Summary of Linear Algebra Topics

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1 Linear algebra in coordinates

(1) A **basis** for a vector space V is a linearly independent subset $S \subset V$ such that $\mathrm{Span}(S) = V$. Here $\mathrm{Span}(S)$ denotes the set of all possible (*finite*) linear combination of vectors in S.

(2) Any vector space possesses a basis. The proof of this fact for arbitrary vector spaces requires a tool from set theory called Zorn's lemma.

(3) It can be shown that any two bases of a vector space have the same cardinality (size). This common cardinality is called the **dimension** of the vector space.

(4) Let V be a finite dimensional vector space of dimension n. Fix an ordered basis $B = \{b_1, \ldots, b_n\}$ for V. Then to any $v \in V$, we can associate a column vector $[v]_B$ called the **coordinate vector of v relative to B** as follows:

$$v = c_1 b_1 + \cdots + c_n b_n \longleftrightarrow [v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

(5) Let V and B be as in the previous part. Then to any linear function $T\colon V \to V$ we can associate an $n \times n$ matrix $[T]_B$ called the **matrix of T relative to B** with the following property:

$$[T]_B [v]_B = [T(v)]_B.$$

Letting $v = b_i$, for $i = 1, \ldots, n$, this shows in particular that the matrix $[T]_B$ can be constructed column by column by applying T to the basis elements in B and then expressing the outputs as coordinate vectors relative to B:

$$[T]_B = \Big[[T(b_1)]_B \vdots \cdots \vdots [T(b_n)]_B \Big]$$

(6) Let $B_1 = \{b_1, \ldots, b_n\}$ and $B_2 = \{b'_1, \ldots, b'_n\}$ be two ordered bases for the same vector space V. Then there exists a unique matrix $P_{B_1 \to B_2}$ called the **transition matrix** from B_1 to B_2 with the property that for for any $v \in V$

$$P_{B_1 \to B_2} [v]_{B_1} = [v]_{B_2}.$$

One can construct this matrix column by column by writing basis vectors in B_1 relative to B_2:

$$P_{B_1 \to B_2} = \left[[b_1]_{B_2} \vdots \cdots \vdots [b_n]_{B_2} \right].$$

Note that $P^{-1}_{B_1 \to B_2} = P_{B_2 \to B_1}$.

Proposition 1. Let V be a vector space with bases B_1 and B_2. Then for any $v \in V$, the following diagram commutes:

$$\begin{array}{ccc} [v]_{B_1} & \xrightarrow{P_{B_1 \to B_2}} & [v]_{B_2} \\ \Big\downarrow{\scriptstyle [T]_{B_1}} & & \Big\downarrow{\scriptstyle [T]_{B_2}} \\ [T(v)]_{B_1} & \xrightarrow{P_{B_1 \to B_2}} & [T(v)]_{B_2} \end{array}$$

That is,

$$[T]_{B_2} P_{B_1 \to B_2} [v]_{B_1} = P_{B_1 \to B_2} [T]_{B_1} [v]_{B_1}$$

or equivalently

$$[T]_{B_1} = P^{-1}_{B_1 \to B_2} [T]_{B_2} P_{B_1 \to B_2}.$$

(1) Two matrices A and B are said to be **similar** if there exists a matrix P such that $A = PBP^{-1}$.

(2) Similarity is an equivalence relation on the set of all $n \times n$ matrices.

(3) The proposition above shows that two complex $n \times n$ matrices M and N are similar if and only if there exists a linear transformation $T \colon \mathbb{C}^n \to \mathbb{C}^n$ such that $M = [T]_{B_1}$ and $N = [T]_{B_2}$ for two bases B_1 and B_2 of $\mathbb{C}^n$.

2 Eigenvalues and Eigenvectors

(1) Let A be a complex $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is called an **eigenvalue** of A if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. Such a vector x is called an **eigenvector** for λ.

(2) We define the **characteristic polynomial** of a matrix A by $\chi_A(\lambda) = \det(\lambda I - A)$.

(3) χ_A is a monic polynomial of degree n and λ is an eigenvalue of A if and only if it is a root of χ_A. It follows, by the fundamental theorem of algebra, that A has n complex eigenvalues (counting multiplicities).

(4) The constant term in χ_A, namely $\chi_A(0)$, equals $(-1)^n \det A$ and the coefficient of λ^{n-1} equals $-\operatorname{tr} A$.

(5) $\det A = \lambda_1 \cdots \lambda_n$ where λ_i are eigenvalues of A counting with multiplicity.

(6) $\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$.

(7) Suppose $\chi_A(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}$ where λ_i are distinct complex numbers. Then we call n_i the **(algebraic) multiplicity** of λ_i.

(8) The **geometric multiplicity** of an eigenvalue λ_i is defined to be the dimension of its eigenspace E_{λ_i}:

$$E_{\lambda_i} = \{x \mid Ax = \lambda_i x\}.$$

(9) It can be shown that the algebraic multiplicity of any eigenvalue is larger than or equal to its geometric multiplicity.

Proposition 2. Any two similar matrices have the same characteristic polynomials, and hence the same eigenvalues, determinant, and trace.

3 Diagonalization

A square matrix A is said to be **diagonalizable** if there exist a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. It follows that

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad P = [v_1 \vdots \cdots \vdots v_n]$$

where λ_i are eigenvalues of A with eigenvectors v_i, that is, $Av_i = \lambda_i v_i$.

Proposition 3 (Necessary and Sufficient Condition)**.** An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Equivalently, a matrix A is diagonalizable if and only if the algebraic multiplicity of each eigenvalue of A equals its geometric multiplicity.

Proposition 4 (Sufficient Condition)**.** An $n \times n$ matrix is diagonalizable if it has n distinct eigenvalues.

Theorem 1 (Spectral Theorem–Real Version)**.** If A is a (real) symmetric matrix, then all eigenvalues of A are real. Moreover, A is **orthogonally diagonalizable** in the sense that there exist a diagonal matrix D and an orthogonal matrix P such that $A = PDP^{-1}$. Conversely, if a matrix is orthogonally diagonalizable, then it is symmetric.