MAT472–Symplectic Matrices

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Exercise 1. Let A and B be $m \times k$ and $k \times n$ **complex** matrices, respectively.

(a) Prove that $(AB)^T = B^T A^T$. (*Hint:* Recall that, by definition, for any matrix C, we have $[C^T]_{ij} = C_{ji}$.)

(b) Verify that $\overline{AB} = \overline{A}\ \overline{B}$.

(c) If A and B have the same size, so that $A + B$ makes sense, then $(A+B)^T = A^T + B^T$ and $\overline{A+B} = \overline{A} + \overline{B}$.

(d) If A is a square matrix with complex entries, then $\det(\overline{A}) = \overline{\det(A)}$.

(e) If A is two-by-two complex matrix, show that $\det(A^T) = \det(A)$. Read the proof of $\det(A) = \det(A^T)$ for $n \times n$ matrices in a linear algebra book, for instance, *Introduction to Linear Algebra* by Strang.

Exercise 2. Show, by giving an example, that in contrast to the case for complex matrices, when A and B are quaternion matrices, in general

$$(AB)^T \neq B^T A^T, \quad \overline{AB} \neq \overline{A}\,\overline{B}.$$

Prove that for quaternion matrices A and B (of the same size)

$$\overline{AB}^T = \overline{B}^T \overline{A}^T.$$

Note: Recall that we have the following relation between the quaternion conjugation of a quaternion q and complex conjugation of its complex Cayley matrix representation M_q:

$$\begin{aligned} q \in \mathbb{H} &\leftrightarrow M_q \in M_2(\mathbb{C}) \\ \overline{q} &\leftrightarrow M_q^* \end{aligned}$$

In other words, quaternion conjugation is equivalent to Hermitian transpose of the complex matrix. This is immediate from the fact that

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \leftrightarrow \overline{q} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$$

and

$$q = (a + \mathbf{i}b) + \mathbf{j}(c - \mathbf{i}d) \leftrightarrow M_q = \begin{bmatrix} a + \mathbf{i}b & -\overline{(c - \mathbf{i}d)} \\ c - \mathbf{i}d & \overline{a + \mathbf{i}b} \end{bmatrix}.$$

This observation about $\overline{q} \leftrightarrow M_q^*$ clearly implies that $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$. Thus we have a proof of $\overline{AB}^T = \overline{B}^T \overline{A}^T$ for 1×1 quaternion matrices! This can be easily extended to the general case by block multiplication of complex Cayley matrices.

Exercise 3. Recall that we defined an "inner product" on $\mathbb{H}^n$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \overline{x_1} y_1 + \cdots + \overline{x_n} y_n = \mathbf{x}^* \mathbf{y}.$$

Use this to show that $Sp(n) = \{A \in M_n(\mathbb{H}) \mid A^* A = I\}$ is the group of all matrices A that leave the quaternionic inner product invariant. That is, $A \in Sp(n)$ if and only if

$$\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle.$$

Exercise 4. Let A be an $n \times n$ matrix with quaternion entries. Define a $2n \times 2n$ matrix M_A with complex entries obtained from A by replacing each entry q of A with the 2×2 complex block M_q.

(a) Show that $M_{A^*} = (M_A)^*$.

(b) Use the previous part to show that if $AA^* = I$, then $M_A M_{A^*} = I$.

Educational Note: In Tapp's book M_A is denoted by $\psi_n(A)$ and some the properties of the map ψ_n, such as its multiplicative property, are explored. This exercise shows that $\psi_n(Sp(n)) = U(2n) \cap \psi_n(GL_n(\mathbb{H}))$. Since ψ_n is injective, by the first isomorphism theorem we get

$$Sp(n) \simeq U(2n) \cap \psi_n(GL_n(\mathbb{H})).$$

Exercise 5. (Stillwell, Page 59) There is an algebraic way of distinguishing matrices C that lie in $\psi_n(M_n(\mathbb{H}))$. In fact a complex matrix $C \in \psi_n(M_n(\mathbb{H}))$ if and only if

$$J_{2n} C J_{2n}^{-1} = \overline{C}$$

where J_{2n} is a block diagonal matrix with n identical blocks

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix J that provides the "complex structure" has the property that $J^2 = -I$.

(a) Suppose that $X \in M_2(\mathbb{C})$. Verify that $X \in \psi_1(M_1(\mathbb{H}))$ if and only if

$$JXJ^{-1} = \overline{X}.$$

Extend this statement to matrices $X \in \psi_n(M_n(\mathbb{H}))$ and conclude that for any such matrix, $\det X \in \mathbb{R}$.

(b) Suppose $A \in Sp(n)$. Conclude from the previous part that $\det \psi_n(A) \in \mathbb{R}$. Combine this with the result of the previous exercise, namely, $\psi_n(A)\psi_n(A)^* = I$, to show that $\det \psi_n(A) = \pm 1$.

(c) Show that if $X \in \psi_n(Sp(n))$, then $X^T J_{2n} X = J_{2n}$. (Warning: This condition involves X^T, not X^*.) *Hint:* Combine unitarity of X, namely $X^* X = I$, with part (a).

Educational Note: It is a remarkable fact that for any $A \in Sp(n)$, $\det \psi_n(A) = 1$ refining part (b) of this exercise. We shall go back to this problem after learning about "path-connectedness" of topological spaces. Part (c) is equivalent to saying that $X \in \psi_n(Sp(n))$ preserves the **symplectic bilinear form** $\omega\colon \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ in the sense that $\omega(X\mathbf{x}, X\mathbf{y}) = \omega(\mathbf{x}, \mathbf{y})$ where

$$\omega(\mathbf{x}, \mathbf{y}) = (x_1 y_1 - x_2 y_2) + \cdots + (x_{2n-1} y_{2n-1} - x_{2n} y_{2n}).$$

Indeed, it it customary to define the **complex symplectic group** $Sp(n, \mathbb{C})$ to be the group of complex matrices that leave the symplectic form invariant:

$$Sp(n, \mathbb{C}) = \{A \in M_{2n}(\mathbb{C}) \mid A^T J_{2n} A = J_{2n}\}.$$

Therefore, by part (a) and also the note following the previous problem,

$$Sp(n) \simeq U(2n) \cap Sp(n, \mathbb{C}).$$