Homework 1 (due on Friday, Jan 26 in class)

Directions:

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

Notes on Writing Proofs:

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

1 Reading assignment

Read pages 26 and 27 of your textbook about the differentiation and integration of *complex-valued* functions of a real variable.

You will learn that "if we differentiate/integrate real and imaginary parts of complex-valued functions separately, and put them back together we will get familiar formulas such as $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$ even if a is a (nonzero) complex number!"

Theorem 1.3 is a neat application of what you will learn from pages 26–27. We will not need this theorem in the course, but I highly encourage you to read its statement and think about it, before taking a look at page 28!

2 Hand in solutions to the following problems

1. Recall that $F\colon \mathbb{R} \to \mathbb{R}$ is called *odd* if $F(-x) = -F(x)$ and it's called *even* if $F(-x) = F(x)$.
   (a) Check that for any $f\colon \mathbb{R} \to \mathbb{R}$, the functions g and h defined by
   $$g(x) = f(x) - f(-x) \quad \text{and} \quad h(x) = f(x) + f(-x)$$
   are odd and even, respectively.
   (b) Use the previous part to conclude that any function $f\colon \mathbb{R} \to \mathbb{R}$ can be written as a sum of an even function and an odd function.

   Educational Note: (For students with background in abstract algebra) Behind the definition of even and odd functions and the decomposition above is the group $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ acting on $\mathbb{R}$ via the map $g \cdot x = (-1)^g x$ for $g \in \mathbb{Z}/2\mathbb{Z}$. This action gives rise to a linear action of $\mathbb{Z}/2\mathbb{Z}$ on the *functions* on $\mathbb{R}$. Fourier analysis studies such actions on function spaces. You are invited to search the Internet for the phrase **group representation** to explore this idea further. See also the practice problems below.

2. In class we saw that the ODE governing a **simple harmonic motion** is $y''(t) + c^2 y(t) = 0$ for a suitable constant c. Let us assume, for simplicity, that $c = 1$, and consider the ODE

$$y''(t) + y(t) = 0. \tag{1}$$

Complete the following steps to prove that the *only* twice differentiable functions H satisfying Equation (1) (i.e., $H''(t) + H'(t) = 0$) are

$$H(t) = a\cos t + b\sin t$$

for constants a and b.

(a) Suppose $y(t)$ is a solution for Equation (1). Define

$$f(t) = y(t)\cos t - y'(t)\sin t$$
$$g(t) = y(t)\sin t + y'(t)\cos t.$$

Check that $y(t) = f(t)\cos t + g(t)\sin t$.

(b) Show that f and g are constant functions. (*Hint*: Use a differentiation.)

(c) Conclude from the previous steps that $y(t)$ must be of the form $a\cos t + b\sin t$ for some constants a and b.

3. (a) (Warm-up) Read the statement of L'Hopital's rule from a calculus textbook to refresh your memory.

(b) Let f be a function with a continuous second derivative.[1] Show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

We shall use this result in the **derivation of the wave equation**.

Hint: Notice that the limit in h is a $\frac{0}{0}$ indeterminate form. Apply L'Hopital's rule, taking into account that the variable of the limit is h, not x!

Educational Note: This problem can also be solved using a Taylor approximation with a remainder. This latter approach has the advantage of allowing generalizations to higher dimensions. Indeed, the **Laplacian** operator can be *defined* similarly in a coordinate-free manner.

4. (Problem 11, page 65) Let $f(x) = |\sin \pi x|$. We note that f is a continuous even function and has period 1.

(a) Show that $\int_0^1 \sin \pi x \, dx = 2/\pi$.

(b) Show that

$$a_n = 2\int_0^1 f(x)\cos 2\pi n x \, dx = \frac{4}{\pi(1-4n^2)}$$

(c) Deduce that the Fourier (cosine) series of f, namely $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2\pi n x$, is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{4n^2 - 1}.$$

[1]Using the notation from analysis, this means that f is in C^2.

3 Practice problems (will not be collected/marked)

1. Verify the following.
   (a) $e^{2\pi in} = 1$ for $n \in \mathbb{Z}$
   (b) $|e^{i\theta}| = 1$ for $\theta \in \mathbb{R}$
   (c) $\overline{e^{i\theta}} = e^{-i\theta}$ for $\theta \in \mathbb{R}$
   (d) $\overline{z} = z$ if and only if $z \in \mathbb{R}$
   (e) $\cos(\pi n) = (-1)^n$ and $\sin(\pi n) = 0$ for $n \in \mathbb{Z}$
2. Is the function $f(x) = \log(x + \sqrt{x^2 + 1})$ even, odd, or neither?
3. (Problem 2, page 63) Do parts (c) and (d) of Problem 2. The result of this exercise is used freely several times in the textbook.
4. In Problem 1 you proved that any function can be written as a sum of an odd function and an even one. Show that this decomposition is *unique*. That is, if $f = e_1 + o_1$ and $f = e_2 + o_2$, then $e_1 = e_2$ and $o_1 = o_2$.
5. (For students with background in abstract algebra) Show that $(m, n) \cdot (x_1, x_2) = ((-1)^m x_1, (-1)^n x_2)$ for $(m, n) \in (\mathbb{Z}/2\mathbb{Z})^2$ and $(x_1, x_2) \in \mathbb{R}^2$ defines an action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{R}^2$. Show that any real-valued (or complex-valued) function on $\mathbb{R}^2$ can be written uniquely as a sum of four functions that are even in both variables; even in x_1 and odd in x_2; odd in x_1 and even in x_2; and odd in both variables.