Homework 3 (due on Tuesday, Feb 13 in class)

Directions:

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

Notes on Writing Proofs:

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

1 Reading assignment

"One of the areas where Fourier series have major applications is **crystalography**. In 1985 the Nobel Prize in Chemistry was given to H. A. Hauptman and J. Karle who developed a new method for calculating some crystallographic constants from their Fourier coefficients, which can be inferred from measurements." A crucial ingredients of their analysis is **Weyl's equidistribution** theorem that we will briefly discuss toward the end of the semester as one of the applications of Fourier analysis.

In the words Bhatia, "If conduction of heat is related to the theory of numbers, and if theorems about numbers are found useful in chemistry, this story has a moral. The boundary between deep and shallow may be sharper than between pure and applied."

I invite you to do some reading around the work of Hauptman and Karle and its connections with Fourier analysis. The expository note "X-ray Crystallography and the Fourier Transform" by Tony Phillips can be a good place to start. http://www.ams.org/publicoutreach/feature-column/fc-2011-10

2 Hand in solutions to the following problems

1. Suppose that f is an integrable function of period 1. Show that if f is an odd function, then $\widehat{f}(n)$ is an odd sequence.

   Educational Note: As you might have guessed, a similar result holds for even functions. Even/odd sequences are defined in exactly the way that you would expect. For clarity, the definition along with an example is included below.

Definition. Consider a doubly-infinite sequence of numbers $\{a_n\}_{n=-\infty}^{\infty}$. We say that this sequence is *odd*, if $a_{-n} = -a_n$ for each $n \in \mathbb{Z}$.
Example. Recall that the sawtooth function[1] has Fourier coefficients $\widehat{s}(n) = \frac{1}{2\pi i n}$, $n \neq 0$, $\widehat{s}(0) = 0$. This is an odd sequence because $\widehat{s}(-n) = -\widehat{s}(n)$.

2. Let f be the function defined by $f(x) = x$ for $-\pi < x < \pi$.
   (a) Calculate the Fourier series of f.
   (b) Explain, using Dirichlet's theorem, why the Fourier series of f is pointwise convergent to f at $\frac{\pi}{2}$.
   (c) Conclude from the previous two parts that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Hint: You may find the formulas for computing the Fourier coefficients of P-periodic functions at the end of Homework 2 helpful.

Educational Note: To find the value of the above series, one can use a **Taylor expansion** and this is where Taylor meets Fourier! To see this, recall that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \quad \text{when } |x| < 1. \tag{2}$$

Since the radius of convergence of the above series is 1, a priori it is not clear that by evaluation of Equation (2) at $x = 1$ we can have

$$\frac{\pi}{4} = \arctan 1 \stackrel{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \tag{3}$$

as $x = 1$ lies on the *boundary* of the interval of convergence of the power series. Luckily, you know from calculus that the *alternating series* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is convergent, so at least the right-hand side of Equation (3) makes sense. A moment's reflection shows that equality in (3) holds if the power series in Equation (2) is left-continuous at $x = 1$. It turns out that left-continuity is implied by the fact that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is convergent. See **Abel's Theorem** (Theorem 4.30).

3. Consider the function of period 1 given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1/2 \\ 0 & \text{if } 1/2 < x < 1 \\ 1/2 & \text{if } x = 1/2 \end{cases}.$$

   (a) Explain, without any calculation, why infinitely many Fourier coefficients of f should be non-zero. (*Hint*: Suppose, for a contradiction, that the Fourier series of f is the *finite* sum $FS(f)(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$. What does Dirichlet's theorem say about the relation between f and $FS(f)$? Why is this a contradiction?)
   (b) Find the Fourier series of f.

[1] By the sawtooth function we always mean, the odd periodic extension of

$$s(x) = \frac{1}{2} - x, \quad 0 < x < 1 \tag{1}$$

of period 1. This periodic extension is sometimes denoted by $s(x) = \frac{1}{2} - \{x\}$ where $\{x\}$ is the *fractional part* of x.

(c) What is the value of the Fourier series at $x = 1/2$? Does this agree with what Dirichlet's theorem say about the point $x = 1/2$? Explain.

Educational Note: The function f above (or some translation of it) is usually referred to as the **square wave** function and it represents a switch which is on for half a second and off for half a second.

4. (a) Let s denote the sawtooth function of Problem 1 and define

$$w(x) = s(x) - s\left(x - \frac{1}{2}\right).$$

Use the properties of the $\widehat{\ }$ operation from your worksheet to compute the Fourier coefficients of w in terms of Fourier coefficients of s.

(b) What is the relation between the Fourier coefficients of w and the Fourier coefficients of the function f defined in Problem 3?

(c) Carefully plot the graph of w over the interval $(0, 1)$.

5. (2 points) (Optional Bonus Problem) Suppose f is an integrable function of period 1. Show that all the *odd* Fourier coefficients of $f(2x)$ are zero.

Challenge: After solving this problem, you might want to think about the Fourier coefficients of $f(kx)$ for any integer k.

3 Practice problems (will not be collected/marked)

1. Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Show that f is differentiable at $x = 0$ by computing the derivative. Conclude that f is differentiable everywhere.

(b) Show that f is not in C^1 although it is differentiable by the previous part.

2. Problem 6, page 64

3. Problem 10, page 64