Matrix Lie Groups–Homework*1

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Please provide the details of your work for each problem. All problems are partial credit.

1. (*4 points*) Suppose $\vec{a}, \vec{b}$, and $\vec{c}$ are three vectors in $\mathbb{R}^3$. In this exercise you will review a few properties of the triple product $\vec{a} \times (\vec{b} \times \vec{c})$. In particular you will see that

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.} \tag{1}$$

(a) Verify (1) for a few special cases like $\vec{a} = \vec{i}, \vec{b} = \vec{j}, \vec{c} = \vec{k}$, and $\vec{a} = \vec{i}, \vec{b} = \vec{i}, \vec{c} = \vec{j}$.

(b) Explain in one short paragraph why it is reasonable to expect $\vec{a} \times (\vec{b} \times \vec{c})$ to be a linear combination of $\vec{b}$ and $\vec{c}$. In other words, explain why $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector that lies in the plane through $\vec{b}$ and $\vec{c}$. (*Hint*: $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.)

(c) Prove that Equation (1) holds for all $\vec{a}, \vec{b}, \vec{c}$. (*Hint:* A short solution is possible, if you observe that both sides of (1) are "linear" in $\vec{a}$, $\vec{b}$, and $\vec{c}$.)

(d) Is the triple product associative? Either prove $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, or show that it is false by giving an example.

(e) Verify that the following "cyclic" property holds for the triple cross product:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c} + \vec{b} \times (\vec{a} \times \vec{c}),$$

or equivalently,

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}.}$$

Educational Note: The boxed formula above is called the **Jacobi identity**; we will revisit this when we study **Lie algebras**.

*Submit on Monday, Jan 30 in class.

2. (*1 point*) (Exercise 1.5, Tapp) Suppose we define the determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_n(\mathbb{H})$ by the usual formula $ad - bc$. Show that for $A = \begin{bmatrix} \mathbf{i} & \mathbf{i} \\ \mathbf{j} & \mathbf{j} \end{bmatrix} \in M_n(\mathbb{H})$, we have $\det A \neq 0$ but nevertheless the linear map $f_A \colon \mathbb{H}^2 \to \mathbb{H}^2$ defined by

$$f_A(v) = Av$$

is not invertible. Is f_{A^T}, defined similarly by $f_{A^T}(v) = A^T v$, invertible?

3. (*2 points*) (Exercise 1.18, Tapp) If $c \in \mathbb{H}$ commutes with every element of $\mathbb{H}$, show that $c \in \mathbb{R}$.

4. (*3 points*) In the 1840s, Hamilton spent much effort trying to find a 3-dimensional field of "hypercomplex" numbers, i.e., of symbols of the form $x + \mathbf{i}y + \mathbf{j}z$ with $x, y, z \in \mathbb{R}$ which can be added, subtracted, multiplied, and divided in the same way that complex numbers can. Show that his quest was hopeless: no matter how we define $\mathbf{i}^2$ and $\mathbf{j}^2$, we will not obtain a 3-dimensional system of the desired sort. (*Hint:* Use linear algebra. Let V denote the proposed system. Fix a nonreal element $c \in V$ and let m_c denote the operation of multiplication by c.

(a) Show that $m_c \colon V \to V$ is a linear transformation over the field $\mathbb{R}$.

(b) Show that m_c must have a **real** eigenvalue. (What is the degree of the characteristic equation of m_c?)

(c) Deduce that one can find two nonzero elements of V whose product is zero, which contradicts the desired existence of division in V.)

The statement of this classical problem is borrowed from Roe, *Winding Around*.

The following **Practice Problems** will not be collected/graded.

5. (*0 points*) Show that the set of positive real numbers $\mathbb{R}^{>0}$ is a subgroup of $(\mathbb{R} - \{0\}, \times)$. Show that $\mathbb{R}^{>0}$ is isomorphic to the group $(\mathbb{R}, +)$.

6. (*0 points*) You are given a group G and a subset H of G. In each case, determine whether H is a subgroup of G.

(a) $G = \mathbb{Z}$, $H = \{$ odd integers $\}$

(b) $G = \mathbb{Z}$, $H = \{$ multiples of 3 $\}$

(c) $G = GL_n(\mathbb{R})$, $H = \{A \in GL_n(\mathbb{R}) \mid \det A$ is an integer $\}$

(d) $G = SL_n(\mathbb{R})$, $H = \{A \in SL_n(\mathbb{R}) \mid$ all entries of A are integers $\}$ *Hint:* Recall the formula for A^{-1} in terms of cofactors of A

(e) $G = GL_n(\mathbb{R})$, $H = \{A \in GL_n(\mathbb{R}) \mid \text{ all entries of } A \text{ are rational } \}$

7. (*0 points*) Given any groups G and $g \in G$, define $C_g \colon G \to G$ by $C_g(h) = ghg^{-1}$. Show that C_g is an automorphism of G. Show that the map $C \colon G \to Aut(G)$ defined by $C(g) = C_g$ is a group homomorphism and that the kernel of this map is the center of G.

8. (*0 points*) Let A and B be $m \times k$ and $k \times n$ **complex** matrices, respectively.

(a) Prove that $(AB)^T = B^T A^T$. (*Hint:* Recall that, by definition, for any matrix C, we have $[C^T]_{ij} = C_{ji}$.)

(b) Verify that $\overline{AB} = \overline{A}\ \overline{B}$.

(c) If A and B have the same size, so that $A+B$ makes sense, then $(A+B)^T = A^T + B^T$ and $\overline{A+B} = \overline{A} + \overline{B}$.

(d) If A is a square matrix with complex entries, then $\det(\overline{A}) = \overline{\det(A)}$.

(e) If A is two-by-two complex matrix, show that $\det(A^T) = \det(A)$. Read the proof of $\det(A) = \det(A^T)$ for $n \times n$ matrices in a linear algebra book, for instance, *Introduction to Linear Algebra* by Strang.

9. (*0 points*) Show, by giving an example, that in contrast to the case for complex matrices, when A and B are quaternion matrices, in general

$$(AB)^T \neq B^T A^T, \quad \overline{AB} \neq \overline{A}\ \overline{B}.$$

Prove that for quaternion matrices A and B (of the same size)

$$\overline{AB}^T = \overline{B}^T \overline{A}^T.$$

Note: Recall that we have the following relation between the quaternion conjugation of a quaternion q and complex conjugation of its complex Cayley matrix representation M_q:

$$q \in \mathbb{H} \leftrightarrow M_q \in M_2(\mathbb{C})$$
$$\overline{q} \leftrightarrow M_q^*$$

In other words, quaternion conjugation is equivalent to Hermitian transpose of the complex matrix. This is immediate from the fact that

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \leftrightarrow \overline{q} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$$

and

$$q = (a + \mathbf{i}b) + \mathbf{j}(c - \mathbf{i}d) \leftrightarrow M_q = \begin{bmatrix} a + \mathbf{i}b & -\overline{(c - \mathbf{i}d)} \\ c - \mathbf{i}d & \overline{a + \mathbf{i}b} \end{bmatrix}.$$

This observation about $\overline{q} \leftrightarrow M_q^*$ clearly implies that $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$. Thus we have a proof of $\overline{AB}^T = \overline{B}^T \overline{A}^T$ for 1×1 quaternion matrices! This can be easily extended to the general case by block multiplication of complex Cayley matrices.