

(c) Use the previous part to give a matrix-proof of the fact that $\overline{q_1 q_2} = \overline{q_2}\ \overline{q_1}$. (*Hint:* $(AB)^T = B^T A^T$ and $\overline{AB} = \overline{A}\ \overline{B}$ for complex matrices A and B.)

4 Group Structure on S^3

We saw earlier that the set of complex numbers of absolute value 1 denoted by

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \{(x_1, x_2) \mid \|(x_1, x_2)\| = 1\} \subset \mathbb{R}^2$$

carries two structures:

(1) (algebraic) S^1 is a commutative group under ordinary multiplication in $\mathbb{C}$, and

(2) (geometric) S^1 can be viewed as a circle (i.e., a sphere of dimension 1, and hence the notation S^1.)

It turns out that among all spheres S^n, $n \geq 1$, defined by

$$S^n = \{(x_1, \ldots, x_{n+1}) \mid \|(x_1, \ldots, x_{n+1})\| = 1\} \subset \mathbb{R}^{n+1}$$

only S^1 and S^3 can be equipped with a (continuous) group operation.

Exercise 5. Identify the 3-sphere S^3 with the set of unit quaternions. Then show that $S^3 \subset \mathbb{H}$ forms a group under the quaternionic multiplication. That is, verify that

(a) If q_1 and q_2 are in S^3, then $q_1 q_2$ is also in S^3. (*Hint:* This is an application of Exercise 4.)

(b) $q \in S^3$ implies that $q^{-1} \in S^3$.

One can give an alternative proof of (a) by observing that $|q_1 q_2| = |q_1||q_2|$ for any two quaternions q_1 and q_2. Explain why this multiplicative property of the absolute values is true.

5 Quaternions and Space Rotations

We saw earlier that the members of S^1 can be thought of as rotations of $\mathbb{R}^2$. In this section we want to associate to any unit quaternion a rotation of $\mathbb{R}^3$. The situation is not quite as easy as before however, because if we simply multiply a unit quaternion in S^3 by a vector (or a pure imaginary quaternion) in $\mathbb{R}^3$ we won't necessarily end up with a vector in $\mathbb{R}^3$. To find a solution, let us make a short digression to take a closer look at S^1.

Let v and w be any two complex numbers and consider their images under multiplication by $u \in S^1$. Then

$$\begin{aligned}
\text{distance from } uv \text{ to } uw &= |uv - uw| \\
&= |u(v-w)| && \text{(by distributive law)} \\
&= |u||v-w| && \text{(by multiplicative property of } |\cdot|\text{)} \\
&= |v-w| && \text{(because } |u| = 1\text{)} \\
&= \text{distance from } v \text{ to } w
\end{aligned}$$

In other words, the linear map $m_u \colon \mathbb{R}^2 \to \mathbb{R}^2$ is distance preserving or an **isometry**.

Exercise 6. Prove that any isometry (a linear, distance preserving map) $T\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves the dot product. That is

$$T(u) \cdot T(v) = u \cdot v \text{ for } u, v \in \mathbb{R}^n.$$

In particular conclude that $u \perp v$ if and only if $T(u) \perp T(v)$. (*Hint:* Think about expressing the dot product and norm of vectors in $\mathbb{R}^n$.)

Exercise 7. Let q be a unit quaternion. Show that q can be written in the form

$$q = \cos\theta + u \sin\theta.$$

for some angle θ and a unit pure imaginary quaternion u. (*Hint:* $\cos^2\theta + \sin^2\theta = 1$.)

Now we are ready to state the main result of this section about space rotations.

Theorem. Let q be a unit quaternion in S^3 and express it as $q = \cos\theta + u\sin\theta$ where u is a pure imaginary quaternion in $\mathbb{R}^3$. Then *conjugation by* q on $\mathbb{R}^3$ is rotation through 2θ about u.