Constructing $\aleph_k$-free Structures

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Dedicated to my parents.

Der fliegende Frosch

Wenn einer, der mit Mühe kaum
Gekrochen ist auf einen Baum,
Schon meint, dass er ein Vogel wär,
So irrt sich der.

Wilhelm Busch

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Contents

1 Introduction

1.1 Our aims

Our notation is standard as can be found in [13, 17, 32, 39]. We will write maps on the right.

How to read this thesis

This thesis is not just about two algebraic constructions (to be covered in the next two sections), but also about a number of deeper structural results and achievements. Therefore, while it might seem most natural to read this work in chronological order, we will give additional reading directions for the reader interested in some of the more specific aspects of the proofs.

ZFC and $\aleph_k$-free structures

The reader will most likely be interested in understanding the basic principles of $\aleph_k$-free constructions in ZFC for positive integers $k > 1$, which is precisely what this thesis is about. The already existing $\aleph_k$-free constructions [27, 30, 48] have served here as the starting point of our research, which were carefully analyzed to evaluate the function and importance of each element of the construction. After replacing some obsolete ideas by new ones, we extracted a construction procedure which is considerably simple to handle, paying particular attention to having the algebraic components (e.g. B, G_{X_*X}) separated from the combinatorial ones (e.g. the various closure conditions). The result is this guideline for $\aleph_k$-free constructions, which, similarly to [13, 32, 33], uses an approach via explanatory examples. The reader interested in $\aleph_k$-free constructions should start looking at Chapter 6 to get a first impression of how very little work is actually necessary once the machinery is set up properly. After this, Chapter 6 should be read and verified parallel to Chapters 2 to 5.

The importance of Black Boxes

There is a certain peculiarity about constructions in ZFC: While the prediction of unwanted homomorphisms is usually the imminent consequence of additional set theoretic restrictions (like MA) or can be formulated as a simple universal combinatorial principle (like $\diamondsuit$ in V=L), this seems not to be the case for constructions within ZFC. Here, prediction seems only possible with the help of some highly sophisticated contraptions, called Black Boxes, which must be adapted every single time to the particular algebraic setting of the construction. To complicate matters further, Black Boxes also come in different "flavours", depending on the intended strength of the prediction to be achieved. The generally accepted explanation for this rather technical approach is that predictions in ZFC need more efforts due to the lack of any helpful additional axioms. But are these efforts unavoidable? To the contrary, you might as well argue that a construction taking place in ZFC (like any other common mathematical argument) should show as little set theory as possible, and in this thesis we assert a slightly provocative statement:

All we really need is the Easy Black Box.

Indeed, instead of a more technical General Black Box or Strong Black Box we just use the Easy Black Box, which appears in its basic combinatorial form in Proposition 4.7 and in simple adaptations to the algebraic settings in Lemmas 5.5 and 6.10. This major achievement of simplification comes at a price: We need the significantly stronger Step Lemmas 4.16 and 6.8, which combine a very restrictive choice of correction elements with shift isomorphisms and a completely new closure property to obtain a particularly powerful killing statement. This shows that we may actually reduce and replace elaborate set theoretic arguments by suitable algebraic ones, and we hope that this will make Black Box constructions even more accessible to algebraists in the future.
The reader interested in Easy Black Box constructions should review both the enhanced step lemma in Chapter 4 as well as the revised final construction in Chapter 5 for the special case $k = 1$. In this case our arguments become particularly simple: From Section 4.1 we will adopt the notion of $(1, \alpha)$-closure from Definition 4.2 and the

Lemmas 4.4 and 4.6. Furthermore, the Step Lemma 4.16 reduces drastically, as the Y_*-admissible bijection τ and the case $f > 0$ of its proof become superfluous.

1.2 On ring realizations and prediction principles

Every abelian group G induces canonically a ring $\operatorname{End} G$, its ring of endomorphisms. Looking at these two structures it is imminent to ask:

Q1. *Which rings can be realized as endomorphism rings of an abelian group?*

This question proved highly difficult, spurring a ceaseless flow of new problems and important results as well as revolutionary methods to accomplish them. A first answer was given by Corner's fundamental realization theorem in [4], showing that any countable reduced torsion-free ring is the endomorphism ring of a countable reduced torsion-free abelian group. This result was accomplished by purely algebraic methods driving the use of algebraically independent elements to its climax.
Consequences of Corner's theorem are extensive: On the one hand, they show that the class of abelian groups is highly complex. On the other hand, they provide us with a large family of distinct examples of groups which again can be used as test candidates to answer or refute other problems in turn. Corner himself applied his celebrated realization theorem in [4] to give counterexamples to Kaplansky's test problems. This approach proves successful whenever a question can be answered by realizing an endomorphism ring with extra properties. This, for example, is the case when constructing indecomposable or super-decomposable groups (see [32]), but it has been applied successfully also more recently to construct uniquely transitive groups in [35].
Subsequently, there have been various attempts to generalize and enhance Corner's result. One way is by imposing stricter restrictions on the class of groups in question. Using still purely algebraic methods, this leads to the class of cotorsion-free groups introduced by Göbel in [19]: A group G is called *cotorsion-free* if it is reduced and satisfies $\operatorname{Hom}(J_p, G) = 0$ for all primes p, where J_p denotes the ring of p-adic numbers. Cotorsion-free groups mediate between torsion-free reduced groups and slender groups, and some of their remarkable properties are listed in [32], given in the following

Lemma 1.1 *For a group G the following conditions are equivalent:*

(a) G is cotorsion-free.

(b) G contains no subgroup isomorphic to either the group $\mathbb{Q}$ of rational numbers, a cyclic p-group $\mathbb{Z}_p$, a Prüfer group $\mathbb{Z}(p^\infty)$ or the p-adic numbers J_p (p any prime).

(c) The additive group of $\operatorname{End} G$ *is cotorsion-free.*

Extending Corner's realization theorem, it was shown in [7] that every ring with cotorsion-free additive group is the endomorphism ring of some group G if and only if G is cotorsion-free. The same paper accomplishes another generalization by dropping the restriction on the cardinality of the rings realizable. This needs a significantly new approach to the problem replacing the classical arguments based on algebraically independent elements, which fail for cardinals larger than the continuum $2^{\aleph_0}$. The groundbreaking new idea is utilizing set theory to achieve a short list of test functions which can be used to both predict and kill unwanted endomorphisms. A very tempting way of obtaining such *prediction principles* is adding suitable axioms to ZFC. Most notably, assuming $\mathrm{V}=\mathrm{L}$ by adding Gödel's axiom of constructibility to ZFC implies that Jensen's diamond principle $\diamondsuit_\lambda$ holds for a suitable list of Jensen functions for any regular uncountable cardinal λ. This can be used, for example, to sharpen previous results by constructing λ-free groups G of cardinality λ with $\operatorname{End} G = \mathbb{Z}$ for any not weakly compact regular cardinal λ. Here a group G is called λ-free, if every subgroup of cardinality $<\lambda$ is free. As shown more recently in [49] the condition $\mathrm{V}=\mathrm{L}$ for the existence of $\diamondsuit_\lambda$ can be weakened to $\mathrm{ZFC} + (2^\lambda = \lambda^+ \wedge \operatorname{cf}\lambda > \aleph_0)$ which reveals the true nature of prediction principles:

Prediction principles impose restrictions on cardinals and on cardinal arithmetics.

But how much of this can be actually proven if we insist on staying in ordinary set theory of ZFC? Here the class of $\aleph_1$-free groups has been the subject of extensive research starting with the discovery that the Baer-Specker group $\mathbb{Z}^\omega$ and also all Whitehead groups are $\aleph_1$-free, see [2, 12, 51, 52]. A prediction principle suitable for constructing $\aleph_1$-free groups is Shelah's Black Box which has been used in [5] to realize endomorphism rings of $\aleph_1$-free groups. A byproduct of this particular prediction principle is that the groups constructed are always of size $\lambda^{\aleph_0} = \lambda$ (hence of minimal size $2^{\aleph_0} = \beth_1$) and fail to be even $\aleph_2$-free. While this covers the range of large $\aleph_1$-free groups in ZFC, there has been also significant success in showing a rich structure theory of small $\aleph_1$-free groups. In analogy to Corner's realization theorem, it was proven in [28, 29] that every countable ring with free additive structure is the endomorphism ring of an $\aleph_1$-free group of cardinality $\aleph_1$. Furthermore, $\aleph_1$-separable counterexamples to Kaplansky's test problems of cardinality $\aleph_1$ were constructed in [14] and almost free E-rings of size $\aleph_1$ were given in [31].

This situation changes drastically when passing to groups of higher degrees of freeness and results here are only meager and far less enthusiastic: Examples of non-free $\aleph_k$-free groups of cardinality $\aleph_k$ $(k \in \omega)$ were first given in 1972/74 by Griffith and Hill [34, 38], but every attempt to even construct $\aleph_2$-free groups with any additional algebraic properties has been unsuccessful during the upfollowing 40 years – and much worse, the existence of indecomposable $\aleph_2$-free groups of cardinality $\aleph_2$ has been shown to be undecidable, see [28]. Quite the same, there is evidently no known prediction principle to achieve $\aleph_2$-free groups or any higher degrees of freeness.

Significant progress on this matter has been made only recently. Examples of $\aleph_k$-free groups G of size $\beth_k$ $(k \in \omega)$ with trivial dual $G^* = \mathrm{Hom}(G, \mathbb{Z}) = 0$ were given in [30, 48] and rings with free additive structure were successfully realized as endomorphism rings of $\aleph_k$-free groups $(k \in \omega)$ in [27]. In this thesis, we want to revisit and analyze these three papers and their basic ideas. Our goal is to extract some universal principles directly adaptable to $\aleph_k$-free group constructions $(k \in \omega)$ of minimal size $\beth_k$.

For this we must overcome certain obvious obstacles of these three preceding papers: In [30, 48], it was made a significant use of the fact that homomorphisms into a small group like $\mathbb{Z}$ are considerably simple to predict, and although the essential ideas of the freeness-lemma and the Easy Black Box appear already in this early work the provided arguments seem far too weak to apply to other situations. In [27], new arguments for predicting arbitrary group endomorphisms were developed, based on an exceedingly complex and technical analysis of the underlying quotient group structure of the groups in question. Copying these concepts, even to slightly modified situations, is undesirably tedious and the task of attacking more advanced and difficult group constructions seems impossible. Consequently, due to these difficulties, we also did not manage to construct $\aleph_k$-free groups of minimal size $\beth_k(\mu)$ but had to settle for some larger cardinal $\beth_k^+(\mu)$ instead, where $\beth_k(\mu)$ and $\beth_k^+(\mu)$ are defined inductively by $\beth_0(\mu) = \mu, \beth_{n+1}(\mu) = 2^{\beth_n(\mu)}$ and $\beth_0^+(\mu) = \mu, \beth_{n+1}^+(\mu) = (2^{\beth_n^+(\mu)})^+$. More precisely, the main result of [27] is

Main Theorem 1.2 (ZFC) *Let R be a cotorsion-free $\mathbb{S}$-ring and A an R-algebra with $|A| \leq \mu$ and free R-module A_R. If $\lambda = \beth_k^+(\mu)$ for some positive integer k, then we can construct an $\aleph_k$-free A-module G of cardinality λ with R-endomorphism algebra* $\operatorname{End}_R G = A$.

In a first step, we want to sharpen this result while at the same time slightly restricting our conditions on the ring R.

Main Theorem 1.3 (ZFC) *Let R be an $\mathbb{S}$-ring with $\pi R \cap R = \{0\}$ for some $\pi \in \widehat{R}$ and A an R-algebra with $|A| \leq \mu$ and A_R an $\aleph_k$-free R-module for some positive integer k. If*

$$\lambda = \begin{cases} \beth_k(\mu) \\ \beth_k(\mu)^+ \end{cases} \text{for } \beth_k(\mu) \begin{cases} \text{regular,} \\ \text{singular,} \end{cases}$$

then we can construct an $\aleph_k$-free A-module G of cardinality λ with R-endomorphism algebra $\operatorname{End}_R G = A$.

Remark 1.4 *Assuming that A is countable, then the smallest examples of $\aleph_k$-free A-modules G in Theorem 1.3 have size $|G| = \beth_k$.*

The proof of this new result will not only drastically shorten and simplify our previous arguments from [27], but we also intend to isolate all the tools necessary for $\aleph_k$-free constructions while presenting them in such a transparent way that their purpose and interaction becomes evident. A consequence of this will be a considerably simple general agenda to turn automatically any known $\aleph_1$-free Black Box construction into an $\aleph_k$-free Black Box construction ($k \in \omega$) and we expect future constructions in ZFC to either take place directly in a general $\aleph_k$-free setting or to include the $\aleph_k$-free case as a mere footnote. With other words, we intend to settle the case of $\aleph_k$-free constructions ($k \in \omega$), and we will demonstrate this by including the construction for $\aleph_k$-free E-rings as Chapter 6 of this thesis.

This of course is not the end to the research of μ-free constructions in ZFC, but opens the door for attempting $\aleph_\omega$-free group constructions next. Or, if we want to follow the ideas of [50], we may as well formulate even more ambitious goals by reformulating our question in the form of two conjectures of set theoretic interest:

Q2. *For which cardinals $\lambda \geq \mu$ does the* trivial dual conjecture $\mathrm{TDU}_{\lambda\mu}$ *hold: There exists some μ-free group G of cardinality λ with $G^* = 0$?*

Q3. *For which cardinals $\lambda \geq \mu$ does the* trivial endomorphism conjecture $\mathrm{TED}_{\lambda\mu}$ *hold: There exists some μ-free group G of cardinality λ with* $\mathrm{End}\, G = \mathbb{Z}$*?*

We will also drop λ and write TDU_μ and TED_μ if we are not interested in the actual size of the cardinal $\lambda \geq \mu$. Results concerning $\mathrm{TDU}_{\lambda\mu}$ and $\mathrm{TED}_{\lambda\mu}$ in ZFC can be summarized as follows:

- $\mathrm{TED}_{\lambda\mu}$ implies $\mathrm{TDU}_{\lambda\mu}$.
- $\mathrm{TED}_{\aleph_1,\aleph_1}$ holds according to [28, 29] while $\mathrm{TDU}_{\aleph_1,\aleph_1}$ was shown in [11].
- $\mathrm{TDU}_{\lambda\lambda}$ and $\mathrm{TED}_{\lambda\lambda}$ are undecidable for $\lambda \geq \aleph_2$ regular: λ-free groups G of cardinality λ with $\mathrm{End}\, G = \mathbb{Z}$ can be constructed under $\mathrm{V} = \mathrm{L}$ using $\diamondsuit_\lambda$, while assuming $\mathrm{ZFC} + \mathrm{MA} + (\lambda < 2^{\aleph_0})$ every $\aleph_2$-free group of cardinality λ is separable, see [28]. Furthermore, an even stronger statement of undecidability was given in [41] by showing that every $\aleph_{\omega^2+1}$-free group of size $\aleph_{\omega^2+1}$ is free in a suitable forcing model of set theory.
- $\mathrm{TDU}_{\lambda\lambda}$ fails for singular λ as every λ-free group of cardinality λ is free by Shelah's singular compactness theorem, see [46].
- $\mathrm{TDU}_{\beth_k,\aleph_k}$ ($k \in \omega$) holds by [30, 48].

- $\mathrm{TED}_{\beth_k^+,\aleph_k}$ ($k \in \omega$) holds by Main Theorem 1.3.

In [50], it was shown that $\mathrm{TDU}_{\aleph_\omega}$ holds in almost every model of set theory and that a failure of $\mathrm{TDU}_{\aleph_\omega}$ imposes very strict restrictions on cardinal arithmetics which seem unlikely to be met. The question whether $\mathrm{TDU}_{\aleph_\omega}$ holds in ZFC is still open.

1.3 On E-rings

The research of E-rings and related structures originates from Problem 45 (p. 232) in Fuchs [16], a question concerning rings A and the endomorphism ring of their additive structure A^+:

Characterize the rings A with $\operatorname{End} A^+ \cong A$.

Rings which satisfy this general isomorphism relation have later on been called generalized E-rings to distinguish them from E-rings, a notion introduced by Schultz in his fundamental paper [45]. His definition, formulated in the context of R-algebras over commutative rings R with 1, reads as follows.

Definition 1.5 *(a) An R-algebra A is called an* $E(R)$-algebra *and its R-module structure A_R is called an* $E(R)$-module *if the evaluation map* $\operatorname{End}_R A \longrightarrow A$ $(\varphi \mapsto 1\varphi)$ *is an R-algebra isomorphism. Furthermore, $E(\mathbb{Z})$-algebras are called E-rings and $E(\mathbb{Z})$-modules are called E-groups.*

(b) An R-algebra A is called a generalized $E(R)$-algebra *if* $\operatorname{End}_R A \cong A$.

A generalized $E(R)$-algebra is an $E(R)$-algebra if and only if it is a commutative R-algebra. Fundamental results on $E(R)$-algebras and $E(R)$-modules and their classification can be found in [20, 40, 42, 43]. The existence of arbitrarily large E-rings is a combined result from [10, 15], where Faticoni [15] covers the case of ranks $\leq 2^{\aleph_0}$, while Dugas, Mader, Vinsonhaler [10] covers the case of all larger cardinals $\lambda^{\aleph_0} = \lambda$, which was soon thereafter generalized to E-groups [6]. First examples of non-commutative (and therefore proper) generalized $E(R)$-algebras were given in [25]. For a survey and some more recent results on $E(R)$-algebras we refer to [32, 53].

The research interest in $E(R)$-algebras was fueled by the reappearing of these algebras in connection with seemingly completely unrelated problems: They often work as catalysts, facilitating a variety of difficult R-module constructions by adding an adequate R-algebra structure. This was used to construct nilpotent groups of class 2 in [9], superdecomposable modules in [18] and uniquely transitive modules in [21, 22, 23, 35]. Furthermore, E-rings allow for a natural interpretation as localizations in the context of topological algebra, see [3]. And finally, $E(R)$-algebras prove to be a very suitable test case for probing the applicability and strength of new construction methods, expanding for instance the scope of classical Black Box constructions in [10] as well as of absolute constructions in [26, 36, 37]. This time we want to add as new result

Main Theorem 1.6 (ZFC) *Let R be a cotorsion-free $\mathbb{S}$-ring and $\pi \in \widehat{R}$ some transcendental element over R, let $|R| \leq \mu$ be a cardinal and k be a positive integer. Then with*

$$\lambda = \begin{cases} \beth_k(\mu) \\ \beth_k(\mu)^+ \end{cases} \text{for } \beth_k(\mu) \begin{cases} \text{regular} \\ \text{singular} \end{cases}$$

we can construct an $\aleph_k$-free $E(R)$-algebra A of cardinality λ.

This theorem can be viewed as a direct continuation of Dugas, Mader, Vinsonhaler [10], generalizing their result from the $\aleph_1$-free case to the $\aleph_k$-free case.

2 The basics for the construction

2.1 Set theoretic preliminaries

The Easy Black Box in Chapter 4 depends on a finite sequence of cardinals satisfying some cardinal conditions. Thus, we will fix a positive integer k and a sequence $\overline{\lambda} = \langle \lambda_1, \dots, \lambda_k \rangle$ of cardinals such that

(i) $|\, A \,| < \lambda_1 = \lambda_1^{|\, A\,|}$ and

(ii) $\lambda_{\ell+1} = \lambda_{\ell+1}^{\lambda_\ell}$ for $1 \leq \ell < k$.

This implies that $\lambda_{\ell+1}^+ = (\lambda_{\ell+1}^+)^{\lambda_\ell}$ for $1 \leq \ell < k$; see the Hausdorff formula [39, p. 57, (5.22)].

If λ is a cardinal, then ${}^{\omega\uparrow}\lambda$ will denote all *order preserving* maps $\eta : \omega \to \lambda$, which we also call *infinite branches* on λ, while ${}^{\omega\uparrow>}\lambda$ denotes the family of all order preserving *finite branches* $\eta : n \to \lambda$ on λ, where the natural number n, λ and ω (the first infinite ordinal) are considered as sets, e.g. $n = \{0, \dots, n-1\}$, thus the finite branch η has length n.

Moreover, we associate with $\overline{\lambda}$ two sets Λ and Λ_*. Thus, let

$$\Lambda = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow}\lambda_k. \tag{2.1}$$

For the second set we replace the m-th (and only the m-th) coordinate ${}^{\omega\uparrow}\lambda_m$ by the finite branches ${}^{\omega\uparrow>}\lambda_m$, i.e., we define

$$\Lambda_{m*} = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_k \text{ for } 1 \leq m \leq k \text{ and let } \Lambda_* = \dot{\bigcup}_{1\leq m \leq k} \Lambda_{m*}. \tag{2.2}$$

The elements of Λ, Λ_* will be written as sequences $\overline{\eta} = (\eta_1, \dots, \eta_k)$ with $\eta_\ell \in {}^{\omega\uparrow}\lambda$ or $\eta_\ell \in {}^{\omega\uparrow>}\lambda$ (for $1 \leq \ell \leq k$), respectively.

With each member of Λ we can associate a subset of Λ_*:

Definition 2.1 *If $\overline{\eta} = (\eta_1, \dots, \eta_k) \in \Lambda$ and $1 \leq m \leq k, n < \omega$, then let $\overline{\eta} \upharpoonright \langle m, n \rangle$ be the following element in Λ_{m*} (thus in Λ_*)*

$$(\overline{\eta} \upharpoonright \langle m, n \rangle)_\ell = \begin{cases} \eta_\ell & \text{if } 1 \leq \ell \neq m \leq k \\ \eta_m \upharpoonright n & \text{if } \ell = m. \end{cases}$$

We associate with $\overline{\eta}$ its support $[\overline{\eta}] = \{\overline{\eta}\restriction\langle m,n\rangle \mid 1 \leq m \leq k, n < \omega\}$ *which is a countable subset of Λ_*. If $S \subseteq \Lambda$, then the support of S is the set $[S] = \bigcup_{\overline{\eta}\in S}[\overline{\eta}] \subseteq \Lambda_*$.*

2.2 Algebraic preliminaries

Let R be a commutative ring with 1 and let $\mathbb{S} \subseteq R\setminus\{0\}$ be a countable multiplicatively closed subset containing 1 such that the following holds.

(i) The elements of $\mathbb{S}$ are not zero-divisors, i.e. if $s \in \mathbb{S}, r \in R$ and $sr = 0$, then $r = 0$.

(ii) $\bigcap_{s\in\mathbb{S}} sR = 0$.

We also say that R is an $\mathbb{S}$-ring. If (i) holds, then R is $\mathbb{S}$-*torsion-free* and if (ii) holds, then R is $\mathbb{S}$-*reduced*, see [32]. To ease notations, we use the letter $\mathbb{S}$ only if we want to emphasize that the argument depends on it. If M is an R-module, then these definitions naturally carry over to M. Finally, we enumerate $\mathbb{S} = \{s_n \mid n < \omega\}$ and put $q_n = \prod_{i<n} s_i$; thus, $q_0 = 1$ and $q_{n+1} = q_n s_n$.
If $G \subseteq M$, then G is $\mathbb{S}$-*pure* in M if $G \cap sM \subseteq sG$ for all $s \in \mathbb{S}$. If $G \subseteq M$ are $\mathbb{S}$-torsion-free R-modules, then G_* denotes the smallest, unique $\mathbb{S}$-pure submodule of M containing G, and we write $G \subseteq_* M$ if G is $\mathbb{S}$-pure in M.
Slightly strengthening [13] (by $\mathbb{S}$-purity) we call an R-module M κ-*free* if there is a family $\mathcal{C}_M$ of $\mathbb{S}$-pure R-submodules of M satisfying the following *Hill conditions*.

(i) Every element of $\mathcal{C}_M$ is a $< \kappa$-generated free R-submodule of M.

(ii) Every subset of M of cardinality $< \kappa$ is contained in an element of $\mathcal{C}_M$.

(iii) $\mathcal{C}_M$ is closed under unions of well-ordered chains of length $< \kappa$.

We say that $\mathcal{C}_M$ is $< \kappa$-*closed*. This definition applies for regular cardinals, in particular for $\kappa = \aleph_k$, which is the case we are interested in.
The $\mathbb{S}$-topology of an $\mathbb{S}$-reduced R-module M is generated by the basis sM $(s \in \mathbb{S})$ of neighbourhoods of 0. It is Hausdorff on M and we consider the $\mathbb{S}$-completion $\widehat{M}$ of M;

see [32] for elementary facts on the elements of $\widehat{M}$. The R-module M is *cotorsion-free* (with respect to $\mathbb{S}$) if M is $\mathbb{S}$-reduced and $\mathrm{Hom}_R(\widehat{R}, M) = 0$.

We also fix an R-algebra A with $\aleph_k$-free R-module A_R and consider A-modules. Some obvious observations with respect to R and A are:

Observation 2.2 *Let R be an $\mathbb{S}$-ring and A be an R-algebra with $\aleph_k$-free R-module A_R. Then the following holds:*

(a) A is an $\mathbb{S}$-ring.

(b) Every $\aleph_k$-free A-module is an $\aleph_k$-free R-module.

Proof. (a) is trivial. For (b), observe that every free A-module $F = \bigoplus_{i\in I} Ae_i$ is an $\aleph_k$-free R-module as witnessed by the family $\mathcal{C}_F = \bigcup_{J\subseteq I,\,|J|<\kappa} \{\bigoplus_{i\in J} A_i e_i \mid A_i \in \mathcal{C}_A\}$. Hence, if G is an $\aleph_k$-free A-module with witnessing family $\mathcal{C}_G$, then it is also an $\aleph_k$-free R-module as witnessed by the family $\bigcup_{F\in\mathcal{C}_G} \mathcal{C}_F$. ∎

Given an R-algebra A, we first define (similarly to the Black Box in [5]), the basic, free A-module B, which is

$$B = \bigoplus_{\overline{\nu}\in\Lambda_*} Ae_{\overline{\nu}}.$$

Definition 2.3 *If $X_* \subseteq \Lambda_*$, then we define $B_{X_*} = \bigoplus_{\overline{\nu}\in X_*} Ae_{\overline{\nu}}$ as the canonical summand of B associated with X_*.*

Every element $b \in \widehat{B}$ has a natural (Λ_*-)*support* $[b] \subseteq \Lambda_*$ which are those $\overline{\nu} \in \Lambda_*$ contributing to the canonical sum-representation $b = \sum_{\overline{\nu}\in\Lambda_*} b_{\overline{\nu}} e_{\overline{\nu}}$ with coefficients $0 \neq b_{\overline{\nu}} \in \widehat{A}$. Thus let $[b] = \{\overline{\nu} \in \Lambda_* \mid b_{\overline{\nu}} \neq 0\}$. Furthermore, we introduce the abbreviations $[b]_{\overline{\nu}} = b_{\overline{\nu}}$ for all $\overline{\nu} \in \Lambda_*$ and $[b]_{X_*} = \sum_{\overline{\nu}\in X_*} b_{\overline{\nu}} e_{\overline{\nu}}$ for all $X_* \subseteq \Lambda_*$. Note that $[b]$ is at most countable. If $S \subseteq \widehat{B}$, then the Λ_*-support of S is the set $[S] = \bigcup_{b\in S}[b]$. As in the earlier Black Boxes (see [32]), we use conditions on the support (given by the prediction) to select particular elements from $\widehat{B}$ added to B to get the final structure M, such that

$$B \subseteq M \subseteq_* \widehat{B}.$$

We will use B, Λ_*, Λ to define the Easy Black Box for cotorsion-free $\aleph_k$-free A-modules in Chapter 5.

3 $\aleph_k$-free A-modules

3.1 The case of finite correction

Let R be an $\mathbb{S}$-ring, A a cotorsion-free R-algebra, and let $B = \bigoplus_{\overline{\nu}\in\Lambda_*} Ae_{\overline{\nu}}$ be the A-module freely generated by $\{e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*\}$.

Next we choose particular elements from $\widehat{B}$. If $\overline{\eta} \in \Lambda$ and $i < \omega$, then we call

$$y_{\overline{\eta}i} = \sum_{n=i}^{\infty} \frac{q_n}{q_i} \Big(\sum_{m=1}^{k} e_{\overline{\eta}\restriction\langle m,n\rangle}\Big)$$

a *branch element associated with* $\overline{\eta}$. In particular let

$$y_{\overline{\eta}} = y_{\overline{\eta}0} = \sum_{n=0}^{\infty} q_n \Big(\sum_{m=1}^{k} e_{\overline{\eta}\restriction\langle m,n\rangle}\Big).$$

Given $\overline{\eta} \in \Lambda$, we also choose $b_{\overline{\eta}} \in B$, $\pi_{\overline{\eta}} = \sum_{n=0}^{\infty} q_n r_n \in \widehat{R}$ and let $\pi_{\overline{\eta}i} = \sum_{n=i}^{\infty} \frac{q_n}{q_i} r_n$. Then we define *branch-like elements* $y'_{\overline{\eta}i}$ by adding some *correction* to our branch-elements, namely

$$y'_{\overline{\eta}i} = \pi_{\overline{\eta}i} b_{\overline{\eta}} + y_{\overline{\eta}i}.$$

In particular we have $y'_{\overline{\eta}} = y'_{\overline{\eta}0} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}}$. It will be very often sufficient to choose $\pi_{\overline{\eta}} \in \{\pi, 0\}$ for some fixed element $\pi \in \widehat{R}$.

Definition 3.1 *(a) A triple $(X_*, X, \mathfrak{F})$ is called* Λ-closed*, if the following holds.*

(i) $X \subseteq \Lambda$ and $X_* \subseteq \Lambda_*$.

(ii) $[\overline{\eta}] \subseteq X_*$ for all $\overline{\eta} \in X$.

(iii) $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid [\pi_{\overline{\eta}} b_{\overline{\eta}}] \subseteq X_*, \overline{\eta} \in X\}$ is a family of branch-like elements.

(b) Similarly, a pair (X_*, X) is called Λ-closed*, if conditions (i) and (ii) hold.*

Definition 3.2 *The construction of the A-module G_{X_*X}:*
If $(X_*, X, \mathfrak{F})$ is Λ-closed, then we let

$$G_{X_*X} = \langle B_{X_*}, Ay'_{\overline{\eta}i} \mid \overline{\eta} \in X, i < \omega\rangle = \langle B_{X_*}, Ay'_{\overline{\eta}} \mid \overline{\eta} \in X\rangle_* \subseteq \widehat{B}.$$

The final R-modules in Theorem 1.3 are of the form described in Definition 3.2. We start with a simple and basic general observation which will be of some prominent use in Chapter 4.

Observation 3.3 (Recognition) *Let $g \in \widehat{B}$ and let $S_* \subseteq \Lambda_*$ be a finite set. If $g \in \langle B, Ay'_{\overline{\eta}} \mid \overline{\eta} \in \Lambda\rangle$ for a suitable choice of $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in B, \overline{\eta} \in \Lambda\}$ then there exist some unique $b \in B$ and $a_{\overline{\eta}} \in A$ ($\overline{\eta} \in \Lambda$) with $g = b + \sum_{\overline{\eta} \in \Lambda} a_{\overline{\eta}} y'_{\overline{\eta}}$. Furthermore, the elements $a_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda$) are entirely determined by $[g]_{\Lambda_* \setminus S_*}$ and independent of the choice of $\mathfrak{F}$.*

Proof. Observe that S_*, $[b]$, $[\pi_{\overline{\eta}} b_{\overline{\eta}}]$ and $[\overline{\eta}] \cap [\overline{\eta}']$ are finite for $\overline{\eta} \neq \overline{\eta}' \in \Lambda$ while $[\overline{\eta}]$ and $[\overline{\eta}']$ are infinite. Consequently, $a_{\overline{\eta}} \neq 0$ if and only if $[y_{\overline{\eta} i}] \subseteq [g] \setminus S_*$ for some $i < \omega$. Thus, from $[g]_{\Lambda_* \setminus S_*}$ we can read off all those $\overline{\eta}$ with $[y_{\overline{\eta} i}] \subseteq [g] \setminus S_*$ in a first step and then determine $a_{\overline{\eta}}$ accordingly from the coefficients appearing in $[[g]_{\Lambda_* \setminus S_*}]_{[y_{\overline{\eta} i}]}$. This argument does not make any use of the correction elements $\pi_{\overline{\eta}} b_{\overline{\eta}}$ and hence, it is independent of the particular choice of $\mathfrak{F}$. ∎

As a consequence of this observation, every element $g \in G_{X_* X}$ has apart from its Λ_*-support also a Λ-*support* $[g]_\Lambda \subseteq \Lambda$, which consists of those $\overline{\eta} \in \Lambda$ contributing coefficients $a_{\overline{\eta}} \neq 0$ to the canonical sum-representation $sg = b + \sum_{\overline{\eta} \in \Lambda} a_{\overline{\eta}} y'_{\overline{\eta}}$ with $s \in \mathbb{S}$, $b \in B$ and $a_{\overline{\eta}} \in A$. This representation of g becomes unique if we divide by s and allow coefficients from $A\mathbb{S}^{-1}$. Thus let $[g]_\Lambda = \{\overline{\eta} \in \Lambda \mid a_{\overline{\eta}} \neq 0\}$ and $[S]_\Lambda = \bigcup_{g \in S} [g]_\Lambda$ for all $S \subseteq \widehat{B}$. Furthermore, we introduce the abbreviations $[g]_{\overline{\eta}} = a_{\overline{\eta}} s^{-1}$ for all $\overline{\eta} \in \Lambda$ and $[g]_X = \sum_{\overline{\eta} \in X} a_{\overline{\eta}} s^{-1} y'_{\overline{\eta}}$ for all $X \subseteq \Lambda$. Note that $[g]_\Lambda$ is always finite.
We next prove some general basic properties of the R-modules $G_{X_* X}$. We start by showing that the previously defined R-modules are cotorsion-free and discussing the relation between this notion and $\aleph_k$-freeness.

Observation 3.4 *Let A be a cotorsion-free R-algebra.*

(a) If the $\mathbb{S}$-ring R is a countable principal ideal domain and G an $\aleph_1$-free R-module, then G is cotorsion-free.

(b) If R is an $\mathbb{S}$-ring and G is the R-module G_{X_*X} as in Definition 3.2, then G is cotorsion-free.

Proof. (a) In this case we can apply [32, p. 426, Proposition 12.3.2] (replacing the $R \setminus \{0\}$-topology by any $\mathbb{S}$-topology). Thus G is $\mathbb{S}$-cotorsion-free if and only if the quotient-field $Q(R)$, the modules R/pR and $\widehat{R}_p$ for primes p with $pR \cap \mathbb{S} \neq \emptyset$ do not embed into G. Assuming that G is $\aleph_1$-free as an R-module, by $|\, R/pR\,|, |\, Q(R)\,| < \aleph_1$ it remains to show that $\widehat{R}_p$ does not embed into G. We can choose $\pi \in \widehat{R}$ to be transcendental over R (see [32, p. 16, Theorem 1.1.20]) and consider the R-submodule $\langle 1R, \pi R\rangle_* \subseteq \widehat{R}$ which has rank 2 and is indecomposable by Baer's theorem (see [17, Vol. 2, p. 123, Theorem 88.1]). If $\widehat{R}$ embeds into G, then also $\langle 1R, \pi R\rangle_*$ embeds into G. Since G is $\aleph_1$-free, the countable R-module $\langle 1R, \pi R\rangle_*$ would be a free R-module of rank 2, which is a contradiction.

(b) Suppose $\varphi : \widehat{R} \longrightarrow G$ is a non-trivial R-homomorphism. Then $1\varphi \neq 0$ because G is reduced and φ is continuous. Choose $n < \omega$ with

$$q_n(1\varphi) = b + \sum_{\overline{\eta} \in I} a_{\overline{\eta}} y'_{\overline{\eta}} \tag{3.1}$$

such that $b \in B$, $0 \neq a_{\overline{\eta}} \in A$ for all $\overline{\eta} \in I \subseteq X$. Moreover, I is finite. For $\pi \in \widehat{R}$ with $\pi\varphi \in G$ we also have $n \leq n' < \omega$ with

$$q_{n'}(\pi\varphi) = b' + \sum_{\overline{\eta} \in I'} a'_{\overline{\eta}} y'_{\overline{\eta}} \tag{3.2}$$

such that $b' \in B$, $0 \neq a'_{\overline{\eta}} \in A$ for all $\overline{\eta} \in I'$. Moreover, I' is also finite.

Comparing (3.1) and (3.2) we get

$$q_{n'}(\pi\varphi) = \frac{q_{n'}}{q_n}\pi[b + \sum_{\overline{\eta} \in I} a_{\overline{\eta}} y'_{\overline{\eta}}] = b' + \sum_{\overline{\eta} \in I'} a'_{\overline{\eta}} y'_{\overline{\eta}}.$$

If $\overline{\eta} \neq \overline{\eta}' \in \Lambda$, then $[\overline{\eta}] \cap [\overline{\eta}']$ is finite and comparing coefficients gives $I = I'$, $\frac{q_{n'}}{q_n}\pi a_{\overline{\eta}} = a'_{\overline{\eta}}$ for all $\overline{\eta} \in I$ and therefore also $\frac{q_{n'}}{q_n}\pi b = b'$.

Using the $\mathbb{S}$-purity of $A \subseteq_* \widehat{A}$, it is immediate that $\frac{q_{n'}}{q_n}\widehat{A} \cap A = \frac{q_{n'}}{q_n}A$. Hence, $\pi a_{\overline{\eta}} \in A$ $(\overline{\eta} \in I)$ and $\pi b \in B$.

If $I \neq \emptyset$, then we can choose a homomorphism $\varphi_1 : \widehat{R} \longrightarrow A$ $(\pi \mapsto \pi a_{\overline{\eta}})$ which is not the zero-homomorphism, a contradiction (because A is cotorsion-free).
If $I = \emptyset$, then $b \neq 0$, $b = \sum_{\overline{\nu} \in J} a_{\overline{\nu}} e_{\overline{\nu}}$ with $J \neq \emptyset$ and $a_{\overline{\nu}} \neq 0$ $(\overline{\nu} \in J \subseteq X_*)$. Choose any $\overline{\nu} \in J$. Similarly, we get a homomorphism $\varphi_2 : \widehat{R} \longrightarrow A$ $(\pi \mapsto \pi a_{\overline{\nu}})$ which is not the zero-homomorphism, which is a final contradiction, showing that G is cotorsion-free. ∎

If S is any set, then $[S]^{<\aleph_0}$ denotes the collection of all finite subsets of S.

Freeness-Proposition 3.5 *Let $F : \Lambda \to [\Lambda_*]^{<\aleph_0}$ be any function, $1 \leq f \leq k$ and Ω a subset of Λ of cardinality $\aleph_{f-1}$ with a family of sets $u_{\overline{\eta}} \subseteq \{1, \dots, k\}$ satisfying $|\, u_{\overline{\eta}} \,| \geq f$ for all $\overline{\eta} \in \Omega$. Then we can find an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_{f-1} \rangle$ of Ω, $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha < \omega$ $(\alpha < \aleph_{f-1})$ such that*

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F \text{ for all } n \geq n_\alpha,$$

where $\Omega_\alpha = \{\overline{\eta}^\beta \mid \beta \leq \alpha\}$.

Proof. The proof follows by induction on f. We begin with $f = 1$, so $|\, \Omega \,| = \aleph_0$. Let $\Omega = \{\overline{\eta}^\alpha \mid \alpha < \omega\}$ be an enumeration without repetitions. From $1 = f \leq |\, u_{\overline{\eta}} \,|$ follows $u_{\overline{\eta}} \neq \emptyset$ and we can choose any $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ for all $\alpha < \omega$. If $\alpha \neq \beta < \omega$, then $\overline{\eta}^\alpha \neq \overline{\eta}^\beta$ and there is $n_{\alpha\beta} \in \omega$ such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \neq \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle$ for all $n \geq n_{\alpha\beta}$. Since $\bigcup \Omega_\alpha F$ is finite, we may enlarge $n_{\alpha\beta}$, if necessary, such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \bigcup \Omega_\alpha F$ for all $n \geq n_{\alpha\beta}$. If $n_\alpha = \max_{\beta < \alpha} n_{\alpha\beta}$, then $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F$ for all $n \geq n_\alpha$. Hence case $f = 1$ is settled and we let $f' = f + 1$ and assume that the proposition holds for f.
Let $|\, \Omega \,| = \aleph_f$ and choose an $\aleph_f$-filtration $\Omega = \bigcup_{\delta < \aleph_f} \Omega_\delta$ with $\Omega_0 = \emptyset$ and $|\, \Omega_{\delta+1} \setminus \Omega_\delta \,| = \aleph_{f-1}$ $(\delta < \aleph_f)$. The next crucial idea comes from [48] based on the construction of elementary submodels: We can also assume that the chain $\{\Omega_\delta \mid \delta < \aleph_f\}$ is *closed*, meaning that for any $\delta < \aleph_f$, $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$ and $\overline{\eta} \in \Omega$ with

$$\{\eta_m \mid 1 \leq m \leq k\} \subseteq \{\nu_m, \nu'_m, \nu''_m \mid \overline{\nu}'' \in \overline{\nu} F \cup \overline{\nu}' F, 1 \leq m \leq k\}$$

follows $\overline{\eta} \in \Omega_\delta$. Thus, if $\overline{\eta} \in \Omega_{\delta+1} \setminus \Omega_\delta$, then the set

$$u^*_{\overline{\eta}} = \{1 \leq \ell \leq k \mid \exists n < \omega, \overline{\mu} \in \Omega_\delta \text{ such that } \overline{\eta} \upharpoonright \langle \ell, n \rangle = \overline{\mu} \upharpoonright \langle \ell, n \rangle \text{ or } \overline{\eta} \upharpoonright \langle \ell, n \rangle \in \overline{\mu} F\}$$

is empty or a singleton. Otherwise there are $n, n' < \omega$ and distinct $1 \leq \ell, \ell' \leq k$ with $\overline{\eta}\upharpoonright\langle \ell, n\rangle \in \{\overline{\nu}\upharpoonright\langle \ell, n\rangle\} \cup \overline{\nu}F$ and $\overline{\eta}\upharpoonright\langle \ell', n'\rangle \in \{\overline{\nu}'\upharpoonright\langle \ell', n'\rangle\} \cup \overline{\nu}'F$ for certain $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$. Hence $\{\eta_m \mid 1 \leq m \leq k\} \subseteq \{\nu_m, \nu'_m, \nu''_m \mid \nu''_m \in \overline{\nu}F \cup \overline{\nu}'F, 1 \leq m \leq k\}$, and the closure property implies the contradiction $\overline{\eta} \in \Omega_\delta$.
If $\delta < \aleph_f$, then let $D_\delta = \Omega_{\delta+1} \setminus \Omega_\delta$ with $|\, D_\delta \,| = \aleph_{f-1}$ and $u'_{\overline{\eta}} := u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ must have size $\geq f' - 1 = f$. Thus, the induction hypothesis applies to $\{u'_{\overline{\eta}} \mid \overline{\eta} \in D_\delta\}$ for each $\delta < \aleph_f$ and we find an enumeration $\overline{\eta}^{\delta\alpha}$ $(\alpha < \aleph_{f-1})$ of D_δ as in the proposition. Finally, we put these chains for each $\delta < \aleph_f$ together with the induced ordering to get an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_f \rangle$ of Ω satisfying the proposition. ■

Freeness-Lemma 3.6 *The module G_{X_*X} from Definition 3.2 is $\aleph_k$-free as A-module.*

Proof. Any subset H of G_{X_*X} is contained in the pure A-submodule

$$G_{\Omega_*\Omega} = \langle Ae_{\overline{\nu}}, Ay'_{\overline{\eta}} \mid \overline{\nu} \in \Omega_*, \overline{\eta} \in \Omega \rangle_* \subseteq G_{X_*X},$$

where $\Omega = [H]_\Lambda$ and $\Omega_* = [H] \cup \bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}] \cup \bigcup_{\overline{\eta}\in\Omega}[\pi_{\overline{\eta}} b_{\overline{\eta}}]$ with $|\, \Omega_* \,|, |\, \Omega \,| \leq \aleph_0 \cdot |\, H \,|$. Without loss of generality, we may assume $\Omega_* = \bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}] \cup \bigcup_{\overline{\eta}\in\Omega}[\pi_{\overline{\eta}} b_{\overline{\eta}}]$ and write

$$G_{\Omega_*\Omega} = G_\Omega = \langle Ae_{\overline{\eta}\upharpoonright\langle m,n\rangle}, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}} \mid \overline{\eta} \in \Omega, \overline{\nu} \in [\pi_{\overline{\eta}} b_{\overline{\eta}}], 1 \leq m \leq k, n < \omega \rangle_* \subseteq G_{X_*X}$$

as $Ae_{\overline{\nu}}$ is a direct summand of $G_{\Omega_*\Omega}$ for all $\overline{\nu} \in \Omega_* \setminus \left(\bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}] \cup \bigcup_{\overline{\eta}\in\Omega}[\pi_{\overline{\eta}} b_{\overline{\eta}}]\right)$.
Thus, in order to show $\aleph_k$-freeness of G_{X_*X}, we will consider any $\Omega \subseteq X$ of size $|\, \Omega \,| < \aleph_k$ and show the freeness of the module G_Ω. We may assume that $|\, \Omega \,| = \aleph_{k-1}$. Let $F : \Lambda \to [\Lambda_*]^{<\aleph_0}$ be any map which assigns to $\overline{\eta} \in X$ the set $\overline{\eta}F = [\pi_{\overline{\eta}} b_{\overline{\eta}}]$.
By Proposition 3.5 (putting simply $u_{\overline{\eta}} = \{1, \ldots, k\}$ for all $\overline{\eta} \in \Omega$) we can express

$$G_\Omega = \langle e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha n} \mid \alpha < \aleph_{k-1}, \overline{\nu} \in \overline{\eta}^\alpha F, 1 \leq m \leq k, n < \omega \rangle_A,$$

where $\langle \ldots \rangle_A$ denotes the A-module generated by $\langle \ldots \rangle$, and we find a sequence of pairs (ℓ_α, n_α) with $1 \leq \ell_\alpha \leq k, n_\alpha < \omega$ such that for $n \geq n_\alpha$

$$\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta\upharpoonright\langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F. \tag{3.3}$$

Let $G_\alpha = \langle e_{\overline{\eta}^\gamma \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\gamma n} \mid \gamma < \alpha, \overline{\nu} \in \overline{\eta}^\gamma F, 1 \le m \le k, n < \omega\rangle_A$ for any $\alpha \le \aleph_{k-1}$; thus $G_0 = \{0\}$ and $G_{\aleph_{k-1}} = G_\Omega$, and if $\alpha < \aleph_{k-1}$, then

$$G_{\alpha+1} = G_\alpha + \langle e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha n} \mid \overline{\nu} \in \overline{\eta}^\alpha F, 1 \le m \le k, n < \omega\rangle_A$$

$$= G_\alpha + \langle e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} \mid n < n_\alpha\rangle_A + \langle y'_{\overline{\eta}^\alpha n} \mid n \ge n_\alpha\rangle_A$$

$$+\langle e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}} \mid \overline{\nu} \in \overline{\eta}^\alpha F, 1 \le \ell_\alpha \ne m \le k, n < \omega\rangle_A.$$

Hence, any element in $G_{\alpha+1}$ can be represented as a sum of the form

$$g + \sum_{n<n_\alpha} a_n e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} + \sum_{n \ge n_\alpha} a_n y'_{\overline{\eta}^\alpha n} + \sum_{n<\omega} \sum_{\ell_\alpha \ne m \le k} a_{mn} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle} + \sum_{\overline{\nu} \in \overline{\eta}^\alpha F} a_{\overline{\nu}} e_{\overline{\nu}},$$

where $g \in G_\alpha$ and all coefficients $a_n, a_{mn}, a_{\overline{\nu}}$ are from A.
Moreover, the summands involving the $e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$'s have disjoint supports. Now condition (3.3) applies recursively. Hence, assuming the above sum is zero, then by disjointness of supports (identifying $e_{\overline{\nu}}$ ($\overline{\nu} \in \overline{\eta}^\alpha F$) with one of the $e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$s if possible and merging all $e_{\overline{\eta}^\alpha} \upharpoonright \langle m, n\rangle, e_{\overline{\nu}} \in G_\alpha$ into g,) it also follows that all the coefficients $a_n, a_{mn}, a_{\overline{\nu}}$ and consequently also g must be zero. This shows that $G_{\alpha+1} = G_\alpha \oplus \bigoplus_{b \in B_\alpha} Ab$ for

$$B_\alpha = \{e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,j\rangle}, y_{\overline{\eta}^\alpha \ell}, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}} \mid$$

$$j < n_\alpha, \ell \ge n_\alpha, 1 \le \ell_\alpha \ne m \le k, n < \omega, \overline{\nu} \in \overline{\eta}^\alpha F\} \setminus G_\alpha.$$

Thus, $G_\Omega = \bigoplus_{\alpha<\aleph_{k-1}} \bigoplus_{b \in B_\alpha} Ab$ is a free A-module. The $\aleph_k$-freeness of G_{X_*X} is now immediate from the existence of the $< \aleph_k$-closed family $\mathcal{C} = \{G_{\Omega_*\Omega} \mid |\,\Omega_*\,|, |\,\Omega\,| < \aleph_k\}$ of free, pure submodules of G_{X_*X}. ∎

Remark 3.7 *If A is an $\aleph_k$-free R-module, then combining Lemma 3.6 with Observation 2.2 we obtain that the module G_{X_*X} from Definition 3.2 is also $\aleph_k$-free as R-module.*

3.2 The case of infinite correction

Branch elements and branch-like elements will serve the purpose of killing undesired partial endomorphisms with the help of a step lemma as described in Chapter 4. Crucial here is the appropriate choice of correction elements from $\widehat{B} \setminus B$ for our branch-like elements. The method described in Section 3.1 achieves this goal by using correction elements of the type $\pi_{\overline{\eta}} b_{\overline{\eta}}$ for suitable $\mathbb{S}$-adic numbers $\pi_{\overline{\eta}} \in \widehat{R}$ which have finite supports $[\pi_{\overline{\eta}} b_{\overline{\eta}}] \subseteq [b_{\overline{\eta}}]$ and are easy to handle within the constructions. This considerably simple setting covers the vast majority of possible problem situations including all the constructions discussed within this thesis. Nevertheless, there are situations where the use of $\mathbb{S}$-adic numbers is either unsuitable or undesirable. This, for example, is the case when constructing separable groups with predefined endomorphism rings between a free group $B = \bigoplus_{\alpha<\kappa} \mathbb{Z}e_\alpha$ and $\overline{B} = \widehat{B} \cap \prod_{\alpha<\kappa} \mathbb{Z}e_\alpha$, see [8, 24, 44]. Here $\pi_{\overline{\eta}} b_{\overline{\eta}} \notin \overline{B}$ prohibits the use of $\mathbb{S}$-adic numbers. Such constructions can be realized using correction elements of infinite support, which need more careful and elaborate definitions and proofs. We will discuss all the necessary changes in this section.

Definition 3.8 *The* λ_k-norm.

(a) For $\eta \in {}^{\omega\geq}\lambda_k$ *let* $\|\eta\| = \sup_{\ell<\lg\eta}(\ell\eta + 1) \in \lambda_k$*, where* $\lg\eta$ *denotes the length of* η*; in particular* $\|\alpha\| = \alpha + 1$ *for* $\alpha \in \lambda_k$ *and* $\|\emptyset\| = 0$ *by default.*

(b) For $\overline{\eta} \in \Lambda$ *let* $\|\overline{\eta}\| = \|\eta_k\|$*, and for* $\overline{\nu} \in \Lambda_*$ *let* $\|\overline{\nu}\| = \|\nu_k\|$.

(c) For $X \subseteq \Lambda$ *put* $\|X\| = \sup_{\overline{\eta}\in X}\|\overline{\eta}\|$*. Similarly,* $\|X\| = \sup_{\overline{\nu}\in X}\|\overline{\nu}\|$ *if* $X \subseteq \Lambda_*$.

(d) If $b \in \widehat{B}$*, then* $\|b\| = \|[b]\|$*, and for* $S \subseteq \widehat{B}$*, let* $\|S\| = \sup_{b\in S}\|b\|$.

Definition 3.9 *A sequence of elements* $(b_{\overline{\eta}i})_{i<\omega}$ *is called* regressive *with respect to* $\overline{\eta} \in \Lambda$*, if the following holds.*

(i) $\|b_{\overline{\eta}0}\| < 0\eta_k$.

(ii) $b_{\overline{\eta}i} - s_i b_{\overline{\eta},i+1} \in B$ *for all* $i < \omega$*, i.e.* $(b_{\overline{\eta}i})_{i<\omega}$ *is a* divisibility chain.

(iii) $[b_{\overline{\eta}i}] \subseteq [b_{\overline{\eta}0}]$ *for all* $i < \omega$.

Evidently, every element $b_{\overline{\eta}} \in \widehat{B}$ allows for a suitable sequence $(b_{\overline{\eta}i})_{i<\omega}$ of elements $b_{\overline{\eta}i} \in \widehat{B}$ such that conditions (ii) and (iii) hold with $b_{\overline{\eta}0} = b_{\overline{\eta}}$. Given $\overline{\eta} \in \Lambda$, we define *branch-like elements* by adding some regressive *correction* $(b_{\overline{\eta}i})_{i<\omega}$ to our branch-elements,

$$y'_{\overline{\eta}i} = b_{\overline{\eta}i} + y_{\overline{\eta}i}.$$

In particular, we have $y'_{\overline{\eta}} = y'_{\overline{\eta}0} = b_{\overline{\eta}} + y_{\overline{\eta}}$. Thus, Definitions 3.1 and 3.2 change minimally as follows

Definition 3.10 *(a) A triple $(X_*, X, \mathfrak{F})$ is called* Λ-closed*, if the following holds.*

(i) $X \subseteq \Lambda$ and $X_* \subseteq \Lambda_*$.

(ii) $[\overline{\eta}] \subseteq X_*$ for all $\overline{\eta} \in X$.

(iii) $\mathfrak{F} = \{y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in \widehat{B}_{X_*}, \overline{\eta} \in X\}$ is a family of branch-like elements $y'_{\overline{\eta}}$ with regressive sequences $(b_{\overline{\eta}i})_{i<\omega}$.

(b) Similarly, a pair (X_*, X) is called Λ-closed*, if conditions (i) and (ii) hold.*

Definition 3.11 *The construction of the A-module G_{X_*X}:*
If $(X_*, X, \mathfrak{F})$ is Λ-closed, then we let

$$G_{X_*X} = \langle B_{X_*}, Ay'_{\overline{\eta}i} \mid \overline{\eta} \in X, i < \omega\rangle = \langle B_{X_*}, Ay'_{\overline{\eta}} \mid \overline{\eta} \in X\rangle_* \subseteq \widehat{B}.$$

For the equivalence of the two representations above, observe that condition (ii) of Definition 3.9 implies $b_{\overline{\eta}} - q_i b_{\overline{\eta}i} \in B$ for all $i < \omega$, thus making $(b_{\overline{\eta}i})_{i<\omega}$ a divisibility chain of $b_{\overline{\eta}} + B$ in $\widehat{B}/B$. Observation 3.3 needs to be reformulated to allow more refined support recognition arguments inherent to the Step Lemmas 4.15 and 4.16.
We introduce the notation $\Lambda^{\geq n}_{m*} = \{\overline{\nu} \in \Lambda_{m*} \mid \lg \eta_m \geq n\}$.

Observation 3.12 (Recognition) *Let be $g \in \widehat{B}$, $1 \leq m \leq k$ and $n < \omega$. If $g \in \langle B, Ay'_{\overline{\eta}} \mid \overline{\eta} \in \Lambda\rangle$ for a suitable choice of $\mathfrak{F} = \{y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in \widehat{B}, \overline{\eta} \in \Lambda\}$ with regressive sequences $(b_{\overline{\eta}i})_{i<\omega}$, then the following holds:*

(a) There exist some unique $b \in B$ and $a_{\overline{\eta}} \in A$ $(\overline{\eta} \in \Lambda)$ with $g = b + \sum_{\overline{\eta}\in\Lambda} a_{\overline{\eta}} y'_{\overline{\eta}}$.

(b) If $a_{\overline{\eta}} \neq 0$, then $\|\overline{\eta}\| \leq \|g\|$.

(c) The elements $a_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda)$ are entirely determined by $[g]_{\Lambda^{\geq n}_{m*}}$ and for $\|\overline{\eta}\| = \|g\|$ the coefficient $a_{\overline{\eta}}$ is independent of the choice of $\mathfrak{F}$.

Proof. Observe that $[b]$, $[b_{\overline{\eta}}] \cap [\overline{\eta}]$, $[b_{\overline{\eta}}] \cap [\overline{\eta}']$ and $[\overline{\eta}] \cap [\overline{\eta}']$ are finite for $\overline{\eta} \neq \overline{\eta}' \in \Lambda$ with $\|\,\overline{\eta}\,\| \leq \|\,\overline{\eta}'\,\|$, while $[\overline{\eta}] \cap \Lambda^{\geq n}_{m*}$ and $[\overline{\eta}'] \cap \Lambda^{\geq n}_{m*}$ are infinite. Consequently

(α) If $a_{\overline{\eta}} = 0$ for all $\overline{\eta} \in \Lambda$, then $[g] = [b]$ is finite.

(β) Otherwise, $[g]$ and $[g] \cap \Lambda^{\geq n}_{m*}$ are both infinite. Furthermore, if $\overline{\eta} \in \Lambda$ with $\|\,\overline{\eta}\,\| = \|\,g\,\|$, then $a_{\overline{\eta}} \neq 0$ if and only if $[y_{\overline{\eta}i}] \subseteq [g]$ for some $i < \omega$.

Now, (b) is an immediate consequence of (β), while (a) and (c) are proven by transfinite induction over $\|\,g\,\|$: If $[g] \cap \Lambda^{\geq n}_{m*}$ is finite, then $a_{\overline{\eta}} = 0$ for all $\overline{\eta} \in \Lambda$ combining (α) and (β), and we are done. If $[g] \cap \Lambda^{\geq n}_{m*}$ is infinite, then using (β) we can read off in a first step all those $\overline{\eta}$ with $\|\,\overline{\eta}\,\| = \|\,g\,\|$ and $[y_{\overline{\eta}i}] \cap \Lambda^{\geq n}_{m*} \subseteq [[g]_{\Lambda^{\geq n}_{m*}}]$ and then determine $a_{\overline{\eta}}$ accordingly from the coefficients appearing in $[[g]_{\Lambda^{\geq n}_{m*}}]_{[y_{\overline{\eta}i}]}$. After that, proceed with

$$g' = g - \sum_{\substack{\overline{\eta} \in \Lambda \\ \|\,\overline{\eta}\,\| = \|\,g\,\|}} a_{\overline{\eta}} y'_{\overline{\eta}}\,.$$

Furthermore, as this first step does not make any use of the correction elements $b_{\overline{\eta}}$, it is independent of the particular choice of $\mathfrak{F}$. ∎

Once more, a well-defined Λ-support is a consequence of Observation 3.12, and Observation 3.4 remains true. The equivalent to the Freeness-Proposition 3.5 for the case of infinite support corrections needs once again some more elaborate restrictions. If S is any set, then $[S]^{\leq \aleph_0}$ denotes the collection of all countable subsets of S.

Definition 3.13 *A function $F : \Lambda \to [\Lambda_*]^{\leq \aleph_0}$ is called* regressive *if $\|\,\overline{\eta}F\,\| < 0\eta_k$ for all $\overline{\eta} \in \Lambda$.*

Freeness-Proposition 3.14 *Let $F : \Lambda \to [\Lambda_*]^{\leq \aleph_0}$ be a regressive function, let be $1 \leq f \leq k$ and let Ω be a subset of Λ of cardinality $\aleph_{f-1}$ with a family of subsets $u_{\overline{\eta}} \subseteq \{1, \ldots, k\}$ satisfying $|\,u_{\overline{\eta}}\,| \geq f$ for all $\overline{\eta} \in \Omega$.*

(a) Then we can find an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \gamma \rangle$ of Ω for some ordinal number $\aleph_{f-1} \leq \gamma < \aleph_f$, $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha < \omega$ ($\alpha < \gamma$) such that

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F \text{ for all } n \geq n_\alpha,$$

where $\Omega_\alpha = \{\overline{\eta}^\beta \mid \beta \leq \alpha\}$.

(b) For $f > 1$, we can choose $\gamma = \aleph_{f-1}$.

Proof. The proof follows by induction on f. We begin with $f = 1$, so $|\,\Omega\,| = \aleph_0$. Let $\Omega = \{\overline{\eta}^\alpha \mid \alpha < \gamma\}$ be an enumeration without repetitions such that $0\eta_k^\beta \leq 0\eta_k^\alpha$ and $0\eta_k^\alpha < 0\eta_k^{\alpha+\omega}$ for all $\beta \leq \alpha < \gamma$. From $1 = f \leq |\,u_{\overline{\eta}}\,|$ follows $u_{\overline{\eta}} \neq \emptyset$ and we can choose any $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ for all $\alpha < \gamma$. If $\beta < \alpha < \gamma$, then $\overline{\eta}^\alpha \neq \overline{\eta}^\beta$ and there is $n_{\alpha\beta} \in \omega$ such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \neq \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n\rangle$ for all $n \geq n_{\alpha\beta}$. Furthermore, if $\beta < \alpha < \gamma$ and $0\eta_k^\beta < 0\eta_k^\alpha$, then also $\|\,\overline{\eta}^\beta F\,\| < 0\eta_k^\alpha$ and $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n\rangle\} \cup \overline{\eta}^\beta F$ for all $n \geq 1$ as witnessed by the entry $0\eta_k^\alpha$. Hence, the trivial choice $n_{\alpha\beta} = 1$ does suffice in the case of $0\eta_k^\beta < 0\eta_k^\alpha$. Otherwise, if $\beta \leq \alpha < \gamma$ and $0\eta_k^\beta = 0\eta_k^\alpha$, then $\|\,\overline{\eta}^\beta F\,\| < 0\eta_k^\alpha$ still holds giving $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \overline{\eta}^\beta F$ for all $n \geq 1$. In particular,

$$n_\alpha = 1 + \left(\max_{\substack{\beta<\alpha \\ 0\eta_k^\beta = 0\eta_k^\alpha}} n_{\alpha\beta} \right) \in \omega$$

is a well-defined number as there exist only finitely many $\beta \leq \alpha < \gamma$ with $0\eta_k^\beta = 0\eta_k^\alpha$ by the choice of our list Ω, and $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F$ for all $n \geq n_\alpha$. Hence, case $f = 1$ is settled and we let $f' = f + 1$ and assume that the proposition holds for f. The induction step is the same as in the proof of Proposition 3.5. ∎

The concluding result of this section is once again

Freeness-Lemma 3.15 *The module G_{X_*X} from Definition 3.11 is $\aleph_k$-free as A-module.*

Proof. The proof is the same as for Lemma 3.6 replacing Proposition 3.5 by Proposition 3.14. Observe here that $F : \Lambda \to [\Lambda_*]^{\leq\aleph_0}$ can be chosen to be regressive with $\overline{\eta}F = [b_{\overline{\eta}0}]$ for all $\overline{\eta} \in X$ by condition (i) of Definition 3.9. ∎

4 The step lemma

4.1 More closure conditions

The whole construction in Section 5 will be based on Λ-closed triples $(X_*, X, \mathfrak{F})$ and their induced A-modules G_{X_*X}. While Λ-closure is still a concept dictated by nothing more than the simple wish of having a well-defined definition of modules G_{X_*X}, this will neither suffice for proving the step lemma nor for the later Easy Black Box. For this purpose, we will introduce stronger closure conditions motivated by model theoretic considerations first to be found in [30, 48]. Central here will be the idea of having pairwise closed triples (compare Proposition 3.5) as well as refinements added in [27]. We start with some simple and intuitive conventions for handling Λ-closed triples $(X_*, X, \mathfrak{F})$.

Definition 4.1 *Let $(X_*^1, X^1, \mathfrak{F}^1)$ and $(X_*^2, X^2, \mathfrak{F}^2)$ be Λ-closed triples.*

(a) We write $(X_*^1, X^1, \mathfrak{F}^1) \subseteq (X_*^2, X^2, \mathfrak{F}^2)$ if $X_*^1 \subseteq X_*^2$, $X^1 \subseteq X^2$ and $\mathfrak{F}^1 \subseteq \mathfrak{F}^2$ holds.

(b) If $\pi^1_{\overline{\eta}} b^1_{\overline{\eta}} = \pi^2_{\overline{\eta}} b^2_{\overline{\eta}}$ for all $\overline{\eta} \in X^1 \cap X^2$, then

- $(X_*^1, X^1, \mathfrak{F}^1) \cap (X_*^2, X^2, \mathfrak{F}^2) = (X_*^1 \cap X_*^2, X^1 \cap X^2, \mathfrak{F}^1 \cap \mathfrak{F}^2)$ *and*
- $(X_*^1, X^1, \mathfrak{F}^1) \cup (X_*^2, X^2, \mathfrak{F}^2) = (X_*^1 \cup X_*^2, X^1 \cup X^2, \mathfrak{F}^1 \cup \mathfrak{F}^2)$

denote the canonically induced Λ-closed triples.

Similarly, further notations for Λ-closed triples $(X_*, X, \mathfrak{F})$ can be defined componentwise.

Definition 4.2 *Let $(X_*, X, \mathfrak{F}) \subseteq (Y_*, Y, \mathfrak{G})$ be Λ-closed triples.*

(a) For each $\overline{\eta} \in \Lambda$, let

$$u_{\overline{\eta}}(X_*) = \{1 \leq m \leq k \mid \overline{\eta} \restriction \langle m, n \rangle \in X_* \text{ for arbitrarily large } n < \omega\}.$$

(b) For each $1 \leq f \leq k$ and $\alpha \in \lambda_k^o = \{\gamma \in \lambda_k \mid \mathrm{cf}(\gamma) = \omega\}$, let

$$Y_{X_*Xf\alpha} = \{\overline{\eta} \in Y \setminus X \mid |\, u_{\overline{\eta}}(X_*)\,| \geq f \text{ and } \|\,\overline{\eta}\,\| = \alpha\}.$$

We will write Y_{X_*X} instead of $Y_{X_*Xf\alpha}$ if the pair (f, α) is clear form the context.

(c) *Let $1 \leq f \leq k$ and $\alpha \in \lambda_k$. Then the triple $(X_*, X, \mathfrak{F})$ is called (f, α)-*closed *with respect to $(Y_*, Y, \mathfrak{G})$, if the following holds.*

(i) *If $\overline{\eta} \in Y$ with $|\, u_{\overline{\eta}}(X_*)\, | \geq f$ and $\| \overline{\eta} \| \neq \alpha$, then $\overline{\eta} \in X$.*

*Furthermore, $(X_*, X, \mathfrak{F})$ is called f-*closed *with respect to $(Y_*, Y, \mathfrak{G})$, if the condition $\| \overline{\eta} \| \neq \alpha$ is dropped from (i).*

(d) *The triple $(X_*, X, \mathfrak{F})$ is called* pairwise closed *with respect to $(Y_*, Y, \mathfrak{G})$ if the following holds.*

(ii) *If $\overline{\eta} \in Y$ with $\overline{\eta} \upharpoonright \langle m, n \rangle, \overline{\eta} \upharpoonright \langle m', n' \rangle \in X_*$ for some $1 \leq m < m' \leq k$, $n, n' < \omega$, then $\overline{\eta} \in X$.*

(e) *If $(Y_*, Y) = (\Lambda_*, \Lambda)$, then we might as well omit $(Y_*, Y, \mathfrak{G})$ and just say that $(X_*, X, \mathfrak{F})$ is (f, α)-closed, f-closed or pairwise closed, respectively.*

By definition, the notion of being f-closed is identical with the notion of being (f, α)-closed for any $\alpha \in \lambda_k$ with $\operatorname{cf} \alpha \neq \omega$ and hence, the former is a special case of the latter. Furthermore, observe that the notion of being pairwise closed is restricted to the case $k > 1$. The following simple properties of (f, α)-closed and pairwise closed triples will be crucial for proving the step lemma in Section 4.3.

Observation 4.3 *Let $(X_*, X, \mathfrak{F})$, $(X^i_*, X^i, \mathfrak{F}^i)$, $(Y_*, Y, \mathfrak{G})$ and $(Z_*, Z, \mathfrak{H})$ be Λ-closed triples. Then the following holds.*

(a) *If $(X_*, X, \mathfrak{F})$ is f-closed with respect to $(Y_*, Y, \mathfrak{G})$, then $(X_*, X, \mathfrak{F})$ is also (f, α)-closed with respect to $(Y_*, Y, \mathfrak{G})$. Furthermore, $Y_{X_* X f \alpha} = \emptyset$ holds for all $\alpha \in \lambda_k$.*

(b) *If $1 \leq f_1 \leq f_2 \leq k$ and $(X_*, X, \mathfrak{F})$ is (f_1, α)-closed (f_1-closed) with respect to $(Y_*, Y, \mathfrak{G})$, then $(X_*, X, \mathfrak{F})$ is also (f_2, α)-closed (f_2-closed) with respect to $(Y_*, Y, \mathfrak{G})$. Furthermore, $Y_{X_*,\, X,\, f_2,\, \alpha} \subseteq Y_{X_*,\, X,\, f_1,\, \alpha}$ holds.*

(c) *If $(X_*, X, \mathfrak{F})$ is (f, α)-closed (f-closed) with respect to $(Y_*, Y, \mathfrak{G})$ and $(Y_*, Y, \mathfrak{G})$ is (f, α)-closed (f-closed) with respect to $(Z_*, Z, \mathfrak{H})$, then $(X_*, X, \mathfrak{F})$ is also (f, α)-closed (f-closed) with respect to $(Z_*, Z, \mathfrak{H})$.*

(d) *If* $(X_*, X, \mathfrak{F})$ *is* k*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*, then* $|[\overline{\eta}] \setminus X_*| < \aleph_0$ *for* $\overline{\eta} \in Y$ *implies* $\overline{\eta} \in X$*. Similarly, if* $(X_*, X, \mathfrak{F})$ *is* (k, α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*, then* $|[\overline{\eta}] \setminus X_*| < \aleph_0$ *for* $\overline{\eta} \in Y$ *and* $\|\overline{\eta}\| \neq \alpha$ *implies* $\overline{\eta} \in X$*.*

Here, the condition $|[\overline{\eta}] \setminus X_*| < \aleph_0$ *is equivalent to saying that* $[y_{\overline{\eta}i}] \subseteq X_*$ *for some* $i < \omega$*.*

(e) *The triple* $(X_*, X, \mathfrak{F})$ *is* 1*-closed with respect to* $(Y_*, Y, \mathfrak{G})$ *if and only if* $|[\overline{\eta}] \cap X_*| = \aleph_0$ *for* $\overline{\eta} \in Y$ *implies* $\overline{\eta} \in X$*. Similarly,* $(X_*, X, \mathfrak{F})$ *is* $(1, \alpha)$*-closed with respect to* $(Y_*, Y, \mathfrak{G})$ *if and only if* $|[\overline{\eta}] \cap X_*| = \aleph_0$ *for* $\overline{\eta} \in Y$ *and* $\|\overline{\eta}\| \neq \alpha$ *implies* $\overline{\eta} \in X$*.*

(f) *If* $(X_*, X, \mathfrak{F})$ *is pairwise closed with respect to* $(Y_*, Y, \mathfrak{G})$*, then* $(X_*, X, \mathfrak{F})$ *is also* 2*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*.*

(g) *If the triples* $(X^i_*, X^i, \mathfrak{F}^i)$ *are pairwise closed with respect to* $(Y_*, Y, \mathfrak{G})$ *such that* $(X^i_*, X^i, \mathfrak{F}^i) \subseteq (X^{i+1}_*, X^{i+1}, \mathfrak{F}^{i+1})$ *for all* $i < \omega$ *and if* $(Z_*, Z, \mathfrak{H})$ *is* (f, α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})$ *for some* $1 \leq f < k$*, then*

$$(Y'_*, Y', \mathfrak{G}') = \bigcup_{i<\omega} (X^i_*, X^i, \mathfrak{F}^i) \cup (Z_*, Z, \mathfrak{H})$$

exists and it is $(f+1, \alpha)$*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*. Furthermore, the inclusion* $Y_{Y'_*Y'} \subseteq Y_{Z_*Z}$ *holds.*

(h) *If* $(X^i_*, X^i, \mathfrak{F}^i)$ *is* (f_i, α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})$ *for* $i \in \{1, 2\}$ *such that* $f_1 + f_2 - 1 \leq k$ *holds, then* $(Y'_*, Y', \mathfrak{G}') = (X^1_*, X^1, \mathfrak{F}^1) \cup (X^2_*, X^2, \mathfrak{F}^2)$ *exists and it is* $(f_1 + f_2 - 1, \alpha)$*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*. Furthermore, the inclusion* $Y_{Y'_*Y'} \subseteq Y_{X^1_*X^1} \cup Y_{X^2_*X^2}$ *holds.*

Proof. Claims (a), (b), (c) and (f) are obvious from Definition 4.2. Claims (d) and (e) are immediate as well, as $|[\overline{\eta}] \setminus X_*| < \aleph_0$ implies $|\, u_{\overline{\eta}}(X_*)\,| = k$, while $|[\overline{\eta}] \cap X_*| = \aleph_0$ implies $|\, u_{\overline{\eta}}(X_*)\,| > 0$.

For (g), let $\overline{\eta} \in Y$ be such that $|\, u_{\overline{\eta}}(Y'_*)\,| \geq f + 1$. If $|\, u_{\overline{\eta}}(Z_*)\,| \geq f$ holds, then $\overline{\eta} \in Z \subseteq Y'$ as $(Z_*, Z, \mathfrak{H})$ is (f, α)-closed, and we are done. Otherwise, if $|\, u_{\overline{\eta}}(Z_*)\,| < f$, then $\left|\, u_{\overline{\eta}}(\bigcup_{i<\omega} X^i_*)\,\right| \geq 2$ and there exist $1 \leq m_1 < m_2 \leq k$, $i_1, i_2, n_1, n_2 < \omega$ with $\overline{\eta}\upharpoonright\langle m_1, n_1\rangle \in X^{i_1}_*$ and $\overline{\eta}\upharpoonright\langle m_2, n_2\rangle \in X^{i_2}_*$. Thus, for $i = \max\{i_1, i_2\}$ holds

$\overline{\eta}\upharpoonright\langle m_1,n_1\rangle, \overline{\eta}\upharpoonright\langle m_2,n_2\rangle \in X^i_*$ and $\overline{\eta}\in X^i \subseteq Y'$ as $(X^i_*, X^i, \mathfrak{F}^i)$ is pairwise closed. Similarly $Y_{Y'_*Y'} \subseteq Y_{Z_*Z}$ follows.
The proof of claim (h) is similar to the proof of (g). ∎

For later applications, it will be enough to recall a simple hierarchy of closure conditions: pairwise closed triples and 1-closed triples are 2-closed, while every 2-closed triple is k-closed. Furthermore, every f-closed triple is (f,α)-closed. These closure conditions of course are accompanied by their respective induced closures, some properties of which are covered subsequently.

Lemma 4.4 *If $(X_*, X, \mathfrak{F})$ is k-closed with respect to $(Y_*, Y, \mathfrak{G})$, then $G_{Y_*Y} \cap \widehat{B}_{X_*} = G_{X_*X}$ holds.*

Proof. The inclusion $G_{X_*X} \subseteq G_{Y_*Y} \cap \widehat{B}_{X_*}$ is obvious. For $G_{Y_*Y} \cap \widehat{B}_{X_*} \subseteq G_{X_*X}$, let some $g \in G_{Y_*Y} \cap \widehat{B}_{X_*}$ be given, and we need to show $g \in G_{X_*X}$. Now, $[g]_\Lambda \subseteq X$ is immediate as Observation 3.3 shows $|\, u_{\overline{\eta}}(X_*)\,| = k$ for all $\overline{\eta} \in [g]_\Lambda$, and $g \in G_{X_*X}$ follows.
In the more general context of Section 3.2 the arguments for $G_{Y_*Y} \cap \widehat{B}_{X_*} \subseteq G_{X_*X}$ must be refined slightly, resulting in an induction on $|\,[g]_\Lambda\,|$ for the chosen element $g \in G_{Y_*Y} \cap \widehat{B}_{X_*}$: If $|\,[g]_\Lambda\,| = 0$, then $g \in B_{X_*} \subseteq G_{X_*X}$ is immediate. If, on the other hand, $|\,[g]_\Lambda\,| > 0$, then $\overline{\eta} \in X$ follows for all $\overline{\eta} \in [g]_\Lambda$ with $\|\,\overline{\eta}\,\| = \|\,g\,\|$ from combining Observation 3.12(β) and Observation 4.3(d). Thus, g can be reduced by these elements $y'_{\overline{\eta}}$ modulo G_{X_*X} and the induction step applies. ∎

Theorem 4.5 *(a) Let $(X_*, X, \mathfrak{F})$ be Λ-closed and $k \geq 2$. Then $|\,X\,| = |\,\mathfrak{F}\,|$ and $0 \leq |\,X\,| \leq |\,X_*\,| \leq \aleph_0 \cdot |\,X\,|$.*

(b) If $(Y_*, Y, \mathfrak{G})$ is Λ-closed, $G \subseteq G_{Y_*Y}$, $1 \leq f \leq k$ and $\alpha \in \lambda_k$, then there exists a uniquely determined triple $C_{f\alpha}(G) = (X_*, X, \mathfrak{F}) \subseteq (Y_*, Y, \mathfrak{G})$ which is minimal under inclusion and such that $(X_*, X, \mathfrak{F})$ is (f,α)-closed with $G \subseteq G_{X_*X}$. Moreover, $|\,X_*\,|, |\,X\,|, |\,\mathfrak{F}\,| \leq |\,G\,|^{\aleph_0}$ for $f = 1$ and $|\,X_*\,|, |\,X\,|, |\,\mathfrak{F}\,| \leq \aleph_0 \cdot |\,G\,|$ for $f > 1$.

Similarly, we define $C_{f\alpha}(Z_*, Z, \mathfrak{H}) = C_{f\alpha}(G_{Z_*Z}) \subseteq (Y_*, Y, \mathfrak{G})$ for any triple $(Z_*, Z, \mathfrak{H}) \subseteq (Y_*, Y, \mathfrak{G})$.

(c) *If* $(Y_*, Y, \mathfrak{G})$ *is* Λ*-closed and* $G \subseteq G_{Y_*Y}$*, then there exists a uniquely determined triple* $PC(G) = (X_*, X, \mathfrak{F}) \subseteq (Y_*, Y, \mathfrak{G})$ *which is minimal under inclusion and such that* $(X_*, X, \mathfrak{F})$ *is pairwise closed with* $G \subseteq G_{X_*X}$*. Moreover,* $|X_*|, |X|, |\mathfrak{F}| \leq \aleph_0 \cdot |G|$.
Furthermore, we define $PC(Z_*, Z, \mathfrak{H}) = PC(G_{Z_*Z}) \subseteq (Y_*, Y, \mathfrak{G})$ *for any triple* $(Z_*, Z, \mathfrak{H}) \subseteq (Y_*, Y, \mathfrak{G})$.

Proof. Claim (a) is trivial.
(b) The triple $(X_*, X, \mathfrak{F})$ is uniquely determined by the closure of the sets $[G] \subseteq \Lambda_*$ and $[G]_\Lambda \subseteq \Lambda$ under Definition 3.1 and Definition 4.2(c). More precisely, $C_{f\alpha}(G) = (X_*, X, \mathfrak{F})$ is the minimal triple characterized by the following properties:

(i) $[G] \subseteq X_*$ and $[G]_\Lambda \subseteq X$.

(ii) $[\overline{\eta}] \subseteq X_*$ and $[\pi_{\overline{\eta}} b_{\overline{\eta}}] \subseteq X_*$ for all $\overline{\eta} \in X$.

(iii) $\overline{\eta} \in X$ for all $\overline{\eta} \in Y$ with $|u_{\overline{\eta}}(X_*)| \geq f$ and $\|\overline{\eta}\| \neq \alpha$.

For $f = 1$, the cardinal $|G|^{\aleph_0}$ is a consequence of condition (iii), which becomes $\aleph_0 \cdot |G|$ for $f > 1$ as (iii) then is a weaker statement than being pairwise closed.
Claim (c) follows similarly to (b) replacing Definition 4.2(c) by Definition 4.2(d). ∎

Theorem 4.5(b) still holds, when replacing (f, α)-closed by f-closed resulting in an accompanying closure $C_f(G)$. Just recall here that being f-closed is a special case of being (f, α)-closed.
We now can also formulate and prove an easy extension of Lemma 4.4.

Lemma 4.6 *Let* $(X_*, X, \mathfrak{F})$ *be* (k, α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*, such that*

$$[\overline{\eta}] \subseteq X_* \text{ and } [\pi_{\overline{\eta}} b_{\overline{\eta}}] \subseteq X_* \text{ for all } \overline{\eta} \in Y_{X_*X}.$$

Then $G_{Y_*Y} \cap \widehat{B}_{X_*} = G_{X_*, X \cup Y_{X_*X}}$ *holds.*

Proof. Just observe that $C_k(G_{X_*X}) = (X_*, X \cup Y_{X_*X}, \mathfrak{F} \cup \mathfrak{H})$ holds for the family $\mathfrak{H} = \{y'_{\overline{\eta}} \mid \overline{\eta} \in Y_{X_*X}\}$ and apply Lemma 4.4. ■

4.2 The Easy Black Box

If δ is an ordinal with $\mathrm{cf}(\delta) = \omega$, then let be

$$\Gamma_\delta = \{\eta \in {}^{\omega\uparrow}\delta \mid \sup \eta = \delta\}, \text{ and if } \eta \in {}^{\omega\uparrow}\delta, \text{ then } [\eta] = \{\eta \restriction n \mid n < \omega\} \subseteq {}^{\omega\uparrow>}\delta.$$

Of very crucial importance for the subsequent step lemma will be the following general

Proposition 4.7 (The Easy Black Box) *For each cardinal $\lambda \geq \aleph_0$ and set Ξ of cardinality $\leq \lambda^{\aleph_0}$ there is a family $\langle g_\eta \mid \eta \in {}^{\omega\uparrow}\lambda\rangle$ with the following properties.*

(i) $g_\eta : [\eta] \longrightarrow \Xi$.

(ii) For each map $g : {}^{\omega\uparrow>}\lambda \longrightarrow \Xi$ there exists some $\eta \in {}^{\omega\uparrow}\lambda$ with $g_\eta \subseteq g$.

Proof. (see [30, p. 55, Lemma 2.3], which we outline for the convenience of the reader.) Since $|\Xi| \leq \lambda^{\aleph_0} = |{}^{\omega}\lambda|$, we can fix an embedding $\pi : \Xi \hookrightarrow {}^{\omega}\lambda$. And since $|{}^{\omega>}\lambda| = \lambda$, there is also a list ${}^{\omega>}\lambda = \langle \mu_\alpha \mid \alpha < \lambda\rangle$ with enough repetitions for each $\eta \in {}^{\omega>}\lambda$, i.e. $\{\alpha < \lambda \mid \mu_\alpha = \eta\} \subseteq \lambda$ is unbounded. Moreover, we define for each $n < \omega$ a coding map

$$\pi_n : {}^{n}\Xi \longrightarrow {}^{n^2}\lambda \subseteq {}^{\omega>}\lambda \quad (\overline{\varphi} = \langle \varphi_0, \ldots, \varphi_{n-1}\rangle \mapsto \overline{\varphi}\pi_n = (\varphi_0\pi \restriction n)^\wedge \ldots {}^\wedge(\varphi_{n-1}\pi \restriction n)).$$

Finally let $X \subseteq {}^{\omega\uparrow}\lambda$ be the collection of all order preserving maps $\eta : \omega \longrightarrow \lambda$ such that the following holds:

$$\exists\, \overline{\varphi} = \langle \varphi_i \mid i < \omega\rangle \in {}^{\omega}\Xi \text{ with } (\overline{\varphi} \restriction n)\pi_n = \mu_{n\eta} \text{ for all } n < \omega. \tag{4.1}$$

By definition of π_n it follows that $\overline{\varphi}$ is uniquely determined by (4.1). (Just note that $\mu_{n\eta}$ determines $\varphi_m\pi \restriction n$ for all $m < n$.)
We now prove the two statements of the proposition. For (i) we consider any $\eta \in {}^{\omega\uparrow}\lambda$. If $\eta \notin X$, then we can choose arbitrary elements $g_\eta(\eta \restriction n) \in \Xi$, and if $\eta \in X$, then we choose the uniquely determined sequence $\overline{\varphi}$ from (4.1) and let $g_\eta(\eta \restriction n) = \varphi_n$.

For (ii) we consider some $g : {}^{\omega\uparrow>}\lambda \longrightarrow \Xi$. In this case we must define a suitable $\eta = \langle \alpha_n \mid n < \omega \rangle \in {}^{\omega\uparrow}\lambda$. Using that the list of μ_αs is unbounded, we can choose inductively $\alpha_n > \alpha_{n-1}$ with $\langle g(\eta \restriction m) \mid m < n \rangle \pi_n = \mu_{\alpha_n}$ for all $n < \omega$.
Finally we check statement (ii). Using (4.1) we will find that the sequence η belongs to X:
If $\overline{\varphi} = \langle g(\eta \restriction i) \mid i < \omega \rangle \in {}^{\omega}\Xi$, then we have $(\overline{\varphi} \restriction n)\pi_n = \langle g(\eta \restriction m) \mid m < n \rangle \pi_n = \mu_{\alpha_n} = \mu_{n\eta}$ for all $n < \omega$, and $g_\eta(\eta \restriction n) = \varphi_n = g(\eta \restriction n)$ for all $n < \omega$ is immediate. ∎

For demonstrational purposes we do mention as well the following simple generalization of the Easy Black Box.

Proposition 4.8 *For each cardinal $\lambda \geq \aleph_0$ and set Ξ of cardinality $\leq \lambda^{\aleph_0}$ there is a family $\langle g_\eta \mid \eta \in {}^{\omega\uparrow}\lambda \rangle$ with the following properties.*

(i) $g_\eta : [\eta] \longrightarrow \Xi$.

(ii) For each map $g : {}^{\omega\uparrow>}\lambda \longrightarrow \Xi$ and ordinals $\xi_0, \xi_1 \in \lambda$ there exists some $\eta \in {}^{\omega\uparrow}\lambda$ with $0\eta = \xi_0$, $\| \eta \| = \sup_{i \in \omega} i\eta > \xi_1$ and $g_\eta \subseteq g$.

Proof. Slightly changing the proof of Proposition 4.7 we define $X \subseteq {}^{\omega\uparrow}\lambda$ as the collection of all order preserving maps $\eta : \omega \longrightarrow \lambda$ such that

$$\exists\, \overline{\varphi} = \langle \varphi_i \mid i < \omega \rangle \in {}^{\omega}\Xi \text{ with } (\overline{\varphi} \restriction n)\pi_n = \mu_{n\eta} \text{ for all } 1 \leq n < \omega.$$

For (ii) we construct $\eta = \langle \alpha_n \mid n < \omega \rangle \in X$ inductively choosing $\alpha_0 = \xi_0$ and $\alpha_n > \max\{\alpha_{n-1}, \xi_1\}$ with $\langle g(\eta \restriction m) \mid m < n \rangle \pi_n = \mu_{\alpha_n}$ for all $1 \leq n < \omega$. ∎

Remark 4.9 *A common feature of all known Black Boxes is that they construct a family of predictions or traps whose members are uniquely determined by some related branches $\eta \in {}^{\omega\uparrow}\lambda$ using a suitable coding map. Thus, claim (ii) can be interpreted as a general statement that we are free to choose the norm $\| \eta \|$ arbitrarily large and the starting value 0η arbitrarily fixed for the predicting branch η. This can be used to avoid Solovay's theorem in the case of singular cardinal constructions.*

4.3 The main result

We first define some standard supports used in the Step Lemma 4.15.

Definition 4.10 *If* $0 \leq f < k$ *and* $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$, *then we put*

- $\Lambda^{\overline{\xi}} = \{\overline{\eta} \in \Lambda \mid \overline{\eta} \restriction (f,k] = \overline{\xi}\}$
- $\Lambda_*^{\overline{\xi}} = \{\overline{\nu} \in \Lambda_* \mid \overline{\nu} \restriction (f,k] = \overline{\xi}\}$
- *If* $f < i \leq k$, *then* $\Lambda_{i*}^{\overline{\xi}} = \{\overline{\nu} \in \Lambda_{i*} \mid \nu_i \trianglelefteq \xi_i, \nu_m = \xi_m$ *for all* $f < m \neq i \leq k\}$
- $\Lambda_{\overline{\xi}*} = \dot{\bigcup}_{f<i\leq k} \Lambda_{i*}^{\overline{\xi}}$.

We next introduce and discuss the concept of ordinal content.

Definition 4.11 *(a) For* $\overline{\nu} \in \Lambda_* \cup \Lambda$, *we define the* ordinal content

$$\operatorname{orco} \overline{\nu} = \bigcup \{\operatorname{Im} \nu_m \mid 1 \leq m \leq k\}.$$

(b) If $S \subseteq \Lambda_* \cup \Lambda$, *then* $\operatorname{orco} S = \bigcup_{\overline{\nu} \in S} \operatorname{orco} \overline{\nu}$.

(c) If $S, T \subseteq \lambda_k$ *and* $\tau : S \longrightarrow T$ *is a bijection, then* τ *extends canonically to a bijection* $\tau : {}^{\omega\geq}S \longrightarrow {}^{\omega\geq}T$ *and for* $\overline{\nu} \in \Lambda_* \cup \Lambda$ *with* $\operatorname{orco} \overline{\nu} \subseteq S$ *we define* $\overline{\nu}\tau = (\nu_1\tau, \ldots, \nu_k\tau)$.

(d) If $X_* \subseteq \Lambda_*$, *then we say that a bijection* $\tau : S \longrightarrow T$ *is* X_*-admissible *if the conditions* $\operatorname{orco} X_* \subseteq S$ *and* $X_*\tau \subseteq \Lambda_*$ *hold.*

(e) If $\tau : S \longrightarrow T$ *is an* X_**-admissible bijection, then* τ *extends canonically to an* A*-module monomorphism* $\tau : \widehat{B}_{X_*} \longrightarrow \widehat{B}_{\Lambda_*} = \widehat{B}$, *which we call* shift-isomorphism *(onto its image). If the bijection* τ *fails to be* X_**-admissible, it may still be extended to a canonical shift-isomorphism* τ *with* $\operatorname{Dom} \tau = \widehat{B}_{X_*}$, *which may prove helpful under circumstances as described in the proof of Step Lemma 4.16.*

We want to show that X_*-admissible maps are compatible with many notions from the last chapter.

Observation 4.12 *(a) If* $X \subseteq \Lambda$ *and* $\tau : S \longrightarrow T$ *is an* $[X]$*-admissible bijection, then* $X\tau \subseteq \Lambda$*.*

(b) If $(X_*, X, \mathfrak{F})$ *with* $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid [\pi_{\overline{\eta}} b_{\overline{\eta}}] \subseteq X_*, \overline{\eta} \in X\}$ *is* Λ*-closed and* τ *is* X_**-admissible, then* $(X_*, X, \mathfrak{F})\tau = (X_*\tau, X\tau, \mathfrak{F}\tau)$ *with the family* $\mathfrak{F}\tau = \{y'_{\overline{\eta}} = (\pi_{\overline{\eta}\tau^{-1}} b_{\overline{\eta}\tau^{-1}})\tau + y_{\overline{\eta}} \mid \overline{\eta} \in X\tau\}$ *is also* Λ*-closed and for the associated A-modules holds* $G_{X_*\tau,\, X\tau} = (G_{X_*X})\tau$*.*

(c) If $(X_*, X, \mathfrak{F})$ *is* f*-closed (pairwise closed) with respect to* $(Y_*, Y, \mathfrak{G})$ *and* τ *is* Y_**-admissible, then* $(X_*, X, \mathfrak{F})\tau$ *is also* f*-closed (pairwise closed) with respect to* $(Y_*, Y, \mathfrak{G})\tau$*.*

(d) If $(X_*, X, \mathfrak{F})$ *is* Λ*-closed,* $G \subseteq G_{X_*X}$ *and* τ *is* X_**-admissible, then* $(C_f(G))\tau = C_f(G\tau)$ *holds for the closures* $C_f(G)$ *with respect to* $(X_*, X, \mathfrak{F})$ *and* $C_f(G\tau)$ *with respect to* $(X_*, X, \mathfrak{F})\tau$*.*

(e) If $(X_*, X, \mathfrak{F})$ *is* Λ*-closed,* $G \subseteq G_{X_*X}$ *and* τ *is* X_**-admissible, then* $(PC(G))\tau = PC(G\tau)$ *holds for the closures* $PC(G)$ *with respect to* $(X_*, X, \mathfrak{F})$ *and* $PC(G\tau)$ *with respect to* $(X_*, X, \mathfrak{F})\tau$*.*

(f) Let $(X_*, X, \mathfrak{F})$ *be* (f, α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})$ *and let* τ *be* Y_**-admissible with the following additional properties.*

(i) $[\overline{\eta}] \subseteq X_*$ *and* $[b_{\overline{\eta}}] \subseteq X_*$ *for all* $\overline{\eta} \in Y_{X_*X}$*.*

(ii) $\tau \restriction \mathrm{orco}\{\eta_k \mid \overline{\eta} \in Y_{X_*X}\} = \mathrm{id}$.

Then $(X_*, X, \mathfrak{F})\tau$ *is* (f, α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})\tau$ *and* $(Y\tau)_{X_*\tau, X\tau} = (Y_{X_*X})\tau$ *holds. Furthermore, also* $(C_f(G_{X_*X}))\tau = C_f(G_{X_*\tau, X\tau})$ *holds for the closures* $C_f(G_{X_*X})$ *with respect to* $(X_*, X, \mathfrak{F})$ *and* $C_f(G_{X_*\tau, X\tau})$ *with respect to* $(X_*, X, \mathfrak{F})\tau$*.*

Proof. Since all statements are obvious, for its illustration we only show (a).
If $\overline{\eta} \in X$, then $[\overline{\eta}] \subseteq [X]$ and $[\overline{\eta}]\tau \subseteq \Lambda_*$. In particular, $\overline{\eta}\upharpoonright\langle m,n\rangle\tau \in \Lambda_*$ for any $1 \leq m \leq k, n < \omega$. Thus, $(\eta_m \upharpoonright n)\tau \in {}^{\omega\uparrow>}\lambda_m$, and $\eta_m\tau \in {}^{\omega\uparrow}\lambda_m$ and $\overline{\eta}\tau \in \Lambda$ follows.
For the proof of claim (f) observe that $(Y\tau)_{X_*\tau,X\tau} = (Y_{X_*X})\tau$ is a direct consequence of the identity

$$\{\overline{\eta} \in (Y\tau) \setminus (X\tau) \mid |\, u_{\overline{\eta}}(X_*\tau)\,| \geq f\}$$

$$= \{\overline{\eta} \in Y \setminus X \mid |\, u_{\overline{\eta}}(X_*)\,| \geq f\}\tau = (Y_{X_*X})\tau.$$

We refer to Proposition 4.6 for $(C_f(G_{X_*X}))\tau = C_f(G_{X_*\tau,X\tau})$. Observe here that once again $C_f(G_{X_*X}) = (X_*, X \cup Y_{X_*X}, \mathfrak{F} \cup \mathfrak{H})$ holds with $\mathfrak{H} = \{y'_{\overline{\eta}} \mid \overline{\eta} \in Y_{X_*X}\}$. ■

Remark 4.13 *It is noteworthy that the most incompatible condition with X_*-admissible maps is the regressiveness condition (i) of Definition 3.9 in the case of infinite support corrections. Therefore, when transferring Observation 4.12 to the setting of Section 3.2 the triple $(X_*, X, \mathfrak{F})\tau$ must also be assumed Λ-closed.*

In the case of finite support corrections prevalent in this thesis, killing unwanted homomorphisms is very often an easy task thanks to the strict algebraic structure of the modules G_{X_*X} as will be demonstrated by the following lemma.

Remark 4.14 *From this point on we will assume $\pi R \cap R = \{0\}$ as additional condition of the $\mathbb{S}$-ring R to keep the arguments of the subsequent step lemmas particularly simple. Observe, that $\pi R \cap R = \{0\}$ implies that R is cotorsion-free. Furthermore, if $\pi R \cap R = \{0\}$ holds, then every $\aleph_k$-free R-module A is cotorsion-free with $\pi A \cap A = \{0\}$ and $\pi B \cap B = \{0\}$. The existence of π depends on R and follows for a large class of rings R of size $< 2^{\aleph_0}$ from [32, Theorem 1.21, p. 17]. This includes the case $R = \mathbb{Z}$.*

Step Lemma 4.15 *Let R be an $\mathbb{S}$-ring with $\pi R \cap R = \{0\}$ for some $\pi \in \widehat{R}$ and assume that the following parameters are given.*

(i) $(Y_*, Y, \mathfrak{G})$ *with* $\mathfrak{G} = \{y'''_{\overline{\eta}} = \pi''_{\overline{\eta}} b''_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y\}$ *is a Λ-closed triple with* $\pi''_{\overline{\eta}} \in \{\pi, 0\}$ *for all* $\overline{\eta} \in Y$.

(ii) $\overline{\xi} \in \Lambda$ *and* $z \in B$.

(iii) $\varphi : B_{[\overline{\xi}]\cup[z]} \longrightarrow G_{Y_*Y}$ *is a homomorphism with* $z\varphi \notin B_{Y_*}$.

Then $\pi_{\overline{\xi}} \in \{\pi, 0\}$ *can be chosen such that the* Λ*-closed triple* $(X_*, X, \mathfrak{F})$ *with*

$$X_* = [\overline{\xi}] \cup [z],\ X = \{\overline{\xi}\},\ \mathfrak{F} = \{y'_{\overline{\xi}}\} \text{ and } y'_{\overline{\xi}} = \pi_{\overline{\xi}} z + y_{\overline{\xi}}$$

has the following property.

If $(Z_*, Z, \mathfrak{H})$ *with* $\mathfrak{H} = \{y''_{\overline{\eta}} = \pi'_{\overline{\eta}} b'_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z\}$ *is a* Λ*-closed triple with* $\pi'_{\overline{\eta}} \in \{\pi, 0\}$ *for all* $\overline{\eta} \in Z$ *and* τ *is some* Y_**-admissible bijection with* $(Y_*, Y, \mathfrak{G})\tau \subseteq (Z_*, Z, \mathfrak{H})$*, then*

$$\varphi\tau : B_{[\overline{\xi}]\cup[z]} \longrightarrow G_{Y_*\tau, Y\tau} \text{ does not extend to a homomorphism } \psi : G_{X_*X} \longrightarrow G_{Z_*Z}.$$

Proof. Set $y^i_{\overline{\xi}} = i \cdot \pi z + y_{\overline{\xi}}$ for $i \in \{0, 1\}$. Now, towards a contradiction, assume that there exist Y_*-admissible bijections τ^i, Λ-closed triples $(Y_*, Y, \mathfrak{G})\tau^i \subseteq (Z^i_*, Z^i, \mathfrak{H}^i)$ and extensions ψ^i such that

$$y^i_{\overline{\xi}}\psi^i \in G_{Z^i_* Z^i} \text{ holds for } i \in \{0, 1\}. \tag{4.2}$$

As furthermore $z\varphi \in G_{Y_*Y}$, there exists some $s \in \mathbb{S}$ such that

$$sz\varphi \in \langle B_{Y_*}, Ay'''_{\overline{\eta}} \mid y'''_{\overline{\eta}} \in \mathfrak{G}\rangle \text{ and } sy^i_{\overline{\xi}}\psi^i \in \langle B_{Z^i_*}, Ay''_{\overline{\eta}} \mid y''_{\overline{\eta}} \in \mathfrak{H}^i\rangle \text{ for } i \in \{0, 1\} \tag{4.3}$$

hold simultaneously. From (4.3) follows with Observation 3.3 that

$$\text{only finitely many coefficients of } sy^i_{\overline{\xi}}\psi^i \in \widehat{B} \text{ are from } \langle \pi A, A\rangle \setminus A \text{ for } i \in \{0, 1\}. \tag{4.4}$$

The homomorphism $\varphi : B_{[\overline{\xi}]\cup[z]} \longrightarrow G_{Y_*Y}$ extends uniquely (by continuity) to a homomorphism $\widehat{\varphi} : \widehat{B}_{[\overline{\xi}]\cup[z]} \longrightarrow \widehat{B}_{Y_*}$ with $y^i_{\overline{\xi}} \in \widehat{B}_{[\overline{\xi}]\cup[z]}$. Thus, $sy^i_{\overline{\xi}}\psi^i = sy^i_{\overline{\xi}}\widehat{\varphi}\tau^i$ holds and applying the shift-isomorphism $(\tau^i)^{-1}$ to (4.4) implies again that

$$\text{only finitely many coefficients of } sy^i_{\overline{\xi}}\widehat{\varphi} \in \widehat{B} \text{ are from } \langle \pi A, A\rangle \setminus A \text{ for } i \in \{0, 1\}.$$

Hence, only finitely many coefficients of

$$sy^1_{\overline{\xi}}\widehat{\varphi} - sy^0_{\overline{\xi}}\widehat{\varphi} = s \cdot (y^1_{\overline{\xi}} - y^0_{\overline{\xi}})\widehat{\varphi} = s \cdot (\pi z)\widehat{\varphi} = \pi \cdot (sz\varphi) \in \widehat{B}$$

are from $\langle \pi A, A\rangle \setminus A$ as well. Then again, from $z\varphi \notin B_{Y_*}$ follows that $[z\varphi]_\Lambda \neq \emptyset$, and (4.3) together with Observation 3.3 implies that $\pi \cdot (sz\varphi)$ contains infinitely many coefficients from $\pi A \setminus \{0\} = \pi A \setminus A$, a contradiction.

Thus, statement (4.2) has to be wrong for at least one $i \in \{0,1\}$ independently of the choice of $(Z^i_*, Z^i, \mathfrak{H}^i)$, τ^i and ψ^i. Once we have chosen such an $i \in \{0,1\}$, the claim of the theorem will hold for the choice $\pi_{\overline{\xi}} = i \cdot \pi$. ∎

We now prove the central step lemma of this chapter.

Step Lemma 4.16 *Let R be an $\mathbb{S}$-ring with $\pi R \cap R = \{0\}$ for some $\pi \in \widehat{R}$ and assume that the following parameters are given.*

(i) $0 \leq f < k$ *and* $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$ *with* $\alpha = \| \xi_k \| = \sup_{\ell<\omega}(\ell\xi_k + 1)$.

(ii) $z \in B$ *with* $[z]_{\overline{\nu}} = 1_A$ *for some* $\overline{\nu} \in \Lambda_*$.

(iii) $G_1 = G_1(\overline{\xi}) = B_{\Lambda_{\overline{\xi}*} \cup [z]}$.

(iv) $(Y_*, Y, \mathfrak{G})$ *with* $\mathfrak{G} = \{y'''_{\overline{\eta}} = \pi''_{\overline{\eta}} b''_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y\}$ *is a* Λ*-closed triple with* $[z] \subseteq Y_*$ *and* $\pi''_{\overline{\eta}} \in \{\pi, 0\}$ *for all* $\overline{\eta} \in Y$.

(v) $\varphi : G_1 \longrightarrow G = G_{Y_*Y}$ *is a homomorphism with* $z\varphi \notin Az$.

Then $\pi_{\overline{\eta}} \in \{\pi, 0\}$ $(\overline{\eta} \in \Lambda^{\overline{\xi}})$ *can be chosen such that the* Λ*-closed triple* $(X_*, X, \mathfrak{F})$ *with*

$$X_* = \Lambda_{\overline{\xi}*} \cup \Lambda^{\overline{\xi}}_* \cup [z],\ X = \Lambda^{\overline{\xi}},\ \mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in X\}$$

and the induced A-module $G_2 = G_{X_*X}$ *have the following property.*
If $(Z_*, Z, \mathfrak{H})$ *is a* Λ*-closed triple with* $\mathfrak{H} = \{y''_{\overline{\eta}} = \pi'_{\overline{\eta}} b'_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z\}$ *and* τ *is a* Y_**-admissible bijection such that*

(vi) $\pi'_{\overline{\eta}} \in \{\pi, 0\}$ *for all* $\overline{\eta} \in Z$,

(vii) $\tau \restriction \operatorname{orco} z = \operatorname{id}$,

(viii) $(Y_*, Y, \mathfrak{G})\tau \subseteq (Z_*, Z, \mathfrak{H})$ *with induced A-modules* $G_3 = G_{Y_*\tau, Y\tau} \subseteq G_4 = G_{Z_*Z}$,

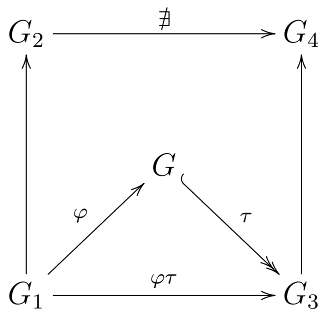
(ix) $(Y_*, Y, \mathfrak{G})\tau$ *is* $(k-f, \alpha)$*-closed with respect to* $(Z_*, Z, \mathfrak{H})$,

(x) $\pi'_{\overline{\eta}} b'_{\overline{\eta}} \in \{\pi z, 0\}$ *for all* $\overline{\eta} \in Z_{Y_*\tau, Y\tau}$,

then

$$\varphi\tau : G_1 \longrightarrow G_3 \text{ does not extend to a homomorphism } \psi : G_2 \longrightarrow G_4.$$

The mappings in the step lemma can be visualized by the following diagram, where arrows without a name are inclusions.



Proof. The step lemma is shown by induction on f.

The case $f = 0$

If $f = 0$, then $\overline{\xi} \in \Lambda$ and the basic sets satisfy

$$\Lambda_*^{\overline{\xi}} = \emptyset, \Lambda_{\overline{\xi}*} = [\overline{\xi}], X_* = \Lambda_{\overline{\xi}*} \cup \Lambda_*^{\overline{\xi}} \cup [z] = [\overline{\xi}] \cup [z], X = \Lambda^{\overline{\xi}} = \{\overline{\xi}\},$$

and the corresponding A-modules are $G_1 = B_{\Lambda_{\overline{\xi}*} \cup [z]} = B_{[\overline{\xi}] \cup [z]}$ and

$$G_2 = G_{X_*X} = \langle B_{X_*}, Ay'_{\overline{\xi}} \rangle_* = \langle B_{[\overline{\xi}] \cup [z]}, Ay'_{\overline{\xi}} \rangle_* = \langle G_1, Ay'_{\overline{\xi}} \rangle_* = \langle G_1, Ay'_{\overline{\xi} i} \mid i < \omega \rangle \subseteq_* \widehat{B}.$$

Now, if $z\varphi \notin B_{Y_*}$ then Step Lemma 4.15 applies and $\pi_{\overline{\eta}}$ can be chosen such that $\varphi : G_1 = B_{[\overline{\xi}] \cup [z]} \longrightarrow G = G_{Y_*Y}$ does not extend. Thus, without loss of generality we may assume in the following that

$$z\varphi \in B_{Y_*}. \tag{4.5}$$

Set $y_{\overline{\xi}}^i = i \cdot \pi z + y_{\overline{\xi}}$ for $i \in \{0, 1\}$. Now, towards a contradiction, assume that there exist suitable Y_*-admissible bijections τ^i, Λ-closed triples $(Y_*, Y, \mathfrak{G})\tau^i \subseteq (Z_*^i, Z^i, \mathfrak{H}^i)$ and extensions ψ^i such that

$$y_{\overline{\xi}}^i \psi^i \in G_{Z_*^i Z^i} \text{ holds for } i \in \{0, 1\}. \tag{4.6}$$

Thus, there exists some $s \in \mathbb{S}$ such that

$$s y_{\overline{\xi}}^i \psi^i \in \langle B_{Z_*^i}, Ay''_{\overline{\eta}} \mid y''_{\overline{\eta}} \in \mathfrak{H}^i \rangle \text{ holds for } i \in \{0, 1\}$$

simultaneously with accompanying representations

$$sy^i_{\overline{\xi}}\psi^i = b^i + \sum_{\substack{\overline{\eta}\in Z^i\\ y''_{\overline{\eta}}\in\mathfrak{H}^i}} a^i_{\overline{\eta}} y''_{\overline{\eta}} \tag{4.7}$$

for suitable elements $b^i \in B_{Z^i_*}$ and coefficients $a^i_{\overline{\eta}} \in A$ ($\overline{\eta} \in Z^i$).

The homomorphism $\varphi : B_{[\overline{\xi}]\cup[z]} \longrightarrow G_{Y_*Y}$ extends uniquely (by continuity) to a homomorphism $\widehat{\varphi} : \widehat{B}_{[\overline{\xi}]\cup[z]} \longrightarrow \widehat{B}_{Y_*}$ with $y^i_{\overline{\xi}} \in \widehat{B}_{[\overline{\xi}]\cup[z]}$. Thus, $y^i_{\overline{\xi}}\psi^i = y^i_{\overline{\xi}}\widehat{\varphi}\tau^i$ holds with $\widehat{\varphi}\tau^i : \widehat{B}_{[\overline{\xi}]\cup[z]} \longrightarrow \widehat{B}_{Y_*\tau^i}$. Furthermore, as

$$\text{for each } \overline{\eta} \in [y^i_{\overline{\xi}}\psi^i]_\Lambda \subseteq Z^i \text{ holds } [y_{\overline{\eta}j}] \subseteq [y^i_{\overline{\xi}}\psi^i] \subseteq Y_*\tau^i \text{ for some } j < \omega \tag{4.8}$$

by Observation 3.3 and as $(Y_*, Y, \mathfrak{G})\tau^i \subseteq (Z^i_*, Z^i, \mathfrak{H}^i)$ is (k,α)-closed by condition (ix), we may apply Observation 4.3(d) to obtain that

$$\text{either } \overline{\eta} \in Y\tau^i \text{ or } \overline{\eta} \in Z^i_{Y_*\tau^i, Y\tau^i} \text{ holds for all } a^i_{\overline{\eta}} \neq 0. \tag{4.9}$$

Setting $\Delta^i = Z^i_{Y_*\tau^i, Y\tau^i}$, we can combine these facts to rewrite (4.7) as

$$sy^i_{\overline{\xi}}\widehat{\varphi}\tau^i = b^i + \sum_{\substack{\overline{\eta}\in Y\tau^i\\ y''_{\overline{\eta}}\in\mathfrak{H}^i}} a^i_{\overline{\eta}} y''_{\overline{\eta}} + \sum_{\substack{\overline{\eta}\in \Delta^i\\ y''_{\overline{\eta}}\in\mathfrak{H}^i}} a^i_{\overline{\eta}} y''_{\overline{\eta}}\,.$$

For all $\overline{\eta} \in Y$, $y''_{\overline{\eta}\tau^i} = y'''_{\overline{\eta}}\tau^i$ holds, and for all $\overline{\eta} \in \Delta^i$, $y''_{\overline{\eta}} \in \mathfrak{H}^i$ holds $y''_{\overline{\eta}} = \pi'^i_{\overline{\eta}} z + y_{\overline{\eta}}$ with $\pi'^i_{\overline{\eta}} \in \{\pi, 0\}$ by conditions (vi) and (x). Thus,

$$sy^i_{\overline{\xi}}\widehat{\varphi}\tau^i = b^i + \sum_{\overline{\eta}\in Y} a^i_{\overline{\eta}\tau_i} y'''_{\overline{\eta}}\tau^i + \sum_{\overline{\eta}\in\Delta^i} a^i_{\overline{\eta}}(\pi'^i_{\overline{\eta}} z + y_{\overline{\eta}})\,. \tag{4.10}$$

Furthermore, from (4.8) and (4.9) follows, for all $\overline{\eta} \in \Delta^i$ with $a^i_{\overline{\eta}} \neq 0$, that $[y_{\overline{\eta}j}] \subseteq Y_*\tau^i$ for some $j < \omega$, hence $\operatorname{orco}\overline{\eta} \subseteq \operatorname{Im}\tau^i$, $\overline{\eta}(\tau^i)^{-1} \in \Lambda$ is well-defined and $(\pi'^i_{\overline{\eta}}z + y_{\overline{\eta}})(\tau^i)^{-1} = \pi'^i_{\overline{\eta}}z + y_{\overline{\eta}(\tau^i)^{-1}}$ holds. Finally, from (4.10) follows that also $\operatorname{orco} b^i \subseteq \operatorname{Im}\tau^i$ must hold, and $b^i(\tau^i)^{-1} \in B$ is well-defined. Thus we may apply $(\tau^i)^{-1}$ to (4.10) to obtain

$$sy^i_{\overline{\xi}}\widehat{\varphi} = b^i(\tau^i)^{-1} + \sum_{\overline{\eta}\in Y} a^i_{\overline{\eta}\tau_i} y'''_{\overline{\eta}} + \sum_{\overline{\eta}\in\Delta^i} a^i_{\overline{\eta}}(\pi'^i_{\overline{\eta}} z + y_{\overline{\eta}(\tau^i)^{-1}}) \text{ for } i \in \{0,1\}$$

and subtraction of these two equations gives

$$\begin{aligned}
\pi\cdot(sz)\varphi &= \pi\cdot(sz)\widehat{\varphi} = s\cdot(\pi z)\widehat{\varphi} = s(y^1_{\overline{\xi}}\widehat{\varphi} - y^0_{\overline{\xi}}\widehat{\varphi})\\
&= (b^1(\tau^1)^{-1} - b^0(\tau^0)^{-1}) + (\sum_{\overline{\eta}\in\Delta^1}\pi'^1_{\overline{\eta}}a^1_{\overline{\eta}} - \sum_{\overline{\eta}\in\Delta^0}\pi'^0_{\overline{\eta}}a^0_{\overline{\eta}})\,z\\
&\quad + \sum_{\overline{\eta}\in Y}(a^1_{\overline{\eta}\tau_1} - a^0_{\overline{\eta}\tau_0})y'''_{\overline{\eta}} + (\sum_{\overline{\eta}\in\Delta^1} a^1_{\overline{\eta}} y_{\overline{\eta}(\tau^1)^{-1}} - \sum_{\overline{\eta}\in\Delta^0} a^0_{\overline{\eta}} y_{\overline{\eta}(\tau^0)^{-1}})\,.
\end{aligned} \tag{4.11}$$

Because of (4.5), the support of the left-hand side of equation (4.11) is finite, and applying Observation 3.3 for $S_* = [z\varphi]$, the branch elements on the right-hand side of equation (4.11) must cancel each other. This in particular implies $a^1_{\overline{\eta}\tau_1} = a^0_{\overline{\eta}\tau_0}$ for all $\overline{\eta} \in Y$, and equation (4.11) reduces to

$$\pi \cdot (sz)\varphi = (b^1(\tau^1)^{-1} - b^0(\tau^0)^{-1}) + (\sum_{\overline{\eta}\in\Delta^1} \pi'^1_{\overline{\eta}} a^1_{\overline{\eta}} - \sum_{\overline{\eta}\in\Delta^0} \pi'^0_{\overline{\eta}} a^0_{\overline{\eta}})\, z.$$

With

$$b = b^1(\tau^1)^{-1} - b^0(\tau^0)^{-1} \in B \text{ and } \pi a = \sum_{\overline{\eta}\in\Delta^1} \pi'^1_{\overline{\eta}} a^1_{\overline{\eta}} - \sum_{\overline{\eta}\in\Delta^0} \pi'^0_{\overline{\eta}} a^0_{\overline{\eta}} \in \pi A$$

we can write $\pi \cdot (sz)\varphi = b + \pi a z$ or

$$\pi \cdot [s(z\varphi) - az] = b. \tag{4.12}$$

Using $\pi B \cap B = \{0\}$ (see Remark 4.13) it follows that $s(z\varphi) = az$. Comparing coefficients, we conclude with the help of condition (ii) that $a = [az]_{\overline{\nu}} \in sA$. Hence, $a = sa'$ for some $a' \in A$ and $z\varphi = a'z \in Az$ as B is $\mathbb{S}$-torsion-free, a contradiction to condition (v).

Thus, statement (4.6) has to be wrong for at least one $i \in \{0, 1\}$, independently of the choice of $(Z^i_*, Z^i, \mathfrak{H}^i)$, τ^i and ψ^i. Given such an $i \in \{0, 1\}$, the claim of the theorem will hold for the choice $\pi_{\overline{\xi}} = i \cdot \pi$.

The case $f > 0$

Now suppose that $f > 0$, and the lemma is already shown for $f - 1$. Let $\lambda = \lambda_f$ and $\theta = \lambda_{f-1}$ (setting $\lambda_0 = |\,A\,|$), so $\theta < \lambda$. The A-modules G_1 and $G = G_{Y_*Y}$ are given, where $G_1 = B_{\Lambda_{\overline{\xi}*} \cup [z]}$ is free. Furthermore, by construction we have $\left|\,\Lambda_{\overline{\xi}*}\,\right| = |\,X_*\,| = |\,X\,| = \lambda^{\aleph_0} = \lambda$. Thus, as $|\,A\,| < \lambda_1 \leq \lambda_f = \lambda$ also $|\,G_1\,| = |\,G_2\,| = \lambda$ holds.

Let $\langle \eta_\alpha \mid \alpha < \lambda \rangle$ be an enumeration of $\Gamma = {}^{\omega\uparrow}\lambda$ without repetitions. For $\nu \in {}^{\omega\uparrow>}\lambda$ and $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$, we define

$$\Lambda_{\nu\overline{\xi}*} = \{\overline{\nu} \in \Lambda_{f*} \mid \nu_f \trianglelefteq \nu, \overline{\nu}\restriction(f,k] = \overline{\xi}\}, G_{1\nu} = B_{\Lambda_{\nu\overline{\xi}*}} \text{ and } \mathcal{G} = \{G_{1\nu} \mid \nu \in {}^{\omega\uparrow>}\lambda\}.$$

Clearly, $|\,G_{1\nu}\,| = \theta$ and $|\,\mathcal{G}\,| = \lambda$.

Let $(Y'_*, Y', \mathfrak{G}') = C_{k-f}(\langle G_1\varphi, Az\rangle) \subseteq (Y_*, Y, \mathfrak{G})$ be the $(k-f)$-closure of $\langle G_1\varphi, Az\rangle$ as defined in Theorem 4.5(b) with $|\, Y'_* \,|, |\, Y' \,|, |\, \mathfrak{G}' \,| \leq |\, G_1\varphi \,|^{\aleph_0} \leq |\, G_1 \,|^{\aleph_0} = \lambda$. In particular, $|\operatorname{orco} Y'_* \,| \leq \lambda = \lambda_f$, and we can find $\Delta \subseteq \lambda_{f+1} \setminus \operatorname{orco} Y'_*$ with $|\, \Delta \,| = \lambda$. Until now we have used sequences $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_k \rangle$ (as in Section 2.1) based on cardinals λ_ℓ (which are ordinals and hence particular sets). In order to have a predefined small universe for the construction of suitable A-modules, we must now pass to sets of ordinals. Extending Section 2.1, we define a sequence $\overline{\lambda'} = \langle \lambda'_1, \ldots, \lambda'_k \rangle$ of sets of ordinals by $\lambda'_\ell = (\operatorname{orco} Y'_*) \dot\cup \Delta$ for all $1 \leq \ell \leq k$. Similarly to the old definition for Λ we now set $\Lambda' = {}^\omega\lambda'_1 \times \cdots \times {}^\omega\lambda'_k$ and $\Lambda'_{m*} = {}^\omega\lambda'_1 \times \cdots \times {}^{\omega>}\lambda'_m \times \cdots \times {}^\omega\lambda'_k$ for any $1 \leq m \leq k$. In contrast to the definition of Λ, we do not utilize the ordering on λ'_ℓ (as a set of ordinals). Again, put $\Lambda'_* = \dot\bigcup_{1 \leq m \leq k} \Lambda'_{m*}$. Now we are ready to define a relatively small A-module $\mathbb{V}$ into which we send interesting submodules by shift-isomorphisms for their predictions. Let $\mathbb{V} = \widehat{\bigoplus_{\overline{\nu} \in \Lambda'_*} Ae'_{\overline{\nu}}}$, which is the $\mathbb{S}$-adic completion of the free A-module $\bigoplus_{\overline{\nu} \in \Lambda'_*} Ae'_{\overline{\nu}}$, thus a canonical $\widehat{A}$-module. Moreover, let $\mathcal{H} = \{H \subseteq \mathbb{V} \mid H \text{ is an } A\text{-submodule}, |\, H \,| \leq \theta\}$. The cardinalities of these new structures are immediate due to Section 2.1 (ii).

$$|\, \Lambda' \,| = |\, \Lambda'_* \,| = \lambda^{\aleph_0} = \lambda, |\, \mathbb{V} \,| = \lambda^{\aleph_0} = \lambda, |\, \mathcal{H} \,| = \lambda^\theta = \lambda_f^{\lambda_{f-1}} = \lambda.$$

Now we can also give the exact definition of a trap. The notion of a trap comes from [5]; it is designed to 'catch' small unwanted homomorphisms and is derived from particular elementary submodels.

Definition 4.17 *A tuple*

$$(G, H, P, Q, \mathcal{R}, \psi)$$

is a trap *(for the step lemma) if $G \in \mathcal{G}, H \in \mathcal{H}, \psi : G \longrightarrow H$ is an R-homomorphism, $P \subseteq \Lambda'_*, Q \subseteq \Lambda'$ and $\mathcal{R} \subseteq \mathbb{V}$ are subsets such that $|\, P \,|, |\, Q \,|, |\, \mathcal{R} \,| \leq \theta$. Let Ξ be the family of all traps $(G, H, P, Q, \mathcal{R}, \psi)$.*

Next, we determine the size of Ξ, which clearly is $|\, \Xi \,| = |\, \mathcal{G} \,| \cdot |\, \mathcal{H} \,| \cdot |\, \Lambda'_* \,|^\theta \cdot |\, \Lambda' \,|^\theta \cdot |\, \mathbb{V} \,|^\theta \cdot |\, H \,|^\theta = \lambda \cdot \lambda^\theta \cdot \theta^\theta = \lambda$. Thus, we can consider the Easy Black Box as stated in Proposition 4.7, but with respect to this new crucial family Ξ of traps:

The Easy Black Box 4.18 *There is a family* $\langle g_\eta \mid \eta \in {}^{\omega\uparrow}\lambda\rangle$ *with* $g_\eta : [\eta] \longrightarrow \Xi$ *such that, for each map* $g : {}^{\omega\uparrow >}\lambda \longrightarrow \Xi$*, there exists some* $\eta \in {}^{\omega\uparrow}\lambda$ *with* $g_\eta \subseteq g$*.*

The construction of G_2

We repeat the arguments as given in [27, p. 34–43]: In order to construct the desired $\aleph_k$-free A-module G with $\operatorname{End} G = A$, we must find particular generators of G which will be branch-like elements having a summand with the ring element $\pi \in \widehat{R}$ as factor, which will prevent unwanted endomorphisms. The A-module G_2 is (a weak form of) an elementary submodel of G; thus it is not surprising that we must determine these factors first for G_2.

For $\alpha < \lambda$ and $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$ as in condition (i) of the Step Lemma 4.16 let $\overline{\xi}_\alpha \in {}^{\omega\uparrow}\lambda_f \times \cdots \times {}^{\omega\uparrow}\lambda_k$ be defined by $(\overline{\xi}_\alpha)_f = \eta_\alpha \in \Gamma$ and $(\overline{\xi}_\alpha)\restriction(f,k] = \overline{\xi}$.

Next we will choose (recursively) the elements $\pi_{\overline{\eta}} \in \{\pi, 0\}$ for $\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$ and for each $\alpha < \lambda$. Since $X = \Lambda^{\overline{\xi}} = \dot{\bigcup}_{\alpha<\lambda}\Lambda^{\overline{\xi}_\alpha}$ (by the definition of Γ), in the end we will have constructed a family of ring elements $\pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}}$) from $\{\pi, 0\}$ as needed for the triple $(X_*, X, \mathfrak{F})$ from above. Hence, G_2 will be determined by $G_2 = G_{X_*X}$ and the family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in X\}$.

Let $\alpha < \lambda$ and $(G_{\alpha n}, H_{\alpha n}, P_{\alpha n}, Q_{\alpha n}, \mathcal{R}_{\alpha n}, \psi_{\alpha n}) = g_{\eta_\alpha}(\eta_\alpha \restriction n) \in \Xi$ be the traps given by the Easy Black Box 4.18. A special choice of $\pi_{\overline{\eta}}$ for $\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$ is only needed in particular situations of these traps, namely, when they represent the local version of an unwanted endomorphism of G, and this will fortunately be only the case when we get support from the results of the last sections. Otherwise we may put $\pi_{\overline{\eta}} = 0$.

Next we specify these conditions when $\pi_{\overline{\eta}} \in \{\pi, 0\}$ must (seriously) be chosen (for killing maps):

We must do some book-keeping by using the results from Sections 4.1, if there exist a Λ-closed triple $(Z_*^\dagger, Z^\dagger, \mathfrak{H}^\dagger)$ with $\mathfrak{H}^\dagger = \{y''^\dagger_{\overline{\eta}} = \pi'^\dagger_{\overline{\eta}} b'^\dagger_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z^\dagger\}$, further Λ-closed triples $(Z_*^{n\dagger}, Z^{n\dagger}, \mathfrak{H}^{n\dagger}) \subseteq (Z_*^\dagger, Z^\dagger, \mathfrak{H}^\dagger)$ for $n < \omega$, a Y_*-admissible bijection $\tau^\dagger$ and a shift-homomorphism $\sigma^\dagger$ with the following properties.

$(A)^\dagger$ $\pi'^\dagger_{\overline{\eta}} \in \{\pi, 0\}$ for all $\overline{\eta} \in Z^\dagger$.

$(B)^\dagger$ $\tau^\dagger \restriction \operatorname{orco} z = \mathrm{id}$.

$(C)^\dagger$ $(Y_*, Y, \mathfrak{G})\tau^\dagger \subseteq (Z_*^\dagger, Z^\dagger, \mathfrak{H}^\dagger)$.

$(D)^\dagger$ $(Y_*, Y, \mathfrak{G})\tau^\dagger$ is $(k-f, \alpha)$-closed with respect to $(Z_*^\dagger, Z^\dagger, \mathfrak{H}^\dagger)$.

$(E)^\dagger$ $\pi_{\overline{\eta}}^{\prime\dagger} b_{\overline{\eta}}^{\prime\dagger} \in \{\pi z, 0\}$ for all $\overline{\eta} \in Z^\dagger_{Y_*\tau^\dagger, Y\tau^\dagger}$.

$(F)^\dagger$ $(Z_*^{n\dagger}, Z^{n\dagger}, \mathfrak{H}^{n\dagger})$ is pairwise closed with respect to $(Z_*^\dagger, Z^\dagger, \mathfrak{H}^\dagger)$.

$(G)^\dagger$ $(Z_*^{n\dagger}, Z^{n\dagger}, \mathfrak{H}^{n\dagger}) \subseteq (Z_*^{n+1,\dagger}, Z^{n+1,\dagger}, \mathfrak{H}^{n+1,\dagger})$

$(H)^\dagger$ $\sigma^\dagger : \operatorname{orco}(Y_*'\tau^\dagger \cup \bigcup_{n<\omega} Z_*^{n\dagger}) \longrightarrow (\operatorname{orco} Y_*') \dot{\cup} \Delta$ is injective.

$(I)^\dagger$ $\sigma^\dagger \restriction \operatorname{orco} Y_*'\tau^\dagger = (\tau^\dagger)^{-1} \restriction \operatorname{orco} Y_*'\tau^\dagger$.

$(J)^\dagger$ $G_{\alpha n} = G_{1, \eta_\alpha \restriction n}$.

$(K)^\dagger$ $H_{\alpha n} = G_{Z_*^{n\dagger} Z^{n\dagger}}\sigma^\dagger$.

$(L)^\dagger$ $(P_{\alpha n}, Q_{\alpha n}, \mathcal{R}_{\alpha n}) = (Z_*^{n\dagger}, Z^{n\dagger}, \mathfrak{H}^{n\dagger})\sigma^\dagger$.

$(M)^\dagger$ The maps $\psi_{\alpha n} : G_{\alpha n} \longrightarrow H_{\alpha n}$ $(n < \omega)$ extend each other, so that

$$\psi_\alpha = \bigcup_{n<\omega} \psi_{\alpha n} \text{ and } G_\alpha = \bigcup_{n<\omega} G_{1,\eta_\alpha \restriction n} = \bigcup_{n<\omega} G_{\alpha n} \text{ are well-defined.} \tag{4.13}$$

$(N)^\dagger$ $\psi_{\alpha n}(\sigma^\dagger)^{-1} \restriction (G_{\alpha n} \cap G_1(\overline{\xi})) = \varphi\tau^\dagger \restriction (G_{\alpha n} \cap G_1(\overline{\xi}))$.

From $(G)^\dagger$ and $(K)^\dagger$ follows $G_{Z_*^{n\dagger} Z^{n\dagger}} \subseteq G_{Z_*^{n+1,\dagger} Z^{n+1,\dagger}}$ and $H_{\alpha n} \subseteq H_{\alpha, n+1}$, making

$$\psi_\alpha(\sigma^\dagger)^{-1} : G_\alpha \longrightarrow \bigcup_{n<\omega} G_{Z_*^{n\dagger} Z^{n\dagger}}$$

a well-defined homomorphism of R-modules, while

$$\varphi\tau^\dagger : G_1(\overline{\xi}) \longrightarrow G_{Y_*'\tau^\dagger, Y'\tau^\dagger}$$

is well-defined by the definition of $(Y_*', Y', \mathfrak{G}')$. Furthermore, from $(N)^\dagger$ we have

$$\psi_\alpha(\sigma^\dagger)^{-1} \restriction (G_\alpha \cap G_1(\overline{\xi})) = \varphi\tau^\dagger \restriction (G_\alpha \cap G_1(\overline{\xi})).$$

Thus, as $G_1(\overline{\xi}_\alpha) \subseteq G_\alpha + G_1(\overline{\xi})$, both homomorphisms $\psi_\alpha(\sigma^\dagger)^{-1}$ and $\varphi\tau^\dagger$ generate a common well-defined extension $\varphi^\dagger$ on $G_1(\overline{\xi}_\alpha)$, and with the help of the Λ-closed triple

$$(\star)^{\dagger}\ \ (Y_*''^{\dagger}, Y''^{\dagger}, \mathfrak{G}''^{\dagger}) = \bigcup_{n<\omega}(Z_*^{n\dagger}, Z^{n\dagger}, \mathfrak{H}^{n\dagger}) \cup (Y_*', Y', \mathfrak{G}')\tau^{\dagger} \subseteq (Z_*^{\dagger}, Z^{\dagger}, \mathfrak{H}^{\dagger})$$

we have

$$\varphi^{\dagger} : G_1(\overline{\xi}_{\alpha}) \longrightarrow \bigcup_{n<\omega} G_{Z_*^{n\dagger}Z^{n\dagger}} + G_{Y_*'\tau^{\dagger},\, Y'\tau^{\dagger}} = G_{Y_*''^{\dagger}Y''^{\dagger}}$$

satisfying $z\varphi^{\dagger} = z\varphi\tau^{\dagger} \notin Az$ with (v) and $(B)^{\dagger}$. Furthermore, observe that the triple $(Y_*''^{\dagger}, Y''^{\dagger}, \mathfrak{G}''^{\dagger})$ suffices $[z] \subseteq Y_*''^{\dagger}$ by the definition of $(Y_*', Y', \mathfrak{G}')$ as well as $\pi_{\overline{\eta}}'^{\dagger} \in \{\pi, 0\}$ for all $\overline{\eta} \in Y''^{\dagger}$ by $(A)^{\dagger}$. We now apply the induction hypothesis of the step lemma. Replacing $f, \overline{\xi}, \alpha, z, G_1(\overline{\xi}), (Y_*, Y, \mathfrak{G}), \varphi$ accordingly by

$$f-1, \overline{\xi}_{\alpha}, \alpha, z, G_1(\overline{\xi}_{\alpha}), (Y_*''^{\dagger}, Y''^{\dagger}, \mathfrak{G}''^{\dagger}), \varphi^{\dagger} \tag{4.14}$$

the Step Lemma 4.16 holds for $f-1$, and the existence of elements $\pi_{\overline{\eta}} \in \{\pi, 0\}$ $(\overline{\eta} \in \Lambda^{\overline{\xi}_{\alpha}})$ follows. Notice that now all of $\{\pi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^{\overline{\xi}}\}$ and $\mathfrak{F} = \{y_{\overline{\eta}}' = \pi_{\overline{\eta}}z + y_{\overline{\eta}} \mid \overline{\eta} \in X\}$ are known. This finishes the construction of G_2.

G_2 satisfies the Step Lemma 4.16 for f.

We finally must show that the family $\mathfrak{F} = \{y_{\overline{\eta}}' = \pi_{\overline{\eta}}z + y_{\overline{\eta}} \mid \overline{\eta} \in X\}$, and thus G_2, is as required in the Step Lemma 4.16. We will prove this by contradiction.
Suppose that $(Z_*^{\ddagger}, Z^{\ddagger}, \mathfrak{H}^{\ddagger})$ is a Λ-closed triple with $\mathfrak{H}^{\ddagger} = \{y_{\overline{\eta}}''^{\ddagger} = \pi_{\overline{\eta}}'^{\ddagger} b_{\overline{\eta}}'^{\ddagger} + y_{\overline{\eta}} \mid \overline{\eta} \in Z^{\ddagger}\}$ and $\tau^{\ddagger}$ is an Y_*-admissible bijection such that

$(A)^{\ddagger}$ $\pi_{\overline{\eta}}'^{\ddagger} \in \{\pi, 0\}$ for all $\overline{\eta} \in Z^{\ddagger}$,

$(B)^{\ddagger}$ $\tau^{\ddagger} \restriction \operatorname{orco} z = \operatorname{id}$,

$(C)^{\ddagger}$ $(Y_*, Y, \mathfrak{G})\tau^{\ddagger} \subseteq (Z_*^{\ddagger}, Z^{\ddagger}, \mathfrak{H}^{\ddagger})$,

$(D)^{\ddagger}$ $(Y_*, Y, \mathfrak{G})\tau^{\ddagger}$ is $(k-f, \alpha)$-closed with respect to $(Z_*^{\ddagger}, Z^{\ddagger}, \mathfrak{H}^{\ddagger})$ and

$(E)^{\ddagger}$ $\pi_{\overline{\eta}}'^{\ddagger} b_{\overline{\eta}}'^{\ddagger} \in \{\pi z, 0\}$ for all $\overline{\eta} \in Z^{\ddagger}_{Y_*\tau^{\ddagger},\, Y\tau^{\ddagger}}$,

but failing to satisfy the lemma. Thus, the homomorphism

$$\varphi\tau^{\ddagger} : G_1 \longrightarrow G_3 = G_{Y_*\tau^{\ddagger},\, Y\tau^{\ddagger}} \text{ lifts to } \psi^{\ddagger} : G_2 \longrightarrow G_4 = G_{Z_*^{\ddagger}Z^{\ddagger}}, \tag{4.15}$$

see the next diagram.

We now define $(Z'^{\ddagger}_*, Z'^{\ddagger}, \mathfrak{H}'^{\ddagger}) = C_1(PC(\langle G_2\psi^{\ddagger}, Az\rangle)) \subseteq (Z^{\ddagger}_*, Z^{\ddagger}, \mathfrak{H}^{\ddagger})$ to be the 1-closure of the pairwise closure of $\langle G_2\psi^{\ddagger}, Az\rangle$ as defined in Theorem 4.5(b),(c). From Observation 4.3(b), Theorem 4.5, Observation 4.12(d) and the chain of inclusions

$$\langle G_1\varphi\tau^{\ddagger}, Az\rangle \subseteq \langle G_2\psi^{\ddagger}, Az\rangle \subseteq PC(\langle G_2\psi^{\ddagger}, Az\rangle)$$

we have

$(O)^{\ddagger}$ $(Y'_*, Y', \mathfrak{G}')\tau^{\ddagger} = C_{k-f}(\langle G_1\varphi, Az\rangle)\tau^{\ddagger} = C_{k-f}(\langle G_1\varphi\tau^{\ddagger}, Az\rangle) \subseteq (Z'^{\ddagger}_*, Z'^{\ddagger}, \mathfrak{H}'^{\ddagger})$.

$(P)^{\ddagger}$ $\left| Z'^{\ddagger}_* \right|, \left| Z'^{\ddagger} \right|, \left| \mathfrak{H}'^{\ddagger} \right| \leq \left| \langle G_2\psi^{\ddagger}, Az\rangle \right|^{\aleph_0} \leq | G_2 |^{\aleph_0} = \lambda^{\aleph_0} = \lambda$.

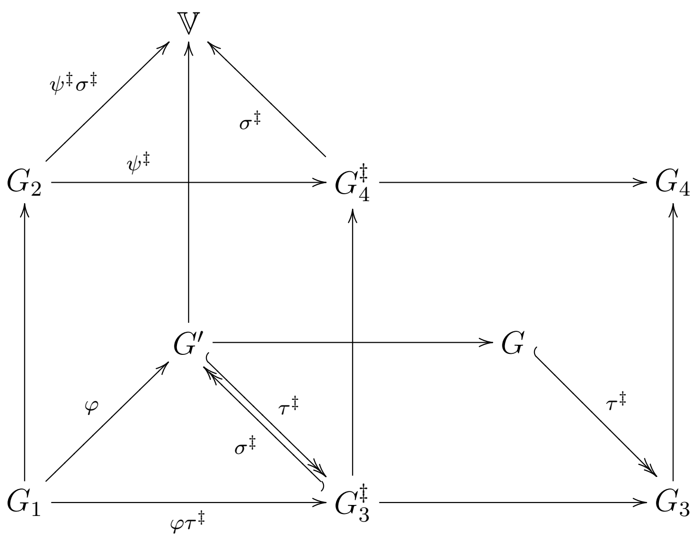
Next we choose an injection $\sigma^{\ddagger}$ with

$(H)^{\ddagger}$ $\sigma^{\ddagger} : \operatorname{orco} Z'^{\ddagger}_* \longrightarrow (\operatorname{orco} Y'_*)\dot{\cup}\Delta$.

$(I)^{\ddagger}$ $\sigma^{\ddagger} \restriction \operatorname{orco} Y'_*\tau^{\ddagger} = (\tau^{\ddagger})^{-1} \restriction \operatorname{orco} Y'_*\tau^{\ddagger}$.

This is possible since $(O)^{\ddagger}$ and $(P)^{\ddagger}$ guarantee $Y'_*\tau^{\ddagger} \subseteq Z'^{\ddagger}_*$ and $\left| Z'^{\ddagger}_* \right| \leq \lambda = | \Delta |$.

Let us pause for a moment and describe the present situation of maps by a diagram. Recall that $G_1 = B_{\Lambda_{\bar{\xi}_*}\cup[z]}$ is a free A-module, $G_2 = G_{X_*X}$, $G_3 = G_{Y_*\tau^{\ddagger}, Y\tau^{\ddagger}}$, $G_4 = G_{Z^{\ddagger}_* Z^{\ddagger}}$ and $G = G_{Y_*Y}$. Naturally, we let $G' = G_{Y'_*Y'}$, $G_3^{\ddagger} = G_{Y'_*\tau^{\ddagger}, Y'\tau^{\ddagger}}$ and $G_4^{\ddagger} = G_{Z'^{\ddagger}_*, Z'^{\ddagger}}$. Thus, we have the following diagram (where arrows with no name are again inclusions).



We want to construct a function $g : {}^{\omega\uparrow>}\lambda \longrightarrow \Xi$ for the use of the Easy Black Box 4.18. For this purpose, choose any $\nu \in {}^{\omega\uparrow>}\lambda$. From the definitions follows $G_{1\nu} \subseteq G_2$, and $\psi^\ddagger \restriction G_{1\nu}$ is a well-defined homomorphism. Let

$$(Z'^{\nu\ddagger}_*, Z'^{\nu\ddagger}, \mathfrak{H}'^{\nu\ddagger}) = PC(G_{1\nu}\psi^\ddagger) \subseteq (Z^\ddagger_*, Z^\ddagger, \mathfrak{H}^\ddagger)$$

be the pairwise closure of $G_{1\nu}\psi^\ddagger$. From Theorem 4.5(c) we have

$(O)^{\nu\ddagger}$ $(Z'^{\nu\ddagger}_*, Z'^{\nu\ddagger}, \mathfrak{H}'^{\nu\ddagger}) = PC(G_{1\nu}\psi^\ddagger) \subseteq PC(\langle G_2\psi^\ddagger, Az\rangle) \subseteq (Z'^\ddagger_*, Z'^\ddagger, \mathfrak{H}'^\ddagger)$.

$(P)^{\nu\ddagger}$ $\left| Z'^{\nu\ddagger}_* \right|, \left| Z'^{\nu\ddagger} \right|, \left| \mathfrak{H}'^{\nu\ddagger} \right| \leq \aleph_0 \cdot \left| G_{1\nu}\psi^\ddagger \right| \leq \aleph_0 \cdot |\, G_{1\nu} \,| = \theta$.

Furthermore, by definition

$(F)^{\nu\ddagger}$ $(Z'^{\nu\ddagger}_*, Z'^{\nu\ddagger}, \mathfrak{H}'^{\nu\ddagger})$ is pairwise closed with respect to $(Z^\ddagger_*, Z^\ddagger, \mathfrak{H}^\ddagger)$.

$(G)^{\nu\ddagger}$ $(Z'^{\nu\restriction((\lg\nu)-1)\,\ddagger}_*, Z'^{\nu\restriction((\lg\nu)-1)\,\ddagger}, \mathfrak{H}'^{\nu\restriction((\lg\nu)-1)\,\ddagger}) \subseteq (Z'^{\nu\ddagger}_*, Z'^{\nu\ddagger}, \mathfrak{H}'^{\nu\ddagger})$.

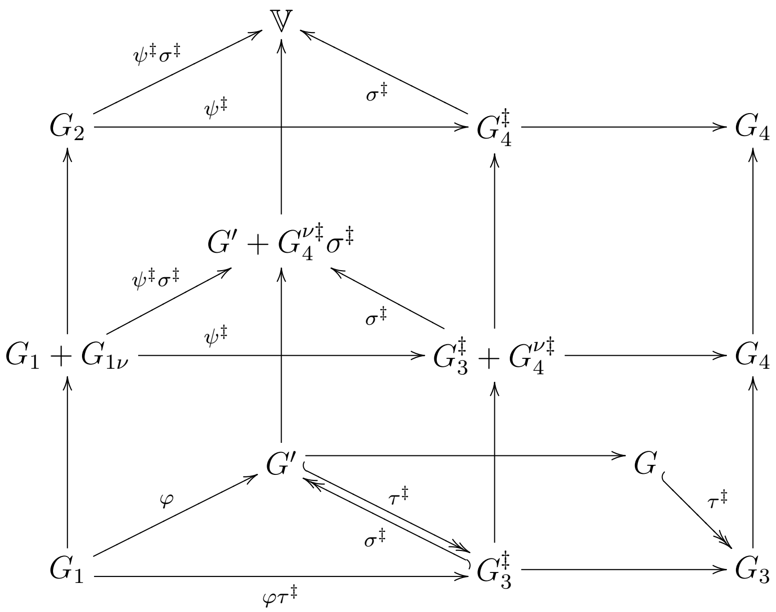
Now we describe a refinement of the last diagram by means of these new results. We naturally put

$$G_4^{\nu\ddagger} = G_{Z'^{\nu\ddagger}_* Z'^{\nu\ddagger}} \subseteq G_4^\ddagger$$

and get the following diagram with the free A-modules

$$G_1 = B_{\Lambda_{\bar{\xi}_*} \cup [z]} \text{ and } G_{1\nu} = B_{\Lambda_{\nu\bar{\xi}_*}}$$

from above, where restrictions of homomorphisms have the same name and inclusions have no name.



We now define the map $g : {}^{\omega\uparrow>}\lambda \longrightarrow \Xi$, which we want to predict, by

$$g(\nu) = (G^\nu, H^\nu, P^\nu, Q^\nu, \mathcal{R}^\nu, \psi^\nu)$$

and the following requirements.

$(J)^{\nu\ddagger}$ $G^\nu = G_{1\nu}$.

$(K)^{\nu\ddagger}$ $H^\nu = G_{Z'^{\nu\ddagger}_* Z'^{\nu\ddagger}}\sigma^\ddagger$.

$(L)^{\nu\ddagger}$ $(P^\nu, Q^\nu, \mathcal{R}^\nu) = (Z'^{\nu\ddagger}_*, Z'^{\nu\ddagger}, \mathfrak{H}'^{\nu\ddagger})\sigma^\ddagger$.

$(Q)^{\nu\ddagger}$ $\psi^\nu = (\psi^\ddagger \restriction G_{1\nu})\sigma^\ddagger : G^\nu \longrightarrow H^\nu$.

From $(P)^{\nu\ddagger}$ follows $|\,P^\nu\,|, |\,Q^\nu\,|, |\,\mathcal{R}^\nu\,| \leq \theta$, and we also have $G^\nu \in \mathcal{G}, H^\nu \in \mathcal{H}$ and $P^\nu \subseteq \Lambda'_*, Q^\nu \subseteq \Lambda', \mathcal{R}^\nu \subseteq \mathbb{V}$. Consequently, $(G^\nu, H^\nu, P^\nu, Q^\nu, \mathcal{R}^\nu, \psi^\nu) \in \Xi$. Finally, by the definition of ψ^ν in $(Q)^{\nu\ddagger}$ and (4.15) follows

$(M)^{\nu\ddagger}$ $\psi^{\nu\restriction((\lg\nu)-1)} \subseteq \psi^\nu$.

$(N)^{\nu\ddagger}$ $\psi^\nu(\sigma^\ddagger)^{-1} \restriction (G^\nu \cap G_1(\overline{\xi})) = \psi^\ddagger \restriction (G^\nu \cap G_1(\overline{\xi})) = \varphi\tau^\ddagger \restriction (G^\nu \cap G_1(\overline{\xi}))$.

Thus, we can apply the Easy Black Box 4.18 and we find some $\eta \in \Gamma$ with $g_\eta \subseteq g$. There is some $\alpha < \lambda$ such that $\eta = \eta_\alpha$. Now, for the construction of $\pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$), the 'serious' case applies as it is witnessed by the choice of

$$(Z^\ddagger_*, Z^\ddagger, \mathfrak{H}^\ddagger), (Z^{n\ddagger}_*, Z^{n\ddagger}, \mathfrak{H}^{n\ddagger}) = (Z'^{\eta_\alpha\restriction n\,\ddagger}_*, Z'^{\eta_\alpha\restriction n\,\ddagger}, \mathfrak{H}'^{\eta_\alpha\restriction n\,\ddagger}), \tau^\ddagger, \sigma^\ddagger$$

as possible candidates for

$$(Z^\dagger_*, Z^\dagger, \mathfrak{H}^\dagger), (Z^{n\dagger}_*, Z^{n\dagger}, \mathfrak{H}^{n\dagger}), \tau^\dagger, \sigma^\dagger.$$

The necessary conditions $(A)^\dagger$ to $(N)^\dagger$ are satisfied by the respective conditions $(A)^\ddagger$ to $(N)^\ddagger$ and $(A)^{\nu\ddagger}$ to $(N)^{\nu\ddagger}$, using $(O)^\ddagger$ and $(O)^{\nu\ddagger}$ for $Y'_*\tau^\ddagger \cup \bigcup_{n<\omega} Z'^{\eta_\alpha\restriction n\,\ddagger} \subseteq Z'^\ddagger_*$. The concluding arguments of this proof are visualized in the following diagram: As in the earlier construction of the ring elements $\pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$) we have

$$(G_{\alpha n}, H_{\alpha n}, P_{\alpha n}, Q_{\alpha n}, \mathcal{R}_{\alpha n}, \psi_{\alpha n}) = g_{\eta_\alpha}(\eta_\alpha \restriction n) \text{ and } \psi_\alpha = \bigcup_{n<\omega} \psi_{\alpha n},$$

and let

$$\varphi^{\ddagger} : G_1(\overline{\xi}_\alpha) \longrightarrow G_{Y_*''^{\ddagger} Y''^{\ddagger}}$$

with

$$(\star)^{\ddagger} \;\; (Y_*''^{\ddagger}, Y''^{\ddagger}, \mathfrak{G}''^{\ddagger}) = \bigcup_{n<\omega} (Z_*^{n\ddagger}, Z^{n\ddagger}, \mathfrak{H}^{n\ddagger}) \cup (Y_*', Y', \mathfrak{G}')\tau^{\ddagger} \subseteq (Z_*^{\ddagger}, Z^{\ddagger}, \mathfrak{H}^{\ddagger})$$

be the well-defined common extension of the homomorphisms $\psi_\alpha(\sigma^{\ddagger})^{-1}$ and $\varphi\tau^{\ddagger}$. By the prediction of the Easy Black Box 4.18, we get the following identities.

(J) $G_{\alpha n} = G^{\eta_\alpha \restriction n} = G_{1, \eta_\alpha \restriction n}$.

(L) $(Z_*^{n\dagger}, Z^{n\dagger}, \mathfrak{H}^{n\dagger})\sigma^{\dagger} = (P_{\alpha n}, Q_{\alpha n}, \mathcal{R}_{\alpha n}) = (P^{\eta_\alpha \restriction n}, Q^{\eta_\alpha \restriction n}, \mathcal{R}^{\eta_\alpha \restriction n}) = (Z_*^{n\ddagger}, Z^{n\ddagger}, \mathfrak{H}^{n\ddagger})\sigma^{\ddagger}$.

(Q) $\psi_{\alpha n} = \psi^{\eta_\alpha \restriction n} = (\psi^{\ddagger} \restriction G_{1, \eta_\alpha \restriction n})\sigma^{\ddagger} = (\psi^{\ddagger} \restriction G_{\alpha n})\sigma^{\ddagger}$ and $\psi_\alpha = (\psi^{\ddagger} \restriction G_\alpha)\sigma^{\ddagger}$.

Now we consider the bijection $\sigma = \sigma^{\dagger}(\sigma^{\ddagger})^{-1}$ of ordinals. Observe that $Az \subseteq G_{Y_*'Y'}$ by definition of $(Y_*', Y', \mathfrak{G}')$. Thus, $[z] \subseteq Y_*'$, and (v), $(A)^{\ddagger}$, $(B)^{\dagger}$, $(B)^{\ddagger}$, $(I)^{\dagger}$, $(I)^{\ddagger}$, $(\star)^{\ddagger}$ yield

$(iv)^{\ddagger}$ $(Y_*''^{\ddagger}, Y''^{\ddagger}, \mathfrak{G}''^{\ddagger})$ is a Λ-closed triple with $[z] \subseteq Y_*''^{\ddagger}$ and $\pi_{\overline{\eta}}'^{\ddagger} \in \{\pi, 0\}$ for all $\overline{\eta} \in Y''^{\ddagger}$.

$(v)^{\ddagger}$ $\varphi^{\ddagger} : G_1(\overline{\xi}_\alpha) \longrightarrow G_{Y_*''^{\ddagger} Y''^{\ddagger}}$ is a homomorphism with $z\varphi^{\ddagger} = z\varphi\tau^{\ddagger} \notin Az$.

$(vii)^{\ddagger}$ $\sigma \restriction \operatorname{orco} z = \mathrm{id}$.

Applying the same arguments,

$$(Y_*', Y', \mathfrak{G}')\tau^{\dagger}\sigma = (Y_*', Y', \mathfrak{G}')\tau^{\dagger}\sigma^{\dagger}(\sigma^{\ddagger})^{-1} = (Y_*', Y', \mathfrak{G}')(\sigma^{\ddagger})^{-1} = (Y_*', Y', \mathfrak{G}')\tau^{\ddagger}$$

holds, while (L) gives $(Z_*^{n\dagger}, Z^{n\dagger}, \mathfrak{H}^{n\dagger})\sigma = (Z_*^{n\ddagger}, Z^{n\ddagger}, \mathfrak{H}^{n\ddagger})$. Hence, $(\star)^{\dagger}$, $(\star)^{\ddagger}$ imply that

$(viii)^{\ddagger}$ $(Y_*''^{\dagger}, Y''^{\dagger}, \mathfrak{G}''^{\dagger})\sigma = (Y_*''^{\ddagger}, Y''^{\ddagger}, \mathfrak{G}''^{\ddagger}) \subseteq (Z_*^{\ddagger}, Z^{\ddagger}, \mathfrak{H}^{\ddagger})$.

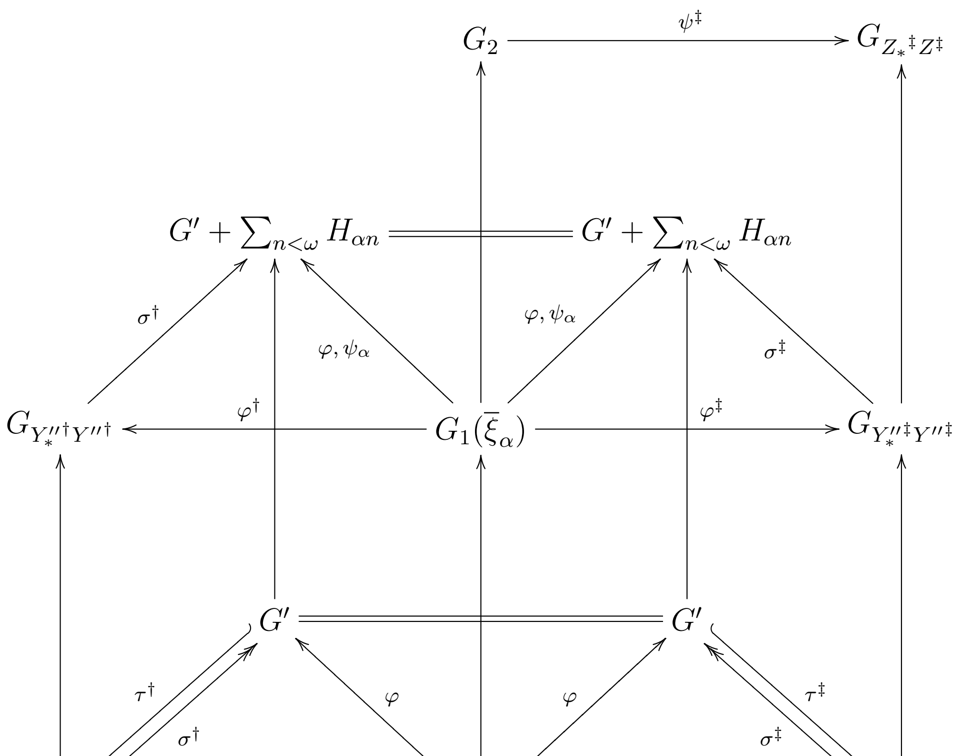
This in particular implies that σ is a $Y_*''^{\dagger}$-admissible bijection. Furthermore, by definition of $(Y_*', Y', \mathfrak{G}')$ and Observation 4.12(c), the triple $(Y_*', Y', \mathfrak{G}')\tau^{\ddagger}$ is $(k-f)$-closed with respect to $(Y_*, Y, \mathfrak{G})\tau^{\ddagger}$, and with $(D)^{\ddagger}$ and Observation 4.3(c) it is $(k-f, \alpha)$-closed with respect to $(Z_*^{\ddagger}, Z^{\ddagger}, \mathfrak{H}^{\ddagger})$. Thus, from $(F)^{\nu\ddagger}$, $(G)^{\nu\ddagger}$ and Observation 4.3(g), it follows that

$(ix)^\ddagger$ $(Y''^\dagger_*, Y''^\dagger, \mathfrak{G}''^\dagger)\sigma$ is $(k-(f-1),\alpha)$-closed with respect to $(Z^\ddagger_*, Z^\ddagger, \mathfrak{H}^\ddagger)$.

Moreover, Observation 4.3(g) yields $Z^\ddagger_{Y''^\dagger_*\sigma, Y''^\dagger\sigma} \subseteq Z^\ddagger_{Y'_*\tau^\ddagger, Y'\tau^\ddagger} \subseteq Z^\ddagger_{Y_*\tau^\ddagger, Y\tau^\ddagger}$ and from $(E)^\ddagger$ follows

$(x)^\ddagger$ $\pi'^\ddagger_{\overline{\eta}} b'^\ddagger_{\overline{\eta}} \in \{\pi z, 0\}$ for all $\overline{\eta} \in Z^\ddagger_{Y''^\dagger_*\sigma, Y''^\dagger\sigma}$.

As seen previously, $\varphi^\dagger\sigma = (\varphi\tau^\dagger)\sigma = \varphi\tau^\ddagger = \varphi^\ddagger$ holds on $G_1(\overline{\xi})$, while $\varphi^\dagger\sigma = (\psi_\alpha(\sigma^\dagger)^{-1})\sigma = \psi_\alpha(\sigma^\ddagger)^{-1} = \varphi^\ddagger$ holds on G_α. Thus, $\varphi^\dagger\sigma = \varphi^\ddagger$. Furthermore, by (4.15) and (Q) we also have $\varphi^\dagger\sigma = \varphi\tau^\ddagger \subseteq \psi^\ddagger$ on $G_1(\overline{\xi})$ and $\varphi^\dagger\sigma = \psi_\alpha(\sigma^\ddagger)^{-1} \subseteq \psi^\ddagger$ on G_α. Hence, $\varphi^\dagger\sigma = \varphi^\ddagger \subseteq \psi^\ddagger$.



The existence of $(Z^\ddagger_*, Z^\ddagger, \mathfrak{H}^\ddagger)$, σ and $\psi^\ddagger$ with $\varphi^\dagger\sigma \subseteq \psi^\ddagger$ contradicts the statement of the step lemma, when we replace $f, \overline{\xi}, \alpha, z, G_1(\overline{\xi}), (Y_*, Y, \mathfrak{G}), \varphi$ by

$$f-1, \overline{\xi}_\alpha, \alpha, z, G_1(\overline{\xi}_\alpha), (Y''^\dagger_*, Y''^\dagger, \mathfrak{G}''^\dagger), \varphi^\dagger. \tag{4.16}$$

In particular this contradicts the choice of $\pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$) at (4.14). Thus the step lemma follows. ■

5 The final construction

In this chapter we construct $\aleph_k$-free R-modules with prescribed endomorphism R-algebras A with the help of Step Lemma 4.16 and some algebraic adaption of our Easy Black Box 4.7. This construction is unprecedented in literature (even in the $\aleph_1$-free case) as we realize a Black Box construction under the weakest assumption of prediction possible. One of the many advantages of our new approach towards $\aleph_k$-free constructions is that we achieve smaller cardinals by replacing the Strong Black Box from [27] by the Easy Black Box, while significantly simplifying the proofs at the same time. This new version of the Black Box needs almost no adjustments, whereas [27] was only manageable with the help of some highly elaborate changes of the settings.
Recall that $\langle \lambda_1, \dots, \lambda_k \rangle$ is the cardinal sequence from Section 2.1 satisfying the cardinal conditions (i) and (ii). In this chapter we will fix the cardinals

$$\lambda = \begin{cases} \lambda_k & \text{for } \lambda_k \text{ regular} \\ \lambda_k^+ & \text{for } \lambda_k \text{ singular} \end{cases}$$

and $\theta = \lambda_{k-1}$ (setting $\lambda_0 = |\,A\,|$). The Easy Black Box requires $|\,A\,| \leq \theta < \lambda = \lambda^\theta$, but this is no further restriction on λ and θ due to assumptions (i) and (ii).
Also recall from Section 2.2 the definition of the free A-module $B = \bigoplus_{\overline{\nu} \in \Lambda_*} Ae_{\overline{\nu}}$ and its $\mathbb{S}$-adic completion $\widehat{B}$. Prediction principles, such as the Easy Black Box, need the notion of traps for capturing the objects to be predicted. For this purpose, we also need an ordering, which will tell us later on which prediction comes first. Thus, we recall from Section 3.2 the notion of λ-norm.

Definition 5.1 *The* λ-norm.

(a) For $\eta \in {}^{\omega\geq}\lambda$, let $\|\,\eta\,\| = \sup_{\ell < \lg \eta}(\ell\eta + 1) \in \lambda$;
in particular, $\|\,\alpha\,\| = \alpha + 1$ for $\alpha \in \lambda$ and $\|\,\emptyset\,\| = 0$ by default.

(b) For $\overline{\eta} \in \Lambda$ let $\|\,\overline{\eta}\,\| = \|\,\eta_k\,\|$, and for $\overline{\nu} \in \Lambda_*$ let $\|\,\overline{\nu}\,\| = \|\,\nu_k\,\|$.

(c) For $X \subseteq \Lambda$ put $\|\,X\,\| = \sup_{\overline{\eta} \in X} \|\,\overline{\eta}\,\|$. Similarly $\|\,X\,\| = \sup_{\overline{\nu} \in X} \|\,\overline{\nu}\,\|$ if $X \subseteq \Lambda_*$.

(d) If $b \in \widehat{B}$, then $\|\,b\,\| = \|\,[b]\,\|$, and for $S \subseteq \widehat{B}$, let $\|\,S\,\| = \sup_{b \in S} \|\,b\,\|$.

The final construction will also involve some weak version of well-ordering for which we recall the idea of regressiveness from Section 3.2.

Definition 5.2 *For $V \subseteq \Lambda$ the family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V\}$ of branch-like elements (from Section 3.1) is* regressive, *if $\| \pi_{\overline{\eta}} b_{\overline{\eta}} \| < \| \overline{\eta} \| = \| y_{\overline{\eta}} \|$ for all $\overline{\eta} \in V$.*

We are now ready to define the traps for our algebraic version of the Easy Black Box. Note that we already used a different kind of traps for the step lemma, which also needs a prediction. For the crucial sets of this definition, we refer back to Definition 4.10.

Definition 5.3 *A quintuple $p = (\eta, V_*, V, \mathfrak{F}, \varphi)$ is a* trap (for the Easy Black Box), *if the following holds.*

(i) $\eta \in {}^{\omega\uparrow}\lambda$.

(ii) $(V_*, V, \mathfrak{F})$ *is Λ-closed.*

(iii) $|V_*|, |V| \leq \theta$.

(iv) $\Lambda_{\langle\eta\rangle *} \subseteq V_*$ *(recall that by definition $\Lambda_{\langle\eta\rangle *} = \{\overline{\nu} \in \Lambda_{k*} \mid \nu_k \lhd \eta, \nu_k \neq \eta\}$).*

(v) $\| \overline{\nu} \| < \| \eta \|$ *for all* $\overline{\nu} \in V_*$, *and* $\| \overline{\eta} \| < \| \eta \|$ *for all* $\overline{\eta} \in V$.

(vi) $\varphi : P \longrightarrow P$ *is an R-endomorphism of the A-module $P = G_{V_*V}$.*

Convention 5.4 *We denote by $\| p \| = \| \eta \| = \| V_* \|$ the* norm of the trap p.

Recall that $\lambda^o = \{\alpha \mid \alpha \in \lambda, \operatorname{cf} \alpha = \omega\}$.

The Easy Black Box 5.5 *Let be $|A| \leq \theta < \lambda = \lambda^\theta$ with λ a regular cardinal. If E is a stationary subset of λ^o, then there are an ordinal $\lambda \leq \lambda^* < \lambda^+$ and a list*

$$p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha) \ (\alpha < \lambda^*) \text{ of traps}$$

with the following properties.

(i) $\| p_\alpha \| \in E$ *for all* $\alpha < \lambda^*$.

(ii) $\| p_\alpha \| \leq \| p_\beta \|$ *for all* $\alpha < \beta < \lambda^*$.

(iii) $\eta_\alpha \neq \eta_\beta$ *for all* $\alpha < \beta < \lambda^*$.

(iv) THE PREDICTION: *For any* Λ*-closed triple* $(\Lambda_*, V, \mathfrak{F})$ *with* $G = G_{\Lambda_* V}$, *any homomorphism* $\varphi \in \mathrm{End}_R G$ *and any set* $S \subseteq \Lambda_*$ *with* $| S | \leq \theta$, *the set*

$$\{\alpha \in E \mid \exists \beta < \lambda^* \text{ with } \| p_\beta \| = \alpha, (V_{\beta *}, V_\beta, \mathfrak{F}_\beta) \subseteq (\Lambda_*, V, \mathfrak{F}), \varphi_\beta \subseteq \varphi, S \subseteq V_{\beta *}\}$$

is stationary.

Remark 5.6 *Like in [27], this particular Black Box will fail to predict arbitrary endomorphisms of* $\widehat{B}$, *due to the fact that we include the* Λ*-closed triple* $(V_*, V, \mathfrak{F})$ *as part of our traps* p *and thus restrict ourselves exclusively to* A*-submodules of* $\widehat{B}$ *of the type* $G_{V_* V}$. *Note that this is only for the convenience of the reader and completely avoidable when defining traps* $p = (\eta, P, \varphi)$ *by the following list of conditions instead.*

(i) $\eta \in {}^{\omega\uparrow}\lambda$.

(ii) $P \subseteq \widehat{B}$ *is an* A*-module.*

(iii) P *is* $\leq \theta$*-generated.*

(iv) $B_{\Lambda_{\langle \eta \rangle *}} \subseteq P$.

(v) $\| b \| < \| \eta \|$ *for all* $b \in P$.

(vi) $\varphi : P \longrightarrow P$ *is an* R*-endomorphism.*

A similar version of the Easy Black Box 5.5 can be defined for traps $p = (\eta, P, \varphi)$. *Furthermore, this Black Box has predecessors like the General Black Box, a prediction principle first introduced by Shelah in [47] and established as a suitable combinatorial principle for general algebraic constructions in Corner, Göbel [5]; see also [32] for another outline of the General Black Box. The major difference to our Easy Black Box is the missing disjointness condition* $\| V_{\alpha *} \cap V_{\beta *} \| < \| p_\alpha \|$ *for all traps* p_α, p_β *with* $\| p_\alpha \| = \| p_\beta \|$. *This will be compensated later on by the Step Lemma 4.16.*

Proof. We call a quintuple $p_n = (\eta^n, V_*^n, V^n, \mathfrak{F}^n, \varphi^n)$ a *partial trap of length n*, if the following holds.

(i) $\eta^n \in {}^{\omega\upharpoonright>}\lambda$ *and* $\lg \eta^n = n$. (5.1)

(ii) $(V_*^n, V^n, \mathfrak{F}^n)$ *is* Λ*-closed.*

(iii) $|\, V_*^n \,|, |\, V^n \,| \leq \theta$.

(iv) $\varphi^n : P^n \longrightarrow \widehat{B}$ *is an R-homomorphism, where* $P^n = G_{V_*^n V^n}$.

Let Ξ be the family of all partial traps and observe that $|\,\Xi\,| = \lambda$. Thus, we can fix a list $\Xi = \langle p^\alpha \mid \alpha < \lambda \rangle$ with repetitions such that $\{\alpha < \lambda \mid p^\alpha = p\} \subseteq \lambda$ is unbounded for all $p \in \Xi$.

For any Λ-closed triple $(\Lambda_*, V, \mathfrak{F})$ with $G = G_{\Lambda_* V}$, any $\varphi \in \operatorname{End}_R G$ and any set $S \subseteq \Lambda_*$ with $|\,S\,| \leq \theta$, we call a sequence $(p_n)_{n<\omega}$ of partial traps (G, φ, S)*-admissible* if the following holds.

(i) $p_n = (\eta^n, V_*^n, V^n, \mathfrak{F}^n, \varphi^n)$ *is a partial trap of length n.* (5.2)

(ii) $\eta^n \subseteq \eta^{n+1}$.

(iii) $\|\, \eta^n \,\|, \|\, V_*^n \,\|, \|\, V^n \,\| < n\eta^{n+1}$.

(iv) $n\eta^{n+1} \in \{\alpha < \lambda \mid p^\alpha = p_n\}$.

(v) $(V_*^n, V^n, \mathfrak{F}^n) \subseteq (V_*^{n+1}, V^{n+1}, \mathfrak{F}^{n+1}) \subseteq (\Lambda_*, V, \mathfrak{F})$.

(vi) $S \subseteq V_*^0$.

(vii) $\{\overline{\nu} \in \Lambda_{k*} \mid \nu_k \trianglelefteq \eta^n\} \subseteq V_*^{n+1}$.

(viii) $\operatorname{Im} \varphi^n \subseteq P^{n+1}$.

(ix) $\varphi^n = \varphi \upharpoonright P^n$.

Evidently, this list can be easily turned into a recursive construction, showing that for any choice of G, φ, S there exists a large number of (G, φ, S)-admissible sequences. For every (G, φ, S)-admissible sequence $(p_n)_{n<\omega}$, we define $p_\omega = (\eta^\omega, V_*^\omega, V^\omega, \mathfrak{F}^\omega, \varphi^\omega)$

by $\eta^\omega = \bigcup_{n<\omega} \eta^n$, $(V_*^\omega, V^\omega, \mathfrak{F}^\omega) = \bigcup_{n<\omega}(V_*^n, V^n, \mathfrak{F}^n)$ and $\varphi^\omega = \bigcup_{n<\omega} \varphi^n$. According to (5.2)(ii), (v) and (ix), this is a well-defined quintuple, which is easily shown to be a trap for the Easy Black Box: Conditions (i) and (v) follow from (5.2)(iii), condition (iv) follows from (5.2)(vii), condition (vi) follows from (5.2)(viii), (ix) and $P^\omega = G_{V_*^\omega V^\omega} = \bigcup_{n<\omega} G_{V_*^n V^n} = \bigcup_{n<\omega} P^n$, and the other conditions of Definition 5.3 are evident from (5.1) and (5.2) as well.

Let $\Xi_{G\varphi S}$ be the family of all traps p_ω generated by some (G, φ, S)-admissible sequence $(p_n)_{n<\omega}$. The set

$$C_{G\varphi S} = \{\| p_\omega \| \mid p_\omega \in \Xi_{G\varphi S}\} \subseteq \lambda$$

is evidently unbounded by our construction. Moreover, we want to show that

$$C_{G\varphi S} \cap E \neq \emptyset. \tag{5.3}$$

For this purpose, we define by transfinite induction on $\alpha < \lambda$ a sequence of traps $p^\alpha \in \Xi_{G\varphi S}$ $(\alpha < \lambda)$ and a family of partial traps $p_\nu \in \Xi$ $(\nu \in {}^{\omega\uparrow>}\lambda)$ such that the following holds.

(i) $\alpha \leq \| p^\alpha \| \in \lambda^o$. (5.4)

(ii) $\| p^\beta \| < \| p^\alpha \|$ *for all* $\beta < \alpha$.

(iii) $\| p^\alpha \| = \sup_{\beta<\alpha} \| p^\beta \|$ *for all* $\alpha \in \lambda^o$.

(iv) $p_\nu = (\eta^\nu, V_*^\nu, V^\nu, \mathfrak{F}^\nu, \varphi^\nu)$ *is a partial trap of length* $\lg \nu$.

(v) $(p_{\eta' \restriction n})_{n<\omega}$ *is a* (G, φ, S)*-admissible sequence for all* $\eta' \in {}^{\omega\uparrow}\lambda$.

(vi) $\| \eta^\nu \| < \| p^\alpha \|$ *for all* $\nu \in {}^{\omega\uparrow>}\alpha$.

(vii) $\| p^{n\nu} \| < \| \eta^\nu \|$ *for all* $\nu \in {}^{\omega\uparrow>}\lambda$ *with* $\lg \nu = n + 1$.

Condition (5.4)(i) is an immediate consequence of (ii) and Definition 5.3. Now let $\alpha < \lambda$ and assume that all $p^\beta \in \Xi_{G\varphi S}$ and $p_\nu \in \Xi$ $(\nu \in {}^{\omega\uparrow>}\beta)$ have been constructed for all $\beta < \alpha$. If $\alpha = \beta + 1$ is a successor ordinal and $\nu \in {}^{\omega\uparrow>}\alpha \setminus {}^{\omega\uparrow>}\beta$, then $n\nu = \beta$ holds for $\lg \nu = n + 1$ and we choose $p_\nu \in \Xi$ according to (5.2) extending $(p_{\nu \restriction k})_{k\leq n}$ in such a way that $\| p^{n\nu} \| = \| p^\beta \| < n\eta^\nu = \| \eta^\nu \|$ holds in addition to (5.2)(iii). Once all the p_ν ($\nu \in {}^{\omega\uparrow >}\alpha$) are constructed, we have $\sup_{\nu \in {}^{\omega\uparrow >}\alpha} \| \eta^\nu \| < \lambda$ as $\left| {}^{\omega\uparrow >}\alpha \right| = \aleph_0 \cdot | \alpha | < \lambda$ and λ is regular. Thus, we can choose $p^\alpha \in \Xi_{G\varphi S}$ such that $\| p^\beta \| < \sup_{\nu \in {}^{\omega\uparrow >}\alpha} \| \eta^\nu \| < \| p^\alpha \|$. If α is a limit ordinal, then ${}^{\omega\uparrow >}\alpha = \bigcup_{\beta<\alpha} {}^{\omega\uparrow >}\beta$ and all the necessary partial traps $p_\nu \in \Xi$ ($\nu \in {}^{\omega\uparrow >}\alpha$) have already been constructed. If $\mathrm{cf}(\alpha) > \omega$, then choose $p^\alpha \in \Xi_{G\varphi S}$ such that $\sup_{\beta<\alpha} \| p^\beta \| \leq \sup_{\nu \in {}^{\omega\uparrow >}\alpha} \| \eta^\nu \| < \| p^\alpha \|$. Otherwise $\mathrm{cf}(\alpha) = \omega$ and we choose any $\eta' \in {}^{\omega\uparrow >}\alpha$ with $\| \eta' \| = \alpha$ and define $p^\alpha \in \Xi_{G\varphi S}$ as the trap determined by the (G, φ, S)-admissible sequence $(p_{\eta' \restriction n})_{n<\omega}$. Thus, from (5.4)(vi) and (vii) follows

$$\| \eta^{\eta' \restriction n} \| < \| p^{n\eta'} \| < \| \eta^{\eta' \restriction (n+1)} \|$$

and with (5.4)(ii) and Convention 5.4

$$\| p^\alpha \| = \sup_{n<\omega} \| \eta^{\eta' \restriction n} \| = \sup_{n<\omega} \| p^{n\eta'} \| = \sup_{\beta<\alpha} \| p^\beta \|.$$

For the sequence of traps $(p^\alpha)_{\alpha<\lambda} \subseteq \Xi_{G\varphi S}$ as defined above the set

$$\overline{C}_{G\varphi S} = \{\sup_{\beta<\alpha} \| p^\beta \| \mid \alpha < \lambda\} \subseteq \lambda$$

is a club with (5.4)(i). Thus, as $E \subseteq \lambda^o$ is stationary, with (5.4)(ii) and (iii) holds

$$\begin{aligned} \emptyset \neq \overline{C}_{G\varphi S} \cap E &\subseteq \{\sup_{\beta<\alpha} \| p^\beta \| \mid \alpha < \lambda, \mathrm{cf}(\alpha) \leq \omega\} \cap E \\ &\subseteq \{\| p^\alpha \| \mid \alpha < \lambda\} \cap E \subseteq C_{G\varphi S} \cap E, \end{aligned}$$

which concludes the proof of (5.3).

Let Ξ_E be the family of all traps p_ω with $\| p_\omega \| \in E$ which are generated by some (G, φ, S)-admissible sequence $(p_n)_{n<\omega}$ for a suitable choice of G, φ and S. Then, by definition, $| \Xi_E | \leq | \Xi |^\omega = \lambda$, and ordering Ξ_E increasingly according to the norm we obtain an ordinal $\lambda \leq \lambda^* < \lambda^+$ and a list $p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)$ $(\alpha < \lambda^*)$ of Ξ_E without repetitions.

We claim that this list satisfies the Easy Black Box: Condition (i) is evident by the definition of Ξ_E, condition (iii) follows from (5.2)(iv) as η^ω uniquely determines the generating sequence $(p_n)_{n<\omega}$, and condition (iv) holds as $C_{G\varphi S} \cap (E \cap C) \neq \emptyset$ is witnessed for any club C by some sequence $(p_n)_{n<\omega}$ with $p_\omega \in \Xi_{G\varphi S} \cap \Xi_E$. ∎

We now want to apply the Easy Black Box 5.5 to derive the Main Theorem 1.3.

Proof. **We first construct the A-module G:**

We will construct a specific regressive family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in B, \overline{\eta} \in V\}$ with $\pi_{\overline{\eta}} \in \{\pi, 0\}$ for all $\overline{\eta} \in V \subseteq \Lambda$ such that the A-module $G = G_{\Lambda_* V}$ generated by the Λ-closed triple $(\Lambda_*, V, \mathfrak{F})$ satisfies Theorem 1.3 and in particular $\operatorname{End}_R G = A$.

Recall that $B = \bigoplus_{\overline{\nu} \in \Lambda_*} Ae_{\overline{\nu}}$ has cardinality λ and that λ is regular. By Solovay's decomposition theorem (see Jech [39, p. 433]) there is a decomposition $\lambda^o = \dot{\bigcup}_{z \in B} E_z$ into stationary sets E_z. For all E_z ($z \in B$), we choose a list of traps p^z_α ($\alpha < \lambda^*_z$) with the help of the Easy Black Box 5.5 and relabel these traps (preserving norms) to get a uniform sequence of traps

$$p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha) \ (\alpha < \lambda^*) \text{ with } \| p_\alpha \| \leq \| p_\beta \| \text{ for all } \alpha < \beta < \lambda^*. \tag{5.5}$$

Recall that $\eta_\alpha \neq \eta_\beta$ for $\alpha \neq \beta$ and that by definition $\Lambda^{\langle \eta_\alpha \rangle} = \{\overline{\eta} \in \Lambda \mid \eta_k = \eta_\alpha\}$. Put $V = \dot{\bigcup}_{\alpha < \lambda^*} \Lambda^{\langle \eta_\alpha \rangle}$. For each $\overline{\eta} \in V$, we must choose $\pi_{\overline{\eta}} \in \{\pi, 0\}$ and $b_{\overline{\eta}} \in B$ with $\| \pi_{\overline{\eta}} b_{\overline{\eta}} \| < \| \overline{\eta} \|$ for the definition of $y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}}$. We will choose these pairs $(\pi_{\overline{\eta}}, b_{\overline{\eta}})$ recursively for $\overline{\eta} \in \Lambda^{\langle \eta_\alpha \rangle}$ and $\alpha < \lambda^*$. Thus, let us assume that for some $\alpha < \lambda^*$ the construction has been conducted successfully for all $\beta < \alpha$. We consider the trap $p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)$ with accompanying family of branch-like elements $\mathfrak{F}_\alpha = \{y'''_{\overline{\eta}} = \pi''_{\overline{\eta}} b''_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V_\alpha\}$ and choose $z \in B$ such that $\| p_\alpha \| \in E_z$. We will only work if the following list of conditions is met.

$$\text{(i) } V_\alpha \subseteq \bigcup_{\substack{\beta < \alpha \\ \| \eta_\beta \| < \| \eta_\alpha \|}} \Lambda^{\langle \eta_\beta \rangle}. \tag{5.6}$$

(ii) $y'''_{\overline{\eta}} = y'_{\overline{\eta}}$ *for all* $\overline{\eta} \in V_\alpha$.

(iii) Either $z = e_{\overline{\nu}}$ *for some* $\overline{\nu} \in V_{\alpha *}$ *or* $z = e_{\overline{\nu}_1} - e_{\overline{\nu}_2}$ *for distinct* $\overline{\nu}_1, \overline{\nu}_2 \in V_{\alpha *}$.

(iv) $z\varphi_\alpha \notin Az$.

In all other cases, we make the trivial choice

$$\pi_{\overline{\eta}} = 0 \text{ and } b_{\overline{\eta}} = z \text{ with } \| \pi_{\overline{\eta}} b_{\overline{\eta}} \| = 0 < \| \overline{\eta} \| \text{ for all } \overline{\eta} \in \Lambda^{\langle \eta_\alpha \rangle}. \tag{5.7}$$

Now assume that the conditions (5.6) hold for some $\alpha < \lambda^*$, thus arriving at the interesting case which needs work. In order to adjust our notations to the preliminaries of the Step Lemma 4.16, we put

$$Y_{\alpha *} = \{\overline{\nu} \in \Lambda_* \mid \|\overline{\nu}\| < \|\eta_\alpha\|\}, Y_\alpha = \bigcup_{\substack{\beta<\alpha \\ \|\eta_\beta\|<\|\eta_\alpha\|}} \Lambda^{\langle \eta_\beta \rangle} \quad \text{and} \quad \mathfrak{G}_\alpha = \{y'_{\overline{\eta}} \mid \overline{\eta} \in Y_\alpha\}.$$

Observe that $\mathfrak{G}_\alpha$ is well-defined since all the branches involved have been determined in previous steps of the construction and that $(Y_{\alpha*}, Y_\alpha, \mathfrak{G}_\alpha)$ is a Λ-closed triple as $\|\pi_{\overline{\eta}} b_{\overline{\eta}}\| < \|\overline{\eta}\|$ for all $\overline{\eta} \in Y_\alpha$ by construction. Furthermore, it is easy to verify that the assumptions of Step Lemma 4.16 hold for

$$f = k-1, \overline{\xi} = \langle \eta_\alpha \rangle, z, (Y_*, Y, \mathfrak{G}) = (Y_{\alpha*}, Y_\alpha, \mathfrak{G}_\alpha), \varphi = \varphi_\alpha \restriction G_1(\overline{\xi}).$$

Assumption (ii) holds due to (5.6)(iii). Furthermore, with Definition 5.3(v) we have $V_{\alpha*} \subseteq Y_{\alpha*}$ and with (5.6)(i) and (ii) follows

$$(V_{\alpha*}, V_\alpha, \mathfrak{F}_\alpha) \subseteq (Y_{\alpha*}, Y_\alpha, \mathfrak{G}_\alpha). \tag{5.8}$$

Thus, in particular, assumptions (iv) and (v) hold, since $\Lambda_{\langle \eta_\alpha \rangle *} \cup [z] \subseteq V_{\alpha*} \subseteq Y_{\alpha*}$ by Definition 5.3(iv) and (5.6)(iii). The other assumptions of Step Lemma 4.16 hold trivially, the lemma applies, and we find elements $\pi_{\overline{\eta}} \in \{\pi, 0\}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}} = \Lambda^{\langle \eta_\alpha \rangle}$), while setting $b_{\overline{\eta}} = z$ with $\|\pi_{\overline{\eta}} b_{\overline{\eta}}\| \leq \|z\| < \|V_{\alpha*}\| = \|\eta_\alpha\| = \|\overline{\eta}\|$ for all $\overline{\eta} \in \Lambda^{\overline{\xi}} = \Lambda^{\langle \eta_\alpha \rangle}$.
Thus all pairs $(\pi_{\overline{\eta}}, b_{\overline{\eta}})$ ($\overline{\eta} \in V$) are constructed resulting in some regressive family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V\}$ with $\pi_{\overline{\eta}} \in \{\pi, 0\}$ for all $\overline{\eta} \in V$ accompanied by its induced A-module $G = G_{\Lambda_* V}$. Also observe that, due to our construction, $b_{\overline{\eta}}$ is uniquely determined by $\|\overline{\eta}\|$:

$$\|\overline{\eta}\| \in E_{b_{\overline{\eta}}} \text{ for all } \overline{\eta} \in V. \tag{5.9}$$

It remains to show that

G is as required in Theorem 1.3:
Clearly, $|G| = \lambda$ and G is an $\aleph_k$-free A-module by the Freeness-Lemma 3.6. Since A acts faithfully on the A-module G, it is also clear that $A \subseteq \operatorname{End}_R G$, where we identify every $a \in A$ with its induced scalar multiplication on G. Thus, it remains to show that $\operatorname{End}_R G \subseteq A$. Let $\psi \in \operatorname{End}_R G$.

First we want to show

Claim 1: If $\overline{\nu} \in \Lambda_*$, then $e_{\overline{\nu}}\psi \in Ae_{\overline{\nu}}$.

Suppose for contradiction that there is some $\overline{\nu} \in \Lambda_*$ with $e_{\overline{\nu}}\psi \notin Ae_{\overline{\nu}}$. Applying the Easy Black Box 5.5 for the stationary set $E_{e_{\overline{\nu}}} \subseteq \lambda^o$ we have for $G = G_{\Lambda_* V}, \psi$ and $S = \{\overline{\nu}\}$ that the set

$$\{\alpha \in E_{e_{\overline{\nu}}} \mid \exists \beta < \lambda^*_{e_{\overline{\nu}}} \text{ with } \| p^{e_{\overline{\nu}}}_\beta \| = \alpha, (V^{e_{\overline{\nu}}}_{\beta *}, V^{e_{\overline{\nu}}}_\beta, \mathfrak{F}^{e_{\overline{\nu}}}_\beta) \subseteq (\Lambda_*, V, \mathfrak{F}), \varphi^{e_{\overline{\nu}}}_\beta \subseteq \psi, \overline{\nu} \in V^{e_{\overline{\nu}}}_{\beta *}\}$$

is stationary.

In particular, there is some $\alpha < \lambda^*$ such that

$$\| p_\alpha \| \in E_{e_{\overline{\nu}}}, (V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha) \subseteq (\Lambda_*, V, \mathfrak{F}), \varphi_\alpha \subseteq \psi \text{ and } \overline{\nu} \in V_{\alpha *}, \tag{5.10}$$

and conditions (5.6) hold for α: From (5.10) and Definition 5.3(v) we have

$$V_\alpha \subseteq V \cap \{\overline{\eta} \in \Lambda \mid \| \overline{\eta} \| < \| \eta_\alpha \|\} = \bigcup_{\substack{\beta < \alpha \\ \| \eta_\beta \| < \| \eta_\alpha \|}} \Lambda^{\langle \eta_\beta \rangle},$$

which yields condition (5.6)(i), while conditions (iii) and (iv) follow from $\| p_\alpha \| \in E_{e_{\overline{\nu}}}$ and the assumption $e_{\overline{\nu}}\varphi_\alpha = e_{\overline{\nu}}\psi \notin Ae_{\overline{\nu}}$. Thus, the non-trivial case of the construction applies and the $\pi_{\overline{\eta}} \in \{\pi, 0\}$ ($\overline{\eta} \in \Lambda^{\langle \eta_\alpha \rangle}$) are chosen according to Step Lemma 4.16. In order to derive the desired contradiction, we set

$$f = k - 1, \overline{\xi} = \langle \eta_\alpha \rangle, z = e_{\overline{\nu}}, (Y_*, Y, \mathfrak{G}) = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha), \varphi = \varphi_\alpha \restriction G_1(\overline{\xi}),$$

$$(Z_*, Z, \mathfrak{H}) = (\Lambda_*, V, \mathfrak{F}) \text{ and } \tau = \mathrm{id}_{\mathrm{orco}\, Y_{\alpha *}}$$

and verify for this choice the missing conditions (vi) to (x) of Step Lemma 4.16: Condition (viii) holds as $(Y_*, Y, \mathfrak{G})\tau = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha) \subseteq (\Lambda_*, V, \mathfrak{F}) = (Z_*, Z, \mathfrak{H})$ by definition. Furthermore, by definition $(Y_*, Y, \mathfrak{G})\tau = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha)$ is $(1, \| \eta_\alpha \|)$-closed with respect to $(\Lambda_*, V, \mathfrak{F})$ since from $\overline{\eta} \in V$ with $| u_{\overline{\eta}}(Y_{\alpha *}) | \geq 1$ and $\| \overline{\eta} \| \neq \| \eta_\alpha \|$ follows

$$\overline{\eta} \in V \cap \{\overline{\eta} \in \Lambda \mid \| \overline{\eta} \| < \| \eta_\alpha \|\} = \bigcup_{\substack{\beta < \alpha \\ \| \eta_\beta \| < \| \eta_\alpha \|}} \Lambda^{\langle \eta_\beta \rangle} = Y_\alpha,$$

and thus, (ix) holds. Finally, we have

$$\begin{aligned} Z_{Y_*\tau, Y\tau} = V_{Y_{\alpha *} Y_\alpha} &= \{\overline{\eta} \in V \setminus Y_\alpha \mid | u_{\overline{\eta}}(Y_{\alpha *}) | \geq 1 \text{ and } \| \overline{\eta} \| = \| \eta_\alpha \|\} \\ &= V \cap \{\overline{\eta} \in \Lambda \mid \| \overline{\eta} \| = \| \eta_\alpha \|\} = \bigcup_{\substack{\beta < \lambda^* \\ \| \eta_\beta \| = \| \eta_\alpha \|}} \Lambda^{\langle \eta_\beta \rangle}. \end{aligned}$$

In particular, $\|\overline{\eta}\| = \|\eta_\alpha\| = \|p_\alpha\| \in E_{e_{\overline{\nu}}}$ and $b_{\overline{\eta}} = e_{\overline{\nu}}$ holds for all $\overline{\eta} \in Z_{Y_*\tau, Y\tau}$, see (5.9). This proves condition (x).

The existence of $(Z_*, Z, \mathfrak{H})$, τ and ψ with $\varphi\tau = \varphi \subseteq \psi$ contradicts the statement of Step Lemma 4.16 for $f = k-1, \overline{\xi} = \langle \eta_\alpha \rangle, z = e_{\overline{\nu}}, (Y_*, Y, \mathfrak{G}) = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha), \varphi = \varphi_\alpha \restriction G_1(\overline{\xi})$ and consequentially the choice of $\pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\langle \eta_\alpha \rangle}$) during our construction.

Claim 2: If $\overline{\nu}_1 \neq \overline{\nu}_2 \in \Lambda_*$, then $(e_{\overline{\nu}_1} - e_{\overline{\nu}_2})\psi \in A(e_{\overline{\nu}_1} - e_{\overline{\nu}_2})$.

This is proven similar to Claim 1. Just replace $e_{\overline{\nu}}$ by $e_{\overline{\nu}_1} - e_{\overline{\nu}_2}$.

From Claim 1 and Claim 2, it is immediate that $\psi \in A$. ∎

6 $\aleph_k$-free E-Rings

Throughout the preceding chapters we have given a full and self-contained construction for realizing a given R-algebra A as endomorphism ring $\operatorname{End}_R G$ of a suitable $\aleph_k$-free R-module G. This construction was executed with particular care in order to keep the proofs and arguments neatly divided into two separate components; one of them dealing entirely with algebraic aspects of the construction, while the other one treats exclusively with set theoretic and model theoretic considerations. The advantage of this approach is that it allows a shortcut for future $\aleph_k$-free constructions as we only need to adjust the considerably small algebraic part while keeping everything else unchanged. We intend to demonstrate this while constructing $\aleph_k$-free E-rings. For the proof of Theorem 1.6 we do mainly refer back to Chapters 2 to 5 of this thesis as a blueprint of the general construction indicating here only those changes necessary to adapt to the new algebraic setting of $E(R)$-algebras.

6.1 The preliminaries

We will fix a positive integer k and a sequence $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_k \rangle$ of cardinals such that

(i) $|\,R\,| \leq \lambda_1 = \lambda_1^{\aleph_0}$ and

(ii) $\lambda_{\ell+1} = \lambda_{\ell+1}^{\lambda_\ell}$ for $1 \leq \ell < k$.

Throughout this chapter we will once again assume the ring R to be a cotorsion-free $\mathbb{S}$-ring. It will be useful not only to talk about $\aleph_k$-free R-modules but also about $\aleph_k$-free R-algebras. Thus, we call a commutative R-algebra A *κ-free* if there is a family $\mathcal{C}_A$ of $\mathbb{S}$-pure R-subalgebras of A satisfying the following modified *Hill conditions*.

(i) Every element of $\mathcal{C}_A$ is a polynomial ring over R with $< \kappa$ free generators.

(ii) Every subset of A of cardinality $< \kappa$ is contained in an element of $\mathcal{C}_A$.

(iii) $\mathcal{C}_A$ is closed under unions of well-ordered chains of length $< \kappa$.

Evidently, every $\aleph_k$-free R-algebra is also an $\aleph_k$-free R-module as defined in Section 2.2.

The basic R-algebra B for our construction is the polynomial ring over R in free commuting variables $e_{\overline{\nu}}$ ($\overline{\nu} \in \Lambda_*$)

$$B = R[e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*] = \bigoplus_{\mathfrak{m} \in \mathfrak{M}(\Lambda_*)} R\mathfrak{m},$$

with its associated free commutative monoid $\mathfrak{M}(\Lambda_*) = \langle e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_* \rangle$ of monomials.

Definition 6.1 *If $X_* \subseteq \Lambda_*$, then we get a canonical R-subalgebra and summand $B_{X_*} = R[e_{\overline{\nu}} \mid \overline{\nu} \in X_*] = \bigoplus_{\mathfrak{m} \in \mathfrak{M}(X_*)} R\mathfrak{m}$ of B with its associated monoid $\mathfrak{M}(X_*) = \langle e_{\overline{\nu}} \mid \overline{\nu} \in X_* \rangle$ of monomials.*

Every element $b \in \widehat{B}$ has a natural (Λ_*-)*support* $[b] \subseteq \Lambda_*$ which is the smallest set such that $b \in B_{[b]}$ holds. Thus let $[b] = \bigcap\{X_* \subseteq \Lambda_* \mid b \in B_{X_*}\}$. Furthermore, for $b = \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda_*)} b_{\mathfrak{m}}\mathfrak{m} \in \widehat{B}$ with coefficients $b_{\mathfrak{m}} \in \widehat{R}$ we introduce the abbreviation $[b]_{\mathfrak{m}} = b_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathfrak{M}(\Lambda_*)$. Note that $[b]$ is at most countable. If $S \subseteq \widehat{B}$, then the Λ_*-support of S is the set $[S] = \bigcup_{b \in S}[b]$. Our goal is to select particular elements from $\widehat{B}$ and to add them to B, such that the resulting final R-algebra A satisfies

$$B \subseteq A \subseteq_* \widehat{B}.$$

6.2 $\aleph_k$-free R-algebras

We will only indicate those definitions and results from Section 3.1 where changes are necessary. We once again define *branch elements* as

$$y_{\overline{\eta}i} = \sum_{n=i}^{\infty} \frac{q_n}{q_i} (\sum_{m=1}^{k} e_{\overline{\eta} \upharpoonright \langle m,n \rangle})$$

for all $\overline{\eta} \in \Lambda$ and $i < \omega$, setting $y_{\overline{\eta}} = y_{\overline{\eta}0}$. Given $\overline{\eta} \in \Lambda$, we also choose $b_{\overline{\eta}} \in B$, $\pi_{\overline{\eta}} = \sum_{n=0}^{\infty} q_n r_n \in \widehat{R}$ and let $\pi_{\overline{\eta}i} = \sum_{n=i}^{\infty} \frac{q_n}{q_i} r_n$. We define *branch-like elements* as $y'_{\overline{\eta}i} = \pi_{\overline{\eta}i} b_{\overline{\eta}} + y_{\overline{\eta}i}$ and $y'_{\overline{\eta}} = y'_{\overline{\eta}0}$. It will be very often sufficient to choose $\pi_{\overline{\eta}} \in \{\pi, 0\}$ for some fixed element $\pi \in \widehat{R}$. Definition 3.1 remains unchanged.

Definition 6.2 *The construction of the R-algebra A_{X_*X}:*
If $(X_*, X, \mathfrak{F})$ is Λ-closed, then we let

$$A_{X_*X} = B_{X_*}[y'_{\overline{\eta}i} \mid \overline{\eta} \in X, i < \omega] = B_{X_*}[y'_{\overline{\eta}} \mid \overline{\eta} \in X]_* \subseteq \widehat{B}$$

be the generated pure R-subalgebra of $\widehat{B}$.

Recognition (Observation 3.3) changes according to this slightly modified definition. We introduce the multiplicative monoid $\mathfrak{M}(\Lambda) = \langle \overline{\eta} \mid \overline{\eta} \in \Lambda \rangle$ generated by $\overline{\eta}$ ($\overline{\eta} \in \Lambda$) as free commutative generators. For any family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in B, \overline{\eta} \in \Lambda\}$ of branch-like elements and any $\mathfrak{m} \in \mathfrak{M}(\Lambda)$, we define $\mathfrak{m}_{\mathfrak{F}}$ as the element of $\widehat{B}$ obtained by substituting any generator $\overline{\eta}$ ($\overline{\eta} \in \Lambda$) in $\mathfrak{m}$ by the respective $y'_{\overline{\eta}} \in \widehat{B}$. Similarly, we define $\mathfrak{m}_{mn} \in \mathfrak{M}(\Lambda_*)$ for $1 \leq m \leq k$, $n < \omega$ as the element of B obtained by substituting each generator $\overline{\eta}$ ($\overline{\eta} \in \Lambda$) in $\mathfrak{m}$ by the respective $e_{\overline{\eta}\upharpoonright\langle m,n\rangle} \in B$.

Observation 6.3 (Recognition) *If $a \in B[y'_{\overline{\eta}} \mid \overline{\eta} \in \Lambda]$ for a suitable choice of $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in B, \overline{\eta} \in \Lambda\}$, then there exist unique elements $a_{\mathfrak{m}} \in B$ ($\mathfrak{m} \in \mathfrak{M}(\Lambda)$) with $a = \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda)} a_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{F}}$. Furthermore, the element $a_{\mathfrak{m}}$ is independent of the choice of $\mathfrak{F}$ for all monomials $\mathfrak{m} \in \mathfrak{M}(\Lambda)$ with the property that $a_{\mathfrak{m}'} = 0$ for all monomials $\mathfrak{m} \neq \mathfrak{m}' \in \mathfrak{M}(\Lambda)$ with $\mathfrak{m} \mid \mathfrak{m}'$.*

Proof. For any representation

$$a = \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda)} a_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{F}}$$

of $a \in B[y'_{\overline{\eta}} \mid \overline{\eta} \in \Lambda]$, almost all coefficients $a_{\mathfrak{m}}$ must be zero and there exists some finite $\Lambda' \subseteq \Lambda$ with $a = \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda')} a_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{F}}$. Now, for $i < \omega$ large enough, we have $[y_{\overline{\eta}i}] \cap [a_{\mathfrak{m}}] = [y_{\overline{\eta}i}] \cap [\pi_{\overline{\eta}} b_{\overline{\eta}}] = [y_{\overline{\eta}i}] \cap [\pi_{\overline{\eta}'} b_{\overline{\eta}'}] = [y_{\overline{\eta}i}] \cap [y_{\overline{\eta}'i}] = \emptyset$ for all $\overline{\eta}, \overline{\eta}' \in \Lambda'$, $\overline{\eta} \neq \overline{\eta}'$ and $\mathfrak{m} \in \mathfrak{M}(\Lambda)$. Thus,

$$[a]_{\mathfrak{n}\mathfrak{m}_{mi}} = [a_{\mathfrak{m}}]_{\mathfrak{n}} \tag{6.1}$$

follows for all monomials $\mathfrak{m} \in \mathfrak{M}(\Lambda')$ and $\mathfrak{n} \in \mathfrak{M}(\Lambda_*)$ with the property that $a_{\mathfrak{m}'} = 0$ for all $\mathfrak{m} \neq \mathfrak{m}' \in \mathfrak{M}(\Lambda')$ with $\mathfrak{m} \mid \mathfrak{m}'$. In the special case when $a = 0$, this immediately implies $a_{\mathfrak{m}} = 0$ for all $1 \neq \mathfrak{m} \in \mathfrak{M}(\Lambda)$ and $a_1 = 0$ follows as well. Furthermore, (6.1) holds for all $\mathfrak{m} \in \mathfrak{M}(\Lambda)$ as we may choose $\Lambda' \subseteq \Lambda$ finite with $\mathfrak{m} \in \mathfrak{M}(\Lambda')$.

Let us now assume that

$$a = \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda)} a^0_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{F}^0} = \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda)} a^1_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{F}^1}$$

holds for suitable families $\mathfrak{F}^i = \{y'^i_{\overline{\eta}} = \pi^i_{\overline{\eta}} b^i_{\overline{\eta}} + y_{\overline{\eta}} \mid b^i_{\overline{\eta}} \in B, \overline{\eta} \in \Lambda\}$, $i \in \{0,1\}$, and let $\mathfrak{m} \in \mathfrak{M}(\Lambda)$ be such that $a^0_{\mathfrak{m}''} = 0$ for all monomials $\mathfrak{m} \neq \mathfrak{m}'' \in \mathfrak{M}(\Lambda)$ with $\mathfrak{m} \mid \mathfrak{m}''$. Towards a contradiction, assume that $a^1_{\mathfrak{m}''} \neq 0$ for some $\mathfrak{m} \neq \mathfrak{m}'' \in \mathfrak{M}(\Lambda)$ with $\mathfrak{m} \mid \mathfrak{m}''$. Then we can choose some $\mathfrak{m} \mid \mathfrak{m}' \in \mathfrak{M}(\Lambda)$ such that $a^0_{\mathfrak{m}'} = 0$, $a^1_{\mathfrak{m}'} \neq 0$ and $a^0_{\mathfrak{m}''} = a^1_{\mathfrak{m}''} = 0$ for all monomials $\mathfrak{m}' \neq \mathfrak{m}'' \in \mathfrak{M}(\Lambda)$ with $\mathfrak{m}' \mid \mathfrak{m}''$. Thus, for any $\mathfrak{n} \in \mathfrak{M}(\Lambda_*)$ with $[a^1_{\mathfrak{m}'}]_{\mathfrak{n}} \neq \emptyset$, it follows that $0 = [a^0_{\mathfrak{m}'}]_{\mathfrak{n}} = [a]_{\mathfrak{n}\mathfrak{m}'_{mi}} = [a^1_{\mathfrak{m}'}]_{\mathfrak{n}} \neq 0$ by (6.1), a contradiction. Similarly, $a^0_{\mathfrak{m}} = a^1_{\mathfrak{m}}$ follows for all $\mathfrak{m} \in \mathfrak{M}(\Lambda)$ such that $a_{\mathfrak{m}'} = 0$ for all $\mathfrak{m} \neq \mathfrak{m}' \in \mathfrak{M}(\Lambda)$ with $\mathfrak{m} \mid \mathfrak{m}'$. ■

As a consequence of this observation, every element $a \in A_{X_*X}$ has a canonical sum representation $sa = \sum_{\mathfrak{m} \in \mathfrak{M}(X)} a_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{F}}$ with $s \in \mathbb{S}$ and $a_{\mathfrak{m}} \in B_{X_*}$, which in general will depend on the family $\mathfrak{F}$ of the generating Λ-closed triple $(X_*, X, \mathfrak{F})$. Furthermore, every element $a \in A_{X_*X}$ has a Λ-*support* $[a]_\Lambda \subseteq \Lambda$, which consists of those $\overline{\eta} \in X$ such that $a_{\mathfrak{m}} \neq 0$ for some monomial $\mathfrak{m} \in \mathfrak{M}(X)$ containing the generator $\overline{\eta}$. According to Observation 6.3, this Λ-support is independent of the particular choice of $\mathfrak{F}$ and we may define

$$[a]_\Lambda = \bigcap \{X' \subseteq \Lambda \mid a \in A_{X'_*X'} \text{ for some } \Lambda\text{-closed triple } (X'_*, X', \mathfrak{F}')\}$$

and $[S]_\Lambda = \bigcup_{a \in S} [a]_\Lambda$ for all $S \subseteq A_{X_*X}$. Note that $[a]_\Lambda$ is always finite. For later use we also mention the following simple consequences of Observation 6.3 and (6.1).

Observation 6.4 *Let $a \in A_{X_*X}$ with canonical sum representation*

$$sa = \sum_{\mathfrak{m} \in \mathfrak{M}(X)} a_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{F}}$$

and $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in B, \overline{\eta} \in X\}$. Then the following holds:

(a) $[y_{\overline{\eta}i}] \subseteq [a]$ for all $\overline{\eta} \in [a]_\Lambda$ and all $i < \omega$ large enough.

(b) $[a_{\mathfrak{m}}] \subseteq [a]$ for all $\mathfrak{m} \in \mathfrak{M}(X)$ such that $a_{\mathfrak{m}'} = 0$ for all $\mathfrak{m}' \neq \mathfrak{m}$ with $\mathfrak{m} \mid \mathfrak{m}'$.

(c) $B_{X_*}[y'_{\overline{\eta}} \mid \overline{\eta} \in X]$ is a free ring extension of B_{X_*} by the variables $y'_{\overline{\eta}}$ $(\overline{\eta} \in X)$.

(d) If $a \neq 0$, then $[a]_{\mathfrak{m}} \in R \setminus \{0\}$ for some $\mathfrak{m} \in \mathfrak{M}(X_*)$.

(e) If $a \in \widehat{B}_{Y_*}$ for some $Y_* \subseteq \Lambda_*$ and $[y'_{\overline{\eta}}] \subseteq Y_*$ for all $\overline{\eta} \in [a]_\Lambda$, then $a_\mathfrak{m} \in B_{Y_*}$ for all $\mathfrak{m} \in \mathfrak{M}(X)$.

Proof. For (a), choose any $\mathfrak{m} \in \mathfrak{M}(X)$ such that $\overline{\eta} \mid \mathfrak{m}$, $a_\mathfrak{m} \neq 0$ and $a_{\mathfrak{m}'} = 0$ for all $\mathfrak{m} \neq \mathfrak{m}' \in \mathfrak{M}(X)$ with $\mathfrak{m} \mid \mathfrak{m}'$ and apply (6.1) to $\mathfrak{m}$ and some $\mathfrak{n} \in \mathfrak{M}(X_*)$ with $[a_\mathfrak{m}]_\mathfrak{n} \neq \emptyset$. Claim (b) is immediate from (6.1) and claim (c) is immediate from Observation 6.3. For (d), take $a \neq 0$, so $sa \neq 0$. If $[sa]_\Lambda \neq \emptyset$, then choose some $\overline{\eta} \in [sa]_\Lambda$ and proceed as in the proof of claim (a) to find some $\mathfrak{m}'' = \mathfrak{n}\mathfrak{m}_{mi} \in \mathfrak{M}(X_*)$ such that $[sa]_{\mathfrak{m}''} \in R \setminus \{0\}$. If $[sa]_\Lambda = \emptyset$, then $0 \neq sa \in B$ and $[sa]_{\mathfrak{m}''} \in R \setminus \{0\}$ for some $\mathfrak{m}'' \in \mathfrak{M}(X_*)$. In both cases, from $[sa]_{\mathfrak{m}''} \in R \setminus \{0\} \subseteq_* \widehat{R}$ follows $[a]_{\mathfrak{m}''} \in R \setminus \{0\}$.
For (e), we may use (b) repeatedly to show $a_\mathfrak{m} \in B_{Y_*}$ for all $\mathfrak{m} \in \mathfrak{M}(X)$ such that $a_{\mathfrak{m}'} = 0$ for all $\mathfrak{m}' \neq \mathfrak{m}$ with $\mathfrak{m} \mid \mathfrak{m}'$, reducing sa by $a_\mathfrak{m}\mathfrak{m}_\mathfrak{F} \in \widehat{B}_{Y_*}$ afterwards. ∎

Observation 6.5 *If R is a cotorsion-free $\mathbb{S}$-ring and A is the R-algebra A_{X_*X} as in Definition 6.2, then A is a cotorsion-free R-module.*

Proof. This is shown similarly to Observation 3.4(b) using Recognition 6.3. ∎

The Freeness-Proposition 3.5 is a completely combinatorial statement and remains unchanged, while Freeness-Lemma 3.6 becomes

Freeness-Lemma 6.6 *The R-algebra A_{X_*X} from Definition 6.2 is an $\aleph_k$-free R-algebra.*

Proof. Any subset H of A_{X_*X} is contained in the pure R-subalgebra

$$A_{\Omega_*\Omega} = R\,[e_{\overline{\nu}}, y'_{\overline{\eta}} \mid \overline{\nu} \in \Omega_*, \overline{\eta} \in \Omega]_* \subseteq A_{X_*X},$$

where $\Omega = [H]_\Lambda$ and $\Omega_* = [H] \cup \bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}] \cup \bigcup_{\overline{\eta}\in\Omega}[\pi_{\overline{\eta}} b_{\overline{\eta}}]$ with $|\,\Omega_*\,|, |\,\Omega\,| \leq \aleph_0 \cdot |\,H\,|$. Without loss of generality, we may assume $\Omega_* = \bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}] \cup \bigcup_{\overline{\eta}\in\Omega}[\pi_{\overline{\eta}} b_{\overline{\eta}}]$ and write

$$A_{\Omega_*\Omega} = A_\Omega = R\left[e_{\overline{\eta}\restriction\langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}} \mid \overline{\eta} \in \Omega, \overline{\nu} \in [\pi_{\overline{\eta}} b_{\overline{\eta}}], 1 \leq m \leq k, n < \omega\right]_* \subseteq A_{X_*X}$$

as the elements from $\Omega_* \setminus \left(\bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}] \cup \bigcup_{\overline{\eta}\in\Omega}[\pi_{\overline{\eta}} b_{\overline{\eta}}]\right)$ generate a free ring extension of A_Ω. Thus, in order to show $\aleph_k$-freeness of A_{X_*X}, we will consider any $\Omega \subseteq X$ of size $|\,\Omega\,| < \aleph_k$ and show that A_Ω is a polynomial ring over R. We may assume that $|\,\Omega\,| = \aleph_{k-1}$. Let $F : \Lambda \to [\Lambda_*]^{<\aleph_0}$ be any map which assigns to $\overline{\eta} \in X$ the set $\overline{\eta}F = [\pi_{\overline{\eta}} b_{\overline{\eta}}]$.

By Proposition 3.5 (putting simply $u_{\overline{\eta}} = \{1, \dots, k\}$ for all $\overline{\eta} \in \Omega$) we can express

$$A_\Omega = R\left[e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha i} \mid \alpha < \aleph_{k-1}, \overline{\nu} \in \overline{\eta}^\alpha F, 1 \le m \le k, n < \omega, i < \omega\right]$$

and we find a sequence of pairs (ℓ_α, n_α) with $1 \le \ell_\alpha \le k, n_\alpha < \omega$ such that for $n \ge n_\alpha$

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F. \tag{6.2}$$

Let $A_\alpha = R\left[e_{\overline{\eta}^\gamma \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\gamma i} \mid \gamma < \alpha, \overline{\nu} \in \overline{\eta}^\gamma F, 1 \le m \le k, n < \omega, i < \omega\right]$ for any $\alpha \le \aleph_{k-1}$; thus, $A_0 = R$, $A_{\aleph_{k-1}} = A_\Omega$, and if $\alpha < \aleph_{k-1}$, then

$$A_{\alpha+1} = A_\alpha\left[e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha i} \mid \overline{\nu} \in \overline{\eta}^\alpha F, 1 \le m \le k, n < \omega, i < \omega\right].$$

We need to check that $A_{\alpha+1}$ is in fact a free ring extension of A_α. Observe that

$$A'_\alpha = A_\alpha\left[e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, j\rangle}, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}} \in A_{\alpha+1} \setminus A_\alpha \mid \overline{\nu} \in \overline{\eta}^\alpha F, j < n_\alpha, 1 \le \ell_\alpha \ne m \le k, n < \omega\right]$$

is indeed a free ring extension of A_α by unused generators of B with

$$A_{\alpha+1} = A'_\alpha\left[y'_{\overline{\eta}^\alpha i} \mid i \ge n_\alpha\right]. \tag{6.3}$$

To see that also (6.3) is a free ring extension, we define ρ_N for $N \ge n_\alpha$ as the canonical R-algebra homomorphism with $\operatorname{Dom} \rho_N = \widehat{B}$ which is induced on the generators $e_{\overline{\nu}}$ by

$$e_{\overline{\nu}}\rho_N = \begin{cases} 0 & \text{if } \overline{\nu} \in [\overline{\eta}^\alpha], \text{unless } \overline{\nu} = \overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \text{ with } n_\alpha \le n \le N, \\ e_{\overline{\nu}} & \text{otherwise,} \end{cases}$$

and extended to $\widehat{B}$ by linearity and continuity. Now obviously

$$A'_\alpha \rho_N \left[e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle} \mid n_\alpha \le n \le N\right]$$

is, with (6.2), a free ring extension of $A'_\alpha \rho_N = A'_\alpha$ by unused generators of B. By replacement

$$A'_\alpha \rho_N \left[\sum_{n=i}^{N} \frac{q_n}{q_i} e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle} \mid n_\alpha \le i \le N\right] = A'_\alpha \rho_N \left[y'_{\overline{\eta}^\alpha i}\rho_N \mid n_\alpha \le i \le N\right]$$

is a ring extension by free generators as well. Thus, $A'_\alpha\left[y'_{\overline{\eta}^\alpha i} \mid n_\alpha \le i \le N\right]$ for all $N \ge n_\alpha$ and also $A_{\alpha+1}$ are ring extensions by free generators. ∎

6.3 The step lemma

This is where our preparation surely applies, and most of the results of Chapter 4 can be reused directly. We mention only those few changes and amendments necessary.

Lemma 6.7 *(a) If the triple* $(X_*, X, \mathfrak{F})$ *is* k*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*, then* $A_{Y_*Y} \cap \widehat{B}_{X_*} = A_{X_*X}$ *holds.*

(b) If $(X_*, X, \mathfrak{F})$ *is* (k, α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*, such that*

$$[\overline{\eta}] \subseteq X_* \text{ and } [\pi_{\overline{\eta}} b_{\overline{\eta}}] \subseteq X_* \text{ for all } \overline{\eta} \in Y_{X_*X},$$

then $A_{Y_*Y} \cap \widehat{B}_{X_*} = A_{X_*,\, X \cup Y_{X_*X}}$ *holds.*

Proof. This is shown similarly to the Lemmas 4.4 and 4.6, replacing Observation 3.3 by Observation 6.4(a) and (e). ∎

Theorem 4.5 also holds when formally replacing $G, G_{X_*X}, G_{Y_*Y}, G_{Z_*Z}$ by $A, A_{X_*X}, A_{Y_*Y}, A_{Z_*Z}$, respectively. In addition to Definition 4.11(e) observe that any X_*-admissible bijection τ also extends canonically to an R-algebra monomorphism $\tau : \widehat{B}_{X_*} \longrightarrow \widehat{B}_{\Lambda_*} = \widehat{B}$. The preparatory Step Lemma 4.15 is not required.

Step Lemma 6.8 *Let* R *be an* $\mathbb{S}$*-ring together with some transcendental element* $\pi \in \widehat{R}$ *over* R *and assume that the following parameters are given.*

(i) $0 \leq f < k$ *and* $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$ *with* $\alpha = \|\xi_k\| = \sup_{\ell<\omega}(\ell\xi_k + 1)$.

(ii) $z \in B$.

(iii) $A_1 = A_1(\overline{\xi}) = B_{\Lambda_{\overline{\xi}*} \cup [z]}$.

(iv) $(Y_*, Y, \mathfrak{G})$ *with* $\mathfrak{G} = \{y'''_{\overline{\eta}} = \pi''_{\overline{\eta}} b''_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y\}$ *is a* Λ*-closed triple with* $[z] \subseteq Y_*$ *and* $\pi''_{\overline{\eta}} \in \{\pi, 0\}$ *for all* $\overline{\eta} \in Y$.

(v) $\varphi : A_1 \longrightarrow A = A_{Y_*Y}$ *is an* R*-module homomorphism with* $z\varphi \notin (z\widehat{B})_* \subseteq \widehat{B}$.

Then $\pi_{\overline{\eta}} \in \{\pi, 0\}$ $(\overline{\eta} \in \Lambda^{\overline{\xi}})$ *can be chosen such that the* Λ*-closed triple* $(X_*, X, \mathfrak{F})$ *with*

$$X_* = \Lambda_{\overline{\xi}*} \cup \Lambda_*^{\overline{\xi}} \cup [z], \quad X = \Lambda^{\overline{\xi}}, \quad \mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in X\}$$

and the induced R*-algebra* $A_2 = A_{X_*X}$ *have the following property.*
If $(Z_*, Z, \mathfrak{H})$ *is a* Λ*-closed triple with* $\mathfrak{H} = \{y''_{\overline{\eta}} = \pi'_{\overline{\eta}} b'_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z\}$ *and* τ *is a* Y_**-admissible bijection such that*

(vi) $\pi'_{\overline{\eta}} \in \{\pi, 0\}$ *for all* $\overline{\eta} \in Z$*,*

(vii) $\tau \restriction (\operatorname{orco} z \cup \operatorname{orco} z\varphi) = \mathrm{id}$*,*

(viii) $(Y_*, Y, \mathfrak{G})\tau \subseteq (Z_*, Z, \mathfrak{H})$ *with induced* R*-algebras* $A_3 = A_{Y_*\tau, Y\tau} \subseteq A_4 = A_{Z_*Z}$*,*

(ix) $(Y_*, Y, \mathfrak{G})\tau$ *is* $(k - f, \alpha)$*-closed with respect to* $(Z_*, Z, \mathfrak{H})$*,*

(x) $\pi'_{\overline{\eta}} b'_{\overline{\eta}} \in \{\pi z, 0\}$ *for all* $\overline{\eta} \in Z_{Y_*\tau, Y\tau}$*,*

then

$$\varphi\tau : A_1 \longrightarrow A_3 \text{ does not extend to a homomorphism } \psi : A_2 \longrightarrow A_4.$$

Proof. The step lemma is shown by induction on f.

The case $f = 0$

If $f = 0$, then $\overline{\xi} \in \Lambda$ and the basic sets satisfy

$$\Lambda_*^{\overline{\xi}} = \emptyset, \Lambda_{\overline{\xi}*} = [\overline{\xi}], X_* = \Lambda_{\overline{\xi}*} \cup \Lambda_*^{\overline{\xi}} \cup [z] = [\overline{\xi}] \cup [z], X = \Lambda^{\overline{\xi}} = \{\overline{\xi}\},$$

and the corresponding R-algebras are $A_1 = B_{\Lambda_{\overline{\xi}*} \cup [z]} = B_{[\overline{\xi}] \cup [z]}$ and

$$A_2 = A_{X_*X} = B_{X_*}[y'_{\overline{\xi}}]_* = B_{[\overline{\xi}] \cup [z]}[y'_{\overline{\xi}}]_* = A_1[y'_{\overline{\xi}}]_* = A_1[y'_{\overline{\xi}i} \mid i < \omega] \subseteq_* \widehat{B}.$$

Set $y^i_{\overline{\xi}} = i \cdot \pi z + y_{\overline{\xi}}$ for $i \in \{0, 1\}$. Towards a contradiction, assume that there exist suitable Y_*-admissible bijections τ^i, Λ-closed triples $(Y_*, Y, \mathfrak{G})\tau^i \subseteq (Z^i_*, Z^i, \mathfrak{H}^i)$ and extensions ψ^i such that

$$y^i_{\overline{\xi}} \psi^i \in A_{Z^i_* Z^i} \text{ holds for } i \in \{0, 1\}. \tag{6.4}$$

Thus, there exists some $s \in \mathbb{S}$ such that

$$sz\varphi \in B_{Y_*}[y'''_{\overline{\eta}} \mid y'''_{\overline{\eta}} \in \mathfrak{G}] \quad \text{and} \quad sy^i_{\overline{\xi}}\psi^i \in B_{Z^i_*}[y'''^i_{\overline{\eta}} \mid y'''_{\overline{\eta}} \in \mathfrak{H}^i] \text{ for } i \in \{0,1\} \tag{6.5}$$

holds simultaneously with accompanying representations

$$sy^i_{\overline{\xi}}\psi^i = \sum_{\mathfrak{m}\in\mathfrak{M}(Z^i)} a^i_{\mathfrak{m}}\mathfrak{m}_{\mathfrak{H}^i} = \sum_{\mathfrak{m}\in\mathfrak{M}([sy^i_{\overline{\xi}}\psi^i]_\Lambda)} a^i_{\mathfrak{m}}\mathfrak{m}_{\mathfrak{H}^i} \tag{6.6}$$

for suitable elements $a^i_{\mathfrak{m}} \in B$ $(\mathfrak{m} \in \mathfrak{M}(Z^i))$.

The R-module homomorphism $\varphi : B_{[\overline{\xi}]\cup[z]} \longrightarrow A_{Y_*Y}$ extends uniquely (by continuity) to a homomorphism $\widehat{\varphi} : \widehat{B}_{[\overline{\xi}]\cup[z]} \longrightarrow \widehat{B}_{Y_*}$ with $y^i_{\overline{\xi}} \in \widehat{B}_{[\overline{\xi}]\cup[z]}$. Thus, $y^i_{\overline{\xi}}\psi^i = y^i_{\overline{\xi}}\widehat{\varphi}\tau^i$ with $\widehat{\varphi}\tau^i : \widehat{B}_{[\overline{\xi}]\cup[z]} \longrightarrow \widehat{B}_{Y_*\tau^i}$ and

$$sy^i_{\overline{\xi}}\widehat{\varphi}\tau^i = \sum_{\mathfrak{m}\in\mathfrak{M}([sy^i_{\overline{\xi}}\psi^i]_\Lambda)} a^i_{\mathfrak{m}}\mathfrak{m}_{\mathfrak{H}^i} \in \widehat{B}_{Y_*\tau^i}. \tag{6.7}$$

Furthermore, with Observation 6.4(a)

$$\text{for each } \overline{\eta} \in [sy^i_{\overline{\xi}}\psi^i]_\Lambda \subseteq Z^i \text{ holds } [y_{\overline{\eta}j}] \subseteq [sy^i_{\overline{\xi}}\psi^i] = [sy^i_{\overline{\xi}}\widehat{\varphi}\tau^i] \subseteq Y_*\tau^i \text{ for some } j < \omega. \tag{6.8}$$

Hence, as $(Y_*, Y, \mathfrak{G})\tau^i \subseteq (Z^i_*, Z^i, \mathfrak{H}^i)$ is (k,α)-closed by condition (ix), we may apply Observation 4.3(d) to obtain that

$$\text{either } \overline{\eta} \in Y\tau^i \text{ or } \overline{\eta} \in Z^i_{Y_*\tau^i,\, Y\tau^i} \text{ applies for every } \overline{\eta} \in [sy^i_{\overline{\xi}}\psi^i]_\Lambda \subseteq Z^i. \tag{6.9}$$

Let be $Z'^i_* = \{\overline{\nu} \in \Lambda_* \mid \operatorname{orco}\overline{\nu} \subseteq \operatorname{Im}\tau^i, \overline{\nu}(\tau^i)^{-1} \in \Lambda_*\}$. For all $\overline{\eta} \in Y\tau^i$, the inclusions $\operatorname{orco} y'''^i_{\overline{\eta}} \subseteq \operatorname{Im}\tau^i$, $[y'''^i_{\overline{\eta}}] \subseteq Z'^i_*$ and $y'''^i_{\overline{\eta}}(\tau^i)^{-1} = y'''_{\overline{\eta}(\tau^i)^{-1}}$ hold with (viii). For all $\overline{\eta} \in Z^i_{Y_*\tau^i,\, Y\tau^i}$, we have $y'''^i_{\overline{\eta}} = \pi'^i_{\overline{\eta}}z + y_{\overline{\eta}}$ with $\pi'^i_{\overline{\eta}} \in \{\pi, 0\}$ and $\|\overline{\eta}\| = \alpha$ by conditions (vi) and (x), while $\overline{\eta}\upharpoonright\langle m,n\rangle \in Y_*\tau^i \subseteq Z'^i_*$ for arbitrarily large $n < \omega$ by the definition of $Z^i_{Y_*\tau^i,\, Y\tau^i}$. Thus, for $\overline{\eta} \in Z^i_{Y_*\tau^i,\, Y\tau^i} \subseteq Z^i \setminus Y\tau^i$, also $[y'''^i_{\overline{\eta}}] \subseteq Z'^i_*$ holds and $y'''^i_{\overline{\eta}}(\tau^i)^{-1} = \pi'^i_{\overline{\eta}}z + y_{\overline{\eta}(\tau^i)^{-1}}$ is well-defined with (vii). Hence, as $sy^i_{\overline{\xi}}\widehat{\varphi}\tau^i \in \widehat{B}_{Y_*\tau^i} \subseteq \widehat{B}_{Z'^i_*}$ by definition and $[y'''^i_{\overline{\eta}}] \subseteq Z'^i_*$ for all $\overline{\eta} \in [sy^i_{\overline{\xi}}\psi^i]_\Lambda$, we have $a^i_{\mathfrak{m}} \in B_{Z'^i_*}$ for all $\mathfrak{m} \in \mathfrak{M}([sy^i_{\overline{\xi}}\psi^i]_\Lambda)$ by Observation 6.4(e) and $a^i_{\mathfrak{m}}(\tau^i)^{-1} \in B$ is well-defined. This results in a representation

$$sy^i_{\overline{\xi}}\widehat{\varphi} = \sum_{\mathfrak{m}\in\mathfrak{M}([sy^i_{\overline{\xi}}\psi^i]_\Lambda)} a^i_{\mathfrak{m}}(\tau^i)^{-1}\cdot\mathfrak{m}_{\mathfrak{H}^i}(\tau^i)^{-1} = \sum_{\mathfrak{m}\in\mathfrak{M}(Y\cup Z^i_{Y_*\tau^i,\, Y\tau^i}(\tau^i)^{-1})} a'^i_{\mathfrak{m}}\mathfrak{m}_{\mathfrak{H}'^i} \tag{6.10}$$

with $a'^i_{\mathfrak{m}} \in B$ for all $\mathfrak{m} \in \mathfrak{M}(Y \cup Z^i_{Y_*\tau^i, Y\tau^i}(\tau^i)^{-1})$ and

$$\mathfrak{H}'^i = \mathfrak{G} \cup \{\pi'^i_{\overline{\eta}\tau^i} z + y_{\overline{\eta}} \mid \overline{\eta} \in Z^i_{Y_*\tau^i, Y\tau^i}(\tau^i)^{-1}\}.$$

Separating the component $\pi'^i_{\overline{\eta}\tau^i} z$ of every occurring branch-like element $\pi'^i_{\overline{\eta}\tau^i} z + y_{\overline{\eta}}$ in (6.10) and rearranging summands leads to some new representation

$$s y^i_{\overline{\xi}} \widehat{\varphi} = \pi z \cdot b^i + \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda)} a''^i_{\mathfrak{m}} \mathfrak{m}_{\mathfrak{H}''} \quad \text{for } i \in \{0, 1\} \tag{6.11}$$

with $a''^i_{\mathfrak{m}} \in B$ for all $\mathfrak{m} \in \mathfrak{M}(\Lambda)$, $\mathfrak{H}'' = \mathfrak{G} \cup \{y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \setminus Y\}$ and

$$b^i \in B[\pi, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda] \subseteq \widehat{B}. \tag{6.12}$$

We next subtract the two equations of (6.11) to obtain

$$\begin{aligned} \pi \cdot (sz)\varphi &= \pi \cdot (sz)\widehat{\varphi} = s \cdot (\pi z)\widehat{\varphi} = s(y^1_{\overline{\xi}}\widehat{\varphi} - y^0_{\overline{\xi}}\widehat{\varphi}) \\ &= \pi z \cdot (b^1 - b^0) + \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda)} (a''^1_{\mathfrak{m}} - a''^0_{\mathfrak{m}}) \cdot \mathfrak{m}_{\mathfrak{H}''} \end{aligned}$$

and

$$\pi \cdot [s(z\varphi) - z(b^1 - b^0)] = \sum_{\mathfrak{m} \in \mathfrak{M}(\Lambda)} (a''^1_{\mathfrak{m}} - a''^0_{\mathfrak{m}}) \cdot \mathfrak{m}_{\mathfrak{H}''}. \tag{6.13}$$

With (6.5) and (6.12) we have $s(z\varphi) - z(b^1 - b^0) \in B[\pi, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda]$. Thus,

$$\big[\pi \cdot [s(z\varphi) - z(b^1 - b^0)]\big]_{\mathfrak{m}'} \in \pi \cdot R[\pi] \subseteq \widehat{R} \quad \text{for all } \mathfrak{m}' \in \mathfrak{M}(\Lambda_*). \tag{6.14}$$

If $\pi \cdot [s(z\varphi) - z(b^1 - b^0)] \neq 0$, then an application of Observation 6.6(d) to the right-hand side of (6.13) gives $[\pi \cdot [s(z\varphi) - z(b^1 - b^0)]]_{\mathfrak{m}'} \in R \setminus \{0\}$ for some $\mathfrak{m}' \in \mathfrak{M}(\Lambda_*)$, a contradiction to (6.14) as $\pi \in \widehat{R}$ is assumed to be transcendental over R.
Hence, $\pi \cdot [s(z\varphi) - z(b^1 - b^0)] = 0$. Using that $[s(z\varphi) - z(b^1 - b^0)]_{\mathfrak{m}'} \in R[\pi] \subseteq \widehat{R}$ for all $\mathfrak{m}' \in \mathfrak{M}(\Lambda_*)$ and that $\pi \in \widehat{R}$ is transcendental over R (and thus, acts faithful on $R[\pi]$), we conclude $s(z\varphi) = z(b^1 - b^0) \in z\widehat{B}$ and $z\varphi \in (z\widehat{B})_* \subseteq \widehat{B}$, a contradiction to (v).
Thus, statement (6.4) has to be wrong for at least one $i \in \{0, 1\}$, independently of the choice of $(Z^i_*, Z^i, \mathfrak{H}^i)$, τ^i and ψ^i. Given such an $i \in \{0, 1\}$, the claim of the theorem will hold for the choice $\pi_{\overline{\xi}} = i \cdot \pi$.

The case $f > 0$

Almost literally the same line of argument applies as in Step Lemma 4.16. The only interesting point to check is whether (4.14) and (4.16) do indeed comply with the respective induction hypotheses. Another noteworthy change is that we set $\lambda_0 = \aleph_0$, arguing rather on the size of the generating sets of our R-algebras than the size of the R-algebras themselves. Thus, for instance, $A_{1\nu} = B_{\Lambda_{\nu\bar{\xi}*}} = R[e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_{\nu\bar{\xi}*}]$ is a ring extension of R by $\leq \theta$ free generators and we define

$$\mathcal{H} = \{H \subseteq \mathbb{V} \mid H \text{ is a } \leq \theta\text{-generated } R\text{-algebra}\}.$$

On (4.14): Note that from the slightly modified statement

$(B)^\dagger$ $\tau^\dagger \restriction (\operatorname{orco} z \cup \operatorname{orco} z\varphi) = \operatorname{id}$

follows $z\varphi^\dagger = z\varphi\tau^\dagger = z\varphi \notin (z\widehat{B})_* \subseteq \widehat{B}$ by (v). Checking all the other conditions is as described in Step Lemma 4.16.

On (4.16): Note the modified statements

$(B)^\ddagger$ $\tau^\ddagger \restriction (\operatorname{orco} z \cup \operatorname{orco} z\varphi) = \operatorname{id}$,

$(v)^\ddagger$ $\varphi^\ddagger : A_1(\overline{\xi}_\alpha) \longrightarrow A_{Y_*''^\ddagger Y''^\ddagger}$ is a homomorphism with $z\varphi^\ddagger = z\varphi\tau^\ddagger = z\varphi \notin (z\widehat{B})_* \subseteq \widehat{B}$,

$(vii)^\ddagger$ $\sigma \restriction (\operatorname{orco} z \cup \operatorname{orco} z\varphi) = \operatorname{id}$.

For (vii)‡, observe $\sigma = \sigma^\dagger(\sigma^\ddagger)^{-1}$ and $\operatorname{orco} z \cup \operatorname{orco} z\varphi \subseteq \operatorname{orco} Y_*'$ and apply $(B)^\dagger$, $(B)^\ddagger$, $(I)^\dagger$ and $(I)^\ddagger$. Checking all the other conditions is as described in Step Lemma 4.16. ∎

6.4 The final construction

Recall that $\langle \lambda_1, \ldots, \lambda_k \rangle$ is the cardinal sequence from Section 6.1 satisfying the cardinal conditions (i) and (ii). We will fix throughout this section the cardinals

$$\lambda = \begin{cases} \lambda_k & \text{for } \lambda_k \text{ regular} \\ \lambda_k^+ & \text{for } \lambda_k \text{ singular} \end{cases}$$

and $\theta = \lambda_{k-1}$ (setting $\lambda_0 = \aleph_0$). Our new algebraic adaption of the Easy Black Box 4.7 reads as follows.

Definition 6.9 *A quintuple* $p = (\eta, V_*, V, \mathfrak{F}, \varphi)$ *is a* trap (for the Easy Black Box)*, if the following holds.*

(i) $\eta \in {}^{\omega\uparrow}\lambda$.

(ii) $(V_*, V, \mathfrak{F})$ *is* Λ*-closed.*

(iii) $|\,V_*\,|, |\,V\,| \leq \theta$.

(iv) $\Lambda_{\langle\eta\rangle *} \subseteq V_*$ *(recall that by definition* $\Lambda_{\langle\eta\rangle *} = \{\overline{\nu} \in \Lambda_{k*} \mid \nu_k \lhd \eta, \nu_k \neq \eta\}$*).*

(v) $\|\,\overline{\nu}\,\| < \|\,\eta\,\|$ *for all* $\overline{\nu} \in V_*$*, and* $\|\,\overline{\eta}\,\| < \|\,\eta\,\|$ *for all* $\overline{\eta} \in V$.

(vi) $\varphi : P \longrightarrow P$ *is an* R*-endomorphism of the* R*-algebra* $P = A_{V_*V}$.

The Easy Black Box 6.10 *Let* $|\,R\,| \leq \lambda = \lambda^{\aleph_0}$ *with* λ *a regular cardinal. If* E *is a stationary subset of* λ^o*, then there are an ordinal* $\lambda \leq \lambda^* < \lambda^+$ *and a list*

$$p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha) \ (\alpha < \lambda^*) \textit{ of traps}$$

with the following properties.

(i) $\|\,p_\alpha\,\| \in E$ *for all* $\alpha < \lambda^*$.

(ii) $\|\,p_\alpha\,\| \leq \|\,p_\beta\,\|$ *for all* $\alpha < \beta < \lambda^*$.

(iii) $\eta_\alpha \neq \eta_\beta$ *for all* $\alpha < \beta < \lambda^*$.

(iv) THE PREDICTION: *For any* Λ*-closed triple* $(\Lambda_*, V, \mathfrak{F})$ *with* $A = A_{\Lambda_* V}$*, any homomorphism* $\varphi \in \mathrm{End}_R\, A$ *and any set* $S \subseteq \Lambda_*$ *with* $|\,S\,| \leq \theta$*, the set*

$$\{\alpha \in E \mid \exists \beta < \lambda^* \text{ with } \|\,p_\beta\,\| = \alpha, (V_{\beta *}, V_\beta, \mathfrak{F}_\beta) \subseteq (\Lambda_*, V, \mathfrak{F}), \varphi_\beta \subseteq \varphi, S \subseteq V_{\beta *}\}$$

is stationary.

Proof. See the proof of the Easy Black Box 5.5. ∎

Our final proof refines a nice and simple idea from [26].

Lemma 6.11 *Let R be a cotorsion-free $\mathbb{S}$-ring and let $\psi \in \mathrm{Hom}_R(B, \widehat{B})$ with*

$$z\psi \in (z\widehat{B})_* \subseteq \widehat{B} \text{ for all } z \in B.$$

Then ψ is the scalar multiplication by some element of $\widehat{B}$.

Proof. Recall that $B = R[e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*]$ is the polynomial ring over R in free commuting variables $e_{\overline{\nu}}$ ($\overline{\nu} \in \Lambda_*$) and that $\widehat{B} \subseteq \widehat{R}[[e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*]]$ is a ring of power series over $\widehat{R}$. By hypothesis on ψ, we find for each $z \in B$ some $s_z \in \mathbb{S}$ and $b_z \in \widehat{B}$ such that $s_z z\psi = z \cdot b_z$. If $\mathfrak{m} \in \mathfrak{M}(\Lambda_*)$ is a monomial and $\overline{\nu} \in \Lambda_*$, then $s_{\mathfrak{m}}\mathfrak{m}\psi = \mathfrak{m} \cdot b_{\mathfrak{m}} = \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{\mathfrak{m}}(e_{\overline{\nu}})$ and $s_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}\mathfrak{m})\psi = e_{\overline{\nu}}\mathfrak{m} \cdot b_{e_{\overline{\nu}}\mathfrak{m}} = e_{\overline{\nu}} \cdot \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}})$ interpreting $\mathfrak{m}, b_{\mathfrak{m}}, b_{e_{\overline{\nu}}\mathfrak{m}} \in \widehat{B}$ as functions of the free generator and variable $e_{\overline{\nu}}$. We now fix $s \in \mathbb{S}$ to compute

$$\begin{aligned} s_{\mathfrak{m}} s_{e_{\overline{\nu}}\mathfrak{m}}(s\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m})\psi &= s_{e_{\overline{\nu}}\mathfrak{m}} s \cdot s_{\mathfrak{m}}\mathfrak{m}\psi - s_{\mathfrak{m}} s_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}\mathfrak{m})\psi \\ &= s_{e_{\overline{\nu}}\mathfrak{m}} s \cdot \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{\mathfrak{m}}(e_{\overline{\nu}}) - s_{\mathfrak{m}} e_{\overline{\nu}} \cdot \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}), \end{aligned}$$

while by hypothesis also $s_{s\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m}}(s\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m})\psi = (s\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m}) \cdot b_{s\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}})$ holds. Thus

$$s_{s\mathfrak{m}-e_{\overline{\nu}}\mathfrak{m}} s_{e_{\overline{\nu}}\mathfrak{m}} s \cdot \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{\mathfrak{m}}(e_{\overline{\nu}}) - s_{s\mathfrak{m}-e_{\overline{\nu}}\mathfrak{m}} s_{\mathfrak{m}} e_{\overline{\nu}} \cdot \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}) = s_{\mathfrak{m}} s_{e_{\overline{\nu}}\mathfrak{m}}(s\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m}) \cdot b_{s\mathfrak{m}-e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}).$$

Substituting $e_{\overline{\nu}} = s$ into this equation of power series we get

$$s_{s\mathfrak{m}-e_{\overline{\nu}}\mathfrak{m}} s_{e_{\overline{\nu}}\mathfrak{m}} s \cdot \mathfrak{m}(s) \cdot b_{\mathfrak{m}}(s) - s_{s\mathfrak{m}-e_{\overline{\nu}}\mathfrak{m}} s_{\mathfrak{m}} s \cdot \mathfrak{m}(s) \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(s) = 0$$

which holds for all $s \in \mathbb{S}$. As $s_{s\mathfrak{m}-e_{\overline{\nu}}\mathfrak{m}} s \cdot \mathfrak{m}(s) \in \mathbb{S} \cdot \mathfrak{M}(\Lambda_*)$ is not a zero-divisor of $\widehat{B}$ this equation reduces further to

$$s_{e_{\overline{\nu}}\mathfrak{m}} \cdot b_{\mathfrak{m}}(s) - s_{\mathfrak{m}} \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(s) = 0 \qquad \text{for all } s \in \mathbb{S}. \tag{6.15}$$

For $s_{e_{\overline{\nu}}\mathfrak{m}} \cdot b_{\mathfrak{m}}(e_{\overline{\nu}}) - s_{\mathfrak{m}} \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}) \in \widehat{B}$ let

$$s_{e_{\overline{\nu}}\mathfrak{m}} \cdot b_{\mathfrak{m}}(e_{\overline{\nu}}) - s_{\mathfrak{m}} \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}) = \sum_{\mathfrak{m}' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\})} c_{\mathfrak{m}'}(e_{\overline{\nu}}) \cdot \mathfrak{m}' \tag{6.16}$$

be the canonical representation with

$$c_{\mathfrak{m}'}(e_{\overline{\nu}}) = \sum_{i=0}^{\infty} r^i_{\mathfrak{m}'} e^i_{\overline{\nu}} \in \widehat{B}_{\{\overline{\nu}\}} \subseteq \widehat{R}[[e_{\overline{\nu}}]] \qquad \text{for all } \mathfrak{m}' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\}) \tag{6.17}$$

and accompanying coefficients $r^i_{\mathfrak{m}'} \in \widehat{R}$. Substituting $e_{\overline{\nu}} = s$ into (6.16) and comparing coefficients with (6.15) we have

$$c_{\mathfrak{m}'}(s) = \sum_{i=0}^{\infty} r^i_{\mathfrak{m}'} s^i = 0 \qquad \text{for all } \mathfrak{m}' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\}) \text{ and } s \in \mathbb{S}. \tag{6.18}$$

We claim that $r^i_{\mathfrak{m}'} = 0$ for all $\mathfrak{m}' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\})$ and $i \in \omega$. Thus, towards a contradiction, assume that $r^j_{\mathfrak{m}''} \neq 0$ for some $\mathfrak{m}'' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\})$ and $j \in \omega$. Choosing $k \in \omega$ minimal with $r^k_{\mathfrak{m}''} \neq 0$ and choosing $s \in \mathbb{S}$ such that $r^k_{\mathfrak{m}''} \notin s\widehat{R}$ it is easy to see that $c_{\mathfrak{m}''}(s) = \sum_{i=0}^{\infty} r^i_{\mathfrak{m}''} s^i \notin s^{k+1}\widehat{R}$. This contradicts (6.18).
With $r^i_{\mathfrak{m}'} = 0$ for all $\mathfrak{m}' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\})$ and $i \in \omega$ we have $c_{\mathfrak{m}'}(e_{\overline{\nu}}) = 0$ ($\mathfrak{m}' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\})$) from (6.17), and from (6.16) follows

$$s_{e_{\overline{\nu}}\mathfrak{m}} \cdot b_{\mathfrak{m}}(e_{\overline{\nu}}) - s_{\mathfrak{m}} \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}) = 0$$

Hence, with $s_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}\mathfrak{m})\psi = e_{\overline{\nu}}\mathfrak{m} \cdot b_{e_{\overline{\nu}}\mathfrak{m}}$ we have

$$s_{\mathfrak{m}} s_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}\mathfrak{m})\psi = e_{\overline{\nu}}\mathfrak{m} \cdot s_{\mathfrak{m}} b_{e_{\overline{\nu}}\mathfrak{m}} = e_{\overline{\nu}}\mathfrak{m} \cdot s_{e_{\overline{\nu}}\mathfrak{m}} b_{\mathfrak{m}}$$

and

$$s_{\mathfrak{m}}(e_{\overline{\nu}}\mathfrak{m})\psi = e_{\overline{\nu}}\mathfrak{m} \cdot b_{\mathfrak{m}} \qquad \text{for all } \mathfrak{m} \in \mathfrak{M}(\Lambda_*) \text{ and } \overline{\nu} \in \Lambda_*. \tag{6.19}$$

Applying (6.19) recursively for all monomials $\mathfrak{m} \in \mathfrak{M}(\Lambda_*)$ gives $s_1\mathfrak{m}\psi = \mathfrak{m} \cdot b_1$ for all $\mathfrak{m} \in \mathfrak{M}(\Lambda_*)$. Hence, $s_1 b\psi = b \cdot b_1$ holds for all $b \in B$. Substituting $b = 1$ gives $s_1\psi = b_1$, and $b\psi = b \cdot 1\psi$ follows for all $b \in B$. Thus, ψ is the scalar multiplication by the element 1ψ. ■

We are now ready for proving the Main Theorem 1.6.

Proof. The construction is basically the same as for the Main Theorem 1.3. The list of necessary conditions for the application of the Step Lemma 6.8 changes as follows.

(i) $V_\alpha \subseteq \bigcup_{\substack{\beta<\alpha \\ \|\eta_\beta\|<\|\eta_\alpha\|}} \Lambda^{\langle \eta_\beta \rangle}$.

(ii) $y'''_{\overline{\eta}} = y'_{\overline{\eta}}$ *for all* $\overline{\eta} \in V_\alpha$.

(iii) $[z] \subseteq V_{\alpha *}$.

(iv) $z\varphi \notin (z\widehat{B})_* \subseteq \widehat{B}$.

It remains to show that the resulting R-algebra $A = A_{\Lambda_* V}$ is as required in Theorem 1.6: Let $\psi \in \operatorname{End}_R A$. Similarly to the proof of Theorem 1.3, we can apply the Easy Black Box 6.10 for the stationary set $E_z \subseteq \lambda^o$ and for $A = A_{\Lambda_* V}, \psi, S = [z]$ to obtain

$$z\psi \in (z\widehat{B})_* \subseteq \widehat{B} \text{ for all } z \in B \subseteq A.$$

Thus, ψ is the scalar multiplication by some $a \in A$ with Lemma 6.11. ∎

6.5 A last remark

Theorem 1.6 allows a generalization along the lines of Dugas, Mader, Vinsonhaler [10].

Theorem 6.12 (ZFC) *Let R be a cotorsion-free $\mathbb{S}$-ring with $\pi \in \widehat{R}$ some transcendental element over R, k be a positive integer, A be a commutative R-algebra with* 1 *and $\aleph_k$-free R-module structure A_R and let $|A| \leq \mu$ be a cardinal. Then with*

$$\lambda = \begin{cases} \beth_k(\mu) \\ \beth_k(\mu)^+ \end{cases} \textit{for } \beth_k(\mu) \begin{cases} \textit{regular} \\ \textit{singular} \end{cases}$$

we can embed A into an $E(R)$-algebra A' of cardinality λ with A'_R an $\aleph_k$-free R-module.

Proof. We use

$$B = A[e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*] = \bigoplus_{\mathfrak{m} \in \mathfrak{M}(\Lambda_*)} A\mathfrak{m},$$

as the basic R-algebra for our construction and proceed as in the proof of Theorem 1.6. Note that A is a cotorsion-free $\mathbb{S}$-ring with $\pi \in \widehat{R} \subseteq \widehat{A}$ a transcendental element over A. There are some noteworthy changes concerning Lemma 6.11: This time, instead of $s\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m}$, we investigate the element $sa\mathfrak{m} - e_{\overline{\nu}}\mathfrak{m}$ for fixed $s \in \mathbb{S}$ and $a \in A$, where (6.15) becomes

$$s_{e_{\overline{\nu}}\mathfrak{m}} \cdot sa \cdot \mathfrak{m}(sa) \cdot b_{a\mathfrak{m}}(sa) - s_{a\mathfrak{m}} \cdot sa \cdot \mathfrak{m}(sa) \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(sa) = 0$$

after substituting $e_{\overline{\nu}} = sa$. Thus, (6.16) becomes

$$s_{e_{\overline{\nu}}\mathfrak{m}} \cdot e_{\overline{\nu}} \cdot \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{a\mathfrak{m}}(e_{\overline{\nu}}) - s_{a\mathfrak{m}} \cdot e_{\overline{\nu}} \cdot \mathfrak{m}(e_{\overline{\nu}}) \cdot b_{e_{\overline{\nu}}\mathfrak{m}}(e_{\overline{\nu}}) = \sum_{\mathfrak{m}' \in \mathfrak{M}(\Lambda_* \setminus \{\overline{\nu}\})} c_{\mathfrak{m}'}(e_{\overline{\nu}}) \cdot \mathfrak{m}'$$

and $s_{e_{\overline{\nu}}\mathfrak{m}} \cdot b_{a\mathfrak{m}} = s_{a\mathfrak{m}} \cdot b_{e_{\overline{\nu}}\mathfrak{m}}$ follows for all $a \in A$, $\mathfrak{m} \in \mathfrak{M}(\Lambda_*)$ and $\overline{\nu} \in \Lambda_*$. ∎

This result shows that even the class of $E(R)$-algebras A of $\aleph_k$-free R-module structure A_R is complicated.

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