Homework 4 (due on Friday, Feb 23 in class)

Directions:

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

Notes on Writing Proofs:

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

1 Reading assignment

Review the following definitions:

- vector space, linear independence, basis (LADE)
- dot product (Calculus III)
- geometric progression (Calculus II). Master formula (0.2) on page 2 of your textbook!

2 Hand in solutions to the following problems

1. Is $f(x) = \sqrt[3]{x}$ a piecewise C^1 function on $[0,1]$? Explain. Answer the same question for the function $g(x) = |x - 1/3| + |x - 1/2|$ on $[0,1]$.
2. (Problem 3(b), page 74) Suppose f and g are integrable functions of period 1. Show that if f is odd and g is even, then $f * g$ is an odd function.
3. Let f be an integrable periodic function of period 1, and let e_n denote the complex-valued function defined by $e_n(x) = e^{2\pi i n x}$.
   (a) Find a formula for the convolution $e_n * f$ in terms of the Fourier coefficients of f.
   (b) Let $T_N(x) = \sum_{n=-N}^{N} t_n e_n(x)$ for a sequence of complex numbers $\{t_n\}_{-N}^{N}$. Observe that T_N is continuous and has period 1. Use your answer from part (a) to find a formula for the convolution $T_N * f$ in terms of the Fourier coefficients of f.

(c) Define $D_N(x) = \sum_{n=-N}^{N} e_n(x)$. Show that

$$S_N(f)(x) = (D_N * f)(x), \tag{1}$$

where $S_N(f)(x) = \sum_{n=-N}^{N} c_n e_n(x)$ is the Nth partial sum of Fourier series of f.

Educational Note: The function T_N above is called a **trigonometric polynomial** of degree N if at least one of t_N, t_{-N} is nonzero. The set $\mathcal{T}_N$ of all trigonometric polynomials of degree at most N is a vector space of dimension $2N+1$ with

$$\mathcal{B} = \{e_n \mid n = -N, \ldots, 0, \ldots, N\}$$

being a basis. (Check this!)

The special trigonometric polynomial D_N which is of importance because of Equation (1) is called the **Dirichlet kernel**. We will take a closer look at this kernel in class in due time and I will explain what the word *kernel* means in this context; for now suffice it to say that it is unrelated to the kernel of linear transformations!

4. (a) (Warm-up) Does the series $\sum_{n=1}^{\infty} \frac{5n^2+n+1}{5n^2-3}$ converge or diverge? Explain. (You might need to refer to your Calculus II notes to refresh your memory about convergence tests for series.)

   (b) Let

$$f(x) = \begin{cases} -\cos x & \text{if } -\pi \le x < 0 \\ \cos x & \text{if } 0 < x < \pi \end{cases}$$

   and extend f to $\mathbb{R}$ as a periodic function with period 2π. Find the real Fourier series of f.

   (c) Differentiate the Fourier series from part (b) term by term.

   (d) Show that the series that you obtained in part (c) is divergent.

Educational Note: This problem shows that some care should be taken in differentiating the Fourier series of functions in general. However, as usual, things are much nicer if we are dealing with the Fourier series of "sufficiently smooth" functions. See Practice Problem 4 below.

3 Practice problems (will not be collected/marked)

1. Suppose f and g are two piecewise C^1 functions on $[0,1]$. Is $f+g$ necessarily a piecewise C^1 function? Prove your assertion. Answer the same question for the product $f \times g$.

2. Prove that $C^\infty(\mathbb{R}) = \bigcap_{k\in\mathbb{N}} C^k(\mathbb{R})$.

3. Suppose that f is a piecewise C^1 function of period 1. Show that if f is continuous at x, then

$$|f(x)| \le \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|.$$

   (The sum of the right-hand side might be $+\infty$.) Show, by giving an example, that if f is not continuous at x, then the above inequality may not hold.

   Hint: Use Dirichlet's Theorem. It turns out the "piecewise C^1" condition is redundant in this problem, but we have not developed the tools to solve this problem in that generality yet.

4. Let f be a C^2 function of period 1. Show that the Fourier series of f can be differentiated term by term, and the series so obtained converges pointwise to f'.

5. Let f, g, and h be continuous functions of period 1. Show that

$$\int_0^1 (h * g)(x) f(x)\, dx = \int_0^1 g(x)(\tilde{h} * f)(x)\, dx$$

where $\tilde{h}$ is the function defined by $\tilde{h}(x) = h(-x)$.

6. Problem 3 parts (a) and (c), page 74