Statistical Signal Processing for Receiver Algorithm Design

August 2022

Dongwoon Bai, Sr. Director

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- Samsung US SOC Lab
- Frequency Offset Estimation with Nuisance Parameters

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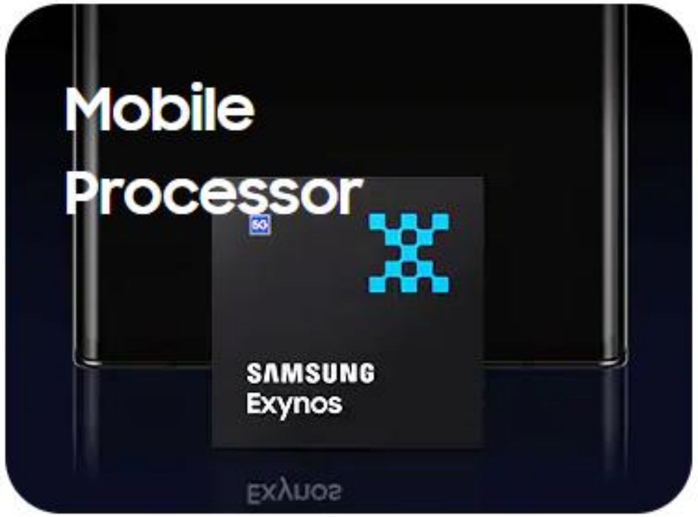

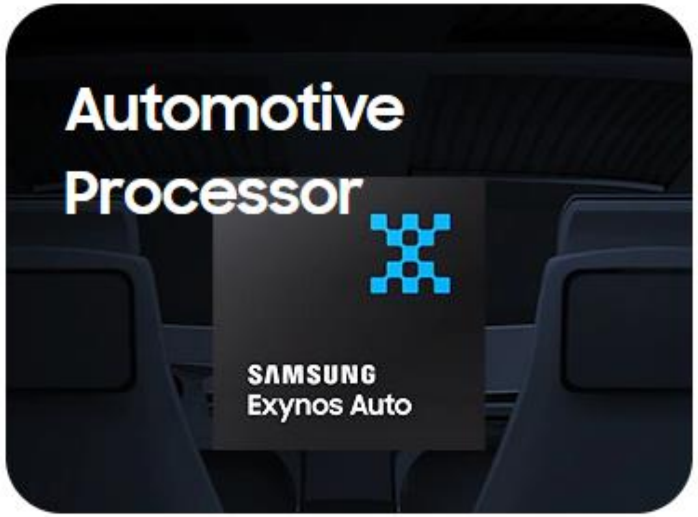

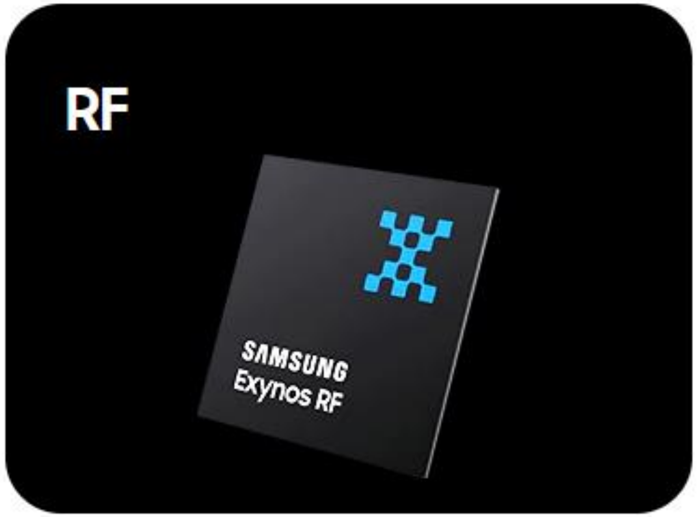

US SOC Lab (San Diego, San Jose, Cedar Rapids, etc)

Cellular & Multimedia

Cellular Modem

Multimedia

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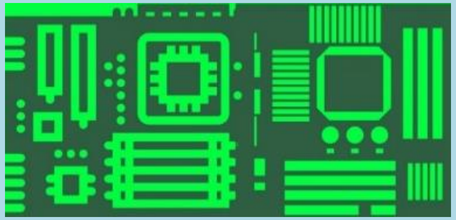

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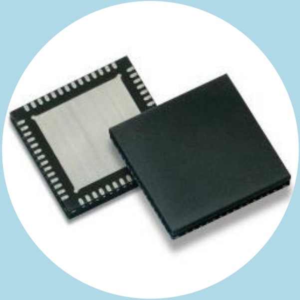


Connectivity

GNSS

Wi-Fi

- Samsung US SOC Lab

- Frequency Offset Estimation with a Nuisance Parameter

  – Synchronization of OFDM systems like 4G/5G (cellular), WiFi, etc.

System Model

- **Frequency offset (ω) estimation problem with a nuisance parameter (r)**

Consider

$$\boldsymbol{Y}_1 = \boldsymbol{r} + \boldsymbol{Z}_1,$$
$$\boldsymbol{Y}_2 = \boldsymbol{r}e^{j\omega} + \boldsymbol{Z}_2,$$

where $\boldsymbol{Z}_n$ is an iid zero-mean circularly symmetric complex Gaussian noise vector with the variance σ^2. ω is the unknown parameter we want to estimate, the frequency offset, and $\boldsymbol{r}$ is a unknown complex vector.

- **Solution given by**

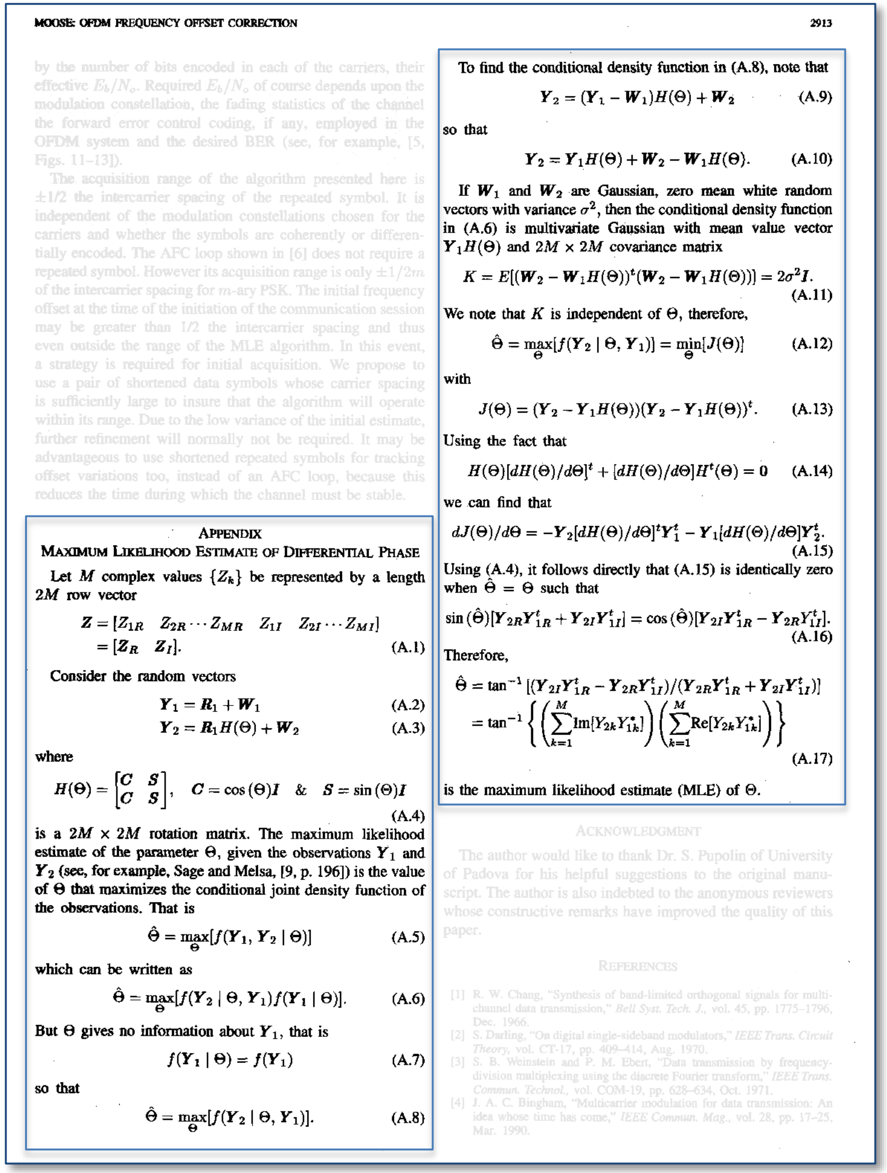
$$\hat{\omega} = \angle \boldsymbol{y}_1^* \boldsymbol{y}_2\,, \qquad \text{for realization, } \boldsymbol{Y}_1 = \boldsymbol{y}_1, \boldsymbol{Y}_2 = \boldsymbol{y}_2,$$

which is claimed to be the maximum likelihood (ML) estimator by the paper:

P. Moose, “A technique for OFDM frequency offset correction,” IEEE Trans. on Comm., vol. 42, no. 10, pp. 2908 –2914, Oct 1994.

Moose’s 1994 Paper

- **Moose’s frequency offset estimator**
  - One of most widely used techniques for OFDM synchronization.
  - Cited by 3580 articles according to Google scholar.
  - However, the proof is wrong and this estimator is not ML, misunderstood as the ML estimator for two decades.



MOOSE: OFDM FREQUENCY OFFSET CORRECTION 2913

by the number of bits encoded in each of the carriers, their effective E_b/N_o. Required E_b/N_o of course depends upon the modulation constellation, the fading statistics of the channel the forward error control coding, if any, employed in the OFDM system and the desired BER (see, for example, [5, Figs. 11–13]).

The acquisition range of the algorithm presented here is ±1/2 the intercarrier spacing of the repeated symbol. It is independent of the modulation constellations chosen for the carriers and whether the symbols are coherently or differentially encoded. The AFC loop shown in [6] does not require a repeated symbol. However its acquisition range is only $\pm 1/2m$ of the intercarrier spacing for m-ary PSK. The initial frequency offset at the time of the initiation of the communication session may be greater than 1/2 the intercarrier spacing and thus even outside the range of the MLE algorithm. In this event, a strategy is required for initial acquisition. We propose to use a pair of shortened data symbols whose carrier spacing is sufficiently large to insure that the algorithm will operate within its range. Due to the low variance of the initial estimate, further refinement will normally not be required. It may be advantageous to use shortened repeated symbols for tracking offset variations too, instead of an AFC loop, because this reduces the time during which the channel must be stable.

APPENDIX

MAXIMUM LIKELIHOOD ESTIMATE OF DIFFERENTIAL PHASE

Let M complex values $\{Z_k\}$ be represented by a length $2M$ row vector

$$\begin{aligned} \boldsymbol{Z} &= [Z_{1R} \quad Z_{2R} \cdots Z_{MR} \quad Z_{1I} \quad Z_{2I} \cdots Z_{MI}] \\ &= [\boldsymbol{Z}_R \quad \boldsymbol{Z}_I]. \end{aligned} \tag{A.1}$$

Consider the random vectors

$$\boldsymbol{Y}_1 = \boldsymbol{R}_1 + \boldsymbol{W}_1 \tag{A.2}$$

$$\boldsymbol{Y}_2 = \boldsymbol{R}_1 H(\Theta) + \boldsymbol{W}_2 \tag{A.3}$$

where

$$H(\Theta) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{S} \\ \boldsymbol{C} & \boldsymbol{S} \end{bmatrix}, \quad \boldsymbol{C} = \cos(\Theta)\boldsymbol{I} \quad \& \quad \boldsymbol{S} = \sin(\Theta)\boldsymbol{I} \tag{A.4}$$

is a $2M \times 2M$ rotation matrix. The maximum likelihood estimate of the parameter Θ, given the observations $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ (see, for example, Sage and Melsa, [9, p. 196]) is the value of Θ that maximizes the conditional joint density function of the observations. That is

$$\hat{\Theta} = \max_{\Theta}[f(\boldsymbol{Y}_1, \boldsymbol{Y}_2 \mid \Theta)] \tag{A.5}$$

which can be written as

$$\hat{\Theta} = \max_{\Theta}[f(\boldsymbol{Y}_2 \mid \Theta, \boldsymbol{Y}_1) f(\boldsymbol{Y}_1 \mid \Theta)]. \tag{A.6}$$

But Θ gives no information about $\boldsymbol{Y}_1$, that is

$$f(\boldsymbol{Y}_1 \mid \Theta) = f(\boldsymbol{Y}_1) \tag{A.7}$$

so that

$$\hat{\Theta} = \max_{\Theta}[f(\boldsymbol{Y}_2 \mid \Theta, \boldsymbol{Y}_1)]. \tag{A.8}$$

To find the conditional density function in (A.8), note that

$$\boldsymbol{Y}_2 = (\boldsymbol{Y}_1 - \boldsymbol{W}_1)H(\Theta) + \boldsymbol{W}_2 \tag{A.9}$$

so that

$$\boldsymbol{Y}_2 = \boldsymbol{Y}_1 H(\Theta) + \boldsymbol{W}_2 - \boldsymbol{W}_1 H(\Theta). \tag{A.10}$$

If $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are Gaussian, zero mean white random vectors with variance σ^2, then the conditional density function in (A.6) is multivariate Gaussian with mean value vector $\boldsymbol{Y}_1 H(\Theta)$ and $2M \times 2M$ covariance matrix

$$K = E[(\boldsymbol{W}_2 - \boldsymbol{W}_1 H(\Theta))^t (\boldsymbol{W}_2 - \boldsymbol{W}_1 H(\Theta))] = 2\sigma^2 \boldsymbol{I}. \tag{A.11}$$

We note that K is independent of Θ, therefore,

$$\hat{\Theta} = \max_{\Theta}[f(\boldsymbol{Y}_2 \mid \Theta, \boldsymbol{Y}_1)] = \min_{\Theta}[J(\Theta)] \tag{A.12}$$

with

$$J(\Theta) = (\boldsymbol{Y}_2 - \boldsymbol{Y}_1 H(\Theta))(\boldsymbol{Y}_2 - \boldsymbol{Y}_1 H(\Theta))^t. \tag{A.13}$$

Using the fact that

$$H(\Theta)[dH(\Theta)/d\Theta]^t + [dH(\Theta)/d\Theta]H^t(\Theta) = 0 \tag{A.14}$$

we can find that

$$dJ(\Theta)/d\Theta = -\boldsymbol{Y}_2[dH(\Theta)/d\Theta]^t \boldsymbol{Y}_1^t - \boldsymbol{Y}_1[dH(\Theta)/d\Theta]\boldsymbol{Y}_2^t. \tag{A.15}$$

Using (A.4), it follows directly that (A.15) is identically zero when $\hat{\Theta} = \Theta$ such that

$$\sin(\hat{\Theta})[\boldsymbol{Y}_{2R}\boldsymbol{Y}_{1R}^t + \boldsymbol{Y}_{2I}\boldsymbol{Y}_{1I}^t] = \cos(\hat{\Theta})[\boldsymbol{Y}_{2I}\boldsymbol{Y}_{1R}^t - \boldsymbol{Y}_{2R}\boldsymbol{Y}_{1I}^t]. \tag{A.16}$$

Therefore,

$$\begin{aligned} \hat{\Theta} &= \tan^{-1}[(\boldsymbol{Y}_{2I}\boldsymbol{Y}_{1R}^t - \boldsymbol{Y}_{2R}\boldsymbol{Y}_{1I}^t)/(\boldsymbol{Y}_{2R}\boldsymbol{Y}_{1R}^t + \boldsymbol{Y}_{2I}\boldsymbol{Y}_{1I}^t)] \\ &= \tan^{-1}\left\{\left(\sum_{k=1}^{M} \mathrm{Im}[Y_{2k}Y_{1k}^*]\right) \Big/ \left(\sum_{k=1}^{M} \mathrm{Re}[Y_{2k}Y_{1k}^*]\right)\right\} \end{aligned} \tag{A.17}$$

is the maximum likelihood estimate (MLE) of Θ.

ACKNOWLEDGMENT

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ML Estimation under a Nuisance Parameter

- **What is ML estimation?**

$$\widehat{\omega}(\boldsymbol{y}) = \arg\max_{\omega} p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega)$$

- **What if an unknown nuisance parameter ($\boldsymbol{r}$) exists?**

$$\widehat{\omega}(\boldsymbol{y}, \boldsymbol{r}) = \arg\max_{\omega} p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega, \boldsymbol{r})$$

  - In general, the maximum $\widehat{\omega}$ is a function of the unknown $\boldsymbol{r}$, which is not a feasible estimator, meaning that we can say ML estimation does not exist.
  - How to take care of a nuisance parameter
    - ML estimation after marginalization of the nuisance parameter $\boldsymbol{r}$ given its prior probability:

$$\hat{\omega} = \arg\max_{\omega} \int_{\boldsymbol{r}} p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega, \boldsymbol{r}) p_{\boldsymbol{R}}(\boldsymbol{r}) d\boldsymbol{r}$$

    - Joint ML estimation if not:

$$(\hat{\omega}, \hat{\boldsymbol{r}}) = \arg\max_{\omega, \boldsymbol{r}} p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega, \boldsymbol{r})$$

    - Conditional inference: Elimination of the nuisance parameter through conditioning is the approach taken in Moose's paper in my understanding.

When Does ML Estimation Exists?

- **Still interested in solving**

$$\hat{\omega}(\boldsymbol{y},\boldsymbol{r}) = \arg\max_{\omega} p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r})$$

- **but looking for a factorization of the pdf between ω and $\boldsymbol{r}$, i.e.,**

$$p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r}) = f(\boldsymbol{y};\omega)\, g(\boldsymbol{y};\boldsymbol{r}), \qquad f, g \geq 0.$$

- **If so, any prior info on r does not change the solution. There is a universally good $\hat{\omega}(\boldsymbol{y},\boldsymbol{r})$ regardless of r, i.e., $\hat{\omega}(\boldsymbol{y}) = \hat{\omega}(\boldsymbol{y},\boldsymbol{r})$. In this case, we can say ML estimation exists.**

$$\hat{\omega}(\boldsymbol{y},\boldsymbol{r}_1) = \arg\max_{\omega} p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r})\Big|_{\boldsymbol{r}=\boldsymbol{r}_1} = \left[\arg\max_{\omega} f(\boldsymbol{y};\omega)\right] g(\boldsymbol{y};\boldsymbol{r})\Big|_{\boldsymbol{r}=\boldsymbol{r}_1}$$

$$= \left[\arg\max_{\omega} f(\boldsymbol{y};\omega)\right] g(\boldsymbol{y};\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{r}_2} = \arg\max_{\omega}\ p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{r}_2} = \hat{\omega}(\boldsymbol{y},\boldsymbol{r}_2),$$

for any $\boldsymbol{r}_1, \boldsymbol{r}_2$, s.t., $g(\boldsymbol{y};\boldsymbol{r}_1), g(\boldsymbol{y};\boldsymbol{r}_2) > 0$.

Moose's Approach

- **Try to eliminate the first equation through conditioning on Y_1.**

$$\boldsymbol{Y}_1 = \boldsymbol{r} + \boldsymbol{Z}_1,$$
$$\boldsymbol{Y}_2 = \boldsymbol{r}e^{j\omega} + \boldsymbol{Z}_2,$$

  - It is claimed the first equation does not give any information on ω, which could be true if $\boldsymbol{r}$ were known.
  - However, it is not true since the first equation gives some information on unknown $\boldsymbol{r}$ and two equations are related through $\boldsymbol{r}$:

$$p_{\boldsymbol{Y}_1,\boldsymbol{Y}_2}(\boldsymbol{y}_1,\boldsymbol{y}_2;\omega,\boldsymbol{r}) = p_{\boldsymbol{Y}_1}(\boldsymbol{y}_1;\boldsymbol{r})\, p_{\boldsymbol{Y}_2}(\boldsymbol{y}_2;\omega,\boldsymbol{r})$$

Conditional Inference

Concept

We say **Strong Ancillarity** holds if we can find $\boldsymbol{T}, \boldsymbol{U}$ s.t.

$$p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega, \boldsymbol{r}) / |J_{\boldsymbol{g}}(\boldsymbol{y})| = p_{\boldsymbol{T},\boldsymbol{U}}(\boldsymbol{t}, \boldsymbol{u}; \omega, \boldsymbol{r}) = p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{t}, \boldsymbol{u}; \omega)\, p_{\boldsymbol{U}}(\boldsymbol{u}; \boldsymbol{r}),$$

where $\boldsymbol{T}$ is a conditional sufficient statistic and $\boldsymbol{U}$ is an ancillary statistic. Then, we have

$$\hat{\omega} = \arg\max_{\omega} p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega, \boldsymbol{r}) = \arg\max_{\omega} p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{t}, \boldsymbol{u}; \omega)$$

We say **Weak Ancillarity** holds if we can find $\boldsymbol{T}, \boldsymbol{U}$ s.t.

$$p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega, \boldsymbol{r}) / |J_{\boldsymbol{g}}(\boldsymbol{y})| = p_{\boldsymbol{T},\boldsymbol{U}}(\boldsymbol{t}, \boldsymbol{u}; \omega, \boldsymbol{r}) = p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{t}, \boldsymbol{u}; \omega)\, p_{\boldsymbol{U}}(\boldsymbol{u}; \omega, \boldsymbol{r}).$$

Then, the same property does not hold any more since

$$\hat{\omega}(\boldsymbol{r}) = \arg\max_{\omega} p_{\boldsymbol{Y}}(\boldsymbol{y}; \omega, \boldsymbol{r}) \neq \arg\max_{\omega} p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{t}, \boldsymbol{u}; \omega).$$

Generalized Transformation

- **What if the transformation is a function of unkown ω?**

$$\begin{aligned} p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r}) &= |J_{\boldsymbol{g}}(\boldsymbol{y};\omega)| p_{\boldsymbol{T},\boldsymbol{U}}(\boldsymbol{g}_{\boldsymbol{T}}(\boldsymbol{y};\omega),\boldsymbol{g}_{\boldsymbol{U}}(\boldsymbol{y};\omega);\omega,\boldsymbol{r}) \\ &= |J_{\boldsymbol{g}^{-1}}(\boldsymbol{t},\boldsymbol{u};\omega)|^{-1} p_{\boldsymbol{T},\boldsymbol{U}}(\boldsymbol{t},\boldsymbol{u};\omega,\boldsymbol{r}) \end{aligned}$$

- **What will happen to strong ancillarity?**

$$\begin{aligned} p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r}) &= |J_{\boldsymbol{g}^{-1}}(\boldsymbol{t},\boldsymbol{u};\omega)|^{-1} p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{t},\boldsymbol{u};\omega)\, p_{\boldsymbol{U}}(\boldsymbol{u};\boldsymbol{r}) \\ &= |J_{\boldsymbol{g}}(\boldsymbol{y};\omega)| p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{g}_{\boldsymbol{T}}(\boldsymbol{y};\omega),\boldsymbol{g}_{\boldsymbol{U}}(\boldsymbol{y};\omega);\omega)\, p_{\boldsymbol{U}}(\boldsymbol{g}_{\boldsymbol{U}}(\boldsymbol{y};\omega);\boldsymbol{r}). \end{aligned}$$

  - ❑ The property of strong ancillarity does not hold any more:

$$\hat{\omega} = \arg\max_{\omega} p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r}) \neq \arg\max_{\omega}\; p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{t},\boldsymbol{u};\omega).$$

  - ❑ We may content with finding a suboptimal solution

$$\hat{\omega} = \arg\max_{\omega} |J_{\boldsymbol{g}}(\boldsymbol{y};\omega)| p_{\boldsymbol{T}|\boldsymbol{U}}(\boldsymbol{g}_{\boldsymbol{T}}(\boldsymbol{y};\omega),\boldsymbol{g}_{\boldsymbol{U}}(\boldsymbol{y};\omega);\omega),$$

where information in $p_U(g_U(\boldsymbol{y};\omega);\boldsymbol{r})$ is ignored.

Correct Derivation – Conditional Inference (1/2)

- **Complete transformation**
  - Independent $\boldsymbol{T}$ and $\boldsymbol{U}$ can be found as

$$\begin{aligned} \boldsymbol{Y}_1 &= \boldsymbol{r} + \boldsymbol{Z}_1, \\ \boldsymbol{Y}_2 &= \boldsymbol{r}e^{j\omega} + \boldsymbol{Z}_2, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \boldsymbol{T} &\triangleq -e^{j\omega}\boldsymbol{Y}_1 + \boldsymbol{Y}_2 = -e^{j\omega}\boldsymbol{Z}_1 + \boldsymbol{Z}_2, \\ \boldsymbol{U} &\triangleq \boldsymbol{Y}_1 + e^{-j\omega}\boldsymbol{Y}_2 = 2\boldsymbol{r} + \boldsymbol{Z}_1 + e^{-j\omega}\boldsymbol{Z}_2. \end{aligned}$$

- **Transformed joint p.d.f.**

$$p_{\boldsymbol{T},\boldsymbol{U}}(\boldsymbol{t},\boldsymbol{u};\omega,\boldsymbol{r}) = p_{\boldsymbol{T}}(\boldsymbol{t})p_{\boldsymbol{U}}(\boldsymbol{u};\boldsymbol{r}),$$

where

$$\begin{aligned} p_{\boldsymbol{T}}(\boldsymbol{t}) &= \frac{1}{(2\pi\sigma^2)^M}\exp\left(-\frac{1}{2\sigma^2}||\boldsymbol{t}||^2\right), \\ p_{\boldsymbol{U}}(\boldsymbol{u};\boldsymbol{r}) &= \frac{1}{(2\pi\sigma^2)^M}\exp\left(-\frac{1}{2\sigma^2}||\boldsymbol{u}-2\boldsymbol{r}||^2\right). \end{aligned}$$

- **The Jacobian determinant**

$$|J_{\boldsymbol{g}}| = 2^{2M}.$$

Correct Derivation – Conditional Inference (2/2)

- **Put them together**

$$\begin{aligned} p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r}) &= |J_{\boldsymbol{g}}(\boldsymbol{t},\boldsymbol{u};\omega)|p_{\boldsymbol{T},\boldsymbol{U}}(\boldsymbol{t},\boldsymbol{u};\omega,\boldsymbol{r}) \\ &= 2^{2M}\frac{1}{(2\pi\sigma^2)^M}\exp\left(-\frac{1}{2\sigma^2}||\boldsymbol{t}||^2\right)\frac{1}{(2\pi\sigma^2)^M}\exp\left(-\frac{1}{2\sigma^2}||\boldsymbol{u}-2\boldsymbol{r}||^2\right) \end{aligned}$$

- **Expressing it in $y=(y_1,y_2)$**

$$\begin{aligned} p_{\boldsymbol{Y}}(\boldsymbol{y};\omega,\boldsymbol{r}) &= 2^{2M}\frac{1}{(2\pi\sigma^2)^M}\exp\left(-\frac{1}{2\sigma^2}||-e^{j\omega}\boldsymbol{y}_1+\boldsymbol{y}_2||^2\right) \\ &\times\frac{1}{(2\pi\sigma^2)^M}\exp\left(-\frac{1}{2\sigma^2}||\boldsymbol{y}_1+e^{-j\omega}\boldsymbol{y}_2-2\boldsymbol{r}||^2\right) \end{aligned}$$

Missing Information

- **The suboptimal solution**

$$\begin{aligned} \hat{\omega} &= \arg\max_{\omega} |J_{\boldsymbol{g}}|p_{\boldsymbol{T}}(\boldsymbol{g}_{\boldsymbol{T}}(\boldsymbol{y}_1,\boldsymbol{y}_2;\omega)) \\ &= \arg\max_{\omega} 2^{2M}\frac{1}{(2\pi\sigma^2)^M}\exp\left(-\frac{1}{2\sigma^2}||-e^{j\omega}\boldsymbol{y}_1+\boldsymbol{y}_2||^2\right) \\ &= \angle\boldsymbol{y}_1^*\boldsymbol{y}_2. \end{aligned}$$

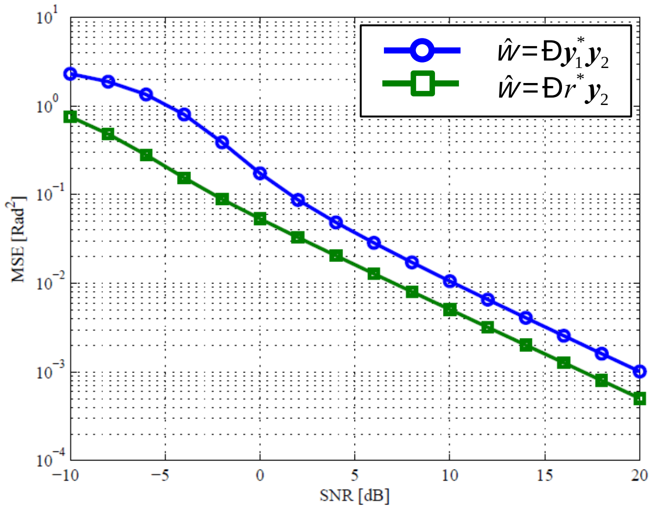
Can We Do Better Than Moose's Estimator $y_1^* y_2$?

- **Counter example showing the ML estimation does not exists: Assume that we know the nuisance parameter, r**
  - If the ML estimation exists, it should not be a function of $\boldsymbol{r}$ even if $\boldsymbol{r}$ is known.

$$Y_1 = r + Z_1,$$
$$Y_2 = re^{j\omega} + Z_2,$$

  - However, the first equation does not give any information on ω and the ML estimation for known $\boldsymbol{r}$ is given by $\hat{\omega} = r^* y_2$ only from the second equation, which does not match with Moose's solution $\hat{\omega} = y_1^* y_2$.
  - Moreover, since the ML estimation is a function of the nuisance parameter $\boldsymbol{r}$, the ML estimation does not exists.
  - The answer is "Yes" if some useful prior knowledge on $\boldsymbol{r}$ can be utilized.

- **Performance comparison**
  - 3dB gap



Reference

- **Correction Paper**
  - Dongwoon Bai, et al, “Comments on ‘A Technique for Orthogonal Frequency Division Multiplexing Frequency Offset Correction’,” *IEEE Transactions on Communications*, Vol. 61, no. 5, pp. 2109-2111, May 2013.

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