Homework 2 (due on Friday, Feb 2 in class)

Directions:

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

Notes on Writing Proofs:

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

1 Reading assignment

Read the short section 3.2, pages 68 and 69, from your textbook about computing Fourier coefficients (and hence the Fourier expansion) for functions of period P, where P is not necessarily 1. The formulas that you will get are of course similar to the case $P = 1$, with the difference that for general P, the formulas will involve P in the denominator in several places. If f is an integrable function of period P, then

$$c_n = \widehat{f}(n) = \frac{1}{P}\int_0^P f(x)e(-nx/P)\,dx \tag{1}$$

$$a_n = \frac{2}{P}\int_0^P f(x)\cos(2\pi nx/P)\,dx \tag{2}$$

$$b_n = \frac{2}{P}\int_0^P f(x)\sin(2\pi nx/P)\,dx. \tag{3}$$

2 Hand in solutions to the following problems

1. Let f be the function of period 1 given by $f(x) = e^{2\pi ax}$ for $0 < x < 1$, where a is a nonzero real number.
   (a) Show that the formula for complex Fourier coefficients of f is
   $$c_n = \frac{e^{2\pi a} - 1}{2\pi(a - in)}$$
   and write down the complex Fourier expansion of f.
   (b) Use your answer from the previous part about c_n to find a_n and b_n and write down the real Fourier series of f.

2. Let $f\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}$ be the function defined by $f(x) = |x|$. We extend f to an even function of period 1.
   (a) Draw the graph of this extended function.
   (b) Explain, without any calculation, why $b_n = 0$ for all n.
   (c) Show that
   $$a_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ \frac{(-1)^n - 1}{(\pi n)^2} & \text{if } n \neq 0 \end{cases}.$$
   (d) What is the Fourier series of f in terms of sines and cosines?
   (e) (Optional Bonus Problem, *+4 points*) Deduce that
   $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
   These results give examples of **an infinite sum of rational numbers which converges to an irrational number**.
   Hint: For this part, you can take it for granted that the Fourier series of f converges to f at every point and then evaluate the Fourier series at $x = 0$.

Historical Note: The problem of evaluating the sum of the infinite series
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$
goes back to the year 1690 when Jacob and Johan Bernoulli determined that the series converges to a sum less than 2 but did not succeed in finding the sum. This problem became famous as the **Basel Problem** after the Swiss town where the Bernoullis lived.

In 1735 Euler, a student of Bernoullis, solved the Basel problem with a method different than the one described above, making the sensational discovery that the sum of the series is $\frac{\pi^2}{6}$. Euler's ideas where later taken up by Riemann who introduced his famous zeta function
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$
in 1859. Of course, the Basel Problem is about finding the value of $\zeta(2)$.

3. (Problem 4, page 64) Let n be an integer. Show that if $c_{-n} = \overline{c_n}$, then a_n and b_n are *real* numbers.

   Note: This is a partial converse for a statement from class, namely that if f is a real-valued integrable function, then $c_{-n} = \overline{c_n}$ for all n.

4. (Problem 9, page 64) Recall the **Riemann–Lebesgue Lemma** for C^1 functions:

Riemann–Lebesgue Lemma. If f has period 1, and f' is continuous, then $\widehat{f}(n) \to 0$ as $n \to \pm\infty$.
To prove this lemma, first we used an integration by parts to obtain
$$\widehat{f}(n) = \frac{1}{2\pi i n} \widehat{f'}(n), \quad n \neq 0. \tag{4}$$
Next we showed that if $|f'(x)| \leq M$ for all x, then
$$|\widehat{f}(n)| \leq \frac{M}{2\pi|n|}, \quad n \neq 0.$$
Since the right side tends to zero as n gets larger and larger in magnitude, so does the left side.

(a) Use Equation (4) twice to show that if f has period 1 and f'' is continuous, then

$$\widehat{f}(n) = \frac{1}{(2\pi i n)^2}\widehat{f''}(n), \quad n \neq 0.$$

Hint: f'' is the first derivative of f'!

(b) Assume $|f''(x)| \leq M$ for all x. Show that $|\widehat{f''}(n)| \leq M$ for all n.

(c) Deduce from the previous two parts that

$$|\widehat{f}(n)| \leq \frac{M}{(2\pi n)^2}, \quad n \neq 0.$$

Educational Note: This problem is once again emphasizing that the "smoother" the function, the "faster" the Fourier coefficients of the function decay. We shall learn more about this in class. See also problem 8 on page 75. Since $\sum_{n=1}^{\infty} 1/n^2 < \infty$, this exercise shows that the Fourier series of f converges "absolutely and uniformly" (indeed to the function f).

3 Practice problems (will not be collected/marked)

1. Alice and Bob are working on a homework problem in Fourier series when Alice observes that

$$e^{2\pi i n x} = (e^{2\pi i n})^x.$$

Then Alice recalls from the practice problems in homework 1 that $e^{2\pi i n} = 1$, so she concludes $e^{2\pi i n x} = 1^x = 1$ for any real number x. However, Bob argues that by Euler's formula

$$e^{2\pi i n x} = \cos(2\pi n x) + i\sin(2\pi n x)$$

which certainly does not equal 1 for some values of x. Alice and Bob feel confused.

(a) Find a real number x for which $e^{2\pi i n x} \neq 1$.

(b) How do you explain this apparent contradiction? Help to clear up the confusion of Alice and Bob.

2. (Problem 7, page 30) This problem is intended to give you practice with **integration of complex-valued functions from Chapter 1**.

Suppose k is an integer. Recall from class that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \int_0^{2\pi} e^{ikx}\,dx = \begin{cases} 2\pi & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

(a) Show that if n is an integer, then

$$(\cos x)^{2n} = \frac{1}{2^{2n}}\sum_{k=0}^{2n}\binom{2n}{k}e^{2(k-n)ix}.$$

You might want to refresh your memory of the statement of the binomial theorem. See Appendix B of your textbook and Equation (B.8) if necessary.

(b) Conclude that

$$\int_0^{2\pi} (\cos x)^{2n}\,dx = \frac{\pi}{2^{2n-1}}\binom{2n}{n} = \frac{\pi(2n)!}{2^{2n-1}n!^2} = 2\pi\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}.$$

Educational Note: This problem is used later in the textbook in the context of the **binomial kernel** (also known as the *de la Vallée Poussin power kernel*) which provides an approximation to a periodic **delta function.**

3. (Problem 13, page 65) Do parts (a)–(d) of Problem 13