Homework 7 (due on Thursday, April 12 in class)

Directions:

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

Notes on Writing Proofs:

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

1 Reading assignment

Review the following from Calculus III to solve problems 3 and 4 below.

- partial derivative, gradient, directional derivative
- chain rule for functions of two variables
- polar coordinates

You might find the review sheet and practice problems from MAT111 on Canvas helpful.

2 Hand in solutions to the following problems

1. (Problem 1, page 145) Let f be a continuous function of period 1.

   [(a)]

   1. Show that if $n > 0$, then $|a_n|^2 + |b_n|^2 = 2|\widehat{f}(n)|^2 + 2|\widehat{f}(-n)|^2$.
   2. Using part (a) and the fact that $a_0 = 2\widehat{f}(0)$, conclude the **Parseval Identity for infinite real Fourier series**
   $$\int_0^1 |f(x)|^2\,dx = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

   Here, as usual, a_n and b_n denote the cosine and sine Fourier coefficients of f, respectively.

   Suggestion: Begin with the Parseval Identity for Fourier series.

2. Let f be the function of period 1 defined by the formula $f(x) = |x|$ for $-\frac{1}{2} \le x \le \frac{1}{2}$. In homework 2 you showed that

$$a_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ \frac{(-1)^n - 1}{(\pi n)^2} & \text{if } n \neq 0 \end{cases}.$$

Apply Parseval's Identity from Problem 1 to f to conclude that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Educational Note: A similar procedure as in Homework 2 applied to the above formula implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Finding a formula for the sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$ or other odd powers of n is a famous unresolved question!

3. (Problems 1 and 2, page 213) In general we expect that a homogeneous linear differential equation of degree 2 will have 2 linearly independent solutions. In this problem we want to show that the only solutions of the differential equation

$$r^2 g''(r) + r g'(r) - n^2 g(r) = 0 \tag{1}$$

which are twice differentiable when $r > 0$ are given by linear combinations of r^n and r^{-n} when $n \neq 0$, and 1 and $\ln r$ when $n = 0$.

Suppose that $g(r)$ satisfies (1), and define $G(r)$ so that $g(r) = r^n G(r)$.

(a) Show that $(2n+1)G'(r) + rG''(r) = 0$.

(b) Write this as $2nG'(r) + (rG'(r))' = 0$. Deduce that $2nG(r) + rG'(r) = c$ for some constant c.

(c) Deduce that G is of the form $G(r) = ar^{-2n} + b$ and hence $g(r) = ar^{-n} + br^n$ when $n > 0$.

(d) Also deduce that when $n = 0$, the only solutions of the ODE (1) are given by linear combinations of 1 and $\ln r$.

4. (Problem 3, page 213) Let $u(x, y)$ be a function with continuous second partial derivatives. Define $v(r, \theta)$ by

$$v(r, \theta) = u(r \cos\theta, r \sin\theta).$$

Use the chain rule to show the following relation:

$$u_{xx} + u_{yy} = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta}.$$

Educational Note: We will use the polar form of the Laplacian Δ in solving the steady-state **heat equation** for the disc.

5. (2 points) (Optional Bonus Problem) Find the value of

$$\min_{a,b,c \in \mathbb{C}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx.$$

Suggestion: You might find it helpful to review Problem 2 from Homework 6.

3 Practice problems (will not be collected/marked)

1. (Problem 4, page 213) Using the notation of Problem 4 show that

$$|u_x|^2 + |u_y|^2 = |v_r|^2 + \frac{1}{r^2}|v_\theta|^2.$$

2. Recall from Homework 2 that if f is a C^2 function of period 1, then $\widehat{f}(n) = \frac{\widehat{f''}(n)}{(2\pi i n)^2}$ for $n \neq 0$. We used this in class, together with the convergence of $\sum 1/n^2$, to conclude that the Fourier series of f converges (to f) absolutely (and uniformly). Now we are in a position to establish the same result for C^1 functions.

   Let f be a C^1 function of period 1 and recall that $\widehat{f}(n) = \frac{\widehat{f'}(n)}{2\pi i n}$ for $n \neq 0$.

   (a) Prove that

$$\sum_{n\neq 0} |\widehat{f}(n)| \leq \left[\sum_{n\neq 0} \frac{1}{(2\pi n)^2}\right]^{1/2} \left[\sum_{n\neq 0} |\widehat{f'}(n)|^2\right]^{1/2}.$$

   (b) Deduce that $\sum_{n\neq 0} |\widehat{f}(n)| \leq C\|f'\| < \infty$ for some constant C.

   (c) Conclude that the Fourier series of f converges absolutely.