**Homework 6 (due on Tuesday, March 27 in class)**

**Directions:**

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

**Notes on Writing Proofs:**

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

# 1 Reading assignment

Read pages 1 and 2 of the Gram–Schmidt handout and complete the solutions of the review problems on pages 2 and 3.

# 2 Hand in solutions to the following problems

1. **True or False** Either prove the statement below or disprove it by a counterexample.

   " Let $W_1$ and $W_2$ be two subspaces of an inner product space $V$. If $W_1 \subseteq W_2$, then $W_2^\perp \subseteq W_1^\perp$."

2. Consider the functions $p_0(x) = 1, p_1(x) = x, p_2(x) = x^2$ in the space of continuous functions $C[-1, 1]$ equipped with the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Use the Gram–Schmidt process to find an orthonormal basis for $W = \text{Span}\{p_0, p_1, p_2\}$.

   **Educational Note:** The orthonormal set that you will find for $W$ contains the first three **Legendre polynomials**. These polynomials are used in solving **Schrödinger's equation** in three dimensions to compute the valences of atoms which explain the periodic table of elements.

3. Let $w_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and define $W = \text{Span}\{w_1, w_2\} \subseteq \mathbb{R}^3$.

   (a) Solve a system of linear equations, as we did in class, to find $W^\perp$.

   (b) Compute the cross product $v = w_1 \times w_2$.

   (c) Verify that your computation from part (a) agrees with $W^\perp = \text{Span}\{v\}$.

4. (a) (Warm-up) Check that for any two real numbers $x$ and $y$,

$$(x+y)^2 \leq 2x^2 + 2y^2.$$

(b) Define the set $V$ of "square-summable" real sequences by

$$V = \left\{ \mathbf{a} = \{a_n\} \;\middle|\; \sum_{n=1}^{\infty} |a_n|^2 < \infty, \text{ and } a_n \in \mathbb{R} \text{ for all } n \right\}.$$

Show that $V$ is closed under the ordinary component-wise addition and scalar multiplication

$$\{a_n\} + \{b_n\} := \{a_n + b_n\} \qquad \text{and } c \cdot \{a_n\} := \{ca_n\}.$$

Conclude that $V$ is a subspace of the vector space of all sequences.
*Hint*: You might find part (a) helpful.

(c) Let $\mathbf{a} = \{a_n\}$ and $\mathbf{b} = \{b_n\}$ be two sequences in $V$. Define

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{n=1}^{\infty} a_n b_n.$$

Show that the right-hand side of the above equality is an (absolutely) convergent series.
*Hint*: Explain why $\sum_{n=1}^{N} |a_n b_n| \leq \left[ \sum_{n=1}^{N} |a_n|^2 \right]^{1/2} \left[ \sum_{n=1}^{N} |b_n|^2 \right]^{1/2}$ holds. Then let $N \to \infty$ and deduce that $\sum_{n=1}^{\infty} |a_n b_n|$ is convergent.

(d) Conclude from the previous part that $\langle \cdot, \cdot \rangle$ defines an inner product on $V$.

**Educational Note:** $V$ is usually denoted by $\ell^2(\mathbb{N})$ or more simply $\ell^2$ and it is an example of a **Hilbert space**. The continuous analog of $\ell^2$, denoted by $L^2$, plays a major role in the context of the mean-square summability of Fourier series.

5. (2 points) (Optional Bonus Problem) Prove or disprove: there is an inner product on $\mathbb{R}^2$ such that its associated norm is

$$\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| = |x_1| + |x_2|$$

for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

*Hint*: You might find the Educational Note proceeding Problem 2, Homework 5 helpful.

6. (2 points) (Optional Bonus Problem, Problem 4, page 344) Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers. Show that at least one of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n a_n}$$

is divergent.

## 3 Practice problems (will not be collected/marked)

1. Let $V_1$ and $V_2$ be two finite-dimensional subspaces of a vector space $V$. Show that

$$\dim \operatorname{Span}(V_1 \cup V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

**Educational Note:** $\operatorname{Span}(V_1 \cup V_2)$ is sometimes called the **sum** of the vector spaces $V_1$ and $V_2$ and is denoted by $V_1 + V_2$.

2. (Problem 2, page 146) Show that if $f$ and $g$ are continuous functions of period 1, then

$$\int_0^1 f(x)g(x)\,dx = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\widehat{g}(-n).$$

3. Show that the equality holds in Cauchy–Schwarz inequality $|\langle u, v\rangle| \leq \|u\|\|v\|$ if and only if the vectors $u$ and $v$ are linearly dependent.