Complex Numbers as Real Matrices

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In class we found the standard matrix of rotation through the angle x in the counter clockwise direction about the origin:

$$\mathrm{Rot}_x = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

The goal of this note is to show you how much mileage we can get from this little calculation.

1 Trigonometry

In trigonometry classes students are usually given some formulas for computing $\cos(x+y)$ and $\sin(x+y)$ in terms of trigonometric values of the angles x and y. Now we will learn how we can use *linear* algebra to justify those formulas about the *nonlinear* trigonometric functions.

We make the following simple observation. A rotation through x followed by a rotation though y is a rotation through $x+y$. For instance, a rotation through $10°$ followed by a rotation though $5°$ is a rotation through $15°$. Thus *geometry* tells us that we must have the *algebraic* relation

$$\mathrm{Rot}_{x+y} = \mathrm{Rot}_x \mathrm{Rot}_y \tag{1}$$

for rotation matrices.

Exercise 1. Do the matrix multiplication on the left side of the equation (**??**) and compare the entries of the resulting matrix with the right side.

(a) Conclude that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Isn't that amazing?

(b) Find a similar identity for $\sin(x+y)$.

2 Complex Numbers

In high school, students are often told that i is a number that solves the equation $x^2 + 1 = 0$, i.e., $i^2 = -1$. This explanation is not really satisfactory, and raises more questions than it answers. Why should such a number 'exist'? Can we just 'invent' numbers this way?

Here you will be given a hint about a possible 'construction' of the imaginary number i, dispelling some of the mystery around i.

Let us decompose the rotation matrix Rot_x as follows:

$$\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = \cos x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin x \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

This should be reminiscent of the familiar Euler formula

$$e^{ix} = \cos x + i \sin x.$$

But what is the imaginary number i in this setting? Remember that i has the property that $i^2 = -1$; therefore, since the identity matrix is the multiplicative unit, we denote it by $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and then we look for a matrix *matrix* $\mathbf{i}$ such that

$$\begin{aligned} \mathbf{i}^2 &= -\mathbf{1} \\ &= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{3}$$

Exercise 2. Let us take the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which appeared in equation (**??**) as a candidate for the matrix $\mathbf{i}$. Check the equation (**??**) holds, namely, that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Does the matrix equation $\mathbf{x}^2 = -\mathbf{1}$ have any solution other than $\mathbf{x} = \mathbf{i}$?

The moral of the above exercise is that any complex number $a + ib$ has a matrix representation

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Exercise 3. Let $a + ib$ be a nonzero complex number.

(a) Write $\frac{1}{a+ib}$ in the form $c+id$ by multiplying the numerator and denominator of $\frac{1}{a+ib}$ by $a - ib$.

(b) Since we can think of $a + ib$ as the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ we should be able to find $\frac{1}{a+ib}$ as in the previous part by computing $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1}$. Check that this is indeed the case.

Now let us say a few words about the absolute value (modulus) of complex numbers.

Exercise 4. This exercise gives a proof of the multiplicative property of the absolute value of complex numbers using linear algebra.

(a) Check that $\det \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2 + b^2$ and conclude that

$$|a + ib|^2 = \det \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

(b) Let $z = a + ib$ and $w = c + id$. Use the previous part to show that

$$|zw| = |z||w|.$$

Hint: Write the above equality as $|zw|^2 = |z|^2|w|^2$ and use the multiplicative property of the determinant function for square matrices Z and W:

$$\det ZW = \det Z \det W.$$