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# LINEAR STABILITY ANALYSIS FOR THE HARTMANN FLOW WITH INTERFACIAL SLIP

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We study linear stability of the Hartmann flow with the interfacial slip between the flowing liquid and the solid wall using the slip length model. To address the effect of the slip length, the eigenvalue problem for the modified Orr-Sommerfeld equation is solved by a MATLAB code using the Chebyshev collocation method. The flow stability is addressed for both symmetric and asymmetric slip conditions, using linear modal and non-modal stability analyses. Using the modal stability analysis, it has been demonstrated that the magnetic field stabilizes the flow, while the increase in slip length significantly increases the critical Reynolds number even for very small slip lengths if compared to the thickness of the Hartmann layer. The non-modal stability analysis, at sub-critical Reynolds numbers, suggests that the slip hardly affects the transient energy growth.

**1. Introduction.** The interfacial slip has been recently identified as an important phenomenon in the duct flows of the so-called Dual Coolant Lead Lithium (DCLL) blanket of the fusion reactor, where the eutectic lead-lithium (PbLi) alloy flows in contact with an insulating flow insert made of silicon carbide (SiC) for heat removal and tritium breeding [1]. Recent experimental studies [2-4] also suggest that the slip phenomenon between the flowing liquid metal and the SiC insert in the blanket would likely occur. An analytical solution for the Hartmann flow with the interfacial slip has been derived in [1] by introducing a slip length. In the same study, it is also observed that the slip length does not affect the Hartmann layer thickness. The classic "SM82" model [5] for quasi-two-dimensional magnetohydrodynamic (MHD) flows is also modified in [1] to take into account the slip effect, and it is observed that the flow demonstrates more irregular behaviour as the slip length increases.

In the present study, we are interested in the effect of the slip length on the stability of the classic Hartmann flow assuming that there is some hydrodynamic slip between the solid walls and the flowing liquid. For the case of hydrodynamic flows, this type of analysis is carried out for the plane Poiseuille flow [6-7] for both symmetric and asymmetric slip conditions. The study by Lauga *et al.* [6] concluded that the slip strongly stabilized the flow, while an extended study by Ling *et al.* [7] showed that the slip played a dual role by either stabilizing or destabilizing the flow, depending on the slip length. For the case of Hartmann flow with no-slip, Lock [8] carried out a stability analysis and found that the critical Reynolds number $\mathrm{Re_c}$ becomes larger than $10^6$ when the Hartmann number Ha is larger than 20. Takashima [9] reexamined the problem in greater detail and found that the critical Reynolds number varies as $\mathrm{Re_c} = 48311.016\,\mathrm{Ha}$, and the critical wave number as $\alpha_c = 0.161531\,\mathrm{Ha}$.

Linear stability analysis (modal) and energy methods are two standard tools for studying the stability of viscous channel flows. Linear stability analysis involves examining the evolution of small perturbations by linearizing the Navier-Stokes equations and yields the Orr–Sommerfeld (O-S) equation. Stability is then determined by examining the eigenvalues of the O-S operator. Energy methods are based on variational techniques and yield conditions for no energy growth for perturbations of arbitrary amplitude. The present study is reduced to the linear stability method, including both modal and non-modal stability analyses.

First, we study the effect of the slip length on the stability of the Hartmann flow by modifying the Orr–Sommerfeld problem to account for the slip effect. A MATLAB code based on the Chebyshev collocation method [10] has been developed and validated against available modal stability results for both hydrodynamic [11] and MHD flows [8, 9] without the slip. After that, the code is applied to the reference slip flow with a magnetic field. The analysis is carried out for both symmetric and asymmetric slip conditions. The results presented in the form of neutral stability curves are compared with analogous results for hydrodynamic flows with and without the slip and also against the classic Hartmann problem without the slip. Second, we perform the non-modal stability analysis to study the effect of the slip length on the transient energy growth at sub-critical Reynolds numbers.

**2. Formulation of the problem.** We consider fully-developed pressure-driven ($\mathrm{d}P/\mathrm{d}x$ is the pressure gradient) flows of an incompressible, viscous, electrically conducting liquid ($\rho$, $\nu$, $\sigma$ are, respectively, the fluid density, kinematic viscosity and the electrical conductivity) between two infinite non-conducting parallel plates in the presence of a uniform external magnetic field $\mathbf{B}_0$ perpendicular to the plates. This classical MHD flow is known as the Hartmann flow. The problem considered here assumes, in addition, a hydrodynamic slip at the flow confining walls, which is characterized by two slip lengths $L_1$ and $L_2$. In this paper, this problem is referred to as the Hartmann flow with the slip. The coordinate origin is taken midway between the plates with the $x$-axis in the direction of the flow and $y$-axis perpendicular to the flow. The plates at $y = \pm b$ are assumed to be electrically insulating. Providing the flow is steady, the problem is governed by two one-dimensional equations for the streamwise component of the velocity $U(y)$ and induced magnetic field $b_x(y)$:

$$0 = 1 + \frac{\mathrm{d}^2 U}{\mathrm{d}y^2} + \mathrm{Ha}\frac{\mathrm{d}b_x}{\mathrm{d}y}, \tag{1}$$

$$0 = \frac{\mathrm{d}^2 b_x}{\mathrm{d}y^2} + \mathrm{Ha}\frac{\mathrm{d}U}{\mathrm{d}y}. \tag{2}$$

The equations are written in the dimensionless form. The velocity is scaled by $[U] = b^2\rho^{-1}\nu^{-1}(-\mathrm{d}P/\mathrm{d}x)$; the induced magnetic field by $[b_x] = [U]\mu_0\sqrt{\sigma\rho\nu}$ ($\mu_0$ is the magnetic permeability); and half of the distance between the plates $b$ is used as the length scale. If we define the dimensionless slip lengths as $\lambda_1 = L_1/b$ and $\lambda_2 = L_2/b$, the slip boundary conditions are

$$y = +1: \quad U + \lambda_1\frac{\mathrm{d}U}{\mathrm{d}y} = 0, \quad b_x = 0; \tag{3a}$$

$$y = -1: \quad U - \lambda_2\frac{\mathrm{d}U}{\mathrm{d}y} = 0, \quad b_x = 0. \tag{3b}$$

The solution for the velocity has been derived in the following form

$$U(y) = (K_1K_2\mathrm{sh}(y\mathrm{Ha}) + K_2\mathrm{ch}(y\mathrm{Ha}) + 1)\,C, \tag{4}$$

where

$$K_1 = \frac{-(\lambda_1 - \lambda_2)\mathrm{Ha}\,\mathrm{sh}(\mathrm{Ha})}{2\mathrm{sh}(\mathrm{Ha}) + (\lambda_1 + \lambda_2)\mathrm{Ha}\cos(\mathrm{Ha})},$$

$$K_2 = \frac{-2}{2\mathrm{ch}(\mathrm{Ha}) + (\lambda_1 + \lambda_2)\mathrm{Ha}\,\mathrm{sh}(\mathrm{Ha}) + (\lambda_1 - \lambda_2)K_1\mathrm{Ha}\,\mathrm{ch}(\mathrm{Ha})},$$

$$C = -\frac{1}{K_2\mathrm{Ha}\,\mathrm{sh}(\mathrm{Ha})}.$$

To study the stability of the flow with the basic velocity profile in the form of Eq. (4), we write the total velocity as the sum of the basic flow and small perturbations. As the total and the basic flow both satisfy the slip conditions, the boundary conditions for the perturbed flow are also of the form of Eq. (3). Following the procedure in [9], the modified Orr–Sommerfeld equation has been derived as

$$\left(\overline{U} - \frac{\omega}{\alpha}\right)\left(D^2 - \alpha^2\right)\phi - D^2\overline{U}\phi = \frac{1}{i\alpha\mathrm{Re}}\left[\left(D^2 - \alpha^2\right)^2\phi - \mathrm{Ha}^2 D^2\phi\right] \tag{5}$$

with the boundary conditions

$$y = +1: \quad \phi = D\phi + \lambda_1 D^2\phi = 0, \tag{6a}$$

$$y = -1: \quad \phi = D\phi - \lambda_2 D^2\phi = 0, \tag{6b}$$

where $\overline{U} = U(y)/U(y=0)$, $\omega = \omega_r + i\omega_i$ is a complex number (with $\omega_i$ being the wave amplification factor), $\alpha$ is the wave number, $D$ is the differentiation matrix, $\phi$ is the stream function, and $\mathrm{Re} = U(y=0)b/\nu$ is the Reynolds number. Equations (5) and (6) govern the eigenvalue problem, which has to be solved for the eigenvalue $\omega$.

A MATLAB code based on the Chebyshev collocation method [10] is modified to calculate the eigenvalues in the reference case. The new code solves Eq. (5) with the slip boundary conditions in the form of Eq. (6). The code has been validated against available hydrodynamic [11] and MHD cases [9]. The results of the stability analysis presented below are restricted in this study to two cases: symmetric ($\lambda_1 = \lambda_2 = \mathrm{const}$) and asymmetric slip ($\lambda_1 = \mathrm{const}$, $\lambda_2 = 0$).

**3. Modal stability analysis.** First, we perform a modal stability analysis. The flow is said to be linearly unstable if there exists at least one eigenvalue with a positive imaginary part, $\omega_i > 0$. The neutral curves $\omega_i(\alpha, \mathrm{Re}) = 0$ for different values of Ha and for both symmetric and asymmetric slip cases are shown in Fig. 1. The area inside the curves corresponds to the linearly unstable flow regime, whereas outside the curves the flow is considered linearly stable. For the case of $\mathrm{Ha} = 0$, the symmetric boundary slip (Fig. 1*a*) appears to shift the neutral curves significantly towards larger values of Re, indicating a strong stabilizing effect. The same behaviour can be observed for the case of asymmetric slip, but it is less pronounced (Fig. 1*b*). In both symmetric and asymmetric cases, when Ha is increased to 10, the value of $\mathrm{Re_c}$ increases enormously (about two orders of magnitude greater if compared to $\mathrm{Ha} = 0$) due to the stabilizing effect of the magnetic field. As shown in Fig. 1*c* for the case of symmetric slip, even very small slip lengths (about one order of magnitude smaller if compared to the case of $\mathrm{Ha} = 0$) have a strong stabilizing effect. Interestingly, the stabilizing effect does not appear to be that strong for the case of asymmetric slip as observed in Fig. 1*d*. The effect of the slip becomes stronger at $\mathrm{Ha} = 20$, as seen in Fig. 1*e* for the case of symmetric slip. If the slip length changes from 0.001 to 0.002, the critical Reynolds number $\mathrm{Re_c}$ increases by one order of magnitude. For the case of asymmetric slip (Fig. 1*f*), there is no or very small change in $\mathrm{Re_c}$ as the slip length is increased. The same behaviour as that of $\mathrm{Ha} = 20$ is observed for higher values of Ha (not shown here) indicating that only the symmetric slip has a strong stabilizing effect on the Hartmann flow.

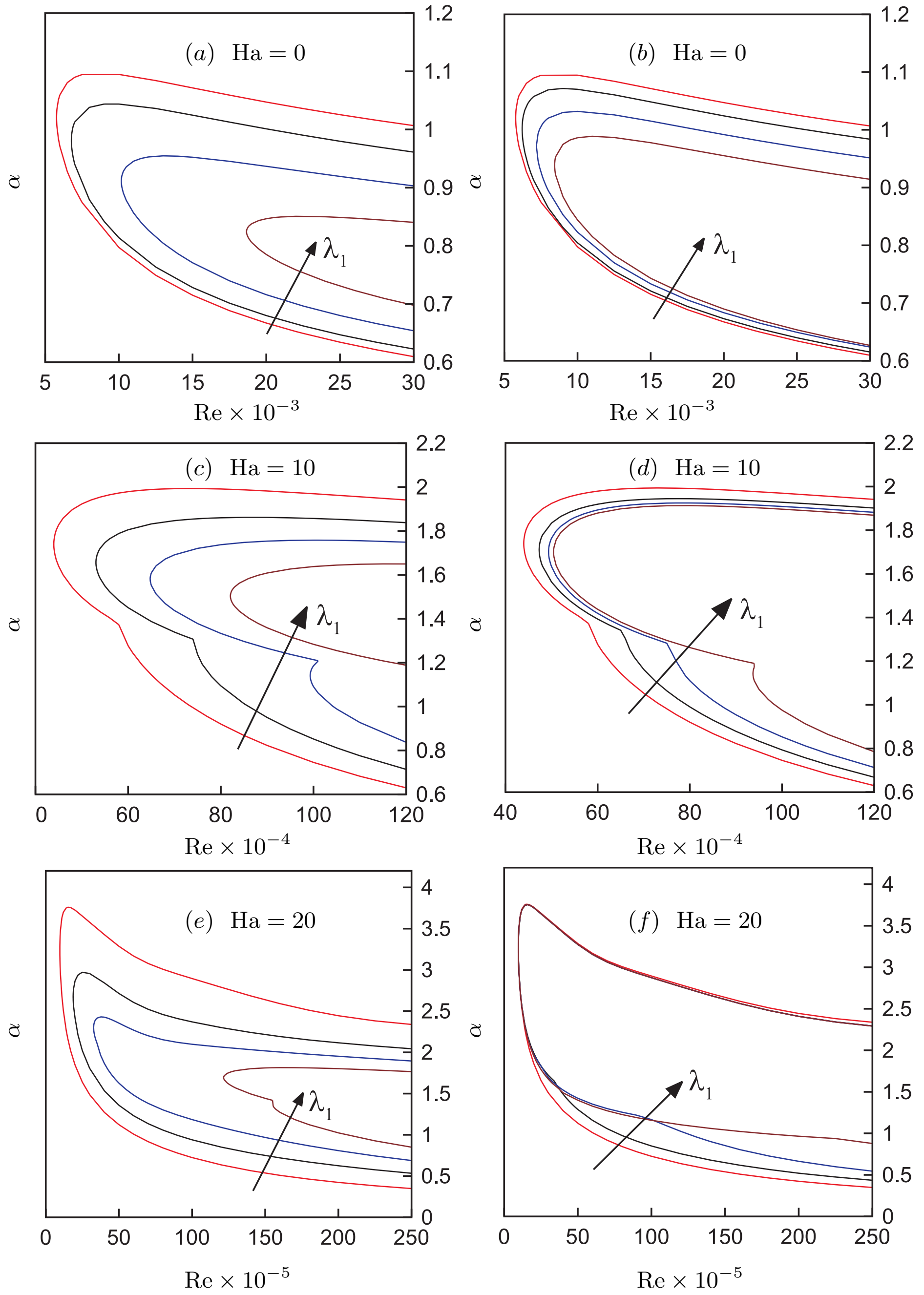


*Fig. 1.* Neutral stability curves for the symmetric (left) and asymmetric (right) slip cases: $(a)$,$(b)$ $\lambda_1 = 0, 0.01, 0.02, 0.03$; $(c)$,$(d)$,$(e)$,$(f)$ $\lambda_1 = 0, 0.001, 0.0015, 0.002$.

With the increase in magnetic field strength, the critical wave number $\alpha_c$ increases. However, $\alpha_c$ is observed to decrease with the increase of the slip length indicating that as the slip length increases, perturbations with smaller wavelengths are suppressed. Fig. 2 shows the variation of $\mathrm{Re}_c$ with the slip length for the cases

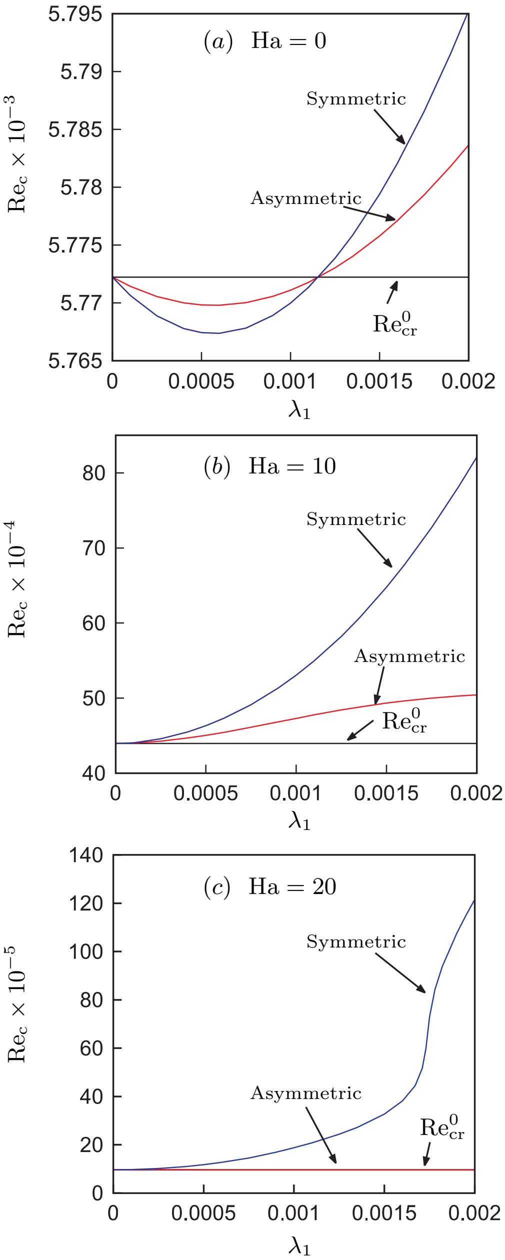


*Fig. 2.* Variation of the critical Reynolds number with the non-dimensional slip length for the symmetric and asymmetric slip cases.

with and without magnetic field. In the absence of the magnetic field (Fig. 2*a*), it appears that the slip length below a certain value destabilizes the flow ($\mathrm{Re}_{cr}^0$ is the no-slip value). Once the magnetic field is applied, this behaviour is not observed anymore. The slip always causes flow stabilization. In the asymmetric slip case, at Ha = 10 (Fig. 2*b*), the slip slightly stabilizes the flow. However, this is not the case once the Ha is increased to 20 the slip does not have any effect on the stability of the flow (Fig. 2*c*). Fig. 3 shows the variation of the critical Reynolds

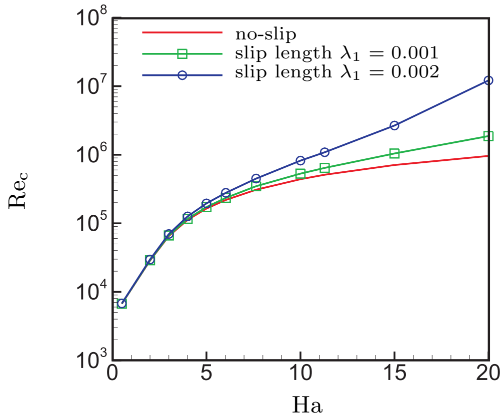


*Fig. 3.* Near exponential increase of the critical Reynolds number with the Hartmann number.

number with the Hartmann number for different slip lengths. At small Hartmann numbers (Ha < 5), there are no visible changes in $\mathrm{Re_c}$ as the slip length increases. At higher Ha, $\mathrm{Re_c}$ appears to increase exponentially with Ha for all values of the slip length.

**4. Non-modal stability analysis.** Modal linear stability analysis (critical Reynolds number $\mathrm{Re_c}$) gives conditions for exponential instability, whereas, as pointed out earlier, energy methods (critical Reynolds number $\mathrm{Re_E}$) provide conditions for no energy growth. A shortcoming of these methods is that they do not indicate if the flow is stable or unstable when the Reynolds number is within the range $\mathrm{Re_E} < \mathrm{Re} < \mathrm{Re_c}$. In this range, the energy of small perturbations decays to zero as $t \to \infty$, but there may be a transient energy growth before the decay starts. This growth occurs in the absence of nonlinear effects and is known to stem from the non-normality (i.e., not self-adjoin) of the Orr–Sommerfeld operator. For such operators, a linear combination of eigenmodes of the operator can undergo a short but possibly intense transient growth. It has been shown for the Hartmann flow [12] that such perturbations can act as finite-amplitude disturbances and destabilize the mean flow well below the linear stability threshold. To investigate various aspects of this transient growth for the present problem, we extend the standard linear modal stability analysis to the non-modal analysis to compute the maximum transient energy growth. Following Ref. [6], we define, for a given Fourier mode ($\hat{u}$), the instantaneous kinetic energy of the flow perturbation as

$$E(t,\alpha,\beta,\hat{u}) \cong \int_{-1}^{1} |\hat{u}(\alpha,y,\beta,t)|^2 \,\mathrm{d}y, \tag{7}$$

which is a function of time ($t$), initial condition ($\hat{u}_0$), streamwise wave number ($\alpha$), and spanwise wave number ($\beta$). The energy growth at the time t maximized over all non-zero initial conditions is defined as

$$G(t,\alpha,\beta) = \max_{\hat{u}_0 \neq 0} \left[ \frac{E(t,\alpha,\beta,\hat{u}_0)}{E(0,\alpha,\beta,\hat{u}_0)} \right], \tag{8}$$

then the maximum transient energy growth possible over all times $G_{\max}(\alpha,\beta)$ is defined as

$$G_{\max}(\alpha,\beta) = \max_{t \geq 0} G(t,\alpha,\beta). \tag{9}$$

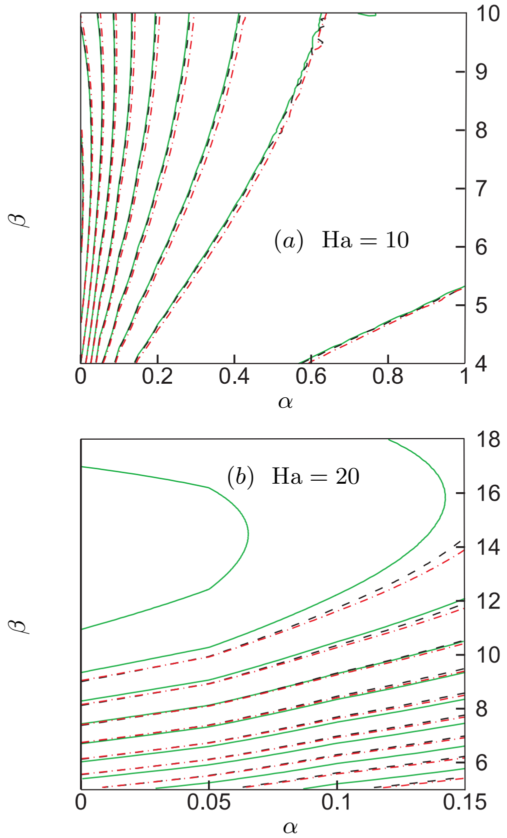


*Fig. 4.* Isovalues of $G_{\max}(\alpha, \beta)$ for Re = 40 000 in three cases: no-slip (solid), symmetric slip with $\lambda_1 = 0.001$ (dash) and with $\lambda_1 = 0.003$ (dash dot). The values of $G_{\max}$ are (*a*) 1000, 3000 to 11000, (*b*) 800 to 2400 in steps of 200 from outer to inner curve.

Fig. 4 shows the isovalues of $G_{\max}(\alpha, \beta)$ at Re = 40 000 for different slip lengths. For the case of Ha = 10 (Fig. 4*a*), the increase in maximum energy growth in the presence of the slip is very small if compared to the no-slip case. As Ha is increased to 20 (Fig. 4*b*), this increase in maximum energy growth with the slip is detectable, but still small. Therefore, the effect of the slip on the maximum energy growth slightly increases with the Hartmann number. For the Hartmann numbers and slip lengths considered in the present study, this increase is small and, therefore, the slip hardly affects the transient energy growth. The optimal maximum transient energy growth for Ha = 10 is obtained at $\alpha = 0$ and $\beta = 6.8$ for both slip and no-slip conditions, whereas for Ha = 20, the corresponding values are $\alpha = 0$ and $\beta = 13.65$. Fig. 5 illustrates the maximum energy growth as a function of time at Re = 40 000 for Ha = 10 (Fig. 5*a*) and Ha = 20 (Fig. 5*b*) for different slip lengths. Again, the increase of optimal growth with the slip length can be observed only when Ha is increased from 10 to 20. As the square root of the maximum energy growth depends linearly on the Reynolds number [13], this suggests that flows with the slip experience higher energy growth compared to no-slip flows, providing other conditions (e.g., the pressure head) are the same. In fact, this behaviour of slightly higher energy growth in the flows with the slip has been observed earlier in the plane Poiseuille flow in [6].

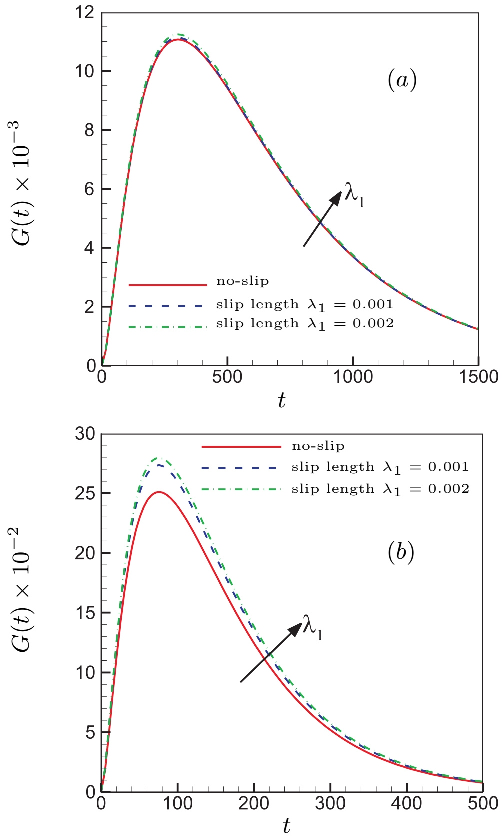


*Fig. 5.* Maximum energy growth rate at Re = 40 000 with different slip lengths in the symmetric slip case. ($a$) $G(\alpha = 0, \beta = 6.8)$ for Ha = 10; ($a$) $G(\alpha = 0, \beta = 13.65)$ for Ha = 20.

**5. Conclusions.** The effect of the slip on the stability of the Hartmann flow is analyzed by performing linear modal and non-modal stability analyses. It is observed that the slip strongly stabilizes the flow even for very small values of the slip length. This behaviour is different from that of hydrodynamic flows, where the slip plays a dual role of either stabilizing or destabilizing the flow depending on the slip length. It is also found that depending on the Hartmann number, the asymmetric slip has a very small or no effect on the flow stability. The stabilizing effect of the slip on the MHD flow is strong even though the slip length is very small if compared to the thickness of the Hartmann layer. The non-modal stability analysis suggests that even though the slip results in an increase of the transient energy growth for Ha > 10, this increase is very small, and it can be said that the slip hardly affects the transient energy growth.

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