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EXPONENTIAL CONVERGENCE FOR MULTIPOLE AND LOCAL EXPANSIONS AND THEIR TRANSLATIONS FOR SOURCES IN LAYERED MEDIA: TWO-DIMENSIONAL ACOUSTIC WAVE*

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Abstract. In this paper, we first derive the multipole expansion (ME) and local expansion (LE) for far fields from wave sources in two-dimensional (2-D) layered media as well as the multipole-to-local translation (M2L) operator, by using the generating function of Bessel functions and Sommerfeld integral representations of Hankel functions. Then, we give a rigorous proof of the exponential convergence of the ME, LE, and M2L. It is shown that the convergence of ME, LE, and M2L for the reaction field components of the 2-D Helmholtz Green's function in layered media depends on the distance between the target charge and an equivalent polarization source. The polarization sources can be used in the implementation of fast multipole methods for wave sources embedded in layered media.

Key words. fast multipole method, multipole expansions, local expansions, Helmholtz equation, layered media, Cagniard–de Hoop transform, equivalent polarization sources

AMS subject classifications. 15A15, 15A09, 15A23

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1. Introduction.

The multipole expansion (ME), local expansion (LE), and multipole-to-local translation (M2L) form the mathematical structure of fast multipole methods (FMMs) for evaluating integral operators associated with the Green's function of Helmholtz equations in wave scattering [11, 3]. The ME for the Green's functions in the free space was based on the Graf's addition theorems for Bessel functions. To extend the FMM for wave scattering in layered media, ME and M2L formulas for Helmholtz equations in a two-dimensional (2-D) half-space domain were proposed in [6]. The derivation in [6] for the ME and M2L made use of an image (point and line images) representation of the Green's function of the half-space domain with an impedance boundary and the MEs, based on the Graf's addition theorem, for the image charges as well as the original source charges. And, it was shown that the ME coefficients used to compress the far field of the source charges in the free space can also be used to compress the far field of the images, thus producing a ME for the Green's functions of the 2-D half-space domain. For the case of the half space with an impedance boundary condition, the image representation of the domain Green's function justifies the truncation order, and thus the exponential convergence, of ME and M2L. Meanwhile, a 2-D heterogeneous FMM was proposed and implemented in [5], [6], giving an $O(N)$ complexity of evaluating the integral operator of low frequency Helmholtz operators for sources in the half space.

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As an image representation of general layered media Green's function may not exist, in this paper, we will present an alternative complete derivation for the ME, LE, and M2L operators for the Green's function in general 2-D layered media by using the generating function of the Bessel functions of the first kind (referred as the Bessel generating function in this paper). Moreover, we will give a rigorous proof of the exponential convergence of the ME, LE, and M2L and local to local (L2L) translation operators for acoustic wave sources in general 2-D layered media. The convergence analysis reveals a very important and practical fact that the convergence of ME, LE, L2L, and M2L for the reaction field component of the Green's function in fact depends on a polarization distance, which is measured between the target and an equivalent polarization source, thus suggesting how the FMM framework should be set for sources and targets in layered media.

The rest of the paper is organized as follows. In section 2, we first give some technical tools crucial to the work in this paper, including the Bessel generating function, which relates plane waves to cylindrical waves and the growth condition of the Bessel functions. A brief discussion of the Green's functions in layered media and their integral representations is also given. Then, the Bessel generating function is used to derive the analytical formula for the ME and LE expansions for sources in 2-D layered media, the M2L and L2L translation operators. The exponential convergence rates for these expansions are validated with some numerical tests. The proofs of exponential convergence rates of the expansions are given in section 3. First, we will give the proof of the exponential convergence of some integral expansions resulting from using the Bessel generating function. The proof is given starting with a special case corresponding to the situation when the far-field location is directly above or below the center of the expansion. Then, the Cagniard–de Hoop transform [4] is introduced so that we can deal with the general case by using complex domain contour integrals. The proof for the error estimate of ME, M2L, etc., introduced in section 2 will follow. A conclusion is given in section 4, while appendices are included for some technical lemmas and proofs of several lemmas from the main text.

2. Far-field expansions and their translations for the 2-D Helmholtz equation in layered media. In this section, we begin with some properties of the Bessel functions of the first kind, which inspires an alternative derivation of the ME of the free space Green's function. These properties will be key to deriving various far-field expansions in layered media. The ME, LE, M2L, and L2L for the layered media will then be derived with error estimates and numerical validations. Also, a feasible FMM framework for sources in layered media is proposed based on the convergence results of the far-field expansions.

2.1. An identity and some estimates on Bessel functions of the first kind. Recalling the Bessel generating function [1, equation (9.1.41)], for any $z, \omega \in \mathbb{C}$ with $\omega \neq 0$,

$$g(z,\omega) = \exp\left(\frac{z}{2}(\omega - \omega^{-1})\right) = \sum_{p=-\infty}^{\infty} J_p(z)\omega^p. \tag{2.1}$$

The identity (2.1) expresses a plane wave function in terms of cylindrical functions, in contrast to the Sommerfeld integral representation of the Green's function, which expresses cylindrical functions in terms of plane waves (2.12). This duality facilitates the derivation of the far-field expansions in this paper.

The above series converges absolutely, which is a corollary of the following lemma. For the rest of this paper, we use the notations $\Re$ and $\Im$ for the real and the imaginary parts of a complex number, respectively.

LEMMA 2.1 (an estimate on Bessel functions of the first kind). *Let p be an integer and $z \geq 0$. With the convention $0^0 = 1$, the following inequality holds:*

$$|J_p(z)| \leq \frac{1}{|p|!}\left(\frac{z}{2}\right)^{|p|}.$$

Proof. When $p \geq 0 > -\frac{1}{2}$, the inequality is given by [1, equation (9.1.62)]. Then, the identity $J_p(z) = (-1)^p J_{-p}(z)$ covers the case $p < 0$. ◻

2.2. The Green's function for 2-D Helmholtz equation in layered media. Consider a horizontally layered medium with L interfaces located at $y = d_l$, $0 \leq l \leq L-1$, arranged from top to bottom as l increases. Each interface $y = d_l$ separates layer l above layer $l+1$, and each layer l is homogeneous with a wave number $k_l > 0$, $0 \leq l \leq L$.

We assume s labels the layer where the source $\mathbf{x}' = (x', y')$ locates, and t labels the layer where the target $\mathbf{x} = (x, y)$ locates, $0 \leq s, t \leq L$.

The layered Green's function $G(\mathbf{x}, \mathbf{x}')$ for the Helmholtz equation is a piecewise smooth function for a source $\mathbf{x}'$ and a target $\mathbf{x}$ from possibly different layers. Within each layer,

$$\Delta G(\mathbf{x}, \mathbf{x}') + k_t^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x}, \mathbf{x}'), \tag{2.2}$$

with two interface conditions at $y = d_l$ of the form

$$[a_t G] = 0, \quad \left[b_t \frac{\partial G}{\partial \mathbf{n}}\right] = 0, \tag{2.3}$$

where the bracket $[\cdot]$ refers to the jump of the quantity inside at the interface, and a_t and b_t are some complex numbers (depending on the layer number t). In typical acoustic wave equations, the parameters can often be reduced such that $a_t = 1$ and b_t are constants depending on each layer media, e.g., the density [13].

Note that the right-hand side of (2.2) is nontrivial only when $\mathbf{x}$ and $\mathbf{x}'$ are in the same layer, i.e., $s = t$. Defining

$$u^{\mathrm{r}}(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}, \mathbf{x}') - \delta_{t,s} G_s^{\mathrm{f}}(\mathbf{x}, \mathbf{x}'), \tag{2.4}$$

here $\delta_{t,s}$ is the Kronecker delta function, and

$$G_s^{\mathrm{f}}(\mathbf{x}, \mathbf{x}') = \frac{\mathrm{i}}{4} H_0^{(1)}\left(k_s |\mathbf{x} - \mathbf{x}'|\right) \tag{2.5}$$

is the free-space Green's function with wave number k_s. u^{r} is called the reaction field using the terminology of electrostatics [2] and satisfies a homogeneous Helmholtz equation with a wave number k_t.

A decomposition of the reaction field u^{r} is given in terms of upward and downward wave propagation components, indicated by the up and down arrow symbols [14]. Suppose the Helmholtz equation in layered media with the interface conditions is well-posed. The reaction field u^{r} can be written in the following summation:

$$\begin{aligned} u^{\mathrm{r}}(\mathbf{x}, \mathbf{x}') &= u_{ts}^{\uparrow\uparrow}\left(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{\uparrow\uparrow}\right) + u_{ts}^{\uparrow\downarrow}\left(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{\uparrow\downarrow}\right) \\ &\quad + u_{ts}^{\downarrow\uparrow}\left(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{\downarrow\uparrow}\right) + u_{ts}^{\downarrow\downarrow}\left(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{\downarrow\downarrow}\right) = \sum_{*\star} u_{ts}^{*\star}\left(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{*\star}\right), \end{aligned} \tag{2.6}$$

where $*, \star \in \{\uparrow, \downarrow\}$ refer to the vertical field propagation directions corresponding to the target and the source, respectively, and each $u_{ts}^{*\star}$ has an integral representation

$$u_{ts}^{*\star}(\mathbf{x},\mathbf{x}';\sigma_{ts}^{*\star}) = \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{*\star}(\mathbf{x},\mathbf{x}',\lambda)\sigma_{ts}^{*\star}(\lambda)d\lambda, \tag{2.7}$$

where the integrand has an exponential factor

$$\mathcal{E}_{ts}^{*\star}(\mathbf{x},\mathbf{x}',\lambda) = e^{-\sqrt{\lambda^2-k_t^2}\tau^*(y-d_t^*)-\sqrt{\lambda^2-k_s^2}\tau^\star(y'-d_s^\star)+\mathrm{i}\lambda(x-x')}, \tag{2.8}$$

and $\sigma_{ts}^{*\star}(\lambda)$ is a coefficient term which does not depend on the coordinates of $\mathbf{x}$ and $\mathbf{x}'$.

In (2.8), we adopt the convention $d_l^\uparrow = d_l$ for $l \neq L$ and $d_l^\downarrow = d_{l-1}$ for $l \neq 0$, $\tau^\uparrow = 1$, and $\tau^\downarrow = -1$. In addition, $d_{-1} = \infty$ and $d_L = -\infty$. These conventions together guarantee $\tau^*(y-d_t^*) > 0$ and $\tau^\star(y'-d_s^\star) > 0$. When d_{-1} or d_L occurs in a component in (2.8), it refers to an incoming wave from $y = \pm\infty$, which is prohibited by the Sommerfeld radiation conditions, and the component itself should vanish. For example, if both $\mathbf{x}$ and $\mathbf{x}'$ are in the top layer, then (2.6) becomes $u^{\mathrm{r}} = u_{00}^{\uparrow\uparrow}$ only.

Appendix B.1 contains the derivation of the decomposition (2.6).

Remark 2.2. The specific form of the exponential term $\mathcal{E}_{ts}^{*\star}(\mathbf{x},\mathbf{x}',\lambda)$ is introduced to ensure that each coefficient term $\sigma_{ts}^{*\star}(\lambda)$ has polynomial growth rate under certain conditions, to be elaborated in Appendix B.3. The polynomial growth of $\sigma_{ts}^{*\star}(\lambda)$ will be needed for the exponential convergence estimate of ME, LE, M2L, and L2L expansions. This specific form also results in a dependence of the exponential convergence on a special "polarization distance" between a source and a target in the layered media, as defined in (2.26) and depicted in Figure 1.

The integrand of (2.7) may have real poles, whose integration should be treated as the limiting case of the field in lossy physical media. To understand the real poles, we first introduce the necessary branch cut for the square roots. For any $z = re^{\mathrm{i}\theta} \in \mathbb{C}$ with $r \geq 0$, $\theta \in [-\pi,\pi)$, define

$$\sqrt{z} = \sqrt{r}e^{\mathrm{i}\frac{\theta}{2}}. \tag{2.9}$$

For each square root $\sqrt{\lambda^2 - k_l^2}$, the corresponding branch cut in the λ-plane is the union of the imaginary axis and the real interval $[-k_l, k_l]$. In a realistic physical case where the medium in layer l is lossy with a perturbed wave number $\tilde{k}_l = k_l + \epsilon_l\mathrm{i}$, $0 < \epsilon_l \ll 1$, the perturbed branch cut is then shown in Figure 2. The branch cut of $\sqrt{\lambda^2 - k_l^2}$ is the limit of the perturbed one as $\epsilon_l \to 0^+$.

Let λ_ν be a real pole of $\sigma_{ts}^{*\star}(\lambda)$ in the integrand of (2.7), which is known as a surface wave pole [9, 12]. Integration across the surface wave pole is understood as the limiting case of the perturbed system with lossy media as mentioned above. For simplicity, suppose $\sigma_{ts}^{*\star}(\lambda) = \sigma(\lambda; k_1, \ldots, k_L)$ is the limit of the perturbed field $\sigma(\lambda; \tilde{k}_1, \ldots, \tilde{k}_L)$ with pole $\tilde{\lambda}_\nu$, and $\tilde{\lambda}_\nu \to \lambda_\nu \in (a,b)$ as all the $\epsilon_l \to 0^+$. Let $\sigma_\nu = \lim_{\lambda\to\lambda_\nu} \sigma(\lambda)(\lambda-\lambda_\nu)$. Given any smooth function $h(\lambda)$, the limiting integral $\int_a^b h(\lambda)\sigma(\lambda)d\lambda$ is evaluated using the formula

$$\begin{aligned}\int_a^b h(\lambda)\sigma(\lambda;\tilde{k}_1,\ldots,\tilde{k}_L)d\lambda \to &\int_a^b \left(h(\lambda)\sigma(\lambda) - \frac{h(\lambda_\nu)\sigma_\nu}{\lambda-\lambda_\nu}\right)d\lambda \\ &+ \text{p.v.}\int_a^b \frac{h(\lambda_\nu)\sigma_\nu}{\lambda-\lambda_\nu}d\lambda \pm \mathrm{i}\pi h(\lambda_\nu)\sigma_\nu.\end{aligned} \tag{2.10}$$

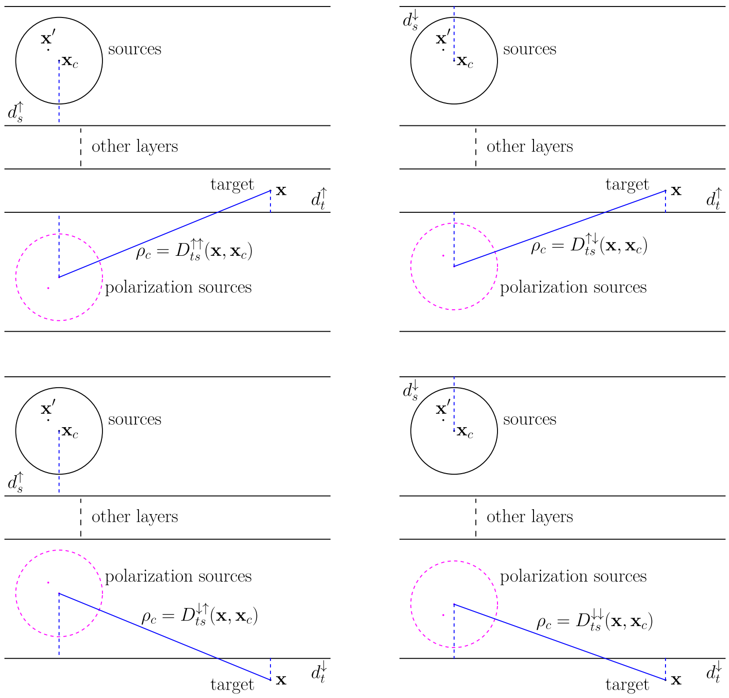

FIG. 1. *The far-field distance ρ_c of the ME in various field propagation directions.*

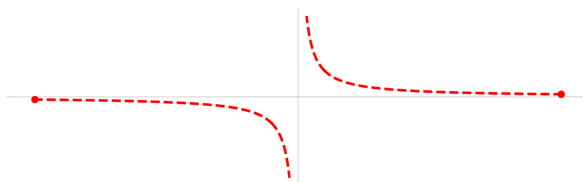

FIG. 2. *The perturbed branch cut starting from $\pm\tilde{k}_l$, where $\tilde{k}_l = k_l + \epsilon_l \mathrm{i}$.*

The $\pm$ sign is positive (or negative) when the perturbed pole $\tilde{\lambda}_\nu \to \lambda_\nu$ from the upper (or the lower) half of the complex plane, and the principal value part vanishes if $(a, b) = (-\infty, +\infty)$.

In a well-posed problem, the poles will be at most of order one, and $\tilde{\lambda}_\nu$ should remain in one side of the half planes as all the perturbation parameters ϵ_l are sufficiently small; otherwise, the limit of the integral does not exist and the field is not well-defined. Also, 0 cannot be a surface wave pole; otherwise the surface wave does not propagate [9, 12].

Remark 2.3. Modes of the layered system are classified as the radiation modes, the guided modes (the real poles), and the leaky modes (the other complex poles) [9].

2.3. The multipole expansions of the free space Green's function revisited. Before introducing the far-field expansions of the layered Green's function, we present an alternative derivation for the well-known ME of the free-space Green's function. Consider N sources with strength q_j placed at locations $\mathbf{x}_j = (x_j, y_j)$, $j = 1, 2, \ldots, N$ within a circle centered at $\mathbf{x}_c = (x_c, y_c)$ with a radius r in the free space $\mathbb{R}^2$, then, the field located at $\mathbf{x}$ due to all sources is given by

$$u^{\mathrm{f}}(\mathbf{x}) = \sum_{j=1}^{N} q_j G^{\mathrm{f}}(\mathbf{x}, \mathbf{x}_j),$$

where G^{f} is the free space Green's function $G^{\mathrm{f}}(\mathbf{x}, \mathbf{x}') = \frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{x}'|)$, k is the wave number, and $H_0^{(1)}$ is the Hankel function of the first kind. A target $\mathbf{x}$ is well-separated from the sources if the distance between $\mathbf{x}$ and the source center $\mathbf{x}_c$ is at least $2r$.

By Graf's addition theorem [1], the free space Green's function for the well-separated sources $\mathbf{x}_j$ and the target $\mathbf{x}$ can be compressed with a multipole expansion

$$u^{\mathrm{f}}(\mathbf{x}) = \frac{\mathrm{i}}{4} \sum_{p=-\infty}^{\infty} \alpha_p H_p^{(1)}(k|\mathbf{x}-\mathbf{x}_c|) e^{\mathrm{i}p\theta_c} \approx \frac{\mathrm{i}}{4} \sum_{|p|<P} \alpha_p H_p^{(1)}(k\rho_c) e^{\mathrm{i}p\theta_c}, \tag{2.11}$$

where $\alpha_p = \sum_{j=1}^{N} q_j J_p(k\rho_j) e^{-\mathrm{i}p\theta_j}$, (ρ_c, θ_c) are the polar coordinates of $\mathbf{x}-\mathbf{x}_c$, (ρ_j, θ_j) are the polar coordinates of $\mathbf{x}_j - \mathbf{x}_c$, and the truncation index P is a constant independent of the number of the sources N [11].

On the other hand, the multipole expansion can also be derived in the frequency domain using (2.1) as follows. Consider one source $\mathbf{x}_j$ and suppose $y - y_j > 0$, $y - y_c > 0$ for simplicity. The interaction between $\mathbf{x}$ and $\mathbf{x}_j$ can be represented by a Sommerfeld integral of plane waves [5],

$$G^{\mathrm{f}}(\mathbf{x}, \mathbf{x}_j) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{x}_j|) = \frac{\mathrm{i}}{4} \frac{1}{\mathrm{i}\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2}(y-y_j)}}{\sqrt{\lambda^2-k^2}} e^{\mathrm{i}\lambda(x-x_j)} d\lambda, \tag{2.12}$$

while each term $H_p^{(1)}(k\rho_c)e^{\mathrm{i}p\theta_c}$ in (2.11) has a similar representation [5]

$$H_p^{(1)}(k\rho_c)e^{\mathrm{i}p\theta_c} = \frac{1}{\mathrm{i}\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2}(y-y_c)}}{\sqrt{\lambda^2-k^2}} e^{\mathrm{i}\lambda(x-x_c)} (-\mathrm{i})^p \left(\frac{\lambda - \sqrt{\lambda^2-k^2}}{k} \right)^p d\lambda. \tag{2.13}$$

These integral forms give an alternative derivation for the multipole expansion of $\frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{x}_j|) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|(\mathbf{x}-\mathbf{x}_c)+(\mathbf{x}_c-\mathbf{x}_j)|)$ with separable plane wave factors in the integrands involving $(\mathbf{x}-\mathbf{x}_c)$ and $(\mathbf{x}_c - \mathbf{x}_j)$,

$$\begin{aligned}
&\frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{x}_j|) \\
&= \frac{\mathrm{i}}{4} \frac{1}{\mathrm{i}\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2}(y-y_j)}}{\sqrt{\lambda^2-k^2}} e^{\mathrm{i}\lambda(x-x_j)} d\lambda \\
&= \frac{\mathrm{i}}{4} \frac{1}{\mathrm{i}\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2}(y-y_c)}}{\sqrt{\lambda^2-k^2}} e^{\mathrm{i}\lambda(x-x_c)} \cdot e^{-\sqrt{\lambda^2-k^2}(y_c-y_j)+\mathrm{i}\lambda(x_c-x_j)} d\lambda \\
&= \frac{\mathrm{i}}{4} \frac{1}{\mathrm{i}\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2}(y-y_c)}}{\sqrt{\lambda^2-k^2}} e^{\mathrm{i}\lambda(x-x_c)} \cdot g\left(k\rho_j, -\mathrm{i}e^{-\mathrm{i}\theta_j} w(\lambda)\right) d\lambda \\
&= \frac{\mathrm{i}}{4} \frac{1}{\mathrm{i}\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2}(y-y_c)}}{\sqrt{\lambda^2-k^2}} e^{\mathrm{i}\lambda(x-x_c)} \cdot \sum_{p=-\infty}^{\infty} J_p(k\rho_j) e^{-\mathrm{i}p\theta_j} \left(-\mathrm{i}w(\lambda)\right)^p d\lambda \\
&= \frac{\mathrm{i}}{4} \sum_{p=-\infty}^{\infty} J_p(k\rho_j) e^{-\mathrm{i}p\theta_j} \cdot \frac{1}{\mathrm{i}\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2}(y-y_c)}}{\sqrt{\lambda^2-k^2}} e^{\mathrm{i}\lambda(x-x_c)} \left(-\mathrm{i}w(\lambda)\right)^p d\lambda \\
&= \frac{\mathrm{i}}{4} \sum_{p=-\infty}^{\infty} J_p(k\rho_j) e^{-\mathrm{i}p\theta_j} \cdot H_p^{(1)}(k\rho_c) e^{\mathrm{i}p\theta_c},
\end{aligned}$$

where

$$w(\lambda) = \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k}. \tag{2.14}$$

The interchangeability of the sum and the integration is verified by the validity of the identity itself, i.e., the Graf's addition theorem.

2.4. The far-field MEs and LEs, translations, and their exponential convergence for the Green's function in layered media. For the sake of convenience, we focus on the interaction between one source and one target unit charge. We will derive far-field expansions for each integral $u_{ts}^{**}(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{**})$ in a natural generalization of the free-space case discussed in subsection 2.3, then show their exponential convergence. The derivation makes use of the following two types of series expansions.

Suppose (ρ_0, θ_0) are the polar coordinates of (x_0, y_0). Denote

$$w_l(\lambda) = \frac{\lambda - \sqrt{\lambda^2 - k_l^2}}{k_l}, \ 0 \le l \le L. \tag{2.15}$$

By using the Bessel generating function (2.1), we have

$$\begin{aligned} e^{-\sqrt{\lambda^2 - k_s^2}\tau^\star y_0 - \mathrm{i}\lambda x_0} &= g\left(k_s\rho_0, -\mathrm{i}e^{\mathrm{i}\tau^\star\theta_0} w_s(\lambda)\right) \\ &= \sum_{p=-\infty}^{\infty} J_p(k_s\rho_0) e^{\mathrm{i}p\tau^\star\theta_0} \cdot (-\mathrm{i}w_s(\lambda))^p, \end{aligned} \tag{2.16}$$

$$\begin{aligned} e^{-\sqrt{\lambda^2 - k_t^2}\tau^* y_0 + \mathrm{i}\lambda x_0} &= g\left(k_t\rho_0, \mathrm{i}e^{\mathrm{i}\tau^*\theta_0} w_t(\lambda)^{-1}\right) \\ &= \sum_{m=-\infty}^{\infty} J_m(k_t\rho_0) e^{\mathrm{i}m\tau^*\theta_0} \cdot \left(\mathrm{i}w_t(\lambda)^{-1}\right)^m. \end{aligned} \tag{2.17}$$

For the ME, we split the difference $\mathbf{x} - \mathbf{x}' = (\mathbf{x} - \mathbf{x}_c) + (\mathbf{x}_c - \mathbf{x}')$, namely, we shift the source $\mathbf{x}'$ to a common source center $\mathbf{x}_c = (x_c, y_c)$, which is assumed to be on the same side of the interface $y = d_s^\star$, i.e., $y_c - d_s^\star$ and $y' - d_s^\star$ have the same sign. Let (ρ_c', θ_c') be the polar coordinates of $\mathbf{x}' - \mathbf{x}_c$. Using (2.16) with $(\rho_0, \theta_0) = (\rho_c', \theta_c')$ and the separability of the plane wave factor $\mathcal{E}_{ts}^{**}(\mathbf{x}, \mathbf{x}')$ (2.7), we get an approximation

$$\begin{aligned} u_{ts}^{**}(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{**}) &= \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{**}(\mathbf{x}, \mathbf{x}', \lambda) \sigma_{ts}^{**}(\lambda) d\lambda \\ &= \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{**}(\mathbf{x}, \mathbf{x}_c, \lambda) e^{-\sqrt{\lambda^2 - k_s^2}\tau^\star(y' - y_c) + \mathrm{i}\lambda(x_c - x')} \sigma_{ts}^{**}(\lambda) d\lambda \\ &= \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{**}(\mathbf{x}, \mathbf{x}_c, \lambda) \sigma_{ts}^{**}(\lambda) \sum_{p=-\infty}^{\infty} J_p(k_s\rho_c') e^{\mathrm{i}p\tau^\star\theta_c'} (-\mathrm{i}w_s(\lambda))^p d\lambda \\ &\approx \sum_{|p|<P} I_p^{**}(\mathbf{x}, \mathbf{x}_c) M_p^\star(\mathbf{x}', \mathbf{x}_c), \end{aligned} \tag{2.18}$$

where the expansion function

$$I_p^{**}(\mathbf{x}, \mathbf{x}_c) = \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{**}(\mathbf{x}, \mathbf{x}_c, \lambda) \sigma_{ts}^{**}(\lambda) (-\mathrm{i}w_s(\lambda))^p d\lambda, \tag{2.19}$$

and the ME cofficient

$$M_p^{\star}(\mathbf{x}', \mathbf{x}_c) = J_p(k_s\rho_c')e^{\mathrm{i}p\tau^{\star}\theta_c'}. \tag{2.20}$$

For the LE, we split the difference $\mathbf{x}-\mathbf{x}' = \left(\mathbf{x}-\mathbf{x}_c^l\right) + \left(\mathbf{x}_c^l - \mathbf{x}'\right)$, namely, we shift the target $\mathbf{x}$ to a common target (local) center $\mathbf{x}_c^l = (x_c^l, y_c^l)$, which is assumed to be on the same side of the interface $y = d_t^*$. Let (ρ^l, θ^l) be the polar coordinates of $\mathbf{x}-\mathbf{x}_c^l$. Using (2.17) with $(\rho_0, \theta_0) = (\rho^l, \theta^l)$ and the separability of the plane wave factor $\mathcal{E}_{ts}^{*\star}(\mathbf{x}, \mathbf{x}')$ (2.7), we get an approximation,

$$\begin{aligned} &u_{ts}^{*\star}(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{*\star}) \\ &= \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}', \lambda)\sigma_{ts}^{*\star}(\lambda) \sum_{m=-\infty}^{\infty} J_m(k_t\rho^l)e^{\mathrm{i}m\tau^*\theta^l} \cdot \left(\mathrm{i}w_t(\lambda)^{-1}\right)^m d\lambda \\ &\approx \sum_{|m|<M} L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x}')K_m^*(\mathbf{x}, \mathbf{x}_c^l), \end{aligned} \tag{2.21}$$

where the expansion function

$$K_m^*(\mathbf{x}, \mathbf{x}_c^l) = J_m(k_t\rho^l)e^{\mathrm{i}m\tau^*\theta^l}, \tag{2.22}$$

and the LE coefficient

$$L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x}') = \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}', \lambda)\sigma_{ts}^{*\star}(\lambda)\left(\mathrm{i}w_t(\lambda)^{-1}\right)^m d\lambda. \tag{2.23}$$

Now, the M2L can be derived directly by using the splitting $\mathbf{x}_c^l - \mathbf{x}' = \left(\mathbf{x}_c^l - \mathbf{x}_c\right) + (\mathbf{x}_c - \mathbf{x}')$ in $L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x}')$, i.e.,

$$\begin{aligned} &L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x}') \\ &= \int_{-\infty}^{\infty} \mathcal{E}_{st}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c, \lambda)\sigma_{st}^{*\star}(\lambda)\left(\mathrm{i}w_t(\lambda)^{-1}\right)^m \sum_{p=-\infty}^{\infty} J_p(k_s\rho_c')e^{\mathrm{i}p\tau^{\star}\theta_c'} \cdot (-\mathrm{i}w_s(\lambda))^p d\lambda \\ &\approx \sum_{|p|<P} A_{mp}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c)M_p^{\star}(\mathbf{x}', \mathbf{x}_c), \end{aligned} \tag{2.24}$$

where the translation coefficients $A_{mp}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c)$ are given by

$$A_{mp}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c) = \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c, \lambda)\sigma_{ts}^{*\star}(\lambda)(-\mathrm{i}w_s(\lambda))^p\left(\mathrm{i}w_t(\lambda)^{-1}\right)^m d\lambda.$$

The L2L shifts the local center $\mathbf{x}_c^l$ in each integral $L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x}')$ to a new local center $\tilde{\mathbf{x}}_c^l = (\tilde{x}_c^l, \tilde{y}_c^l)$. Let $(\tilde{\rho}, \tilde{\theta})$ be the polar coordinates of $\tilde{\mathbf{x}}_c^l - \mathbf{x}_c^l$. Using (2.17) with $(\rho_0, \theta_0) = (\tilde{\rho}, \tilde{\theta})$,

$$\begin{aligned} &L_m^{*\star}(\tilde{\mathbf{x}}_c^l, \mathbf{x}') \\ &= \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}', \lambda)\sigma_{ts}^{*\star}(\lambda)\left(\mathrm{i}w_t(\lambda)^{-1}\right)^m \sum_{p=-\infty}^{\infty} J_p(k_t\tilde{\rho})e^{\mathrm{i}p\tau^*\tilde{\theta}} \cdot \left(\mathrm{i}w_t(\lambda)^{-1}\right)^p d\lambda \\ &\approx \sum_{|p+m|<P} J_p(k_t\tilde{\rho})e^{\mathrm{i}p\tau^*\tilde{\theta}} \int_{-\infty}^{\infty} \mathcal{E}_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}', \lambda)\sigma_{ts}^{*\star}(\lambda)\left(\mathrm{i}w_t(\lambda)^{-1}\right)^m\left(\mathrm{i}w_t(\lambda)^{-1}\right)^p d\lambda \\ &= \sum_{|p|<P} L_p^{*\star}(\mathbf{x}_c^l, \mathbf{x}')K_{p-m}^*(\tilde{\mathbf{x}}_c^l, \mathbf{x}_c^l). \end{aligned} \tag{2.25}$$

Remark 2.4. The use of plane waves for expressing the MEs and LEs was first proposed in the new version of FMMs for the Laplace and Helmholtz equations in free space [7], [8] to reduce the MEs to LEs translation cost.

Polarization distance. Before we present the main result of this paper on the convergence of the series expansions above, we introduce the concept of "polarization distance" unique to the interaction in layered media. Given layer indices s, t and direction marks $*, \star \in \{\uparrow, \downarrow\}$, for a target $\mathbf{x}_1 = (x_1, y_1)$ and a source $\mathbf{x}_2 = (x_2, y_2)$, the polarization distance is defined as

$$D_{ts}^{*\star}(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{(x_1 - x_2)^2 + (\tau^*(y_1 - d_t^*) + \tau^\star(y_2 - d_s^\star))^2}, \tag{2.26}$$

provided both $\tau^*(y_1 - d_t^*) > 0$ and $\tau^\star(y_2 - d_s^\star) > 0$. (Note that the polarization distance is not symmetric with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$.) This distance is in fact the distance between the target $\mathbf{x}_1$ and an equivalent polarization source for the source point $\mathbf{x}_2$. (See (2.35) for its definition and Figure 1 for an illustration of the locations of the polarization sources for different reaction components.)

THEOREM 2.5 (exponential convergence of far-field expansions in layered media). *Suppose the integral $u_{ts}^{*\star}(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{*\star})$ is derived from a well-posed Helmholtz problem in layered media as in* (2.6). *Then, we have the truncation error of ME* (2.18)

$$\left| u_{ts}^{*\star}(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{*\star}) - \sum_{|p|<P} I_p^{*\star}(\mathbf{x}, \mathbf{x}_c) M_p^\star(\mathbf{x}', \mathbf{x}_c) \right| \le c^{\mathrm{ME}}(P) \left(\frac{|\mathbf{x}' - \mathbf{x}_c|}{D_{ts}^{*\star}(\mathbf{x}, \mathbf{x}_c)} \right)^P, \tag{2.27}$$

the truncation error of LE (2.21)

$$\left| u_{ts}^{*\star}(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{*\star}) - \sum_{|m|<M} L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x}') K_m^*(\mathbf{x}, \mathbf{x}_c^l) \right| \le c^{\mathrm{LE}}(M) \left(\frac{|\mathbf{x} - \mathbf{x}_c^l|}{D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}')} \right)^M, \tag{2.28}$$

the truncation error of M2L (2.24) *for each LE coefficient*

$$\left| L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x}') - \sum_{|p|<P} A_{mp}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c) M_p^\star(\mathbf{x}', \mathbf{x}_c) \right| \le c_m^{\mathrm{M2L}}(P) \left(\frac{|\mathbf{x}' - \mathbf{x}_c|}{D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c)} \right)^P, \tag{2.29}$$

and the truncation error of L2L (2.25) *for each LE coefficient*

$$\left| L_m^{*\star}(\tilde{\mathbf{x}}_c^l, \mathbf{x}') - \sum_{|p|<P} L_p^{*\star}(\mathbf{x}_c^l, \mathbf{x}') K_{p-m}^*(\tilde{\mathbf{x}}_c^l, \mathbf{x}_c^l) \right| \le c_m^{\mathrm{L2L}}(P) \left(\frac{|\tilde{\mathbf{x}}_c^l - \mathbf{x}_c^l|}{D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}')} \right)^P \tag{2.30}$$

for some functions $c^{\mathrm{ME}}(\cdot)$, $c^{\mathrm{LE}}(\cdot)$, $c_m^{\mathrm{M2L}}(\cdot)$, and $c_m^{\mathrm{L2L}}(\cdot)$ having polynomial growth rates, provided that for some given $\mu > 1$, the far-field conditions measured with the polarization distances,

$$\begin{aligned} D_{ts}^{*\star}(\mathbf{x}, \mathbf{x}_c) &\ge \mu|\mathbf{x}' - \mathbf{x}_c|, \quad D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}') \ge \mu|\mathbf{x} - \mathbf{x}_c^l|, \\ D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c) &\ge \mu|\mathbf{x}' - \mathbf{x}_c|, \quad D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}') \ge \mu|\tilde{\mathbf{x}}_c^l - \mathbf{x}_c^l|, \end{aligned} \tag{2.31}$$

hold, respectively. If all the sources, targets, and centers involved above are bounded by a given box, the distances from every center to its nearby interface have a given nonzero lower bound, and there exist $0 < \rho_m \le \rho_M$ such that

$$\rho_m \le D_{ts}^{*\star}(\mathbf{x}, \mathbf{x}_c), D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}'), D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}_c), D_{ts}^{*\star}(\mathbf{x}_c^l, \mathbf{x}') \le \rho_M,$$

then the functions $c^{\rm ME}(\cdot)$, $c^{\rm LE}(\cdot)$, $c_m^{\rm M2L}(\cdot)$, *and* $c_m^{\rm L2L}(\cdot)$ *can be chosen to be determined by these bounds, without dependence on the actual positions of the source locations.*

The proof to be given in section 3 will be special cases of a general convergence result of the Bessel-type expansions in Theorem 3.9.

2.5. Numerical validation of exponential convergence. Here we present some numerical examples showing the exponential convergence rates of MEs and LEs. Consider a 3-layer problem with a source $\mathbf{x}'$ and a target $\mathbf{x}$ both in the middle layer. We examine the reaction field component

$$u_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}') = \int_{-\infty}^{\infty} e^{\mathrm{i}\lambda(x-x') - \sqrt{\lambda^2-k_1^2}(d_0-y) - \sqrt{\lambda^2-k_1^2}(d_0-y')} \sigma_{11}^{\downarrow\downarrow}(\lambda) d\lambda. \tag{2.32}$$

Suppose the source center $\mathbf{x}_c$ and the target center $\mathbf{x}_c^l$ are in the middle layer, and the far-field conditions $D_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}_c) > |\mathbf{x}'-\mathbf{x}_c|$ and $D_{11}^{\downarrow\downarrow}(\mathbf{x}_c^l,\mathbf{x}') > |\mathbf{x}-\mathbf{x}_c^l|$ are met. For the reaction component $u_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}')$, the relative error of the ME at source center $\mathbf{x}_c$, and that of the LE at target center $\mathbf{x}_c^l$ are defined for a given truncation index P

$$\begin{aligned} e_P^{\rm ME} &= \left| u_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}') - \sum_{|p|<P} I_p^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}_c) M_p^{\downarrow}(\mathbf{x}',\mathbf{x}_c) \right| \Big/ |u_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}')|, \\ e_P^{\rm LE} &= \left| u_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}') - \sum_{|p|<P} L_p^{\downarrow\downarrow}(\mathbf{x}_c^l,\mathbf{x}') K_p^{\downarrow}(\mathbf{x},\mathbf{x}_c^l) \right| \Big/ |u_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}')|. \end{aligned} \tag{2.33}$$

For comparison, we define the reference exponential convergence ratio

$$r_{\rm ME} = \frac{|\mathbf{x}'-\mathbf{x}_c|}{D_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}_c)}, \quad r_{\rm LE} = \frac{|\mathbf{x}-\mathbf{x}_c^l|}{D_{11}^{\downarrow\downarrow}(\mathbf{x}_c^l,\mathbf{x}')}. \tag{2.34}$$

Take $d_0 = 0.5$, $d_1 = -0.5$, $k_0 = 2$, $k_1 = 3$, $k_2 = 4.7$, $a_0 = a_1 = a_2 = 1$, $b_0 = 2$, $b_1 = 3$, $b_2 = 4.7$, the source center $\mathbf{x}_c = (0,0)$, and the target center $\mathbf{x}_c^l = (0.6, 0.2)$. The closed form of $\sigma_{11}^{\downarrow\downarrow}(\lambda)$ is given in (B.13). $\sigma_{11}^{\downarrow\downarrow}(\lambda)$ has a pair of real poles at $\lambda = \pm k_1$. If we consider the perturbed wave numbers in each layer $\tilde{k}_l = k_l + \epsilon_l \mathrm{i}$, $0 < \epsilon_l \ll 1$, then the perturbed real poles are $\pm(k_1 + \epsilon_1 \mathrm{i})$ with positive and negative imaginary part, respectively. Hence (2.10) will be used to evaluate the integrals.

We select three target-source pairs for numerical testing: case (1) $\mathbf{x} = (0.5, 0.3)$, $\mathbf{x}' = (0.3, 0.4)$; case (2) $\mathbf{x} = (0.5, 0.4)$, $\mathbf{x}' = (-0.1, -0.3)$; case (3) $\mathbf{x} = (0, 0.2)$, $\mathbf{x}' = (-0.1, 0.2)$. For each pair we compute and plot the relative errors of ME and LE of $u_{11}^{\downarrow\downarrow}(\mathbf{x},\mathbf{x}')$ for $P = 3, 4, \ldots, 12$ in Figure 3. Then we compare the results with the reference exponential convergence rates indicated by the corresponding colored dashed lines with slopes $\log_{10} r_{\rm ME}$ and $\log_{10} r_{\rm LE}$, respectively. The comparison shows that the relative errors decay at the expected exponential rates determined by the polarization distance.

2.6. An FMM framework for sources in layered media. In the far-field conditions (2.31) of the convergence results, the polarization distances $D_{ts}^{*\star}$ play the role of the far-field distances as in the free-space cases for the FMM implementation.

Polarization sources. To make use of this fact for the setup of FMM, we define a bijective linear mapping for each source point $\mathbf{x}_2$ by

$$\mathcal{P}_{ts}^{*\star} : \mathbf{x}_2 = (x_2, y_2) \mapsto \tilde{\mathbf{x}}_2 = (x_2, d_t^* - \tau^*\tau^\star (y_2 - d_s^\star)) \tag{2.35}$$

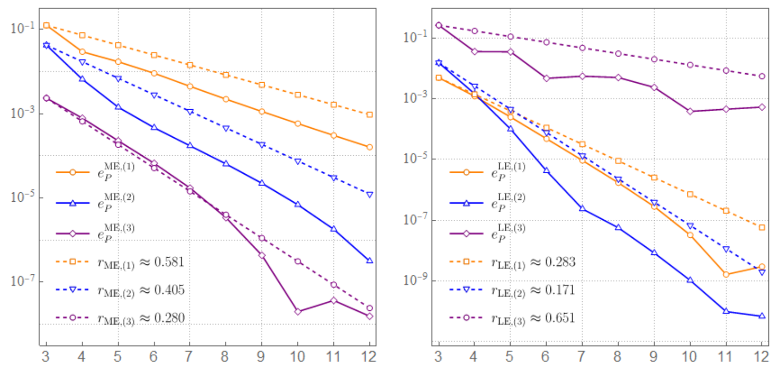

FIG. 3. *Relative errors of ME and LE for* $P = 3, 4, \ldots, 12$ *for three cases, compared to the reference exponential convergence rates indicated by* $r_{\mathrm{ME,(k)}}, r_{\mathrm{LE,(k)}}$*, case* $(k) = 1, 2, 3$.

provided $\tau^\star(y_2 - d_s^\star) > 0$. It is straightforward to see that

$$D_{ts}^{\star\star}(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathcal{P}_{ts}^{\star\star}(\mathbf{x}_2)\|\,; \tag{2.36}$$

here $\|\cdot\|$ is the Euclidean norm. Figure 1 shows how $\mathcal{P}_{ts}^{\star\star}$ maps the sources to their equivalent polarization sources.

The FMM for layered media can be set up to evaluate each reaction component $u_{ts}^{\star\star}$ as follows: $\mathcal{P}_{ts}^{\star\star}$ maps the source layer s to a neighboring layer (below or above) of the target layer t, where all the far-field distances become Euclidean as in (2.36). Then, to calculate the interaction due to any of the reaction component $u_{ts}^{\star\star}(\mathbf{x}, \mathbf{x}'; \sigma_{ts}^{\star\star})$, we simply move the source charges to the locations of their corresponding "equivalent polarization sources."An implementation for the Helmholtz equation and the Laplace equation in 3-D layered media based on this approach are given in [15], [16], respectively.

3. The convergence estimate on Bessel-type expansions.

In this section, we will give convergence estimates on general Bessel-type expansions, of which Theorem 2.5 will be a special case.

The Bessel-type expansions are defined as follows. Let $k > 0$, (ρ, θ), (ρ', θ') be the polar coordinates of $\mathbf{x} = (x, y)$ and $\mathbf{x}' = (x', y')$, respectively. Suppose $y > 0$, $y + y' > 0$, and $\rho > \rho' \geq 0$. For simplicity, define

$$\Psi(\lambda) \equiv \Psi(\mathbf{x},\lambda) = e^{-\sqrt{\lambda^2-k^2}y+\mathrm{i}\lambda x}, \quad \Psi'(\lambda) \equiv \Psi'(\mathbf{x}',\lambda) = e^{-\sqrt{\lambda^2-k^2}y'-\mathrm{i}\lambda x'}. \tag{3.1}$$

Then, we claim the *pointwise Bessel-type expansion* for a given $\lambda_\nu \in \mathbb{C}$,

$$e^{-\sqrt{\lambda_\nu^2-k^2}(y+y')+\mathrm{i}\lambda_\nu(x-x')} = \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p\theta'}\Psi(\mathbf{x},\lambda_\nu)\left(-\mathrm{i}w(\lambda_\nu)\right)^p \tag{3.2}$$

and the *integral Bessel-type expansion* over $\lambda \in (a, b)$, $-\infty \leq a < b \leq +\infty$,

$$\int_a^b e^{-\sqrt{\lambda^2-k^2}(y+y')+\mathrm{i}\lambda(x-x')}f(\lambda)d\lambda = \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p\theta'}F_p(x, y), \tag{3.3}$$

where $f(\lambda)$ is a complex function defined on (a,b) satisfying certain conditions to be specified later, and $F_p(x,y)$ is the expansion function

$$F_p(x,y) = \int_a^b \Psi(\mathbf{x},\lambda)\,(-\mathrm{i}w(\lambda))^p\, f(\lambda)d\lambda.$$

3.1. Convergence of pointwise Bessel-type expansions. We first present the convergence of (3.2).

LEMMA 3.1. *Let $\mu > 1$, $k > 0$. Suppose (ρ', θ') are the polar coordinates of (x', y'). Suppose $x \in \mathbb{R}$, $y \in \mathbb{R}^+$ satisfying $\rho = \sqrt{x^2+y^2} > \mu\rho' \geq 0$ and $x \cdot \Im\lambda_\nu \geq 0$. Then, the Bessel-type expansion* (3.2) *holds with truncation error estimate*

$$\left|\sum_{|p|\geq P} J_p(k\rho')e^{\mathrm{i}p\theta'}\Psi(\mathbf{x},\lambda_\nu)\,(-\mathrm{i}w(\lambda_\nu))^p\right| \leq \frac{2\mu}{\mu-1}\left(\frac{\rho'}{\rho}\right)^P \tag{3.4}$$

for any

$$P \geq e(|\lambda_\nu| + k/2)\rho. \tag{3.5}$$

Proof. The equality of (3.2) is given by the Bessel generating function (2.1)

$$\begin{aligned} e^{-\sqrt{\lambda_\nu^2-k^2}(y+y')+\mathrm{i}\lambda_\nu(x-x')} &= \Psi(\lambda_\nu)g\left(k\rho', -\mathrm{i}e^{\mathrm{i}\theta'}w(\lambda_\nu)\right) \\ &= \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p\theta'}e^{-\sqrt{\lambda_\nu^2-k^2}y+\mathrm{i}\lambda x}\,(-\mathrm{i}w(\lambda_\nu))^p. \end{aligned} \tag{3.6}$$

With the given conditions, $|\exp(-\sqrt{\lambda_\nu^2-k^2}y+\mathrm{i}\lambda_\nu x)| \leq 1$, $|w(\lambda_\nu)| \leq (2|\lambda_\nu|+k)/k$. Hence for each p, using Lemma 2.1,

$$\left|J_p(k\rho')e^{\mathrm{i}p\theta'}e^{-\sqrt{\lambda_\nu^2-k^2}y+\mathrm{i}\lambda_\nu x}\,(-\mathrm{i}w(\lambda_\nu))^p\right| \leq \frac{1}{|p|!}\left(\frac{k\rho'}{2}\right)^{|p|}\left(\frac{2|\lambda_\nu|+k}{k}\right)^{|p|}.$$

For $|p| \geq e(|\lambda_\nu| + k/2)\rho$, using Stirling's formula [10],

$$|p|! \geq \left(\frac{|p|}{e}\right)^{|p|} \geq \left(\left(|\lambda_\nu| + \frac{k}{2}\right)\rho\right)^{|p|},$$

we have

$$\left|J_p(k\rho')e^{\mathrm{i}p\theta'}e^{-\sqrt{\lambda_\nu^2-k^2}y+\mathrm{i}\lambda_\nu x}\,(-\mathrm{i}w(\lambda_\nu))^p\right| \leq \left(\frac{\rho'}{\rho}\right)^{|p|},$$

which will give the estimate of the truncation error after summing over $|p| \geq P$. ∎

3.2. Special cases of the integral Bessel-type expansion. First, we will prove (3.3) for a more general setting when the integral is defined on a bounded curve.

LEMMA 3.2. *Let $\mu > 1$, $k > 0$. Let (ρ, θ) and (ρ', θ') be the polar coordinates of $\mathbf{x} = (x, y)$ and $\mathbf{x}' = (x', y')$, respectively. Suppose $y > 0$, $\rho > \mu\rho' \geq 0$. Let $\kappa \subset \mathbb{C}$ be a complex contour which is parameterized as*

$$\kappa: \quad \lambda = \lambda(s) = a(s) + b(s)\mathrm{i},\ 0 \leq s \leq 1, \tag{3.7}$$

where $a(s)$ and $b(s)$ are real differentiable functions. Suppose $x \cdot b(s) \geq 0$ for any $s \in [0,1]$. Let $f(\lambda)$ be a complex function on κ satisfying a convergence condition

$$\int_0^1 \left| f(\lambda(s))\sqrt{a'(s)^2 + b'(s)^2} \right| ds = S < \infty. \tag{3.8}$$

Then, the series expansion

$$E_\kappa = \int_\kappa \Psi(\lambda)\Psi'(\lambda) f(\lambda) d\lambda = \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p\theta'} \int_\kappa \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p f(\lambda)d\lambda \tag{3.9}$$

holds with a truncation error estimate

$$\left| \sum_{|p|\geq P} J_p(k\rho')e^{\mathrm{i}p\theta'} \int_\kappa \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p f(\lambda)d\lambda \right| \leq \frac{2\mu S}{\mu - 1}\left(\frac{\rho'}{\rho}\right)^P \tag{3.10}$$

for any $P \geq e(\lambda_M + k/2)\rho$, where $\lambda_M = \max_{\lambda\in\kappa}|\lambda|$.

Proof. Using the results from the proof of Lemma 3.1, for $\lambda \in \kappa$,

$$\left| e^{-\sqrt{\lambda^2-k^2}y + \mathrm{i}\lambda x} \right| \leq 1,\ |w(\lambda)|^{\pm 1} \leq \frac{2\lambda_M + k}{k},$$

so for each p, using Lemma 2.1,

$$\begin{aligned} &\int_0^1 \left| J_p(k\rho')e^{\mathrm{i}p\theta'}\Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p f(\lambda)\left(a'(s) + b'(s)\mathrm{i}\right)\right| ds \\ \leq &\frac{1}{|p|!}\left(\frac{k\rho'}{2}\right)^{|p|} \cdot 1 \cdot \left(\frac{2\lambda_M + k}{k}\right)^{|p|} \cdot S. \end{aligned}$$

Hence, using the Bessel generating function (2.1) and Fubini's theorem,

$$\begin{aligned} E_\kappa &= \int_0^1 \Psi(\lambda)\Psi'(\lambda)f(\lambda)\left(a'(s) + b'(s)\mathrm{i}\right) ds \\ &= \sum_{p=-\infty}^{\infty}\int_0^1 J_p(k\rho')e^{\mathrm{i}p\theta'}\Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p f(\lambda)\left(a'(s) + b'(s)\mathrm{i}\right) ds \\ &= \sum_{p=-\infty}^{\infty}\int_\kappa J_p(k\rho')e^{\mathrm{i}p\theta'}\Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p f(\lambda)d\lambda, \end{aligned}$$

we obtain the equality of (3.9). When $|p| \geq e(\lambda_M + k/2)\rho$, using Stirling's formula [10], $|p|! \geq (|p|/e)^{|p|}$, we can show that each integral

$$\left|\int_\kappa J_p(k\rho')e^{\mathrm{i}p\theta'}\Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p f(\lambda)d\lambda\right| \leq \frac{1}{|p|!}\left(\frac{k\rho'}{2}\cdot\frac{2\lambda_M + k}{k}\right)^{|p|} S \leq S\left(\frac{\rho'}{\rho}\right)^{|p|}.$$

By adding up the bounds for $|p| \geq P$ we get a truncation error estimate with the following bound:

$$\left| \sum_{|p|\geq P} J_p(k\rho')e^{\mathrm{i}p\theta'} \int_\kappa \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p f(\lambda)d\lambda \right| \leq \sum_{|p|\geq P} S\left(\frac{\rho'}{\rho}\right)^{|p|} \leq \frac{2\mu S}{\mu - 1}\left(\frac{\rho'}{\rho}\right)^P. \qquad \square$$

A similar result on a bounded real interval follows immediately.

LEMMA 3.3. *Let $\mu > 1$, $k' \geq k > 0$. Let (ρ, θ) and (ρ', θ') be the polar coordinates of $\mathbf{x} = (x, y)$ and $\mathbf{x}' = (x', y')$, respectively. Suppose $y > 0$, $\rho > \mu\rho' \geq 0$, and the function $f(\lambda)$ on $[-k', k']$ satisfies $\int_{-k'}^{k'} |f(\lambda)| d\lambda = S < +\infty$, then the integral Bessel-type expansion* (3.3) *holds on $[-k', k']$ with truncation error estimate*

$$\left| \sum_{|p| \geq P} J_p(k\rho') e^{\mathrm{i}p\theta'} F_p \right| \leq \frac{2\mu S}{\mu - 1} \left(\frac{\rho'}{\rho} \right)^P \tag{3.11}$$

for any $P \geq ek'\rho$.

Proof. The same proof of Lemma 3.2 can be applied by using the following estimate instead:

$$|w(\lambda)|^{\pm 1} = \left| \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right|^{\pm 1} \leq \frac{2k'}{k}$$

for any $\lambda \in [-k', k']$, which gives the necessary lower bound of P. ∎

Next, we consider the special case $(x, y) = (0, \rho)$ in the Bessel-type expansion (3.3) over an infinite interval.

LEMMA 3.4. *Let $\mu > 1$, $k' \geq k > 0$, $x', y' \in \mathbb{R}$, $\rho > \mu\rho' \geq 0$. Suppose $f(\lambda)$ is a continuous function on $[k', \infty)$ such that $|f(\lambda)| \leq C\lambda^K$ for some given positive constant C and nonnegative integer K. For the integral*

$$E_p^+ = \int_{k'}^{\infty} e^{-\sqrt{\lambda^2 - k^2}\rho} \left(-\mathrm{i}w(\lambda)\right)^p f(\lambda) d\lambda, \ p \in \mathbb{Z}, \tag{3.12}$$

we have the estimate

$$\left| E_p^+ \right| \leq \int_{k'}^{\infty} e^{-\sqrt{\lambda^2 - k^2}\rho} w(\lambda)^p |f(\lambda)| d\lambda \leq 3C\left(|p| + K\right)! \left(\frac{2}{\rho} \right)^{K+1} \left(\frac{k\rho}{2} \right)^{-|p|} \tag{3.13}$$

for any $|p| \geq (k\rho)^2/4 + 1 - K$. In addition, the Bessel-type expansion (3.3) *holds with $(x, y) = (0, \rho)$ on the interval (k', ∞) and the truncation error is given by*

$$\left| \int_{k'}^{\infty} e^{-\sqrt{\lambda^2 - k^2}(\rho + y') + \mathrm{i}\lambda(-x')} f(\lambda) d\lambda - \sum_{|p| < P} J_p(k\rho') e^{\mathrm{i}p\theta'} E_p^+ \right| \leq c(P, \rho) \left(\frac{\rho'}{\rho} \right)^P \tag{3.14}$$

$\forall P \geq (k\rho)^2/4 + 1 - K$, where

$$c(P, \rho) = 6C(K+1)! \left(\frac{2\mu}{\rho(\mu - 1)} \right)^{K+1} (P + K)^K. \tag{3.15}$$

Proof. We will first consider the estimate in (3.13). Notice that for $\lambda \geq k$ we have $\sqrt{\lambda^2 - k^2} \leq \lambda$ and $0 \leq \lambda - \sqrt{\lambda^2 - k^2} \leq k \leq \lambda \leq \lambda + \sqrt{\lambda^2 - k^2}$, $|E_p^+| \leq Ck^{K+1} I_p$, where

$$I_p = \int_k^{\infty} \frac{e^{-\sqrt{\lambda^2 - k^2}\rho}}{\sqrt{\lambda^2 - k^2}} \left(\frac{\lambda + \sqrt{\lambda^2 - k^2}}{k} \right)^{M+1} d\lambda \tag{3.16}$$

and $M = |p| + K$. With the substitution $v = (\lambda + \sqrt{\lambda^2 - k^2})/k$,

$$\begin{aligned} I_p &= \int_1^\infty e^{\frac{k\rho}{2}(-v+v^{-1})} v^M dv \\ &\le \int_1^\infty e^{\frac{k\rho}{2}(-v)} \left(\sum_{j=0}^{M-1} \frac{1}{j!}\left(\frac{k\rho}{2}v^{-1}\right)^j + \frac{1}{M!}\left(\frac{k\rho}{2}v^{-1}\right)^M e^{\frac{k\rho}{2}v^{-1}} \right) v^M dv \\ &\le \sum_{j=0}^{M-1} \frac{1}{j!}\left(\frac{k\rho}{2}\right)^j \int_0^\infty e^{\frac{k\rho}{2}(-v)} v^{M-j} dv + \frac{1}{M!}\left(\frac{k\rho}{2}\right)^M \int_1^\infty e^{\frac{k\rho}{2}(-v+1)} dv \\ &= \sum_{j=0}^{M-1} \frac{(M-j)!}{j!}\left(\frac{k\rho}{2}\right)^{2j-M-1} + \frac{1}{M!}\left(\frac{k\rho}{2}\right)^{M-1} \\ &= M!\left(\frac{k\rho}{2}\right)^{-M-1} \sum_{j=0}^{M} c_j, \end{aligned}$$

where

$$c_j = \frac{(M-j)!}{M!j!}\left(\frac{k\rho}{2}\right)^{2j}, \ j = 0, \dots, M. \tag{3.17}$$

One can verify $c_0 = 1$, $c_1 = (k\rho/2)^2/M \le (M-1)/M$. For $1 \le j \le M-2$, we have $c_{j+1}/c_j = (k\rho)^2/4(j+1)(M-j) \le 1/2$. For c_M we have $c_M/c_{M-1} = (k\rho)^2/(4M) \le 1$. By summation, $\sum_{j=0}^M c_j \le c_0 + 2c_1 \le 1 + 2(M-1)/M \le 3$, so

$$|E_p^+| \le Ck^{K+1} I_p \le 3Ck^{K+1} M!\left(\frac{k\rho}{2}\right)^{-M-1} \le 3C\,(|p|+K)!\left(\frac{2}{\rho}\right)^{K+1}\left(\frac{k\rho}{2}\right)^{-|p|}. \tag{3.18}$$

To get the expansion (3.3) with $(x, y) = (0, \rho)$ on $[k', \infty)$, by using Lemma 2.1 and (3.18), we have

$$\begin{aligned} &\int_{k'}^\infty \left| J_p(k\rho')e^{\mathrm{i}p\theta'} e^{-\sqrt{\lambda^2-k^2}\rho} (-\mathrm{i}w(\lambda))^p f(\lambda) \right| d\lambda \\ &\le \frac{1}{|p|!}\left(\frac{k\rho'}{2}\right)^{|p|} \cdot 3C\,(|p|+K)!\left(\frac{2}{\rho}\right)^{K+1}\left(\frac{k\rho}{2}\right)^{-|p|} \\ &= 3C\frac{(|p|+K)!}{|p|!}\left(\frac{2}{\rho}\right)^{K+1}\left(\frac{\rho'}{\rho}\right)^{|p|}. \end{aligned}$$

Now, as in (3.6), by using Fubini's theorem, we have the expansion of (3.14),

$$\begin{aligned} &\int_{k'}^\infty e^{-\sqrt{\lambda^2-k^2}(\rho+y')+\mathrm{i}\lambda(-x')} f(\lambda) d\lambda \\ &= \int_{k'}^\infty \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p\theta'} e^{-\sqrt{\lambda^2-k^2}\rho} (-\mathrm{i}w(\lambda))^p f(\lambda) d\lambda \\ &= \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p\theta'} E_p^+ \end{aligned}$$

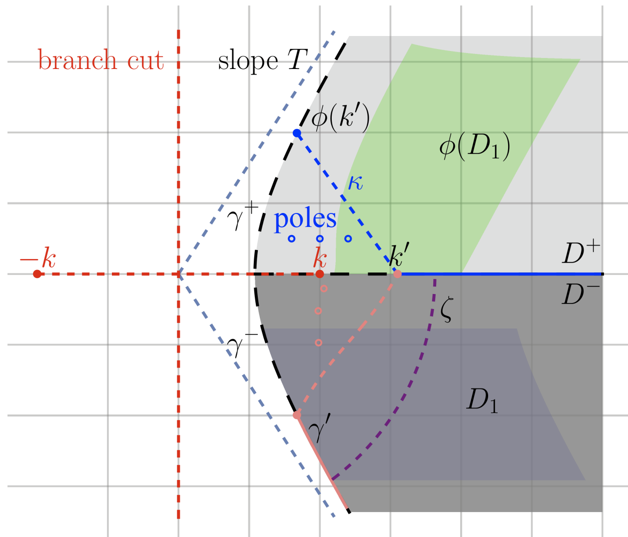

FIG. 4. *The mapping $\phi : D^- \to D^+$. The shadowed regions D_1 and $\phi(D_1)$ are for illustration.*

with a P-term truncation error for $P \geq (k\rho)^2/4 + 1 - K$,

$$\left| \sum_{|p|\geq P} J_p(k\rho')e^{ip\theta'} E_p^+ \right| \leq \sum_{|p|\geq P} 3C \frac{(|p|+K)!}{|p|!} \left(\frac{2}{\rho}\right)^{K+1} \left(\frac{\rho'}{\rho}\right)^{|p|} \leq c(P,\rho) \left(\frac{\rho'}{\rho}\right)^P . \qquad \square$$

Remark 3.5. The bound of $|E_p^+|$ in Lemma 3.4 is shown as an analogue of the asymptotic behavior of $H_n^{(1)}(x) \sim (n-1)!(x/2)^n/(i\pi)$ for $x > 0$ as $n \to \infty$ [1, equation (9.3.1)].

3.3. Convergence of general integral Bessel-type expansions. In order to obtain the convergence estimate of the integral Bessel-type expansion (3.3) on an infinite interval, we will take two steps. First, the Cagniard–de Hoop transform [4] will be used to convert the general (x, y) case to the $(0, \rho)$ case as discussed in Lemma 3.4, namely, the complex factor $e^{-\sqrt{\lambda^2-k^2}y+i\lambda x}$ in (3.3) is converted to $e^{-\sqrt{\lambda^2-k^2}\rho}$. Second, we deform the new complex contour of integration as a result of the transform to the real axis; see the illustration in Figure 4.

3.3.1. The Cagniard–de Hoop transform. Given positive real numbers x, y, and T satisfying $x < Ty$. Letting (ρ, θ) be the polar coordinates of (x, y). Letting $\beta = \frac{\pi}{2} - \theta \in (0, \frac{\pi}{2})$, then $y + xi = \rho e^{i\beta}$.

Define an open set

$$\Omega = \{z \in \mathbb{C} : \Re z > 0, z \notin (0, k]\} . \tag{3.19}$$

Then the holomorphic Cagniard–de Hoop mapping [4] $\phi : \Omega \to \mathbb{C}$ is given by

$$\phi(z) = z\cos\beta + i\sqrt{z^2 - k^2}\sin\beta. \tag{3.20}$$

Consider the right branch of the hyperbola

$$\Gamma = \left\{ a + bi : a, b \in \mathbb{R}, \frac{a}{\cos\beta} = \sqrt{\frac{b^2}{\sin^2\beta} + k^2} \right\} \tag{3.21}$$

with its vertex $k\cos\beta$ on the real axis, and the upper and lower parts of Γ denoted as γ^+ and γ^-, respectively, i.e.,

$$\Gamma = \gamma^+ \cup \gamma^- \cup \{k\cos\beta\}. \tag{3.22}$$

We can easily verify that $\phi((k,+\infty)) = \gamma^+$, and $\phi(\gamma^-) = (k,+\infty)$, namely,

$$\gamma^+ = \phi((k,+\infty)), \quad \gamma^- = \phi^{-1}((k,+\infty)), \tag{3.23}$$

where

$$\phi^{-1}(w) = w\cos\beta - \mathrm{i}\sqrt{w^2-k^2}\sin\beta. \tag{3.24}$$

Usually, Γ is known as the Cagniard–de Hoop contour. The two straight lines passing the origin with slopes $\pm\tan\beta$ are the asymptotes of $\gamma^{\pm}$, respectively.

Define regions to the right of Γ in the first and the fourth quadrant, respectively, by

$$D^{\pm} = \{z + t : z \in \gamma^{\pm}, t \in \mathbb{R}^+\}. \tag{3.25}$$

$D^{\pm}$ are isomorphic as the following lemma shows.

LEMMA 3.6. *$\phi|_{D^-}$ is a bijection to D^+ with inverse $\phi^{-1}|_{D^+}$ given by* (3.24).

Proof. See Appendix A. ∎

3.3.2. The general Bessel-type expansion. With the above preparation, we can now prove the expansion (3.3) when $f(\lambda)$ has a polynomial bound in Ω and $|\lambda|$ is sufficiently large and $\Im\lambda/\Re\lambda$ is bounded. To be specific, we make the following assumptions.

Assumption 3.7. Given $T > 0$, $\epsilon_0 > 0$. Let $f(\lambda)$ be a complex function with branch points $\pm k_0, \ldots, \pm k_L$, even, and meromorphic in $\mathbb{C}$ excluding the branch cuts of $\sqrt{\lambda^2 - k_l^2}$, $0 \le l \le L$, with poles of order up to one. Also, we assume the following:

- $f(\lambda)$ has a decomposition

$$f(\lambda) = \sum_{r=1}^{n_r} \frac{f_r}{\lambda - \lambda_r} + \bar{f}(\lambda) \text{ and } \bar{f}(\lambda) = \sum_{c=1}^{n_c} \frac{f_c}{\lambda - \lambda_c} + \bar{\bar{f}}(\lambda); \tag{3.26}$$

  here $\lambda_r \neq 0$ are all the real poles of $f(\lambda)$ with residue f_r, and λ_c are all the (complex) poles of $f(\lambda)$ in the region

$$\Omega_T^+ = \{a + b\mathrm{i} : a > 0, 0 < b < aT\} \tag{3.27}$$

  with residue f_c, respectively. Further, we suppose the complex poles in Ω_T^+ have a given bound λ_M, i.e., $\lambda_M \ge \max_{1\le c\le n_c} |\lambda_c|$.
- $\left|\bar{\bar{f}}(\lambda)\right| \le C\left(1 + |\lambda|^K\right)$ for any $\lambda \in \Omega_T^+ \cup \mathbb{R}^+$ satisfying $\min_{0\le l\le L} |\lambda - k_l| \ge \epsilon_0$; here $C > 0$ and $K \in \mathbb{N} \cup \{0\}$ are given integer constants.
- For $k' = 4k_M + 2\lambda_M + 2\epsilon_0$, $k_M = \max\{k_1, \ldots, k_L\}$, $S = \int_{-k'}^{k'} \left|\bar{f}(\lambda)\right| d\lambda < +\infty$.

LEMMA 3.8. *Let $\mu > 1$, $T > 0$, $\epsilon_0 > 0$ be some given constants, and the function $f(\lambda)$ satisfy Assumption* 3.7, *and $\bar{f}(\lambda)$ is so defined with the real poles removed from $f(\lambda)$, (ρ, θ) and (ρ', θ') are the polar coordinates of (x, y) and (x', y'), respectively, and $\rho > \mu\rho' \ge 0$. Suppose $y > 0$, $y + y' > 0$, and $|x| < Ty$. Then, the integral*

Bessel-type expansion (3.3) *holds on the interval* (k',∞) *(by replacing the original* $f(\lambda)$*) with* $\bar{f}(\lambda)$*, with a truncation error estimate for a finite* P*-term truncation*

$$\left|\sum_{|p|\geq P} J_p(k\rho')e^{\mathrm{i}p\theta'}\int_{k'}^{\infty}\Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p\bar{f}(\lambda)d\lambda\right| \leq c_+(P,\rho)\left(\frac{\rho'}{\rho}\right)^P \tag{3.28}$$

for any sufficiently large $P\geq m_+(\rho)$*. Here,* $m_+(\rho)$ *is an (at most) quadratic function of* ρ*, and* $c_+(P,\rho)$ *is a function having polynomial growth rate in* P*.*

Proof. If $x=0$, then $y=\sqrt{x^2+y^2}=\rho$, and

$$|\bar{f}(\lambda)|\leq\sum_{c=1}^{n_c}\frac{|f_c|}{\Im\lambda_c}+C\left(1+|\lambda|^K\right)\leq C_1\left(1+|\lambda|^K\right)$$

for $\lambda\in(k',\infty)$; here $C_1>0$ is a constant only depending on $f(\lambda)$. By Lemma 3.4, we can choose

$$m(\rho)=\left(\frac{k\rho}{2}\right)^2+1-K,\quad c_+(P,\rho)=6C_1(K+1)!\left(\frac{2\mu}{\rho(\mu-1)}\right)^{K+1}(P+K)^K.$$

If $x\neq 0$, without a loss of generality, we assume $x>0$, since the case $x<0$ will follow by taking complex conjugates. Let $\beta=\frac{\pi}{2}-\theta$, then $\tan\beta\in(0,T)$. Let κ be the segment from $\phi(k')$ to k' (see Figure 4); here ϕ is the Cagniard–de Hoop mapping defined in (3.20). One can verify that the length of κ is bounded by $\sqrt{2}k'$ and that $\lambda_M+k_M\leq|\lambda|\leq\sqrt{2}k'$ and $|\lambda-k_l|>\epsilon_0$ for $\lambda\in\kappa$, $0\leq l\leq L$. Defining

$$E=\int_{\kappa\cup(k',\infty)}\Psi(\lambda)\Psi'(\lambda)\bar{f}(\lambda)d\lambda,\quad G=\int_{\kappa}\Psi(\lambda)\Psi'(\lambda)\bar{f}(\lambda)d\lambda, \tag{3.29}$$

we will discuss the expansions for E and G, separately, then give the integral Bessel-type expansion for $E-G$. On κ we have the bound of $\bar{f}(\lambda)$ given by

$$|\bar{f}(\lambda)|\leq C\left(1+|\lambda|^K\right)+\sum_{n=1}^{n_c}\frac{|f_c|}{|\lambda-\lambda_c|}\leq C\left(1+(\sqrt{2}k')^K\right)+\sum_{c=1}^{n_c}\frac{|f_c|}{k_M}:=C_2. \tag{3.30}$$

Thus by Lemma 3.2, the Bessel-type expansion for G is given by

$$G=\sum_{p=-\infty}^{\infty}J_p(k\rho')e^{\mathrm{i}p\theta'}G_p,\quad G_p=\int_{\kappa}\Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p\bar{f}(\lambda)d\lambda, \tag{3.31}$$

with a truncation error

$$\left|\sum_{|p|\geq P}J_p(k\rho')e^{\mathrm{i}p\theta'}G_p\right|\leq c_\kappa\left(\frac{\rho'}{\rho}\right)^P\ \text{ for } P\geq m_\kappa(\rho); \tag{3.32}$$

here $c_\kappa=2\mu C_2\cdot\sqrt{2}k'/(\mu-1)$, $m_\kappa(\rho)=e(\lambda_M+k/2)\rho$. For the contour $\kappa\cup(k',\infty)$, with the substitution $\lambda=\phi(\lambda')=\lambda'\cos\beta+\mathrm{i}\sqrt{\lambda'^2-k^2}\sin\beta$ we have

$$\sqrt{\lambda^2-k^2}=\frac{\lambda\cos\beta-\phi^{-1}(\lambda)}{\mathrm{i}\sin\beta}=\frac{\phi(\lambda')\cos\beta-\lambda'}{\mathrm{i}\sin\beta}=\sqrt{\lambda'^2-k^2}\cos\beta+\mathrm{i}\lambda'\sin\beta,$$

so

$$\Psi(\lambda) = e^{-(\sqrt{\lambda'^2-k^2}\cos\beta+\mathrm{i}\lambda'\sin\beta)(\rho\cos\beta)+\mathrm{i}(\lambda'\cos\beta+\mathrm{i}\sqrt{\lambda'^2-k^2}\sin\beta)(\rho\sin\beta)} = e^{-\sqrt{\lambda'^2-k^2}\rho}.$$

Similarly, $\Psi'(\lambda) = e^{-\sqrt{\lambda'^2-k^2}\rho'\sin(\theta'-\beta)-\mathrm{i}\lambda'\rho'\cos(\theta'-\beta)}$ and $w(\lambda) = e^{-\mathrm{i}\beta}w(\lambda')$. Hence

$$E = \int_{\phi^{-1}(\kappa)\cup\gamma'} e^{-\sqrt{\lambda'^2-k^2}(\rho+\rho'\sin(\theta'-\beta))-\mathrm{i}\lambda'\rho'\cos(\theta'-\beta)}\tilde{f}(\lambda')d\lambda', \tag{3.33}$$

where $\gamma' = \phi^{-1}((k',\infty))$ is the lower part of γ starting from $\phi^{-1}(k')$ located somewhere on γ^-, and

$$\tilde{f}(\lambda') = \bar{f}(\lambda)\frac{d\lambda}{d\lambda'} = \bar{f}(\phi(\lambda'))\frac{\sqrt{\phi(\lambda')^2-k^2}}{\sqrt{\lambda'^2-k^2}}. \tag{3.34}$$

Since $\phi(\lambda')$ has a polynomial bound, roughly,

$$|\phi(\lambda')| = \left|\lambda'\cos\beta + \mathrm{i}\sqrt{\lambda'^2-k^2}\sin\beta\right| \le |\lambda'| + \sqrt{|\lambda'|^2+k^2} \le 2|\lambda'|+k \tag{3.35}$$

when $\lambda' \in D^-$ and $|\lambda'|$ is sufficiently large, $\tilde{f}(\lambda')$ also has a polynomial bound of $|\lambda'|$.

Next, we proceed to change the contour of the integral E from $\phi^{-1}(\kappa)\cup\gamma'$ back to (k',∞). Let ζ be the counterclockwise arc with radius r connecting $\phi^{-1}(\kappa)\cup\gamma'$ and the real axis, parameterized by $\lambda' = re^{\mathrm{i}\eta}$, where the range of η is a subset of $(-\beta,0)$. On the arc $\zeta : \lambda' = re^{\mathrm{i}\eta}$, as $r\to\infty$, the exponent of the integrand in E satisfies

$$\begin{aligned} -\sqrt{\lambda'^2-k^2}\,(\rho+\rho'\sin(\theta'-\beta)) - \mathrm{i}\lambda'\rho'\cos(\theta'-\beta) &\sim -\lambda' e^{\mathrm{i}\beta}\left(\rho e^{-\mathrm{i}\beta} + \mathrm{i}\rho' e^{-\mathrm{i}\theta'}\right) \\ &\sim r\bar{\rho}\exp\left(\mathrm{i}\left(\eta+\bar{\theta}+\beta+\frac{\pi}{2}\right)\right), \end{aligned}$$

where $(\bar{\rho},\bar{\theta})$ are the polar coordinates of $(x-x', y+y')$, and the rest of the integrand has a polynomial bound. Since $y+y'>0$, $\rho>\rho'$, one can verify $\bar{\theta}\in(0,\pi-\beta)$. Then

$$\Re\left\{r\bar{\rho}\exp\left(\mathrm{i}\left(\eta+\bar{\theta}+\beta+\frac{\pi}{2}\right)\right)\right\} \le r\cdot\max\{-(y+y'), -\rho-\rho'\sin(\theta'-\beta)\}$$

for any $\eta\in(-\beta,0)$, so the integrand on ζ decays exponentially, and the corresponding integral on ζ vanishes as $r\to+\infty$. Also notice that there are no poles of $\tilde{f}(\lambda')$ in $D'\subset D^-$, where D' is the region enveloped by $(k',+\infty)$ and $\phi^{-1}(\kappa)\cup\gamma'$. This is because ϕ is a holomorphic function on D' which maps any possible pole in D' to a pole of f in $\phi(D')$; however, for any $\lambda'\in D'$ and any pole $\lambda_c\in\Omega_T^+$, $|\phi(\lambda')|\ge\lambda_M+k_M>|\lambda_c|$. Hence, by deforming the integration contour in E to the real axis, we have

$$E = E' := \int_{(k',\infty)} e^{-\sqrt{\lambda'^2-k^2}(\rho+\rho'\sin(\theta'-\beta))-\mathrm{i}\lambda'\rho'\cos(\theta'-\beta)}\tilde{f}(\lambda')d\lambda'. \tag{3.36}$$

Now, for $\lambda'\in(k',\infty)$, recalling that

$$\tilde{f}(\lambda) = \frac{\sqrt{\lambda^2-k^2}}{\sqrt{\lambda'^2-k^2}}\left(\bar{\bar{f}}(\phi(\lambda')) + \sum_{c=1}^{n_c}\frac{f_c}{\phi(\lambda')-\lambda_c}\right),$$

for each λ_c we have $|\phi(\lambda')-\lambda_c| \ge |\phi(\lambda')|-|\lambda_c| \ge \sqrt{\lambda'^2-k^2\sin^2\beta}-\lambda_M \ge 3k_M$, so using (3.35), there exists some constant $C_2>0$ such that

$$|\tilde{f}(\lambda')| \le \frac{2|\lambda'|+2k}{3k_M}\left(C\left(1+(2|\lambda'|+k)^K\right) + \sum_{c=1}^{n_c}\frac{|f_c|}{3k_M}\right) \le C_2|\lambda'|^{K+1}.$$

Hence by Lemma 3.4, E' has a series expansion

$$E' = \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p(\theta'-\beta)} \int_{k'}^{\infty} e^{-\sqrt{\lambda'^2-k^2}\rho} \left(-\mathrm{i}w(\lambda')\right)^p \tilde{f}(\lambda')d\lambda' \tag{3.37}$$

with a P-term truncation error estimate

$$\left| \sum_{|p|\geq P} J_p(k\rho')e^{\mathrm{i}p(\theta'-\beta)} \int_{k'}^{\infty} e^{-\sqrt{\lambda'^2-k^2}\rho} \left(-\mathrm{i}w(\lambda')\right)^p \tilde{f}(\lambda')d\lambda' \right| \leq c_{E'}(P,\rho)\left(\frac{\rho'}{\rho}\right)^P \tag{3.38}$$

for $P \geq m_{E'}(\rho) = (k\rho)^2/4 - K$. Here

$$c_{E'}(P,\rho) = 6C_2(K+2)!\left(\frac{2\mu}{\rho(\mu-1)}\right)^{K+2}(P+K+1)^{K+1}. \tag{3.39}$$

In the series (3.37), the pth term is

$$\begin{aligned} & e^{-\mathrm{i}p\beta}\int_{(k,\infty)} e^{-\sqrt{\lambda'^2-k^2}\rho} \left(-\mathrm{i}w(\lambda')\right)^p \tilde{f}(\lambda')d\lambda' \\ &= e^{-\mathrm{i}p\beta}\int_{\phi^{-1}(\kappa)\cup\gamma'} e^{-\sqrt{\lambda'^2-k^2}\rho} \left(-\mathrm{i}w(\lambda')\right)^p \tilde{f}(\lambda')d\lambda' \qquad (3.40) \\ &= \int_{\kappa\cup(k',\infty)} \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda)d\lambda := E_p. \qquad (3.41) \end{aligned}$$

In the above equation, the first equality is obtained by changing the contour, and on the path $\zeta : \lambda' = re^{\mathrm{i}\eta}$ the integrand decays exponentially as $r \to \infty$ as the real part of the exponent

$$\Re\left(-\sqrt{(re^{\mathrm{i}\eta})^2-k^2}\rho\right) \sim \Re(-re^{\mathrm{i}\eta}\rho) \leq -ry,$$

while the remaining parts have polynomial growth rate. The second equality is by the substitution from λ' to λ. In total we have proven the series expansion of E given by $E = \sum_{p=-\infty}^{\infty} J_p(k\rho')e^{\mathrm{i}p\theta'}E_p$ with a P-term truncation error estimate

$$\left| E - \sum_{|p|<P} J_p(k\rho')e^{\mathrm{i}p\theta'}E_p \right| \leq c_{E'}(P,\rho)\left(\frac{\rho'}{\rho}\right)^P \quad \text{for } P \geq m_{E'}(\rho). \tag{3.42}$$

For each p,

$$\begin{aligned} E_p - G_p &= \int_{\kappa\cup(k',\infty)} \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda)d\lambda - \int_{\kappa} \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda)d\lambda \\ &= \int_{k'}^{\infty} \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda)d\lambda, \end{aligned} \tag{3.43}$$

which is the desired expansion function in the Bessel-type expansion (3.3).

Finally, by combining the results (3.32) and (3.42), $\forall P \geq \max\{m_{E'}(\rho), m_\kappa(\rho)\}$,

$$\begin{aligned} &\left| \int_{k'}^{\infty} \Psi(\lambda)\Psi'(\lambda)\bar{f}(\lambda)d\lambda - \sum_{|p|<P} J_p(k\rho')e^{\mathrm{i}p\theta'} \int_{k'}^{\infty} \Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda)d\lambda \right| \\ &\leq \left| E - \sum_{|p|<P} J_p(k\rho')e^{\mathrm{i}p\theta'}E_p \right| + \left| G - \sum_{|p|<P} J_p(k\rho')e^{\mathrm{i}p\theta'}G_p \right| \leq \left(c_{E'}(P,\rho) + c_\kappa\right)\left(\frac{\rho'}{\rho}\right)^P, \end{aligned}$$

which suggests $c_+(P,\rho) = c_{E'}(P,\rho) + c_\kappa$ and $m_+(\rho) = \max\{m_{E'}(\rho), m_\kappa(\rho)\}$. ∎

THEOREM 3.9 (the Bessel-type expansion). *Suppose the conditions of Lemma* 3.8 *are satisfied. Further suppose* $0 < \rho_m < \rho_M$ *are given such that* $\rho \in [\rho_m, \rho_M]$*. Then, the integral Bessel-type expansion* (3.3) *holds with a truncation error estimate*

$$\left| \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k^2}(y+y')+\mathrm{i}\lambda(x-x')} f(\lambda) d\lambda - \sum_{|p|<P} J_p(k\rho') e^{\mathrm{i}p\theta'} F_p \right| \le c(P) \left(\frac{\rho'}{\rho}\right)^P \tag{3.44}$$

for some function $c(\cdot)$ *with polynomial growth rate when* P *is sufficiently large, i.e.,* $P \ge m(\rho_M)$*,* $m(\rho_M)$ *is an at most quadratic function.*

Proof. Consider the decomposition of the integral

$$\begin{aligned} I = & \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k^2}(y+y')+\mathrm{i}\lambda(x-x')} f(\lambda) d\lambda \\ = & \sum_{r=1}^{n_r} \tau_r \mathrm{i}\pi \Psi(\lambda_r) \Psi'(\lambda_r) f_r + \left(\int_{-\infty}^{-k'} + \int_{-k'}^{k'} + \int_{k'}^{\infty} \right) \Psi(\lambda) \Psi'(\lambda) \bar{f}(\lambda) d\lambda \\ := & \sum_{r=1}^{n_r} I_r + I_- + I_0 + I_+; \end{aligned} \tag{3.45}$$

here each $\tau_r = \pm 1$ are determined by the well-posed physical problem (see (2.10)). Each term I_j of the decomposition with index j has the corresponding Bessel-type expansion, $j = 0, 1, \dots, n_r, +, -$. Namely, for each I_r, by Lemma 3.1, by choosing $c_r = 2\pi\mu |f_r|/(\mu - 1)$ and $m_r(\rho) = e(|\lambda_r| + k/2)\rho$, the pointwise Bessel-type expansion (3.2) holds,

$$I_r = \sum_{p=-\infty}^{\infty} J_p(k\rho') e^{\mathrm{i}p\theta'} I_{r,p}, \quad I_{r,p} = \tau_r \mathrm{i}\pi \Psi(\lambda_r) \left(-\mathrm{i}w(\lambda_r)\right)^p$$

with the truncation error for a P-term truncation

$$\left| \sum_{|p|\ge P} J_p(k\rho') e^{\mathrm{i}p\theta'} I_{r,p} \right| \le c_r \left(\frac{\rho'}{\rho}\right)^P \text{ for } P \ge m_r(\rho). \tag{3.46}$$

For I_0, by Lemma 3.3, by choosing $c_0 = 2\pi\mu S/(\mu - 1)$ and $m_0(\rho) = ek'\rho$, the integral Bessel-type expansion (3.3) holds,

$$I_0 = \sum_{p=-\infty}^{\infty} J_p(k\rho') e^{\mathrm{i}p\theta'} I_{0,p}, \quad I_{0,p} = \int_{-k'}^{k'} \Psi(\lambda) \left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda) d\lambda$$

with the truncation error for a P-term truncation

$$\left| \sum_{|p|\ge P} J_p(k\rho') e^{\mathrm{i}p\theta'} I_{0,p} \right| \le c_0 \left(\frac{\rho'}{\rho}\right)^P \text{ for } P \ge m_0(\rho). \tag{3.47}$$

For I_+ and I_-, by choosing the $c_+(P, \rho)$ and $m_+(\rho)$ provided by Lemma 3.8, and $c_-(P, \rho) = c_+(P, \rho)$ and $m_-(\rho) = m_+(\rho)$ due to the symmetry, the integral Bessel-type expansion (3.3) holds as $I_\pm = \sum_{p=-\infty}^{\infty} J_p(k\rho') e^{\mathrm{i}p\theta'} I_{\pm,p}$, where

$$I_{+,p} = \int_{k'}^{\infty} \Psi(\lambda) \left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda) d\lambda, \quad I_{-,p} = \int_{-\infty}^{-k'} \Psi(\lambda) \left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda) d\lambda$$

with the truncation error for a P-term truncation

$$\left|\sum_{|p|\geq P} J_p(k\rho')e^{\mathrm{i}p\theta'}I_{\pm,p}\right| \leq c_{\pm}(P,\rho)\left(\frac{\rho'}{\rho}\right)^P \quad \text{for } P\geq m_{\pm}(\rho). \tag{3.48}$$

For each p, the expansion functions add up to F_p because

$$\begin{aligned} F_p &= \sum_{r=1}^{n_r}\tau_r \mathrm{i}\pi\Psi(\lambda_r)\left(-\mathrm{i}w(\lambda_r)\right)^p f_r + \left(\int_{-\infty}^{-k'} + \int_{-k'}^{k'} + \int_{k'}^{\infty}\right)\Psi(\lambda)\left(-\mathrm{i}w(\lambda)\right)^p \bar{f}(\lambda)d\lambda \\ &= \sum_{r=1}^{n_r} I_{r,p} + I_{-,p} + I_{0,p} + I_{+,p}. \end{aligned}$$

Hence by adding the series expansions up, for any $P \geq m(\rho) := \max_j m_j(\rho)$,

$$\left|I - \sum_{|p|<P} J_p(k\rho')e^{\mathrm{i}p\theta'}F_p\right| \leq c(P,\rho)\left(\frac{\rho'}{\rho}\right)^P,$$

where $c(P,\rho) := \sum_{r=1}^{n_r} c_r + c_0 + c_+(P,\rho) + c_-(P,\rho)$. Since the only dependence of $c(P,\rho)$ on ρ appears in the terms $c_{\pm}(P,\rho)$ which reach their upper bounds at $\rho = \rho_m$, and each $m_j(\cdot)$ is an increasing function, we conclude that by choosing $c(P) := c(P,\rho_m)$, the truncation error estimate (3.44) holds for any $P \geq m(\rho_M)$. ∎

3.4. Proof of Theorem 2.5. We first consider the proof of the ME (2.27) and let $\tilde{x} = x - x_c$, $\tilde{y} = \tau^*(y - d_t^*) + \tau^\star(y_c - d_s^\star)$, $\tilde{x}' = x' - x_c$, $\tilde{y}' = \tau^\star(y' - y_c)$, and

$$f(\lambda) = e^{(\sqrt{\lambda^2-k_s^2}-\sqrt{\lambda^2-k_t^2})\tau^*(y-d_t^*)}\sigma_{ts}^{*\star}(\lambda) \tag{3.49}$$

so that the integral (2.7) can be written as

$$\begin{aligned} u_{ts}^{*\star}(\mathbf{x},\mathbf{x}';\sigma_{ts}^{*\star}) &= \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_t^2}\tau^*(y-d_t^*)-\sqrt{\lambda^2-k_s^2}\tau^\star(y'-d_s^\star)+\mathrm{i}\lambda(x-x')}\sigma_{ts}^{*\star}(\lambda)d\lambda \\ &= \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_s^2}(\tilde{y}+\tilde{y}')+\mathrm{i}\lambda(\tilde{x}-\tilde{x}')}f(\lambda)d\lambda. \end{aligned}$$

With the assumption that the sources, the targets, and the centers are bounded in a given box and that $|y_c - d_s^\star|$ has a nonzero lower bound, there exists fixed $T > 0$ such that $|\tilde{x}| < T\tilde{y}$. By Theorem B.1, $\sigma_{ts}^{*\star}(\lambda)$ has a polynomial bound in the region $\Omega_T = \{a + b\mathrm{i} : a > 0, -aT < b < aT\}$ when $\Re\lambda$ is sufficiently large and has a finite number of poles in Ω_T, which easily imply the same for $f(\lambda)$. With the decomposition (3.26), when neighborhoods of each branch point k_l with a sufficiently small radius $\epsilon_0 > 0$ are excluded from Ω_T, $\bar{\bar{f}}(\lambda)$ is finite and hence has polynomial bound. Replacing x, y, x', y', k in Theorem 3.9 by $\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}', k_s$ finishes the proof of (2.27).

For the LE (2.28), similarly, , $\tilde{x} = x_c^l - x'$, $\tilde{y} = \tau^*(y_c^l - d_t^*) + \tau^\star(y' - d_s^\star)$, $\tilde{x}' = x_c^l - x$, $\tilde{y}' = \tau^*(y - y_c^l)$, $k = k_t$, and $f(\lambda) = e^{(\sqrt{\lambda^2-k_t^2}-\sqrt{\lambda^2-k_s^2})\tau^\star(y'-d_s^\star)}\sigma_{ts}^{*\star}(\lambda)$.

For the M2L (2.29), for each LE coefficient $L_m^{*\star}(\mathbf{x}_c^l, \mathbf{x})$, choose $\tilde{x} = x_c^l - x_c$, $\tilde{y} = \tau^*(y_c^l - d_t^*) + \tau^\star(y_c - d_s^\star)$, $\tilde{x}' = x' - x_c$, $\tilde{y}' = \tau^\star(y' - y_c)$, $k = k_s$, and

$$f(\lambda) = e^{(\sqrt{\lambda^2-k_s^2}-\sqrt{\lambda^2-k_t^2})\tau^*(y_c^l-d_t^*)}\sigma_{ts}^{*\star}(\lambda)\left(\mathrm{i}w_t(\lambda)^{-1}\right)^m.$$

For the L2L (2.30), for each LE coefficient $L_m^{**\star}(\mathbf{x}_c^l, \mathbf{x})$, choose $\tilde{x} = x_c^l - x'$, $\tilde{y} = \tau^*(y_c^l - d_t^*) + \tau^\star(y' - d_s^\star)$, $\tilde{x}' = x_c^l - \tilde{x}_c^l$, $\tilde{y}' = \tau^\star(\tilde{y}_c^l - y_c^l)$, $k = k_t$, and

$$f(\lambda) = e^{(\sqrt{\lambda^2 - k_t^2} - \sqrt{\lambda^2 - k_s^2})\tau^\star(y' - d_s^\star)} \sigma_{ts}^{**\star}(\lambda) \left(\mathrm{i} w_t(\lambda)^{-1}\right)^m .$$

Remark 3.10 (dependence of convergence estimate on the number of layer interfaces L). As pointed out in Theorem B.2, if the interface conditions (2.3) satisfy $a_t, b_t \in \mathbb{R}^+$, $0 \le t \le L$, then each $\sigma_{ts}^{**\star}(\lambda)$ is asymptotically sublinear as $\lambda \to \infty$ regardless of L (the number of the interfaces). When applying Theorem 3.9 and Lemma 3.8, the bounds of $\sigma_{ts}^{**\star}(\lambda)$ are assumed of the same polynomial order as $\lambda \to \infty$. Therefore, as L increases, the required terms for truncation, namely, the $m(\rho_M)$ in the proof of Theorem 3.9 has linear dependence on the distribution of the poles as shown by Lemma 3.1, while the leading term $(k\rho_M/2)^2$ remains unchanged.

4. Conclusion.

Far-field expansions of ME, LE as well as M2L and L2L translation operators are derived and the exponential convergence rates are proven. The analysis shows that the convergence of ME and LE for the reaction field components depends on the distance between the target and the equivalent polarization source. This fact shows how the ME and LE for the layered media can be used in the traditional FMM framework, and such an approach has been implemented for the 3-D Helmholtz equation in [15] and the 3-D Laplace's equation in [16].

In a future work, we will extend the convergence analysis results to the 3-D Helmholtz equation and the 3-D Laplace equation in layered media.

Appendix A. Proof of Lemma 3.6.

We begin with the following two lemmas, which are stated given the same conditions as in Lemma 3.6.

Lemma A.1. *Let $a, b \in \mathbb{R}$ such that $z = a + b\mathrm{i} \in D^-$, then $\Re\phi(z) > 0$, $\Im\phi(z) > 0$.*

Proof. Let $u, v \in \mathbb{R}$ such that $u + v\mathrm{i} = \sqrt{z^2 - k^2}$, then $uv = ab < 0$. With the convention of the branch cut (2.9), we have $u > 0$, so $v < 0$. Recalling that $\beta \in (0, \frac{\pi}{2})$, we have $u \sin\beta - b\cos\beta > 0$ and $\Re\phi(z) = a\cos\beta - v\sin\beta > 0$. For $\Im\phi(z)$, let

$$Q_1 = (a^2 - b^2 - k^2)^2 + 4a^2b^2, \quad Q_2 = (a^2 - b^2 - k^2)\sin^2\beta - 2b^2\cos^2\beta. \tag{A.1}$$

By simple calculation, we have $2u^2\sin^2\beta - 2b^2\cos^2\beta = \sqrt{Q_1}\sin^2\beta + Q_2$, and

$$Q_1 \sin^4\beta - Q_2^2 = b^2\sin^2(2\beta)\left(a^2\cos^{-2}\beta - b^2\sin^{-2}\beta - k^2\right) > 0,$$

so $\sqrt{Q_1}\sin^2\beta = \left|\sqrt{Q_1}\sin^2\beta\right| > |Q_2|$, which implies

$$\Im\phi(z) = b\cos\beta + u\sin\beta = \frac{\sqrt{Q_1}\sin^2\beta + Q_2}{2(u\sin\beta - b\cos\beta)} > 0. \qquad \square$$

Lemma A.2. *If $w \in \gamma^+$, then $\phi(z) \ne w$ for any $z \in D^-$.*

Proof. Suppose for contradiction that $z \in D^-$, $\phi(z) = w$. Since $w \in \gamma^+$, $\exists x_0 \ge k$ such that $w = x_0\cos\beta + \mathrm{i}\sqrt{x_0^2 - k^2}\sin\beta$. Therefore, x_0 and z are distinct roots of the quadratic equation $\lambda^2 - 2\lambda w\cos\beta + w^2 = k^2\sin^2\beta$ of λ. Hence $z = 2w\cos\beta - x_0 = x_0\cos(2\beta) + \mathrm{i}\sqrt{x_0^2 - k^2}\sin(2\beta) \notin D^-$ because $\Im z \ge 0$, a contradiction. $\square$

Proof of Lemma 3.6. Define $\phi' : D^+ \to \mathbb{C}$ by $\phi'(w) = w\cos\beta - \mathrm{i}\sqrt{w^2 - k^2}\sin\beta$. It suffices to show that ϕ' is the inverse of ϕ on D^+, i.e., $\phi^{-1}|_{D^+} = \phi'$. First, we will show that $\phi(D^-) \subset D^+$. By Lemmas A.1 and A.2, $\phi(D^-)$ is a subset of the first quadrant, and it has no intersection with the hyperbola Γ. If $w = \phi(z)$ for some $z \in D^-$ and $w \notin D^+$, when we move z horizontally to the left, eventually z touches Γ and $\phi(z)$ approaches the positive real axis, so the trajectory of $\phi(z)$, which must be continuous because ϕ is holomorphic, crosses Γ in the first quadrant, but it contradicts Lemma A.2 since the intersection must has its inverse in D^-. Similarly (by taking complex conjugates), $\phi'(D^+) \subset D^-$. Second, we will show that ϕ is bijective on D^- with inverse ϕ'. Let $a, b \in \mathbb{R}^+$ such that $z = a + b\mathrm{i} \in D^-$, then $w = \phi(z) \in D^+$ is one of the roots of the quadratic equation of λ

$$\lambda^2 - 2\lambda z \cos\beta + z^2 = k^2 \sin^2\beta. \tag{A.2}$$

Let $u, v \in \mathbb{R}$ such that $\sqrt{z^2 - k^2} = u + v\mathrm{i}$, then $u > 0$, and the pair of roots are given by

$$\lambda_\pm = (a\cos\beta \mp v\sin\beta) + \mathrm{i}(b\cos\beta \pm u\sin\beta). \tag{A.3}$$

By Lemma A.1, $\Im w = \Im\phi(z) > 0$, so $w = \lambda^+$. Conversely, z is the only root of the quadratic equation $\lambda^2 - 2\lambda w\cos\beta + w^2 = k^2\sin^2\beta$ in D^- provided $\phi(z) = w$ by a similar reason, so ϕ is injective and $z = \phi'(w)$. Repeating this step for any $w' \in D^+$ and let $z' = \phi'(w')$, we have ϕ is surjective and $w' = \phi(\phi'(w'))$. ∎

Appendix B. Properties of Green's function in layered media.

As preliminaries for the proofs of the convergence estimates, some properties of Green's function in layered media are discussed, including the decomposition (2.6), the algebraic structure of the reflection/transmission coefficients $\sigma_{ts}^{\ast\star}(\lambda)$, and their polynomial bound in frequency λ.

B.1. Green's functions in layered media.

Suppose a source $\mathbf{x}'$ is in layer s and a target $\mathbf{x}$ is in layer t. Consider the 1-D Fourier transform $x - x' \mapsto \lambda$,

$$G(\mathbf{x}, \mathbf{x}') = \int_{-\infty}^{\infty} e^{\mathrm{i}\lambda(x-x')}\hat{G}(y, y', \lambda)d\lambda. \tag{B.1}$$

In the frequency λ domain, we have the decomposition $\hat{G} = \delta_{t,s}\hat{G}_s^{\mathrm{f}} + \hat{u}^{\mathrm{r}}$, and from (2.12) the free-space part in the frequency domain can be shown as

$$\hat{G}_s^{\mathrm{f}} = \frac{e^{-\sqrt{\lambda^2 - k_s^2}|y-y'|}}{4\pi\sqrt{\lambda^2 - k_s^2}}, \tag{B.2}$$

and the reaction field $\hat{u}^{\mathrm{r}}$ satisfies a homogeneous Helmholtz equation

$$(-\lambda^2 + \partial_{yy})\hat{u}^{\mathrm{r}} + k_t^2\hat{u}^{\mathrm{r}} = 0. \tag{B.3}$$

The solution to this ordinary differential equation has a general form

$$\hat{u}^{\mathrm{r}} = A_{ts}^{\uparrow}(y', \lambda)e^{-\sqrt{\lambda^2 - k_t^2}(y - d_t)} + A_{ts}^{\downarrow}(y', \lambda)e^{-\sqrt{\lambda^2 - k_t^2}(d_{t-1} - y)}, \tag{B.4}$$

where $A_{ts}^{\uparrow}$ and $A_{ts}^{\downarrow}$ do not depend on y within each layer, only on the target and source layer indices t, s. Here, we have assumed

$$d_{-1} = \infty, \quad d_L = -\infty, \tag{B.5}$$

and the corresponding term vanishes as the Sommerfeld radiation condition requires.

The interface condition at $y = d_l$ given by (2.3) is equivalent to

$$[a_t \hat{u}^{\mathrm{r}}] = -\left[\delta_{t,s} a_t \hat{G}_s^{\mathrm{f}}\right], \quad \left[b_t \frac{\partial \hat{u}^{\mathrm{r}}}{\partial y}\right] = \left[\delta_{t,s} b_t \frac{\partial \hat{G}_s^{\mathrm{f}}}{\partial y'}\right] \text{ at } y = d_l, \tag{B.6}$$

where the brackets describe the jump between layer $t = l$ and layer $t = l + 1$. When treated as linear equations for $A_{ts}^{\uparrow}$ and $A_{ts}^{\downarrow}$, using (B.2), the right-hand side of the equation is always a linear combination of $e^{-\sqrt{\lambda^2 - k_s^2}(y' - d_s)}$ and $e^{-\sqrt{\lambda^2 - k_s^2}(d_{s-1} - y')}$ with coefficients not depending on y'. The separation of variable y' implies

$$A_{ts}^{*}(y', \lambda) = \sigma_{ts}^{*\uparrow}(\lambda) e^{-\sqrt{\lambda^2 - k_s^2}(y' - d_s)} + \sigma_{ts}^{*\downarrow}(\lambda) e^{-\sqrt{\lambda^2 - k_s^2}(d_{s-1} - y')}, \quad * \in \{\uparrow, \downarrow\}, \tag{B.7}$$

which proves (2.6) with (B.4).

B.2. The algebraic structure of the reflection/transmission coefficients. Now we make some further observation on the interface conditions to characterize the coefficients $\sigma_{ts}^{*\star}(\lambda)$ in more detail.

With the separation of variables y and y', the interface condition (B.6) can be further expanded as linear equations of $\sigma_{ls}^{*\star}(\lambda)$ and $\sigma_{l+1,s}^{*\star}(\lambda)$:

$$\begin{aligned} -a_l \sigma_{ls}^{\uparrow\star} - a_l e_l \sigma_{ls}^{\downarrow\star} + a_{l+1} e_{l+1} \sigma_{l+1,s}^{\uparrow\star} + a_{l+1} \sigma_{l+1,s}^{\downarrow\star} &= v_{l,s}^{\star}, \\ b_l h_l \sigma_{ls}^{\uparrow\star} - b_l h_l e_l \sigma_{ls}^{\downarrow\star} - b_{l+1} h_{l+1} e_{l+1} \sigma_{l+1,s}^{\uparrow\star} + b_{l+1} h_{l+1} \sigma_{l+1,s}^{\downarrow\star} &= w_{l,s}^{\star}, \end{aligned} \tag{B.8}$$

where $\star \in \{\uparrow, \downarrow\}$, $v_{l,s}^{\uparrow} = \delta_{l,s} a_l / (4\pi h_l)$, $v_{l,s}^{\downarrow} = -\delta_{l+1,s} a_{l+1} / (4\pi h_{l+1})$, $w_{l,s}^{\uparrow} = \delta_{l,s} b_l / (4\pi)$, $w_{l,s}^{\downarrow} = \delta_{l+1,s} b_{l+1} / (4\pi)$, and the coefficients

$$h_t = \sqrt{\lambda^2 - k_t^2} \quad e_t = e^{-h_t(d_{t-1} - d_t)} \quad t = l, l+1. \tag{B.9}$$

As e_0 and e_L vanish in (B.8), these terms will be ignored from the equations.

If we expand all the $2L$ interface conditions into the form (B.8), two linear system for unknowns $\boldsymbol{\sigma}_s^{\uparrow}$, consisting of components $\sigma_{ts}^{*\uparrow}$, and $\boldsymbol{\sigma}_s^{\downarrow}$, consisting of components $\sigma_{ts}^{*\downarrow}$, are then obtained in the form

$$\mathbf{A}(\lambda) \boldsymbol{\sigma}_s^{\uparrow}(\lambda) = \mathbf{b}_s^{\uparrow}(\lambda), \quad \mathbf{A}(\lambda) \boldsymbol{\sigma}_s^{\downarrow}(\lambda) = \mathbf{b}_s^{\downarrow}(\lambda); \tag{B.10}$$

here $\mathbf{A}$ does not depend on the source layer s or the source-related direction $\star$. The functions $\sigma_{ts}^{*\star}(\lambda)$ can be solved from linear systems (B.10) using the Cramer's rule, so the complex roots of $\det \mathbf{A}(\lambda)$ are the poles of each $\sigma_{ts}^{*\star}(\lambda)$.

Consider the field $\mathbb{F}$ of functions of λ defined by field extension from $\mathbb{C}$

$$\begin{aligned} \mathbb{F} &= \mathbb{C}\left(h_t, e_m; 0 \le t \le L, 1 \le m \le L-1\right) \\ &= \mathbb{C}\left(\sqrt{\lambda^2 - k_t^2}, e^{-\sqrt{\lambda^2 - k_m^2}(d_{m-1} - d_m)}; 0 \le t \le L, 1 \le m \le L-1\right), \end{aligned} \tag{B.11}$$

where h_t, e_m are defined in (B.9). Since coefficients of the linear systems (B.10) are all in $\mathbb{F}$ as shown in (B.8), it follows that each

$$\sigma_{ts}^{*\star}(\lambda) \in \mathbb{F}. \tag{B.12}$$

For example, if the Helmholtz equation is equipped with interface conditions (2.3) with each $a_t = 1$, then the linear system for $\boldsymbol{\sigma}_1^{\downarrow}$ is

$$\begin{bmatrix} -1 & e_1 & 1 & 0 \\ 0 & -1 & -e_1 & 1 \\ b_0h_0 & -b_1h_1e_1 & b_1h_1 & 0 \\ 0 & b_1h_1 & -b_1h_1e_1 & b_2h_2 \end{bmatrix} \begin{bmatrix} \sigma_{01}^{\uparrow\downarrow} \\ \sigma_{11}^{\uparrow\downarrow} \\ \sigma_{11}^{\downarrow\downarrow} \\ \sigma_{21}^{\downarrow\downarrow} \end{bmatrix} = \begin{bmatrix} \frac{-1}{4\pi h_1} \\ 0 \\ \frac{k_1}{4\pi} \\ 0 \end{bmatrix},$$

and the coefficient $\sigma_{11}^{\downarrow\downarrow}(\lambda) \in \mathbb{F}$ has closed form

$$\text{(B.13)} \quad \sigma_{11}^{\downarrow\downarrow}(\lambda) = \frac{1}{4\pi h_1} \frac{(b_0h_0 - b_1h_1)(b_2h_2 + b_1h_1)}{(b_0h_0 + b_1h_1)(b_2h_2 + b_1h_1) - e_1^2(b_0h_0 - b_1h_1)(b_2h_2 - b_1h_1)}.$$

B.3. Polynomial bounds of the reflection/transmission coefficients. An alternative point of view on the linear systems (B.10) will reveal a polynomial bound of the functions $\sigma_{ts}^{**}(\lambda)$ in a certain domain in the complex plane. This estimate will be crucial to the error estimates on the far-field expansions.

Take any $k_M \geq \max_{0\leq l\leq L} k_l$ and $T > 0$, and define the open set

$$\text{(B.14)} \qquad \Omega_T = \{a + b\mathrm{i} : a > 0, -aT < b < aT\} \setminus (0, k_M]$$

in the complex plane. Since for the branch cut $\{\lambda | -k_l < \lambda < k_l\}$ for $\sqrt{\lambda^2 - k_l^2}$ is excluded from Ω_T, $\sigma_{ts}^{**}(\lambda)$ is a meromorphic function in Ω_T. We claim there is a polynomial bound of $\sigma_{ts}^{**}(\lambda)$ for $\lambda \in \Omega_T$ having a sufficiently large real part.

THEOREM B.1. *Suppose the function $\sigma(\lambda) \in \mathbb{F}$. Suppose $\forall \epsilon > 0$, $\sigma(\lambda) \ll \exp(\epsilon\lambda)$ as $\lambda \to +\infty$. Then, $\exists k_M' > 0$, $C > 0$, and nonnegative integer K such that $|\sigma(\lambda)| \leq C|\lambda|^K$ when $\lambda \in \Omega_T$ and $\Re\lambda > k_M'$. In addition, $\sigma(\lambda)$ has finitely many poles in Ω_T.*

Proof. Since $\sigma(\lambda) \in \mathbb{F}$, there exist polynomials P_1 and P_2 such that

$$\text{(B.15)} \qquad \sigma(\lambda) = \frac{I_1}{I_2} = \frac{P_1\left(\sqrt{\lambda^2 - k_l^2}, \dots, e^{+\sqrt{\lambda^2 - k_m^2}(d_{m-1}-d_m)}, \dots\right)}{P_2\left(\sqrt{\lambda^2 - k_l^2}, \dots, e^{+\sqrt{\lambda^2 - k_m^2}(d_{m-1}-d_m)}, \dots\right)};$$

here P_1 and P_2 are complex polynomials of the terms in the parentheses, including terms with indices $0 \leq l \leq L$ and $1 \leq m \leq L-1$. To show the asymptotic behavior of I_1 and I_2, we characterize them as elements of a ring $\mathcal{S}$ defined below. Let $\Omega_{T,M} = \{a + b\mathrm{i} \in \Omega_T : a, b \in \mathbb{R}, a > k_M\}$ be an open subset of Ω_T. Define

$$\mathcal{G} = \left\{ g(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^{m-n} : m \in \mathbb{Z},\ c_n \in \mathbb{C},\ c_0 \neq 0,\ g(\lambda) \text{ is holomorphic in } \Omega_{T,M} \right\},$$

which is the collection of holomorphic functions in $\Omega_{T,M}$ such that the number of nonzero terms with positive exponent is finite in the Laurent series of $g(\lambda)$ at ∞.

It follows that each $\sqrt{\lambda^2 - k_l^2} \in \mathcal{G}$, because it has neither a pole nor a branch point in $\Omega_{T,M}$, where $\Re\lambda > k_M \geq k_l$, and

$$\text{(B.16)} \qquad \sqrt{\lambda^2 - k_l^2} = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}(-k_l^2)^n}{2\Gamma(n+1)\Gamma(-n+\frac{3}{2})} \lambda^{1-2n}.$$

Letting $\mathcal{S}$ be the collection of all holomorphic functions $h(\lambda)$ in $\Omega_{T,M}$ in the form

$$\mathcal{S} = \left\{ h(\lambda) = \sum_{q=1}^{Q} e^{A_q \lambda} g_q(\lambda) : Q \geq 0,\ A_1 > \ldots > A_Q \geq 0, \text{ each } g_q \in \mathcal{G} \right\}, \tag{B.17}$$

we claim that $\forall d > 0$, $e^{\sqrt{\lambda^2 - k_i^2} d} \in \mathcal{S}$. To show this, notice that the exponential has neither a pole nor a branch point in $\Omega_{T,M}$ and that $e^{\sqrt{\lambda^2-k_l^2}d} = e^{\lambda d} e^{(\sqrt{\lambda^2-k_l^2}-\lambda)d}$. For the second factor, setting $\mu = \lambda^{-1}$, we have

$$e^{(\sqrt{\lambda^2-k_l^2}-\lambda)d} = \exp\left(\sum_{n=0}^{\infty} \frac{\sqrt{\pi}(-k_l^2)^{n+1} d}{2\Gamma(n+2)\Gamma(-n+\frac{1}{2})} \mu^{2n+1} \right), \tag{B.18}$$

which is regular in a neighborhood of $\mu = 0$. Therefore, the Laurent series at 0 in the μ-plane has zero principle part, which implies $e^{(\sqrt{\lambda^2-k_l^2}-\lambda)d} \in \mathcal{G}$ and $e^{\sqrt{\lambda^2-k_l^2}d} \in \mathcal{S}$.

It is obvious that $\mathcal{G} \subset \mathcal{S}$, and $\mathcal{S}$ is a ring with function addition and multiplication. For any function $h(\lambda) = \sum_{q=1}^{Q} e^{A_q\lambda} g_q(\lambda) \in \mathcal{S}$ which is not identical to 0, if the leading term of $g_1(\lambda)$ is $B\lambda^m$, then

$$h(\lambda) \sim e^{A_1\lambda} B\lambda^m \tag{B.19}$$

as $\Re\lambda \to \infty$. This is because in $\Omega_{T,M}$, $\Re\lambda \leq |\lambda| \leq \sqrt{1+T^2}\Re\lambda$, the limit as $|\lambda| \to \infty$ and the limit as $\Re\lambda \to \infty$ happen together. As $|\lambda| \to \infty$, each $g_q(\lambda) \in \mathcal{G}$ approaches its leading term, in addition, as $\Re\lambda \to \infty$, $\left|e^{A_1\lambda} B\lambda^m\right|$ dominates.

Now we look at $\sigma(\lambda)$. Since I_1, I_2 are polynomials of elements of the ring $\mathcal{S}$, we have $I_1, I_2 \in \mathcal{S}$. Suppose the numerator and the denominator

$$I_1 \sim e^{A_1\lambda} B\lambda^m, \quad I_2 \sim e^{A_1'\lambda} B'\lambda^{m'}$$

as $\Re\lambda \to \infty$, then it immediately follows that

$$\sigma(\lambda) \sim e^{(A_1 - A_1')\lambda} \frac{B}{B'} \lambda^{m-m'} \tag{B.20}$$

and that $A_1 \leq A_1'$ because $\sigma(\lambda) \ll \exp(\epsilon\lambda)$ for any $\epsilon > 0$. As a result, $\sigma(\lambda) = \mathcal{O}(|\lambda|^{m-m'})$ for $\lambda \in \Omega_{T,M}$ as $\Re\lambda \to \infty$, so the polynomial bound can be found for sufficiently large $\Re\lambda > k_M'$ and can be given in terms of $C|\lambda|^K$. This immediately implies that poles of $\sigma(\lambda)$ in Ω_T can only be found for sufficiently small $\Re\lambda$, i.e., in a bounded region. Hence the number of poles must be finite in Ω_T. ∎

The proof above also implies the asymptotic property of $\sigma_{ts}^{**}(\lambda)$ as $\Re\lambda \to \infty$. Indeed, given any two nonzero asymptotic orders $e^{A\lambda}\lambda^m$, where $A, m \in \mathbb{R}$, we can always compare their orders, i.e., the limit of the ratio is either infinity or a real number. The following theorem provides an improved estimate for some usual interface conditions.

THEOREM B.2. *With the conditions in Theorem* B.1, *if the interface conditions of the Helmholtz equation* (2.3) *satisfy* $a_t, b_t \in \mathbb{R}^+$, $0 \leq t \leq L$, *then as* $\Re\lambda \to \infty$, *all coefficient* $\sigma_{ts}^{**}(\lambda) = \mathcal{O}(|\lambda|^{-1})$.

Proof. Without a loss of generality, suppose among all the reflection/transmission coefficients, $\sigma_{ls}^{\uparrow\star}(\lambda)$ has the highest asymptotic order as $\Re\lambda \to \infty$. Suppose

$$\lim_{\Re\lambda\to\infty} \lambda^{-1}/\sigma_{ls}^{\uparrow\star} = 0. \tag{B.21}$$

In (B.8), dividing the first equation by $\sigma_{ls}^{\uparrow\star}$, we have

$$-a_l - a_l e_l \frac{\sigma_{ls}^{\downarrow\star}}{\sigma_{ls}^{\uparrow\star}} + a_{l+1} e_{l+1} \frac{\sigma_{l+1,s}^{\uparrow\star}}{\sigma_{ls}^{\uparrow\star}} + a_{l+1} \frac{\sigma_{l+1,s}^{\downarrow\star}}{\sigma_{ls}^{\uparrow\star}} = \frac{v_{l,s}^{\star}}{\sigma_{l,s}^{\uparrow\star}}. \tag{B.22}$$

Since as $\Re\lambda \to \infty$, $e_l = \exp(-h_l(d_{l-1} - d_l)) \to 0$ and $e_{l+1} \to 0$, and $h_l = \sqrt{\lambda^2 - k_l^2}$ and $h_{l+1} = \sqrt{\lambda^2 - k_{l+1}^2}$ are on the order of λ, by taking the limit of (B.22), we get

$$\lim_{\Re\lambda \to \infty} \frac{\sigma_{l+1,s}^{\downarrow\star}}{\sigma_{ls}^{\uparrow\star}} = \frac{a_l}{a_{l+1}} > 0. \tag{B.23}$$

Dividing the second equation of (B.8) by $h_l \sigma_{ls}^{\uparrow\star}$ and taking the limit, similarly we get

$$\lim_{\Re\lambda \to \infty} \frac{\sigma_{l+1,s}^{\downarrow\star}}{\sigma_{ls}^{\uparrow\star}} = -\frac{b_l}{b_{l+1}} < 0, \tag{B.24}$$

which is a contradiction, implying that (B.21) cannot be true, and thus we reach the conclusion of the theorem. □

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