MAT329–Modes of Convergence

Recall that we say that a sequence of (complex or real) *numbers* $\{a_n\}$ is **convergent** to a number L, if $|a_n - L| \to 0$ as $n \to \infty$. In analysis, one often needs to make sense of the convergence of a sequence of *functions*. For instance, let $S_N(f)$ denote the Nth partial sum of the Fourier series of f:

$$S_N(f)(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}.$$

Of course, $\{S_N(f)\}$ is a sequence of functions and one of the goals of Fourier analysis is to answer whether "$S_N(f) \to f$" as $N \to \infty$, and a step in this project is to make it precise what we mean by the convergence of a sequence of functions. In this note, you will be introduced to two notions of convergence: *pointwise*, and *uniform*. The discussions will be kept short, because these notions are discussed in any good real analysis book.

Perhaps the easiest way to make sense of "$f_n \to f$" is to define it via the convergence of a sequence of numbers by evaluating f_n and f at a fixed *point* x. This gives rise to the *pointwise convergence* of a sequence of functions.

Definition 1. We say that the sequence of functions $\{f_n\}$ is **pointwise convergent** to f, in symbols $f_n \overset{\text{pw}}{\to} f$, if for each point x, the sequence of numbers $f_n(x)$ converges to $f(x)$.

Example 1. The sequence of continuous functions $f_n \colon [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n$ is pointwise convergent to the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

because if $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.

Example 2. Consider the function f defined by

$$f(x) = \begin{cases} 5 & \text{if } x = 1/2 \\ 0 & \text{if } x \ne 1/2 \text{ and } x \in [0,1]. \end{cases}$$

Then the complex Fourier coefficients of (a periodic extension of) f, namely c_n are all equal to 0 (check this!) and hence $S_N(f)$ is the constant function 0. Thus $S_N(f)$ is not pointwise convergent to f at $x = 1/2$.

One of the most basic results in Fourier analysis is Dirichlet's Theorem about (pointwise) convergence of the sequence of partial sums $\{S_N(f)\}$. In order to state this theorem we need to learn about the notion of *piecewise continuously differentiable functions.*

Definition 2. A function f is said to be **piecewise** C^1 on $[a, b]$ if there are finitely many points $a = a_0 < a_1 < \cdots < a_k = b$ such that f is C^1 on each interval (a_j, a_{j+1}) and moreover the limits $f'(a_j^+)$ and $f'(a_j^-)$ exist for all j.

Theorem 1 (Dirichlet)**.** Suppose f is a piecewise C^1 function of period 1. Then

$$S_N(f)(x) \to \frac{f(x^+) + f(x^-)}{2} \text{ as } N \to \infty.$$

In particular, $S_N(f)(x) \to f(x)$ at any point of continuity x.

As a straightforward exercise, you can use Dirichlet's Theorem to explain the pointwise convergence of $S_N(f)(1/2)$ to 0 in Example 2.

Another approach to making sense of "$f_n \to f$" is to treat all points equally or *uniformly.*

Definition 3. We say that the sequence of functions $\{f_n\}$ is **uniformly convergent** to f, in symbols $f_n \rightrightarrows f$, if $\sup_x |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Note that it is clear from the definition that uniform convergence implies pointwise convergence. The converse, however, is not true as the following example illustrates.

Example 3. Consider the sequence of functions $f_n \colon (0, 1) \to \mathbb{R}$ defined by $f_n(x) = \frac{1}{1+nx}$. It is clear that for each point $x \in (0, 1)$, $f_n(x) \to 0$ as $n \to \infty$. Thus, $f_n \stackrel{\text{pw}}{\to} 0$. You can easily check that $\sup_{x\in(0,1)} |f_n(x)| \not\to 0$. So f_n does not converge to the zero function uniformly.

Example 4. According to one of your homework problems, if f belongs to C^2 and is of period 1, then

$$|\widehat{f}(n)| \leq \frac{M}{(2\pi n)^2}, n \neq 0$$

where M is the maximum value of $|f''(x)|$. Using this, one can show (as sketched in class by a comparison with the series $\sum 1/n^2$) that $S_N(f)$ converges uniformly (and absolutely) to a function. It takes a little more work to prove that this limit function is indeed f.

Later, as an easy application of the Cauchy-Schwarz inequality and Parseval identity, we shall see that the conclusion of Example 4 holds for C^1 functions as well. But before that we need to learn about *inner product spaces* and in particular another mode of convergence: the *mean-square convergence.*

There are many theorems governing interchange of various limit processes and we now give one such example. (This is a more detailed version of my answer to a question that Faisal asked last week.)

Theorem 2. Let $\{s_n\}$ be a sequence of integrable functions. If s_n converges uniformly to an integrable function s, then

$$\lim_{n\to\infty}\int_0^1 s_n(x)\,dx = \int_0^1 s(x)\,dx.$$

That is, in this case the limit can be transferred inside the integral.

Proof. Note that

$$\lim_{n\to\infty}\int_0^1 s_n(x)\,dx = \lim_{n\to\infty}\int_0^1 s_n(x) - s(x)\,dx + \int_0^1 s(x)\,dx.$$

Thus it suffices to show that $\lim\limits_{n\to\infty}\int_0^1 s_n(x) - s(x)\,dx = 0$. But

$$\begin{aligned}
\lim_{n\to\infty} |\int_0^1 s_n(x) - s(x)\,dx| &\le \lim_{n\to\infty}\int_0^1 |s_n(x) - s(x)|\,dx && \text{(triangle inequality)}\\
&\le \lim_{n\to\infty}\int_0^1 \underbrace{\sup_x |s_n(x) - s(x)|}_{\text{indepedent of } x}\,dx \\
&= \lim_{n\to\infty}\sup_x |s_n(x) - s(x)| = 0, && \text{(uniform convergence)}
\end{aligned}$$

and this completes the proof. □

Exercise 1. Assume that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[0, 1]$. Show that

$$\int_0^1 \sum_{n=0}^{\infty} f_n(x)\, dx = \sum_{n=0}^{\infty} \int_0^1 f_n(x)\, dx$$

provided that the integrals exist. (*Hint*: Define the nth partial sum of the series by $s_n = \sum_{k=0}^{n} f_k$ and apply the previous theorem to s_n.)

Exercise 2. Consider the sequence $\{f_n\}$ defined by $f_n(x) = x^n$ on $(0, 1)$. Show that f_n is pointwise convergent to a function f. Is the convergence uniform?