Homework 5 (due on Thursday, March 8 in class)

Directions:

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

Notes on Writing Proofs:

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

1 Reading assignment

(a) (Warm-up) Is the series $\sum_{n=0}^{\infty}(-1)^n$ convergent or divergent? What is the sequence of partial sums of this series? That is, find a formula for $s_k = \sum_{n=0}^{k}(-1)^n$.

(b) Read pages 91–93 of your textbook about the notion of **Cesàro summability** of series and its connection with the **Fejér kernel**.

2 Hand in solutions to the following problems

1. Is the set
$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$
a vector space under the addition $+$ and scaling $\cdot$ given by
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} cx_1 \\ x_2 \end{bmatrix}?$$
If yes, justify your answer by checking the axioms of a vector space, otherwise explicitly name a vector space axiom which fails to hold.

2. Let V be an inner product space and let $v, w \in V$. Use the properties of the inner product on V to prove that
$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2. \tag{1}$$
Educational Note: The above identity is known as the **parallelogram law** and asserts that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the two diagonals. It is a well-known fact that although both sides of Equation (1) make sense in any normed space, the identity holds *only* if the norm is induced by an inner product.

3. Show that for a normed space V and any $v, w \in V$ the following inequality holds:
$$\|v\| - \|w\| \leq \|v-w\|.$$
Likewise show that $\|w\| - \|v\| \leq \|v-w\|$. Then conclude that
$$|\|v\| - \|w\|| \leq \|v-w\|$$
where $|\cdot|$ on the left-hand side denotes the ordinary absolute value of real numbers.

4. Let v and w be vectors in a real inner product space V and assume that $\|v\| = 5$ and $\|w\| = 3$.
   (a) What are the smallest and largest values of $\|v-w\|$?
   (b) What are the smallest and largest values of $\langle v, w\rangle$?
   Hint: You might find the result of Problem 3 helpful.

5. (2 points) (Optional Bonus Problem) Let $\mathrm{Mat}(n \times n, \mathbb{C})$ denote the vector space of $n \times n$ matrices with complex entries. Show that $\langle A, B\rangle = \mathrm{Tr}(A\overline{B^T})$ defines an inner product on $\mathrm{Mat}(n \times n, \mathbb{C})$.

   Here $\mathrm{Tr}(X) = x_{11} + x_{22} + \cdots + x_{nn}$ denotes the sum of the diagonal entries of X and B^T is the transpose of B.

3 Practice problems (will not be collected/marked)

1. Let S and T be subsets of a vector space. Prove that
$$\mathrm{Span}(S \cap T) \subseteq \mathrm{Span}\, S \cap \mathrm{Span}\, T.$$
Give an example to show that the equality need not hold.

2. Let W be a non-empty subset of a vector space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Show that W is a subspace of V if and only if W is *closed under addition and scalar multiplication*, i.e.,
   (a) If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$, and
   (b) if $w \in W$ and $c \in \mathbb{F}$, then $cw \in W$.
   Educational Note: This result is saying that other axioms are a vector space are "inherited in W", a fact that becomes quite handy when checking whether a subset W is a subspace.

3. Let V and W be (real or complex) vector spaces and let $T\colon V \to W$ be a linear map. Prove that the set $S = \{v \in V \mid Tv = 0\}$ is a subspace of V.

   Educational Note: S is usually called the **kernel** (or **null space**) of T and is denoted by $\ker T$. Note that S is the "preimage" of the zero vector, that is, $S = T^{-1}(0)$.

4. Let V be an n-dimensional real vector space and let $T\colon V \to \mathbb{R}$ be a nonzero linear map. Prove that $\ker T$ is an $(n-1)$-dimensional subspace of V.

5. Let V_1 and V_2 be subspaces of a vector space V. Show that $V_1 \cup V_2$ is never a subspace of V unless $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.