Homework 8 (due on Tuesday, April 24 in class)

Directions:

- You are strongly encouraged to work in groups and discuss your assignments with your fellow students, but to make sure that you gain a good understanding of the material and to avoid slipping from collaborating to copying, you should write up your solutions on your own. Also make sure to give credit to **all** sources (books, websites, or people) that you consult.
- No late homework assignments will be accepted, but as compensation I will drop your lowest homework score.
- LaTeX use is encouraged (but not required) for writing up solutions for homework assignments. See the "Files" folder on Canvas for a short manual on LaTeX.
- Please leave at least one inch of blank space at the end of your solution for each problem (or each part of a problem when applicable).

Notes on Writing Proofs:

- Clearly state any assumptions you are making.
- Clearly reference any results that you are using (e.g., by Dirichlet's Theorem, or Theorem 3.2) and carefully show each step you are taking.
- Conclude your proof.

And finally read over each sentence of your proof. Does it make sense? Is each sentence complete? Are there any steps left out?

1 Reading assignment

Review the statement of Green's theorem from Calculus III.

2 Hand in solutions to the following problems

1. Let $u(x, y)$ be a function with continuous second partial derivatives. Use the chain rule to show the following relation:

$$u_{xx} + u_{yy} = u_{ss} + u_{tt}$$

where

$$s = x\cos\alpha + y\sin\alpha$$
$$t = -x\sin\alpha + y\cos\alpha$$

for some angle α.

Educational Note: This exercise shows that the Laplacian is "rotation invariant."

2. Solve the first-order PDE $3u_x + 2u_y = 0$ with the condition $u(x, 0) = \sin x$.

3. Consider the one-dimensional wave problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1/2, \quad t > 0, \\ u(0,t) &= 0, \quad u(1/2,t) = 0, \\ u(x,0) &= 2\sin(6\pi x) - 3\sin(22\pi x), \\ u_t(x,0) &= \sin(4\pi x) \end{aligned}$$

(a) Solve this PDE using d'Alembert's formula.
(b) Solve this PDE by a separation of variables (or using the Fourier series form of the solution from class).
(c) (Optional) Check that the answers from both parts are in fact equal.

4. (Adapted from Problem 3, page 132 of the textbook) For a natural number N and $0 \leq r < 1$, define

$$p(x) = \sum_{n=-N}^{N} r^{|n|} e^{2\pi i n x}.$$

(a) Write p in the form $\sum_{n=-N}^{0} + \sum_{n=1}^{N}$ to discard the absolute value sign in the definition of p. Then use the formula for the geometric progression on page 2 of your textbook to find a closed form for each summation.
(b) For $0 \leq r < 1$, the **Poisson kernel** of period 1 is defined by

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n x} = 1 + 2\sum_{n=1}^{\infty} r^n \cos 2\pi n x.$$

Use the previous part to show that

$$P_r(x) = \frac{re^{2\pi i x}}{1 - re^{2\pi i x}} + \frac{1}{1 - re^{-2\pi i x}}.$$

(c) Deduce that

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos 2\pi x + r^2}.$$

(d) (2 points) (Optional Bonus Problem) Show that $P_r(x) > 0$ for all x.

Educational Note: The Poisson kernel appears naturally in the process of solving the steady-state heat equation for the disc.

As $r \to 1^-$, the functions $P_r(x)$ form an approximate identity for the convolution operation. It is clear that $\int_0^1 P_r(x)\, dx = 1$. The Poisson kernel of period 2π which is used very frequently is defined by

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1 - r^2}{1 - 2r\cos x + r^2}.$$

Note that in this case the "average" $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)\, dx = 1$ as in the previous case.

3 Practice problems (will not be collected/marked)

1. (Problem 4, page 220) Suppose that f is a continuously differentiable function of period 1. Show that

$$\int_0^1 |f(x)|^2\, dx \leq \left| \int_0^1 f(x)\, dx \right|^2 + \frac{1}{4\pi^2}\int_0^1 |f'(x)|^2\, dx.$$

Suggestion: Modify the proof of Wirtinger's inequality.

2. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where a_n are complex numbers. Write $f(z) = u(z) + iv(z)$ where u and v are real-valued functions of the variable $z = x + iy$. Thus,

$$u(x+iy) + iv(x+iy) = \sum_{n=0}^{\infty} a_n (x+iy)^n.$$

(a) Differentiate the above series term-by-term and conclude that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

(b) Show that Equation (1) implies $\Delta u = 0$ and $\Delta v = 0$.

Educational Note: Any complex function f which can be expressed as a power series around a point p, say $p = 0$, is called **analytic** (or **holomorphic**) at p. The first part of this problem shows that analytic functions satisfy the **Cauchy–Riemann equations** (1). The second part of the problem says that the real and imaginary parts of analytic functions satisfy the Laplace equation, and hence are harmonic functions. This result has a partial converse that we shall not discuss here. The interested student who wants to study these facts in depth is encouraged to take a course in complex variables.

3. Let $h \in C([0,1])$ be a given continuous function and consider the boundary value problem

$$\begin{cases} f'' = h \\ f(0) = f(1) = 0. \end{cases}$$

Show that $f = Kh$ where K is the operator defined by

$$K(h)(s) = \int_0^1 G(s,t)h(t)\,dt$$

for the **Green function** on $[0,1] \times [0,1]$

$$G(s,t) = \begin{cases} s(t-1) & \text{for } 0 \le s \le t \le 1 \\ t(s-1) & \text{for } 0 \le t \le s \le 1. \end{cases}$$