Uniquely Transitive R-modules

Dissertation

zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften

von

Daniel Herden

vorgelegt beim Fachbereich Mathematik

der Universität Duisburg-Essen, Standort Essen

Essen 2005

Antragssteller:	Daniel Herden, geb. am 11.10.1979 in Münster, Westfalen
Abgabe der Dissertation:	26.04.2005
Zulassung zur Promotion:	02.05.2005
Tag der Disputation:	17.06.2005
Prüfungsausschuß:	
Vorsitzender:	Prof. Dr. W. Lempken
Gutachter:	Prof. Dr. M. Droste Prof. Dr. R. Göbel

God exists since mathematics is consistent,

and the devil exists since we cannot prove it.

Hermann Weyl

Acknowledgement

I would like to thank my supervisor, Rüdiger Göbel, for his invaluable advice. Without his constant support and concern this work could never have been completed.
I am indebted to my family, who enabled me to study and who always encouraged me to continue this work.
I am also grateful to Lutz Strüngmann and Simone Walutis for their instructive advice and support and to my colleagues Nicole Hülsmann, Christian Müller and Sebastian Pokutta for a pleasant working atmosphere and many productive discussions.
Furthermore I would like to thank all my friends for giving me an enjoyable time apart from mathematics.

This work is supported by the project No. I-706-54.6/2001 of the German-Israeli Foundation for Scientific Research & Development.

Dedicated to my parents.

1 Introduction

Emmanuel Dror Farjoun raised in 1997 the following problem:

Do uniquely transitive torsion-free abelian groups exist?

(This question comes from his work on homotopy theory.) Here a torsion-free abelian group $G \neq \mathbb{Z}$ is called (uniquely) transitive if for any ordered pair $(a, b) \in G \times G$ of pure elements there is some (exactly one) automorphism $\varphi \in \operatorname{Aut} G$ mapping a onto b. The group $\mathbb{Z}$ of integers has the pure elements 1 and -1 and the automorphism group $\operatorname{Aut} \mathbb{Z} = \{-1, 1\}$, thus it is uniquely transitive and therefore is excluded from the definition. Note that $\mathfrak{p}G$, the set of all pure elements in G, may be empty, if G is divisible for instance. In order to avoid this and other trivial cases we also require that G is $\aleph_1$-free, hence every countable subset of G is free. Thus every element in G is a multiple of some pure element and $|\mathfrak{p}G| = |G|$ holds. Note that also $|\mathfrak{p}G| = |\operatorname{Aut} G|$ for uniquely transitive groups.

This problem proved to be unexpectedly hard. It is related to classical problems of transitivity and ring realization in commutative algebra.

Questions concerning transitivity have a long and vivid history in algebra. They first occurred in non-commutative algebra, when characterizing the (uniquely) transitive subgroups of the symmetric group $\mathcal{S}(X)$ over some index set X. Here a group $G \subseteq \mathcal{S}(X)$ is called (uniquely) transitive, if for all $x_1, x_2 \in X$ there exists some (exactly one) permutation $\pi \in G$ which maps x_1 onto x_2. In this context Wielandt [40] asked to investigate those partially ordered sets $(P, \leq)$, whose group of order preserving automorphisms acts transitively on all subsets $Q \subseteq P$ of fixed cardinality $|Q| = \kappa$. A full solution to this problem was given by Droste in [9].

Transitivity for modules, in particular for abelian p-groups was introduced in commutative algebra by Kaplansky [34]. Here a p-group G is called (fully-)transitive, if for any pair $(g_1, g_2) \in G \times G$ of elements of the same Ulm-sequence ($U(g_1) \leq U(g_2)$) there exists some automorphism (endomorphism) $\phi \in \operatorname{Aut} G$ ($\phi \in \operatorname{End} G$) which maps g_1 onto g_2. Various construction methods for (fully-)transitive p-groups were obtained by

Carroll and Goldsmith [2], Corner [5], Megibben [35] and Hennecke [30]. Transitivity for torsion-free groups was studied by Hausen, [28] and [29]. Obviously any free group is transitive. Examples of transitive groups, which are $\aleph_1$-free but not separable, were given by Dugas and Hausen [13]. For a construction of arbitrary large transitive $\aleph_1$-free indecomposable groups in L we refer to Dugas and Shelah [15].
We note that all these examples failed to be uniquely transitive groups due to the following serious problem: Suppose $G \subseteq \mathrm{Aut}\, F$ is a subgroup of the full automorphism group of some group F which acts uniquely but not necessarily transitive on F, for instance consider $G = 1$. Then it seems to be a hopeless task to generate a uniquely transitive subgroup of $\mathrm{Aut}\, F$ by adding further automorphisms step by step. The reader of this thesis can easily convince himself that adding just one automorphism α to some G causes severe problems and will destroy almost certainly the uniqueness property by the huge number of new automorphisms obtained by $w(\alpha)$ for any word $w(x) \in \langle G, x\rangle$ with free variable x. Furthermore the requirement $G = \mathrm{Aut}\, F$ is a complicated additional task.

On the other hand the question of realizing endomorphism rings is easy and very appealing: Characterize those rings which are endomorphism rings of groups and realize every suitable ring as endomorphism ring of some group. Thus we will try to borrow methods from this well-studied area.
The first important result concerning countable rings was obtained by Corner [4] showing that every countable reduced torsion-free ring is an endomorphism ring. This result was complemented recently for countable divisible rings and modules of size $\aleph_1$ by Göbel and Shelah [22]. Corner's result was extended to endomorphism rings of large cardinality in Dugas and Göbel [10], [11] and [12]. Finally a comprehensive and uniform realization theorem for endomorphism rings (for torsion-free, torsion and mixed groups) was obtained by Corner and Göbel [6]. Here the notion of cotorsion-freeness, which was introduced earlier by Göbel, plays an important role. All these realization theorems are based on combinatorial results, the Black Boxes, the Diamond Principle or other combinatorial methods that help to construct groups. Again it is easy to

check that the automorphism groups of these abelian groups must fail the uniquely transitivity problem: The constructed groups have much larger cardinality than their endomorphism rings.

A related problem was posed by Fuchs [18] in 1958: Characterize the rings R for which $\mathrm{End}\,(R^+) \cong R$. Schultz [37] gave a partial solution to this problem and introduced the class of E-rings. Here a ring R with unit is called an E-ring, if the canonical endomorphism $\varepsilon : \mathrm{End}\,(R^+) \to R$, $\varphi \mapsto \varphi(1)$ is a bijection. Important results on E-rings are due to Faticoni [17], Dugas, Mader and Vinsonhaler [14] and Strüngmann [39] constructing arbitrary large E-rings. For the construction of almost-free E-rings see Göbel, Shelah [21], Göbel, Strüngmann [25] and Göbel, Shelah, Strüngmann [26]. Here a group is called almost-free, if every subgroup of smaller cardinality can be embedded into a free subgroup. A non-commutative solution to Fuchs' problem is given in Göbel and Shelah [24]. The construction of E-rings is an important example, where the group and its endomorphism ring must be of the same size.

A first attempt to solve Farjoun's problem goes back to Dugas, Shelah [15], a V=L construction mainly based on iterated tensor products and the Diamond Principle. But the resulting groups do not determine the automorphism group and it is not hard to check that the constructed automorphism groups fail the uniqueness property. Thus they will not answer the posed problem.

The first successful construction for uniquely transitive groups was given by Göbel and Shelah [23]. Using iterated pushouts and the Strong Black Box-argument they showed, that assuming ZFC for any successor cardinal $\kappa = \mu^+$ with $\mu = \mu^{\aleph_0}$ there exists an $\aleph_1$-free uniquely transitive group G of cardinality κ. Furthermore, they proved that the endomorphism ring of G is isomorphic to the integral group ring $\mathbb{Z}F$ over a non-commutative free (absolute free) group F of cardinality κ.

In [31] we refined these arguments using some new combinatorial ideas and the Diamond Principle to construct κ-free uniquely transitive groups G of cardinality κ in L for any non-reflecting cardinal κ. Here a group is called κ-free if every subgroup of cardinality less than κ is free. We showed that the automorphism group of G is isomorphic to $\mathbb{Z}F^*$, the unit group of $\mathbb{Z}F$. On the other hand, we could not determine the endomorphism ring of G. Here we will overcome the last problem.

We will develop several new methods to attack Farjoun's problem. On the one hand we will refine the Black-Box-arguments, on the other hand we will obtain a new method which proves extremely helpful in this case and may be relevant for similar questions – this is a mixture of localization arguments that do not destroy $\aleph_1$-freeness and combinatorial methods. Thus we are able to construct examples for uniquely transitive groups which are very rigid in the sense of module theory – the related endomorphism rings are "in some sense minimal", only allowing elements of a related PID S which makes the constructed groups canonical S-modules.

In Chapter 2 we shall present basic set theoretic and algebraic tools, discuss κ-free modules and some Prediction Principles. Furthermore we deduce some general properties of uniquely transitive groups and state two Main Theorems, which will be proved in Part I and Part II.

Part I deals with [23] and [31]. We strengthen the algebraical and combinatorial components from [31], and show in Gödel's constructible universe L that indeed $\operatorname{End} G = \mathbb{Z}F$ holds for the constructed uniquely transitive groups G (see Theorem 2.28 (a)). Also we improve [23] showing that for **any** cardinal κ with $\kappa = \kappa^{\aleph_0}$ there exists an $\aleph_1$-free uniquely transitive group G of cardinality κ, particularly including limit cardinals (see Theorem 2.28 (b)). Thus the smallest example has size $2^{\aleph_0}$. It is interesting to note and helpful to show, that in both cases $\operatorname{rk}(\operatorname{Ker}(\varphi))$ is finite for every endomorphism $0 \neq \varphi \in \operatorname{End} G$.

In Part II we present a new construction for $\aleph_1$-free uniquely transitive groups of arbitrary large cardinality κ using iterated localizations. As indicated above, these groups G are very different from those in [23] and [31]: The groups G are the additive groups of rings S, which are at the same time principal ideal domains and E-rings. We present two versions, one for successor cardinals κ assuming Weak Diamond (see Theorem 2.29 (a)) and one for cardinals κ with $\kappa = \kappa^{\aleph_0}$ assuming ZFC (see Theorem 2.29 (b)). This particularly includes uniquely transitive groups of cardinality $\aleph_1$ assuming ZFC and

$2^{\aleph_0} < 2^{\aleph_1}$.

All these results extend obviously to R-modules over arbitrary cotorsion-free principal ideal domains R. Our notation is standard, maps are written on the right etc., see [16], [19], [20] and [32] for further terminology and unexplained notation.

Contents

2 Basic Tools

This chapter provides an overview of this thesis, including some basic definitions and a first discussion of uniquely transitive modules. We will use most theorems of this chapter later on without further reference.

2.1 Basic Definitions

Module Theory

In this section let M denote a left R-torsion-free R-module over some domain R. Let R^* be the group of units of R.

All definitions in this section are (implicitly) related to the ring R.

We define **Aut** M and **End** M to be the group of automorphisms respectively the ring of endomorphisms of M. Let 1 be the identity and 0 the zero-map naturally defined on M.

The **rank** $\mathrm{rk}\, M$ of the module M is the cardinality of a maximal set $T \subseteq M$ of independent elements in M.

For a submodule U of M we will write $U \subseteq M$. Also subsets T of M will be denoted by $T \subseteq M$. For any subset T of M let $\langle T\rangle$ respectively $\langle T\rangle_R$ be the submodule of M generated by T. For direct summands U of M we write $U \sqsubseteq M$.

A submodule U of M is said to be R**-pure** or RD-pure ($U \subseteq_* M$ or $U \subseteq_{*R} M$) if

$$\forall u \in U, m \in M, r \in R : (rm = u \Longrightarrow \exists u' \in U : ru' = u).$$

For any subset T of M there exists a uniquely determined smallest R-pure submodule of M containing T, the **purification** $\langle T\rangle_*$ or $\langle T\rangle_{*R}$ of T in M: $\langle T\rangle_* = \bigcap\{U \subseteq_* M | T \subseteq U\} = \{m \in M | \exists r \in R \setminus \{0\} : rm \in \langle T\rangle\}$. For submodules U of M we have $U_* = \langle U\rangle_*$. For $U \subseteq_* M$ and any maximal set $T \subseteq U$ of linear independent elements in U we have $U = \langle T\rangle_*$.

We call an element $m \in M$ R**-pure** ($m \in_* M$) if $\forall\, r \in R,\ m' \in M :$
$rm' = m \Longrightarrow r \in R^*$. Let $\mathfrak{p}M$ be the set of all R-pure elements in M.

Set Theory

For any limit ordinal α ($\alpha \in \mathrm{LORD}$) we define the **cofinality** of α to be the smallest cardinal λ, for which there exists a subset $T \subseteq \alpha$ with $|T| = \lambda$ and $\sup(T) := \bigcup_{t\in T} t = \alpha$. We write $\mathrm{cf}(\alpha)$ for the cofinality of α. We call a cardinal κ **regular** if $\mathrm{cf}(\kappa) = \kappa$ and **singular** otherwise. For instance $\aleph_n$ ($n \in \omega$) is regular but $\aleph_\omega$ is singular with $\mathrm{cf}(\aleph_\omega) = \aleph_0$. For more details see [7].

From now on λ will always be a cardinal and $\aleph_0 < \kappa$ an uncountable regular cardinal.

Definition 2.1 *A subset C of λ is called a* **cub** *(closed unbounded set) if the following holds.*

(i) C is **closed** *in λ, i.e. $\forall\ T \subseteq C : (\sup T \in \lambda \Longrightarrow \sup T \in C)$.*

(ii) C is **unbounded** *in λ, i.e. $\sup C = \lambda$.*

For any cardinal λ and cub $C \subseteq \lambda$ we have $\mathrm{cf}(\lambda) \leq |C| \leq \lambda$, because C is unbounded. A subset S of λ is called **stationary** if $S \cap C \neq \emptyset$ for any cub $C \subseteq \lambda$. Stationary sets are always unbounded, but not necessarily closed. A well known example for stationary sets:

Lemma 2.2 *For any cardinal $\aleph_0 \leq \rho < \kappa$ the set $E_\rho := \{\alpha \in \kappa | \mathrm{cf}(\alpha) = \rho\}$ is stationary.*

Proof: See [16, Example II.4.7, p. 37].

We set $\kappa^{\mathbf{o}} := E_{\aleph_0}$.

The intersection of less than $\mathrm{cf}(\lambda)$ cubs in λ is again a cub. For any stationary set $S \subseteq \lambda$ and any cub $C \subseteq \lambda$ the set $S \cap C$ is stationary. For more details see [16, Chapter II.4, p. 35 ff].

We call a subset $E \subseteq \kappa$ **non-reflecting** if $E \cap \alpha$ is not stationary in α for any ordinal $\alpha \in \kappa$ with $\aleph_0 < \mathrm{cf}(\alpha)$. Therefore, given any non-reflecting subset E and any ordinal $\alpha \in \kappa$ with $\aleph_0 < \mathrm{cf}(\alpha)$, there exists a cub C of α with $C \subseteq \alpha \setminus E$. Here the notions "cub" and "stationary subset" are generalized in an obvious way from cardinals to arbitrary limit ordinals. Now a regular cardinal κ is **non-reflecting** if a stationary, non-reflecting subset $E \subseteq \kappa^o$ exists.

2.2 κ-free Modules over PIDs

We will use PID as an abbreviation for principal ideal domain and recall some basic properties of PIDs. Let $\mathrm{P}(R)$ be the set of all prime elements in some PID R and for every prime element p of R let $\widehat{R}_p$ be the p-adic completion of R.

For any PID S and elements $a, b \in S$ we abbreviate "a divides b" by "$a|b$". In particular a and b are **associated** ($a \sim b$) if $a|b$ and $b|a$, i.e. if $ae = b$ for some unit $e \in S^*$. Thus, if $s \in S \setminus (S^* \cup \{0\})$ there exists a representation $s = \prod_{i=1}^n s_i$ as finite product of prime elements. This product is uniquely determined, where we allow substitution of any s_i by an associated element and permutation of the factors.

We recall some elementary properties of modules over PIDs:

Theorem 2.3 *Let S be a PID.*

(a) Submodules of free S-modules are free.

(b) Any finitely generated torsion-free S-module is free.

(c) Free modules are projective: If M/U is free, then $U \sqsubseteq M$.

Proof: See [3, Proposition 10.6.1, p. 283 and Corollary 10.6.2', p. 285] and [19, Lemma 9.4, p. 47].

Theorem 2.4 (Pontryagin)

Let S be a PID and M an S-module with $\operatorname{rk} M \leq \aleph_0$. Then M is free if and only if any submodule $U \subseteq M$ with $\operatorname{rk} U < \aleph_0$ is free.

Proof: See [19, Theorem 19.1, p. 91].

For more details about free modules over PIDs see also [3, Chapter 10.6, p. 283 ff]. The Pontryagin Theorem is important for $\aleph_1$-free modules. More generally we discuss the notion of κ-free modules:

Definition 2.5 *Let R be a PID and κ a cardinal.*

*(a) An R-module M is called κ-**free** if every submodule $U \subseteq M$ with $\operatorname{rk} U < \kappa$ is free.*

(b) An R-module M is called **strongly κ-free** *if M is κ-free and every submodule U with $\operatorname{rk} U < \kappa$ embeds into a submodule U' with $\operatorname{rk} U' < \kappa$ and M/U' κ-free.*

Submodules of κ-free modules are κ-free. Every κ-free module is torsion-free. Every free module is always κ-free. Direct sums of κ-free modules are κ-free. The following lemma obviously holds also for free modules.

Lemma 2.6 *Let A, B and C be R-modules with $C \subseteq B \subseteq A$.*
(a) If B and A/B are $\aleph_1$-free, then also A is $\aleph_1$-free.
(b) If A/B and B/C are $\aleph_1$-free, then also A/C is $\aleph_1$-free.

Proof:
(a): Let $M \subseteq A$ be a submodule of at most countable rank. Then also $(M+B)/B \subseteq A/B$ is a submodule of at most countable rank: $\operatorname{rk}((M+B)/B) \leq \operatorname{rk}(M)$. Therefore $(M+B)/B$ is free. With B and $(M+B)/B$ also $M+B \cong B \oplus ((M+B)/B)$ is $\aleph_1$-free. The submodule $M \subseteq M+B$ of at most countable rank is therefore free. Thus A must be $\aleph_1$-free.
(b): Observe that $A/B \cong (A/C)/(B/C)$; thus with A/B and B/C $\aleph_1$-free (a) yields A/C $\aleph_1$-free. □

Let α be a limit ordinal ($\alpha \in \mathrm{LORD}$). We call $(A_i)_{i\in\alpha}$ an **ascending chain** (short: chain) of R-modules if $A_j \subseteq A_k$ holds for all $j \leq k < \alpha$. A chain $(A_i)_{i\in\alpha}$ is **continuous** if $A_\beta = \bigcup_{\gamma\in\beta} A_\gamma$ for all $\beta \leq \alpha$, $\beta \in \mathrm{LORD}$. We now deduce a characterization of $\aleph_1$-free R-modules by chains.

Lemma 2.7 *Let $\alpha \in \mathrm{LORD}$ and $(A_i)_{i\in\alpha}$ be an ascending chain of R-modules with $A_j \subseteq_* A_k$ for all $j \leq k < \alpha$. Then $A_j \subseteq_* A$ holds for all $j < \alpha$, where $A := \bigcup_{i\in\alpha} A_i$.*

Proof: Given any $a_j \in A_j, a \in A, r \in R$ with $ra = a_j$ the definition of A yields a $j \leq k < \alpha$ with $a \in A_k$. Now $A_j \subseteq_* A_k$ implies $ra'_j = a_j$ for an $a'_j \in A_j$. Therefore $A_j \subseteq_* A$. □

Lemma 2.8 *Let M be a torsion-free R-module of at most countable rank. Then M is free if and only if every R-pure finite rank submodule of M is free.*

Proof:

$\Longrightarrow$: Submodules of free modules are always free.

$\Longleftarrow$: Let U be a finite rank submodule of M. Then $U \subseteq U_*$ embeds into a pure finite rank submodule of M. According to the conditions U_* is free and so also U is free. Now Lemma 2.4 yields that M is free. $\square$

Corollary 2.9 *Let M be a torsion-free R-module. Then M is $\aleph_1$-free if and only if every pure finite rank submodule of M is free.*

Proof:

$\Longrightarrow$: This follows directly from the definition of $\aleph_1$-free.

$\Longleftarrow$: We can apply Lemma 2.8 to any submodule U of at most countable rank deducing U is free. Hence M is $\aleph_1$-free. $\square$

We will also use the following

Corollary 2.10 *Let $\alpha \in$ LORD and $(A_i)_{i\in\alpha}$ be an ascending chain of $\aleph_1$-free R-modules with $A_j \subseteq_* A_k$ for all $j \leq k < \alpha$. Then $A := \bigcup_{i\in\alpha} A_i$ is $\aleph_1$-free.*

Proof: Let $\langle a_1, ..., a_n\rangle_*$ be a pure finite rank submodule of A. There exists a $j < \alpha$ with $a_1, ..., a_n \in A_j$, where $A_j \subseteq_* A$ according to Lemma 2.7. This yields $\langle a_1, ..., a_n\rangle_* \subseteq A_j$ and thus $\langle a_1, ..., a_n\rangle_*$ is free. Therefore according to Lemma 2.8 A is $\aleph_1$-free. $\square$

Corollary 2.11 *Let $\alpha \in$ LORD and $(A_i)_{i\in\alpha}$ be an ascending chain of $\aleph_1$-free R-modules with A_k/A_j $\aleph_1$-free for all $j \leq k < \alpha$. Then for $A := \bigcup_{i\in\alpha} A_i$ holds:*

(a) A is $\aleph_1$-free.

(b) A/A_j is $\aleph_1$-free for all $j < \alpha$.

Proof:

(a): From A_k/A_j $\aleph_1$-free follows $A_j \subseteq_* A_k$. Therefore A is $\aleph_1$-free by Corollary 2.10.

(b): For any j the $\aleph_1$-free R-modules A_k/A_j $(j \leq k < \alpha)$ form an ascending chain with union A/A_j. From $(A_l/A_j)/(A_k/A_j) \cong A_l/A_k$ $\aleph_1$-free for all $j \leq k \leq l < \alpha$ it follows that A/A_j is $\aleph_1$-free . $\square$

We also have a general characterization of κ-free modules by chains.

Lemma 2.12 *Let κ be a regular cardinal and M be an R-module with $|M| = \kappa$ and $|R| < \kappa$.*

(a) Then M is κ-free if and only if there exists a continuous chain $(M_\alpha)_{\alpha\in\kappa}$ of free modules with $M_\alpha \subseteq_* M_\beta \subseteq M$ for all $\alpha \leq \beta < \kappa$ and $M = \bigcup_{\alpha\in\kappa} M_\alpha$.

(b)Then M is strongly κ-free if and only if there exists a continuous chain $(M_\alpha)_{\alpha\in\kappa}$ of free modules with $M/M_{\alpha+1}$ κ-free for all $\alpha \in \kappa$, $M_\alpha \subseteq_* M_\beta \subseteq M$ for all $\alpha \leq \beta < \kappa$ and $M = \bigcup_{\alpha\in\kappa} M_\alpha$.

Proof: See [16, Chapter IV.1, p. 88 ff].

2.3 Prediction Principles

First we define some helpful topological notions.

Trees

For any set S we define $T_S := {}^{\omega>}S := \{\tau : n \to S, n \in \omega\}$ to be the **canonical tree** on S and $\mathrm{Br}\,(T_S) := {}^{\omega}S := \{\tau : \omega \to S\}$, where ω is the first infinite ordinal.
For any $\tau \in T_S$, $\tau : n \to S$ let $l(\tau) := n$ be the length of τ. The set $\{\tau \upharpoonright n : n \leq l(\tau)\}$ of initial segments of τ is called the **finite branch** induced by τ. This set is linearly ordered by inclusion. For any $f \in \mathrm{Br}\,(T_S)$ let $\{f \upharpoonright n : n \in \omega\} \subseteq T_S$ be the **branch** induced by f. We always will identify $\tau \in T_S$ with its finite branch and $f \in \mathrm{Br}\,(T_S)$ with its branch.
A subset U of T_S is called a **tree** or subtree of T_S if it is closed under taking initial segments:

$$\tau \in U \Longrightarrow \tau \subseteq U.$$

Thus canonical trees are trees and every (finite) branch of T_S is a tree, especially a subtree of T_S. The **product** of trees $U' \subseteq T_U, V' \subseteq T_V$ is naturally given by $U' \times V' := \{\tau : n \to U \times V \mid \tau(m) = (\tau_1(m), \tau_2(m))$ for $m < n, \tau_1 \in U', \tau_2 \in V', l(\tau_1) = l(\tau_2) = n \in \omega\}$. Thus in general $T_{U \times V} = T_U \times T_V$ holds.

Norms and Traps

Let $\aleph_0 < \kappa$ be a regular cardinal and set $T := T_{\kappa\times\kappa\times\aleph_0} = T_\kappa \times T_\kappa \times T_{\aleph_0}$. We now can define a **norm** on T setting $||\tau|| := \sup_{i<l(\tau)} \tau_1(i) \in \kappa$ for any $\tau = (\tau_1, \tau_2, \tau_3) \in T_\kappa \times T_\kappa \times T_{\aleph_0}$. The norm extends to arbitrary subsets $S \subseteq T$ taking $||S|| := \sup_{\tau\in S} ||\tau|| \leq \kappa$. Clearly $||S|| \in \kappa$ holds if $|S| < \kappa$, because κ is regular. Thus $||f|| \in \kappa$ for any branch $f \in \mathrm{Br}\,(T)$.
We call $f \in \mathrm{Br}\,(T)$ a **stretched** branch, if $||f \upharpoonright n|| < ||f \upharpoonright (n + 1)||$ for all $n \in \omega$.
In the following $B := \bigoplus_{\tau\in T} B_\tau$ (with arbitrary R-modules B_τ) will be our basic module. If the prime element $p \in R$ is fixed, then let $\widehat{B}$ be the p-adic completion of B. We also set $T_\alpha := T_{\alpha\times\kappa\times\aleph_0}$ and $B_\alpha := \bigoplus_{\tau\in T_\alpha} B_\tau$. For any element $b = \sum_{\tau\in T} b_\tau \in \widehat{B}$ (thus $b_\tau \neq 0$ for only countably many elements τ) let $[b] := \{\tau \in T | b_\tau \neq 0\}$ be the **support**

of b. If $X \subseteq \widehat{B}$, then we write $||X|| := ||[X]||$. Also fix pure elements $e_\tau \in B_\tau$. Moreover we need the following definition.

Definition 2.13 *A pair $p = (f, \varphi)$ is called a* **trap** *if the following properties hold.*

(i) $f : \omega \to \kappa \times \kappa$ *is a stretched branch of the tree* $T_{\kappa\times\kappa}$.

(ii) $||f \upharpoonright n||$ *is a successor ordinal for all* $n \in \omega$.

(iii) $\varphi : P \to P$ *is an R-endomorphism with* $P \subseteq \widehat{B}$ *countably generated.*

(iv) $f \times T_{\aleph_0} \subseteq [P] \subseteq T_{\kappa\times\kappa\times\aleph_0}$ *is a subtree.*

(v) $e_\tau \in P$ *for any* $\tau \in [P]$ *and* $||x|| < ||P||$ *for any* $x \in P$.

(vi) $||p|| := ||f|| = ||P|| \in \kappa^o$.

The General Black Box GBB

With this definitions we now can state the following version of the General Black Box which is (up to some minor changes) a special case of [20, The General Black Box 2.2.27, p. 91]. This is a theorem in ordinary set theory (ZFC).

Theorem 2.14 - The General Black Box

Let $\aleph_0 < \kappa$ be a regular cardinal with $\kappa^{\aleph_0} = \kappa$ and $E \subseteq \kappa^o$ be a stationary subset. Also let $T := T_{\kappa\times\kappa\times\aleph_0}$ and $B := \bigoplus_{\tau\in T} B_\tau$ for R-modules B_τ with $|B_\tau| \leq \kappa$.

Then there exists an ordinal $\kappa^* < \kappa^+$ and a list of traps $p_\alpha = (f_\alpha, \varphi_\alpha)$ $(\alpha \in \kappa^*)$ with the following properties for all $\alpha, \beta \in \kappa^*$.

(i) $||p_\alpha|| \in E$.

(ii) *If* $\beta \leq \alpha$, *then* $||p_\beta|| \leq ||p_\alpha||$.

(iii) *If* $\beta \neq \alpha$, *then* $\mathrm{Br}\,(f_\alpha \times T_{\aleph_0}) \cap \mathrm{Br}\,(f_\beta \times T_{\aleph_0}) = \emptyset$.

(iv) *If* $\beta + 2^{\aleph_0} \leq \alpha$, *then* $\mathrm{Br}\,(f_\alpha \times T_{\aleph_0}) \cap \mathrm{Br}\,[P_\beta] = \emptyset$.

(v) *If $X \subseteq \widehat{B}$ is a countable subset, $C \subseteq \kappa$ a cub and $\varphi \in \mathrm{End}\,\widehat{B}$, then there is an $\alpha \in \kappa^*$ such that the trap p_α catches X, C and φ, i.e. the following holds:*

  (a) $X \subseteq P_\alpha := \mathrm{Dom}\,\varphi_\alpha$.

  (b) $||X|| < ||p_\alpha|| \in C$.

  (c) $\varphi \upharpoonright P_\alpha = \varphi_\alpha$.

Proof: See [20, The General Black Box 2.2.27, p. 91].

Here **ZFC** stands for the axioms of Zermelo-Fraenkel set theory together with the axiom of choice.
The proof of this theorem is remarkably easy though not evident using only naive set theory and combinatorics. Its basic idea is an ingenious recursive definition of branches, which ensures that the occurring supports are sufficiently disjoint.

The Weak Diamond Φ_κ

We first formulate the weak diamond principle. Let $\mathfrak{P}(S)$ be the powerset of all subsets of S for any set S. A continuous chain $A = \bigcup_{\alpha\in\kappa} A_\alpha$ is a κ**-filtration** if $|A_\alpha| < \kappa$ for all $\alpha \in \kappa$.

Definition 2.15 - The Weak Diamond Principle $\Phi_\kappa(E)$

Let E be a stationary subset of a regular uncountable cardinal κ. We will say that the weak diamond principle $\Phi_\kappa(E)$ holds if for any family of partition functions $P_\alpha : \mathfrak{P}(\alpha) \to 2 \quad (\alpha \in E)$ there is a **weak diamond function** *$F : E \to 2$ such that for all $X \subseteq \kappa$ the set $\{\alpha \in E | P_\alpha(X \cap \alpha) = F(\alpha)\}$ is stationary in κ.*

The following theorems are the most important facts about the weak diamond principle.

Theorem 2.16 (Devlin, Shelah) $\Phi_{\aleph_1}(\aleph_1)$ *holds if and only if* $2^{\aleph_0} < 2^{\aleph_1}$.

Proof: See [8].

Theorem 2.17 *Assume $\Phi_\kappa(E)$ holds, let $A = \bigcup_{\alpha\in\kappa} A_\alpha$ and $B = \bigcup_{\alpha\in\kappa} B_\alpha$ be two κ-filtrations and for each $\alpha \in E$ let $P_\alpha : \mathrm{map}(A_\alpha, B_\alpha) \to 2$ be a partition function. Then there is a weak diamond function $\rho : E \to 2$ such that for all $f \in \mathrm{map}(A, B)$ the set $\{\alpha \in E : (f \restriction A_\alpha) \in \mathrm{map}(A_\alpha, B_\alpha) \wedge P_\alpha(f \restriction A_\alpha) = \rho(\alpha)\}$ is stationary in κ.*

Proof: See [20, Theorem 2.1.11, p. 50].

The Diamond $\diamondsuit_\kappa$

Here **V=L** stands for Gödel's universe of set theory.

We first formulate the diamond principle.

Definition 2.18 - The Diamond Principle $\diamondsuit_\kappa(E)$

Let E be a stationary subset of a regular cardinal $\aleph_0 < \kappa$. We will say that the diamond principle $\diamondsuit_\kappa(E)$ holds if there exist $W_\alpha \subseteq \alpha$ $(\alpha \in E)$ such that for any subset $X \subseteq \kappa$ the set $\{\alpha \in E | W_\alpha = X \cap \alpha\}$ is stationary in κ.

The following theorems are the most important facts about the diamond principle.

Theorem 2.19 (V=L) (Jensen)

For any regular cardinal $\kappa > \aleph_0$ and any stationary subset $E \subseteq \kappa$ the diamond principle $\diamondsuit_\kappa(E)$ holds. Moreover, κ is non-reflecting if and only if κ is not weakly compact.

Proof: See [33].

Thus assuming V=L, the diamond principle holds for any regular cardinal execept possibly those which are weakly compact cardinals whose existence is not known.

Theorem 2.20 *Assume $\diamondsuit_\kappa(E)$ holds, let $A = \bigcup_{\alpha\in\kappa} A_\alpha$ and $B = \bigcup_{\alpha\in\kappa} B_\alpha$ be two κ-filtrations. Then there exist so-called* **Jensen-functions**

$$g_\alpha : A_\alpha \to B_\alpha \quad (\alpha \in E)$$

such that for all $f \in \mathrm{map}(A, B)$ the set $\{\alpha \in E : f \restriction A_\alpha = g_\alpha\}$ is stationary in κ.

Proof: See [20, Theorem 2.1.8, p. 49].

2.4 UT-modules over PIDs

Throughout this and the following sections R will always be a **cotorsion-free** PID, i.e. $\mathrm{Hom}\,(\widehat{R}, R) = 0$. For PIDs R this is equivalent to say that R is neither a field nor $R = \widehat{R}_p$ for some $p \in \mathrm{P}(R)$. The restriction to cotorsion-free rings will be discussed in Lemma 2.25.

Notions as torsion-free, pure, etc. will refer to the PID R. Let M be always a torsion-free left R-module. Therefore $M \neq 0$ yields $\aleph_0 \leq |R| \leq |M|$. Recall that $\mathfrak{p}M$ is the set of all pure elements in M.

We want to construct different UT-modules of a given rank κ over a PID R.

Definition 2.21

(a) An R-module M is a **T-module** *(transitive) if for any pair $m_1, m_2 \in \mathfrak{p}M$ of pure elements there exists an automorphism $\varphi \in \mathrm{Aut}\, M$ with $m_1\varphi = m_2$.*

(b) A T-module is a **UT-module** *(uniquely transitive) if φ in (a) is uniquely determined by m_1 and m_2. Thus $\mathrm{Aut}\, M$ acts sharply transitive on $\mathfrak{p}M$.*

Observe that R itself is always a UT-module.

The existence of UT-modules is obvious if R-modules M satisfy $\mathfrak{p}M = \emptyset$. We want to get rid of these trivial cases and concentrate on R-homogeneous modules. Recall that a torsion-free R-module M over the PID R is **R-homogeneous** if for every element $0 \neq m \in M$ there are $r \in R$ and $m' \in \mathfrak{p}M$ with $m = rm'$.

Every homogeneous module M with $|R| < |M|$ satisfies $|\mathfrak{p}M| = |M|$, hence $\mathfrak{p}M$ is "large". Note also that all κ-free modules are homogeneous.

UT-modules have a remarkable property shown next.

Proposition 2.22 *Any homogeneous UT-module over a PID R is indecomposable.*

Proof: Let M be a homogeneous UT-module and suppose, $M = M_1 \oplus M_2$ is decomposable with $M_1 \neq 0$ and $M_2 \neq 0$. Then also M_1 and M_2 are homogeneous modules. Let 1_{M_1} and 1_{M_2} be the identities on M_1 respectively M_2. We now can define two distinct automorphisms $\varphi_1 := 1_{M_1} \oplus 1_{M_2} \in \operatorname{Aut} M$ and $\varphi_2 := 1_{M_1} \oplus (-1) \cdot 1_{M_2} \in \operatorname{Aut} M$ with $x\varphi_1 = x\varphi_2 = x$ for all $x \in \mathfrak{p}M_1$. However $\mathfrak{p}M_1 \neq \emptyset$ is a contradiction. $\square$

The restriction to cotorsion-free PID's will be revealed by the next lemma. If the PID R is not cotorsion-free, then either R is a field or there is a prime p with $R = \widehat{R}_p$.

Definition 2.23 *A PID R with $R = \widehat{R}_p$ for some $p \in P(R) \neq \emptyset$ is called a* **complete discrete valuation domain**.

We recall the following remarkable theorem.

Theorem 2.24 *Let R be a field or a complete discrete valuation domain and $M \neq 0$ be an R-module. Then M has a cyclic direct summand, i.e. $M = Rm \oplus M'$ holds for some $m \in M$, $M' \subseteq M$.*

Proof:
If R is a field, then this is a trivial result of linear algebra.
If R is a complete discrete valuation domain, then apply [34, Section 16, Corollary 1, p. 53]. $\square$

Now we show that the only homogeneous UT-modules over a given field or over a complete discrete valuation domain are copies of R.

Lemma 2.25 *Let R be a field or a complete discrete valuation domain and $M \neq 0$ a torsion-free homogeneous UT-module. Then $M \cong R$ holds.*

Proof: Let $M \neq 0$ be a homogeneous UT-module. Then M is torsion-free and by Theorem 2.24 it follows that $M = Rm \oplus M' \cong R \oplus M'$. Using Proposition 2.22 we conclude $M' = 0$, hence $M \cong R \oplus M' = R$. $\square$

Our main interest in this thesis will be the construction of $\aleph_1$-free and κ-free UT-modules.

Before we can formulate the main theorems, we recall the notions of group rings and

E(R)-algebras for further discussion.

Let F be a free non-commutative (absolute free) group in κ variables with $\aleph_0 \leq |R| < \kappa$ and let RF be the group ring of all polynomials with monomials in F and coefficients in R. Further let $R^* \times F$ be the multiplicative group of all monomials in F with coefficients in R^*.

Lemma 2.26 *Let R be a PID and F an absolute free group. Then the following holds:*
(a) RF has no proper zero divisors;
(b) $(RF)^* = R^* \times F$;
(c) 0 and 1 are the only idempotents of RF.

Proof:
(a) and (b): We sketch the proof.
Following [36] we can define a linear order on F, which is preserved under multiplication in F from the right. Therefore [38, Lemma 45.2 and 45.3, p. 276] applies and RF has no proper zero divisors and $(RF)^* = R^* \times F$ holds. (Here we used that R is a PID, i.e. has only trivial zero divisors.)
(c): Let $f \in RF$ with $f^2 = f$. Thus $f(f-1) = 0$ and from (a) follows $f \in \{0, 1\}$. □

From now on we will view the ring RF also as left respectively right R-module. For any ring S and any faithful right S-module M we will identify $s \in S$ with the induced S-endomorphism $m \mapsto ms$ on M.

Definition 2.27 *An R-algebra S is an* **E(R)-algebra** *if $\operatorname{End}_R S = S$ holds using the above identification.*

An extensive discussion of the notion E(R)-algebra can be found in [39].
We now can formulate the two main theorems of this paper:

Main-Theorem 2.28 *Let R be a cotorsion-free PID and κ be a regular cardinal such that $\aleph_0 \leq |R| < \kappa$. Suppose that F is an absolute free group of cardinality $|F| = \kappa$.*
(a) **(V=L)** *For non-reflecting κ there exists a strongly κ-free UT-module M of cardinality κ with $\operatorname{End} M \cong \mathrm{RF}$ such that every endomorphism $0 \neq \varphi \in \operatorname{End}(M)$ of M*

satisfies $\text{rk}\,(Ker\varphi) < \aleph_0$. *In particular* $\text{End}\,(M)$ *is generated by* $\text{Aut}\,(M)$.

(b) **(ZFC)** *If* $\kappa^{\aleph_0} = \kappa$, *there exists an* $\aleph_1$*-free UT-module* M *of cardinality* κ *with* $\text{End}\,M \cong \text{RF}$ *such that every endomorphism* $0 \neq \varphi \in \text{End}\,(M)$ *of* M *satisfies* $\text{rk}\,(Ker\varphi) < \aleph_0$. *In particular* $\text{End}\,(M)$ *is generated by* $\text{Aut}\,(M)$.

The module M constructed in Main Theorem 2.28 also satisfies $\text{Aut}\,M \cong R^* \times F$ as follows by Lemma 2.26.

Main-Theorem 2.29 *Let* R *be a cotorsion-free PID with* $\widehat{R}$ *of transcendence degree* ≥ 2 *over* R *and let* $\aleph_0 \leq |R| < \kappa$ *be a regular cardinal.*

(a) **($\Phi(\kappa)$)** *For successor cardinals* κ *there exists a PID* S*, which is also an* E*(R)-algebra, such that* ${}_RS$ *is an* $\aleph_1$*-free UT-module of cardinality* κ.

(b) **(ZFC)** *If* $\kappa^{\aleph_0} = \kappa$, *there exists a PID* S*, which is also an* E*(R)-algebra, such that* ${}_RS$ *is an* $\aleph_1$*-free UT-module of cardinality* κ.

Here $\mathbf{\Phi}(\kappa)$ is the weak diamond principle for the cardinal κ. The condition on the transcendence degree of $\widehat{R}$ is natural: In fact every cotorsion-free PID R with $|R| < 2^{\aleph_0}$ has a completion $\widehat{R}$ of transcendence degree $2^{\aleph_0}$ over R, see [20, Theorem 1.3.5, p. 13].

The proof of Main Theorem 2.28 is based on the ranks of the kernels of endomorphisms. Thus we begin with a theorem, showing that arbitrary large finite ranks are possible. In particular M is not torsion-free over its endomorphism ring.

Theorem 2.30 *Let* M *be an* $\aleph_1$*-free UT-module with* $\text{End}\,M \cong RF$. *Then for any* $0 \leq n \in \mathbb{Z}$ *there exists an endomorphism* $0 \neq \varphi \in \text{End}\,M$ *with* $\text{rk}\,(\ker\varphi) \geq n$.

Proof: For any $m, m' \in \mathfrak{p}M$ let $\psi_{mm'}$ be the uniquely determined automorphism mapping m to m'.

Now for any element $0 \neq x \in M$ choose linear independent pure elements $a, b, c \in \mathfrak{p}M$ and $0 \neq r, s \in R$ with $x = ra$ and $ra + b = sc$. We obtain for $\Psi_x := r + \psi_{ab} - s\psi_{ac} \in \text{End}\,M$, that $a\Psi_x = ra + b - sc = 0$. Hence $a \in \ker\Psi_x$ and thus $x = ra \in \ker\Psi_x$.

Using $\text{End}\,M \cong RF$ we observe, that Ψ_x is the sum of three different monomials in $\text{Aut}\,M \cong F$ and so $\Psi_x \neq 0$.

If $x_1, \ldots, x_n$ are independent elements of M we can define inductively an endomorphism $\Psi_{x_1 \ldots x_n} \neq 0$ with $x_1, \ldots, x_n \in \ker \Psi_{x_1 \ldots x_n}$. Put $\Psi_{x_1 \ldots x_i} := \Psi_{x_1 \ldots x_{i-1}} \Psi_{x_i \Psi_{x_1 \ldots x_{i-1}}}$ for any $1 < i \leq n$. We have $x_1 \in \ker \Psi_{x_1}$ and $\Psi_{x_1} \neq 0$ as above.
From $x_1, \ldots, x_{i-1} \in \ker \Psi_{x_1 \ldots x_{i-1}}$ and $x_i \Psi_{x_1 \ldots x_{i-1}} \in \ker \Psi_{x_i \Psi_{x_1 \ldots x_{i-1}}}$ follows
$x_1, \ldots, x_i \in \ker \Psi_{x_1 \ldots x_i}$ for all $i \leq n$. This yields $\mathrm{rk}\,(\ker \Psi_{x_1 \ldots x_n}) \geq n$. Also from $\Psi_{x_1 \ldots x_{i-1}} \neq 0$ and $\Psi_{x_i \Psi_{x_1 \ldots x_{i-1}}} \neq 0$ follows $\Psi_{x_1 \ldots x_i} = \Psi_{x_1 \ldots x_{i-1}} \Psi_{x_i \Psi_{x_1 \ldots x_{i-1}}} \neq 0$, because $\mathrm{End}\, M \cong RF$ has no proper zero divisors. $\square$

Later on we will see, that Theorem 2.30 is not true, if we replace "$n < \aleph_0$" by "$\aleph_0 \leq \lambda$". Finally, we would like to note that Theorem 2.28 (a) sharpens a result by Dugas, Shelah [15] and Theorem 2.28 (b) implies the main result by Göbel, Shelah [23].

2.5 Overview of Part I – The Proof of Main-Theorem 2.28

The proof of Theorem 2.28 is given in Chapters 3 to 5. We will discuss at the same time two cases, the "free case" and the "$\aleph_1$-free case" leading to UT-modules M that are either $|M|$-free or $\aleph_1$-free, respectively. Notions with a star * or in brackets () will always refer to the proof of Theorem 2.28 (b) (the $\aleph_1$-free case) while all other notions will refer to the proof of Theorem 2.28 (a) (the free case).

In Chapter 3 we prepare the Free UT-construction: We will show, how to embed any free ($\aleph_1$-free) R-module M into a free ($\aleph_1$-free) R-module M' such that some subgroup $G \subseteq \operatorname{Aut} M'$ acts uniquely transitive on $\mathfrak{p}M'$. This construction does not use any set theory. But it is designed to pass from partial automorphisms to (total) automorphisms by using module-extensions.

Chapter 4 is a Step-Lemma. This lemma allows us to control $\operatorname{End} M$ by killing undesirable endomorphisms.

In Chapter 5 we will introduce the main construction, which leads to the modules needed for Theorem 2.28. At this stage more set theory comes into play.

2.6 Overview of Part II – The Proof of Main-Theorem 2.29

The proof of Theorem 2.29 is given in the Chapters 6 to 8.

In Chapter 6 we introduce "unit-free algebras", a special class of PIDs, and discuss their properties. In particular it will be emphasized that this class is closed under localization.

Chapter 7 is devoted to another Step-Lemma.

In Chapter 8 we will introduce the main construction, which leads to the modules needed for Theorem 2.29. At this stage set theory comes into play again.

Part I

UT-Modules with Absolute Free Endomorphism Rings

3 The Free UT-Construction

We want to embed any given free ($\aleph_1$-free) R-module M into a free ($\aleph_1$-free) R-module M' in such a way, that some subgroup $G \subseteq \operatorname{Aut} M'$ acts uniquely transitive on $\mathfrak{p}M'$. For this task we will take advantage of the pushout-construction in [23].

As a first step we define some technical tools and discuss their properties.

Definition 3.1 *A map φ is a* **partial automorphism** *of a module M if φ is an isomorphism with $\operatorname{Dom}\varphi \subseteq M$ and $\operatorname{Im}\varphi \subseteq M$.*

For a partial automorphism φ of a module M let φ^{-1} be the inverse map of φ. Note that φ^{-1} is not the inverse of φ in the usual way, i.e. $\varphi\varphi^{-1} = \varphi^{-1}\varphi = 1_M$ holds only if $\operatorname{Dom}\varphi = \operatorname{Im}\varphi = M$. To emphasize this we will call φ^{-1} the **weak inverse** of φ.

Let 0_t be the trivial partial automorphism φ with $\operatorname{Dom}\varphi = \operatorname{Im}\varphi = 0$.

The composition $\varphi\mu$ of two partial automorphisms φ, μ is again a partial automorphism with $\operatorname{Dom}\varphi\mu = (\operatorname{Im}\varphi \cap \operatorname{Dom}\mu)\varphi^{-1}$ and $\operatorname{Im}\varphi\mu = (\operatorname{Im}\varphi \cap \operatorname{Dom}\mu)\mu$. The partial automorphisms of a module M form a monoid with unit element 1, where $1 = 1_M$ is the identity on M.

Definition 3.2 *For any free ($\aleph_1$-free) module M let* **pAut** M (**pAut**$^*M$) *be the set of all partial automorphisms of M with $M/\operatorname{Dom}\varphi$ and $M/\operatorname{Im}\varphi$ free ($\aleph_1$-free).*

Lemma 3.3 *For any module M the set $\operatorname{pAut} M$ (pAut^*M) is a submonoid of the monoid of all partial automorphisms of M with unit element 1. Moreover $\operatorname{pAut} M$ (pAut^*M) is closed under taking weak inverses.*

Proof: Let $\varphi, \mu \in \operatorname{pAut} M$ be partial automorphisms. Then $\varphi\mu$ and φ^{-1} are partial automorphisms. Moreover, $M/\operatorname{Dom}\varphi^{-1} = M/\operatorname{Im}\varphi$ and $M/\operatorname{Im}\varphi^{-1} = M/\operatorname{Dom}\varphi$ are free, and therefore $\varphi^{-1} \in \operatorname{pAut} M$. Thus $\operatorname{pAut} M$ is closed under taking weak inverses. From $\mu \in \operatorname{pAut} M$ follows, that

$$\operatorname{Im}\varphi/(\operatorname{Im}\varphi \cap \operatorname{Dom}\mu) \cong (\operatorname{Im}\varphi + \operatorname{Dom}\mu)/\operatorname{Dom}\mu \subseteq M/\operatorname{Dom}\mu$$

is free. Multiplication by φ^{-1} shows that $\operatorname{Dom}\varphi/(\operatorname{Im}\varphi \cap \operatorname{Dom}\mu)\varphi^{-1} = \operatorname{Dom}\varphi/\operatorname{Dom}\varphi\mu$ is free. From $\varphi \in \operatorname{pAut} M$ follows that $M/\operatorname{Dom}\varphi$ is free and using freeness of $\operatorname{Dom}\varphi/\operatorname{Dom}\varphi\mu$ we have, that $M/\operatorname{Dom}\varphi\mu$ is free. Similarly we observe the freeness of $M/\operatorname{Im}\varphi\mu$; hence $\varphi\mu \in \operatorname{pAut} M$. Obviously $1 \in \operatorname{pAut} M$. So $\operatorname{pAut} M$ is as required. A similar argument applies to $\operatorname{pAut}^* M$. $\square$

The following elementary property of elements in $\operatorname{pAut} M$ will be basic for our investigation.

Lemma 3.4 *Let M be a module, $\varphi \in \operatorname{pAut} M$ $(\operatorname{pAut}^* M)$ and $0 \neq x \in \operatorname{Dom}\varphi$. Then x is pure in M if and only if $x\varphi$ is pure in M, in particular: $x \in \mathfrak{p}M \iff x\varphi \in \mathfrak{p}M$.*

Proof:
"$\Longrightarrow$": Let $x \in \mathfrak{p}M \cap \operatorname{Dom}\varphi$ and write $x\varphi = ry$ for some $y \in M$, $r \in R$. Modulo $\operatorname{Im}\varphi$ we get $0 = x\varphi + \operatorname{Im}\varphi = r(y + \operatorname{Im}\varphi) \in M/\operatorname{Im}\varphi$ which is torsion-free. Therefore $r = 0$ or $y \in \operatorname{Im}\varphi$. From $r = 0$ follows $x = ry = 0$ contradicting $x \in \mathfrak{p}M$. Hence $y \in \operatorname{Im}\varphi$ and $y = x'\varphi$ for some $x' \in M$. Since $x\varphi = ry = (rx')\varphi$ we conclude $x = rx'$, thus $r \in R^*$ and $x\varphi$ is pure in M.
"$\Longleftarrow$": This follows replacing φ by φ^{-1}. $\square$

3.1 The U-Property

Recall that we identify $R \subseteq \mathrm{End}\, M$: Any $r \in R$ is viewed as left scalar multiplication on M. In particular R^* becomes a subgroup of $\mathrm{Aut}\, M$.
For any subset T of $\mathrm{pAut}\, M$ ($\mathrm{pAut}^* M$) let $\langle T \rangle$ denote the submonoid obtained by multiplicative closure of $R^* \cup T \cup T^{-1}$, where $T^{-1} := \{\varphi^{-1} | \varphi \in T\}$.

For Theorem 2.28 we need to construct a module M with $\mathrm{Aut}\, M \cong R^* \times F$, with F an absolute free group of rank $|M|$. Therefore we must assign (partial) automorphisms to the basis elements of absolute free groups. This follows next.

Let $\mathfrak{F} = \{\varphi_t | t \in J\}$ be a basis of the absolute free group F. Then $R^* \times F$ denotes the multiplicative group of all monomials $\langle \mathfrak{F} \rangle$ with coefficients in R^*. If $\varphi = r\varphi' \in R^* \times F$ ($r \in R^*$, $\varphi' \in \langle \mathfrak{F} \rangle$) then $\varphi^{-1} = (r\varphi')^{-1} = r^{-1}\varphi'^{-1} \in R^* \times F$.
Let $\pi : \mathfrak{F} \to \mathrm{pAut}\, M$ ($\mathrm{pAut}^* M$) be any map. We can extend π in a well defined way to a map with domain $R^* \times F$: An element $\mu \in R^* \times F$ has a unique **reduced representation** $\mu = r \cdot \mu_1 \cdots \mu_n$ with $r \in R^*$ and $\mu_1, \cdots, \mu_n \in \mathfrak{F}_\pm := \mathfrak{F} \cup \mathfrak{F}^{-1}$. Set $\pi(\mu) := r \cdot \pi(\mu_1) \cdots \pi(\mu_n)$. Recall $\mathfrak{F}^{-1} := \{\varphi_t^{-1} | t \in J\}$, $\pi(1) := 1 \in \mathrm{pAut}\, M$ and $\pi(\varphi_t^{-1}) := \pi(\varphi_t)^{-1}$ for any $t \in J$.
If $\mu_1, \mu_2 \in R^* \times F$ are elements in reduced representation and if μ is the reduced representation of $\mu_1\mu_2$ then $\pi(\mu_1)\pi(\mu_2) \subseteq \pi(\mu)$ holds as graphs; i.e., $\mathrm{Dom}\, \pi(\mu_1)\pi(\mu_2) \subseteq \mathrm{Dom}\, \pi(\mu)$ and $\pi(\mu) \restriction \mathrm{Dom}\, \pi(\mu_1)\pi(\mu_2) = \pi(\mu_1)\pi(\mu_2)$. However equality will fail in general. Nevertheless, if the formal product $\mu_1\mu_2$ is also reduced, then $\pi(\mu_1)\pi(\mu_2) = \pi(\mu_1\mu_2)$ is satisfied.

We now fix $F := \langle \mathfrak{F} \rangle = \{\varphi_t | t \in J\}$ as an absolute free group F with basis $\mathfrak{F}$ and let $\pi : \mathfrak{F} \to \mathrm{pAut}\, M$ ($\mathrm{pAut}^* M$), which will be identified with its extension
$\pi : R^* \times F \to \mathrm{pAut}\, M$ ($\mathrm{pAut}^* M$) as discussed.
Moreover, RF will denote the induced group ring and $f \in RF$ will be a polynomial in reduced representation $f = \sum_{j=1}^n r_j\mu_j \in RF$, where $0 \neq r_j \in R$ are its coefficients and $\mu_j \in F$ are different monomials.
We extend the map π naturally to RF by putting $\pi(f) : \mathrm{Dom}\, (\pi(f)) \to M$ with $\mathrm{Dom}\, \pi(f) := \bigcap_{j=1}^n \mathrm{Dom}\, \pi(\mu_j) \subseteq M$ and $m\pi(f) := m \sum_{j=1}^n r_j\pi(\mu_j)$ for all $m \in \mathrm{Dom}\, \pi(f)$.

In the sequel we shall always assume $m \in \operatorname{Dom} \pi(f)$ if we write $m\pi(f)$.

Definition 3.5 *Let M be an R-module.*
A map $\pi : \mathfrak{F} \to \operatorname{pAut} M$ $(\operatorname{pAut}^* M)$ has the **U-property** *for M, if the following conditions hold for all reduced polynomials $f = \sum_{j=1}^{n} r_j \mu_j \in RF$ $(1 < n)$:*
(i) $\operatorname{rk}(\operatorname{Ker} \pi(f)) < n - 1$. In particular, if $\{b_i | 1 \leq i \leq m\}$ is a set of independent elements in M and $b_i \pi(f) = 0$ for all i, then $m < n - 1$.
(ii) If $\{b_i | 1 \leq i \leq m\}$ is a set of independent elements in M and $b_i \pi(f) \in \operatorname{Dom} \pi(\varphi)$ holds for all i and some fixed $\varphi \in \mathfrak{F}_{\pm}$, but there are $1 \leq j_i \leq n$ with $b_i \pi(\mu_{j_i}) \notin \operatorname{Dom} \pi(\varphi)$ for all i, then $m < n$.

Condition (i) of the U-property will be crucial showing the uniqueness of UT. It also will estimate the ranks of the kernels of elements in $\operatorname{End} M$ coming from earlier partial automorphisms. Condition (ii) is a technical tool to verify (i) inductively.
Later on the case $n = 2$ of Definition 3.5 (i) will play a special role. It can be reformulated as our **basic U-property** in the form:

If $\varphi, \varphi' \in R^* \times F$ and $x \in \mathfrak{p}M$ with $x\pi(\varphi) = x\pi(\varphi')$, then $\varphi = \varphi'$. **(*)**

Thus $\pi(R^* \times F)$ acts uniquely on M by (*), and for every module M with $\operatorname{Aut} M = \pi(R^* \times F)$ and U-property the automorphism group $\operatorname{Aut} M$ acts uniquely on M.

Observation 3.6 *Definition 3.5 (i) for $n = 2$ is equivalent to* (*).
Proof: One implication is obvious, hence assume (*). Substituting any reduced $f := r\varphi - r'\varphi' \in RF$ with $0 \neq r, r' \in R$ into Definition 3.5 equation $x\pi(f) = 0$ yields $rx\pi(\varphi) = r'x\pi(\varphi')$. With Lemma 3.4 and $x \in \mathfrak{p}M$ follows $x\pi(\varphi), x\pi(\varphi') \in \mathfrak{p}M$, thus $r' = re$ for some $e \in R^*$. Now from $rx\pi(\varphi) = r'x\pi(\varphi')$ follows $x\pi(\varphi) = ex\pi(\varphi')$, thus $\varphi = e\varphi'$ and $r\varphi = r'\varphi'$ by (*), contradicting that f is reduced. $\square$

The above Observation 3.6 shows that for any module M satisfying the U-property the map $\xi : R^* \times F \to \mathfrak{p}M$ $(\varphi \mapsto x\pi(\varphi))$ is injective for any $x \in \mathfrak{p}M$. Thus we have $|\mathfrak{F}| \leq |\langle \mathfrak{F} \rangle| \leq |R^* \times F| \leq |\mathfrak{p}M| \leq |M|$ and therefore $|\mathfrak{F}| \leq |M|$.
We now state a simple test for the basic U-property.

Lemma 3.7 *The map* $\pi : \mathfrak{F} \to \mathrm{pAut}\, M$ $(\mathrm{pAut}^{\,*}M)$ *satisfies the basic U-property if and only if for any* $\varphi \in R^* \times F$, $x \in \mathfrak{p}M$ *with* $x\pi(\varphi) = x$ *it follows that* $\varphi = 1 \in R^* \times F$.

Proof:
"$\Longrightarrow$" : Let π have the basic U-property. Thus $x\pi(\varphi) = x = x\pi(1)$ implies $\varphi = 1$.
"$\Longleftarrow$" : Conversely, assume that any $\varphi \in R^* \times F$, $x \in \mathfrak{p}M$ with $x\pi(\varphi) = x$ is the identity map. If $x' \in \mathfrak{p}M$ and $\varphi'_1, \varphi'_2 \in R^* \times F$ with $x'\pi(\varphi'_1) = x'\pi(\varphi'_2)$, then let φ' be the reduced representation of $\varphi'_1{\varphi'}_2^{-1}$.
We have $x'\pi(\varphi'_1) = x'\pi(\varphi'_2)$, thus $x'\pi(\varphi'_1)\pi({\varphi'}_2^{-1}) = x'$ and $x'\pi(\varphi') = x'$. Hence $1 = \varphi' = \varphi'_1{\varphi'}_2^{-1}$ and $\varphi'_1 = \varphi'_2$. Thus the basic U-property (*) holds for π. $\square$

Given $\pi : \mathfrak{F} \to \mathrm{pAut}\, M$ $(\mathrm{pAut}^{\,*}M)$ with U-property we want to extend at the same time $M \subseteq M'$, $\mathfrak{F} \subseteq \mathfrak{F}'$ and $\pi \subseteq \pi'$ such that $\pi' : \mathfrak{F}' \to \mathrm{pAut}\, M'$ $(\mathrm{pAut}^{\,*}M')$ has the U-property, and $\pi'(R^* \times \langle \mathfrak{F}' \rangle) \subseteq \mathrm{Aut}\, M'$ acts uniquely transitive on $\mathfrak{p}M'$. For this purpose we naturally consider tuples $(M, \mathfrak{F}, \pi)$.

Definition 3.8 *Let* $\mathfrak{K}$ $(\mathfrak{K}^*)$ *the family of all tuples* $\mathfrak{x} = (M, \mathfrak{F}, \pi) =: (M^{\mathfrak{x}}, \mathfrak{F}^{\mathfrak{x}}, \pi^{\mathfrak{x}})$ *satisfying the following properties.*
(i) M *is a free* ($\aleph_1$*-free*) R*-module;*
(ii) $\mathfrak{F} = \{\varphi_t | t \in J^{\mathfrak{x}}\}$ *is the basis of an absolute free group;*
(iii) $\pi : \mathfrak{F} \to \mathrm{pAut}\, M$ $(\mathrm{pAut}^{\,*}M)$ *satisfies the U-property for M.*

We abbreviate $\varphi^{\mathfrak{x}} := \pi(\varphi)$ for $\mathfrak{x} = (M, \mathfrak{F}, \pi)$ and $\varphi \in R^* \times F$; also set $xf^{\mathfrak{x}} := x\pi(f)$ for any polynomial $f \in RF$ and $x \in M$.
It is convenient to introduce the following two **order relations** $\subseteq$ and $\sqsubseteq$ on $\mathfrak{K}$ $(\mathfrak{K}^*)$.

Definition 3.9 *If* $\mathfrak{x}, \mathfrak{y} \in \mathfrak{K}$ $(\mathfrak{K}^*)$, *then* $\mathfrak{x} \subseteq \mathfrak{y}$ *if the following conditions hold.*
(i) $M^{\mathfrak{x}} \subseteq M^{\mathfrak{y}}$;
(ii) $\mathfrak{F}^{\mathfrak{x}} \subseteq \mathfrak{F}^{\mathfrak{y}}$ *and* $\varphi_t^{\mathfrak{x}} \subseteq \varphi_t^{\mathfrak{y}}$ *for all* $t \in J^{\mathfrak{x}}$.
In this case we say that $\pi^{\mathfrak{y}}$ *is a* **weak extension** *of* $\pi^{\mathfrak{x}}$.

Definition 3.10 *If* $\mathfrak{x}, \mathfrak{y} \in \mathfrak{K}$ $(\mathfrak{K}^*)$, *then* $\mathfrak{x} \sqsubseteq \mathfrak{y}$ *if the following conditions hold.*
(i) $M^{\mathfrak{x}} \subseteq M^{\mathfrak{y}}$ *and* $M^{\mathfrak{y}}/M^{\mathfrak{x}}$ *is free* ($\aleph_1$*-free*);
(ii) $\mathfrak{F}^{\mathfrak{x}} \subseteq \mathfrak{F}^{\mathfrak{y}}$ *and* $\varphi_t^{\mathfrak{x}} \subseteq \varphi_t^{\mathfrak{y}}$ *for all* $t \in J^{\mathfrak{x}}$, *hence* $\pi^{\mathfrak{y}}$ *is a weak extension of* $\pi^{\mathfrak{x}}$.

3.2 Adding Baby-Automorphisms

The following method is used to add partial automorphisms, which will make the final M a T-module. In order to avoid conflicts with the U-property we will add partial automorphisms with minimal domain.

Lemma 3.11 (Adding Baby-Automorphisms)
Let $\mathfrak{x} = (M, \mathfrak{F}, \pi) \in \mathfrak{K}$ ($\mathfrak{K}^*$) and $x, y \in \mathfrak{p}M$ with $x\varphi^{\mathfrak{x}} \neq y$ for all $\varphi \in R^* \times F$. Consider the natural isomorphism $\varphi_0^{\mathfrak{y}} : Rx \to Ry$ ($rx \mapsto ry$); let $\mathfrak{F}^{\mathfrak{y}} := \mathfrak{F}^{\mathfrak{x}} \cup \{\varphi_0\}$ and $\pi^{\mathfrak{y}} := \pi^{\mathfrak{x}} \cup \{(\varphi_0, \varphi_0^{\mathfrak{y}})\}$. Then $\mathfrak{x} \sqsubseteq \mathfrak{y} := (M, \mathfrak{F}^{\mathfrak{y}}, \pi^{\mathfrak{y}}) \in \mathfrak{K}$ ($\mathfrak{K}^*$).

Proof: First we have to prove that

$\varphi_0^{\mathfrak{y}} \in \mathrm{pAut}\, M$ ($\varphi_0^{\mathfrak{y}} \in \mathrm{pAut}^*M$).

Proof: We start with the free case.

From $x \in \mathfrak{p}M$ it follows that xR is a pure submodule of the free R-module M, thus splits and $M/\mathrm{Dom}\,\varphi_0^{\mathfrak{y}} = M/Rx$ is free. Similarly $M/\mathrm{Im}\,\varphi_0^{\mathfrak{y}} = M/Ry$ is free and hence $\varphi_0^{\mathfrak{y}} \in \mathrm{pAut}\, M$ follows.

Now we consider the $\aleph_1$-free case.

If $M' := \langle m_j' + Rx | 1 \leq j \leq k\rangle_* \subseteq M/\mathrm{Dom}\,\varphi_0^{\mathfrak{y}} = M/Rx$ has finite rank, then $M'' := \langle m_j', xR | 1 \leq j \leq k\rangle_* \subseteq M$ is a pure submodule of finite rank of the $\aleph_1$-free module M, hence free. From $Rx \subseteq_* M$ follows $Rx \sqsubseteq M''$, so M''/Rx and thus $M' \subseteq M''/Rx$ are free. Hence $M/\mathrm{Dom}\,\varphi_0$ is $\aleph_1$-free by Pontryagin's Theorem. Similarly $M/\mathrm{Im}\,\varphi_0^{\mathfrak{y}} = M/Ry$ is $\aleph_1$-free and thus $\varphi_0^{\mathfrak{y}} \in \mathrm{pAut}^*M$.

It remains to prove that $\pi^{\mathfrak{y}}$ has the basic U-property and finally the U-property.

Proof: We will use Lemma 3.7 and to simplify notation, set $\mathfrak{F}' := \mathfrak{F}^{\mathfrak{y}}$ and $\mu := \varphi_0$ and let $\varphi \in R^* \times \langle \mathfrak{F}' \rangle$, $z \in \mathfrak{p}M$ with $z\varphi^{\mathfrak{y}} = z$. Write φ in the canonical representation $\varphi = r \cdot \xi_1 \mu^{\varepsilon_1} \xi_2 \cdots \mu^{\varepsilon_{k-1}} \xi_k$ with $0 \neq \varepsilon_i \in \mathbb{Z}$, $r \in R^*$ and $\xi_i \in \langle \mathfrak{F} \rangle$, where $\xi_i \neq 1$ except possibly ξ_1 and ξ_k. We must show that $\varphi = 1$. First we claim

$\varepsilon_i = 1 \vee \varepsilon_i = -1$ *for all* $i < k$. **(1)**

Otherwise there exists a factor μ^2 or μ^{-2} in the representation of φ; let μ^2 be a factor of φ without loss of generality. Hence $(\mu^{\mathfrak{y}})^2$ is a factor of $\varphi^{\mathfrak{y}}$. We have $\mathrm{Dom}\,(\mu^{\mathfrak{y}})^2 = (\mathrm{Im}\,\mu^{\mathfrak{y}} \cap \mathrm{Dom}\,\mu^{\mathfrak{y}})(\mu^{\mathfrak{y}})^{-1} = (Ry \cap Rx)(\mu^{\mathfrak{y}})^{-1} = 0$. For otherwise $Ry \cap Rx \neq 0$ for $x, y \in \mathfrak{p}M$ would yield an $s \in R^*$ with $x(s \cdot 1)^{\mathfrak{x}} = sx = y$ contradicting the conditions of the theorem. It follows $(\mu^{\mathfrak{y}})^2 = 0_t$, hence $\varphi^{\mathfrak{y}} = 0_t$ and $\mathrm{Dom}\,\varphi^{\mathfrak{y}} = 0$ contradicting $z \in \mathrm{Dom}\,\varphi^{\mathfrak{y}} \cap \mathfrak{p}M$. So (1) holds.

We now consider the **path** $[z]$ of an element $z \in \mathfrak{p}M$ under the map $\varphi^{\mathfrak{y}}$ which is the sequence of elements:

$$z_0 := z,\ z_1 := rz,\ z_2 := z_1\xi_1^{\mathfrak{y}},\ z_3 := z_2(\mu^{\mathfrak{y}})^{\varepsilon_1},\ \cdots,\ z_{2k} := z_{2k-1}\xi_k^{\mathfrak{y}}.$$

Note that the elements z_i from $[z]$ are pure in M by Lemma 3.4.

If μ^{ε_2} exists in the canonical representation of φ, then $z_3 \in \mathrm{Im}\,(\mu^{\mathfrak{y}})^{\varepsilon_1} \wedge z_4 \in \mathrm{Dom}\,(\mu^{\mathfrak{y}})^{\varepsilon_2}$, hence $z_3, z_4 \in (\mathrm{Dom}\,\mu^{\mathfrak{y}} \cup \mathrm{Im}\,\mu^{\mathfrak{y}}) \cap \mathfrak{p}M = R^*x \cup R^*y$. Without loss of generality we distinguish the following two cases:

Case 1: $z_3, z_4 \in R^*x$ (similarly $z_3, z_4 \in R^*y$). Then $z_3\xi_2^{\mathfrak{x}} = z_3\xi_2^{\mathfrak{y}} = z_4 = sz_3$ for a suitable $s \in R^*$. The U-property of $\pi^{\mathfrak{x}}$ implies $\xi_2 = s$ contradicting that φ is reduced.

Case 2: $z_3 \in R^*x$, $z_4 \in R^*y$ (similarly $z_3 \in R^*y$, $z_4 \in R^*x$). Hence $z_3 = sx, z_4 = s'y$ for some $s, s' \in R^*$. It follows $z_3\xi_2^{\mathfrak{y}} = z_4$, thus $sx\xi_2^{\mathfrak{y}} = s'y$ and $x(ss'^{-1}\xi_2^{\mathfrak{x}}) = x(ss'^{-1}\xi_2^{\mathfrak{y}}) = y$ with $ss'^{-1}\xi_2 \in R^* \times F$ contradicting the assumption of Lemma 3.11. Thus μ^{ε_2} does not exist and φ must be of the form

$$\varphi = r \cdot \xi_1\mu^{\varepsilon_1}\xi_2. \tag{2}$$

Now we assume that μ^{ε_1} exists in the canonical representation of φ. From our assumption $z\varphi^{\mathfrak{y}} = z$ and (1) and (2) follows $r \cdot z\xi_1^{\mathfrak{y}}(\mu^{\mathfrak{y}})^{\varepsilon_1}\xi_2^{\mathfrak{y}} = z$, where $\varepsilon_1 \in \{1, -1\}$. Substituting μ^{-1} for μ if necessary, we can assume $\varepsilon_1 = 1$ and $r \cdot z\xi_1^{\mathfrak{y}}\mu^{\mathfrak{y}}\xi_2^{\mathfrak{y}} = z$ without loss of generality.

Under these restrictions the path $[z]$ of z becomes very special: We have $z_2 = r \cdot z\xi_1^{\mathfrak{y}} \in \mathrm{Dom}\,\mu^{\mathfrak{y}} \cap \mathfrak{p}M = R^*x$, $z_3 = z_2\mu^{\mathfrak{y}} \in \mathrm{Im}\,\mu^{\mathfrak{y}} \cap \mathfrak{p}M = R^*y$ and $z_3\xi_2^{\mathfrak{y}} = z$. Thus we can write $z_2 = sx, z_3 = s'y$, $(s, s' \in R^*)$ and it follows $y = s'^{-1}z_3 = s'^{-1}z(\xi_2^{\mathfrak{y}})^{-1} = s'^{-1}r^{-1}z_2(\xi_1^{\mathfrak{y}})^{-1}(\xi_2^{\mathfrak{y}})^{-1} = z_2(rs'\xi_2^{\mathfrak{y}}\xi_1^{\mathfrak{y}})^{-1} = sx(rs'\xi_2^{\mathfrak{x}}\xi_1^{\mathfrak{x}})^{-1}$ where $s(rs'\xi_2\xi_1)^{-1} \in R^* \times F$. This contradicts the assumptions of the Lemma, hence μ^{ε_1} can not exist and φ must be of the form

$\varphi = r \cdot \xi_1$. **(3)**

From our assumption $z\varphi^{\mathfrak{y}} = z$ and (3) follows $rz\xi_1^{\mathfrak{x}} = rz\xi_1^{\mathfrak{y}} = z$. The basic U-property of $\pi^{\mathfrak{x}}$ implies $\varphi = r \cdot \xi_1 = 1$. Thus $\varphi = 1$, $\pi^{\mathfrak{y}}$ has the basic U-property by Lemma 3.7.

Let us now consider condition (i) of the U-property for $\pi^{\mathfrak{y}}$. **(4)**

Above we established the U-property for $n = 2$ and thus assume $n > 2$.

Now let $f = \sum_{j=1}^{n} r_j \mu_j$ be a polynomial and $\{b_i | 1 \leq i \leq m\}$ be a set of independent elements in M with $b_i f^{\mathfrak{y}} = 0$ for all i.

If μ is used in the representation of f, then $\operatorname{rk} \operatorname{Dom} f^{\mathfrak{y}} \leq \operatorname{rk} \operatorname{Dom} \mu^{\mathfrak{y}} = 1$ and hence $\{b_i | 1 \leq i \leq m\} \subseteq \operatorname{Dom} f^{\mathfrak{y}}$. It follows that $m \leq 1 < n - 1$, as required.

If μ is not used in the representation of f, then $b_i f^{\mathfrak{x}} = b_i f^{\mathfrak{y}} = 0$ for all i. Hence by the U-property of $\pi^{\mathfrak{x}}$ we conclude $m < n - 1$.

Finally, we consider condition (ii) of the U-property for $\pi^{\mathfrak{y}}$.

Let f be a polynomial and $\{b_i | 1 \leq i \leq m\}$ be a set of independent elements in M with $b_i f^{\mathfrak{y}} \in \operatorname{Dom} \varphi^{\mathfrak{y}}$ for all i and some $\varphi \in \mathfrak{F}_{\pm}$. Moreover, assume that there are $1 \leq j_i \leq n$ with $b_i \mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom} \varphi^{\mathfrak{y}}$ for all i.

As above we consider two cases.

If μ is used in the representation of f, then $\operatorname{rk} \operatorname{Dom} f^{\mathfrak{y}} \leq 1$, hence $\{b_i | 1 \leq i \leq m\} \subseteq \operatorname{Dom} f^{\mathfrak{y}}$ and clearly $m \leq 1 < n$.

If μ is not used in the representation of f and $\varphi \notin \{\mu, \mu^{-1}\}$, we can replace $f^{\mathfrak{y}} = f^{\mathfrak{x}}$, $\varphi^{\mathfrak{y}} = \varphi^{\mathfrak{x}}$ and by the U-property of $\pi^{\mathfrak{x}}$ we have $m < n$. Thus assume $\varphi \in \{\mu, \mu^{-1}\}$. In this case $\operatorname{rk} \operatorname{Dom} \mu^{\mathfrak{y}} = 1$. We find an independent set $\{b'_i | 1 \leq i \leq m - 1\} \subseteq \langle b_i | 1 \leq i \leq m \rangle$ with $b'_i f^{\mathfrak{y}} = 0$. From (4) it follows that $m - 1 < n - 1$, hence $m < n$.

It remains to check the remaining conditions of Definitions 3.8 and 3.10, but this is easy, hence $\mathfrak{y} \in \mathfrak{K}$ as stated in Lemma 3.11. $\square$

3.3 The Pushout-Construction

Next we provide arguments, that allow us to pass from partial automorphisms to total automorphisms. We will use pushouts and state their basic properties as

Lemma 3.12 *Let A, B, C be R-modules and $\alpha : C \to A$ and $\beta : C \to B$ homomorphisms. Then there exist homomorphisms $\gamma : A \to D, \delta : B \to D$ and an R-module D satisfying the following conditions:*
(i) $\alpha\gamma = \beta\delta$, i.e. the following diagram of maps **commutes**.

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow[\delta]{} & D \end{array}$$

(ii) Given another commuting diagram,

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ B & \xrightarrow[\delta']{} & D' \end{array}$$

there exists a uniquely determined homomorphism $\phi : D \to D'$ with $\gamma\phi = \gamma'$ and $\delta\phi = \delta'$.

Proof: See [19, Theorem 10.2, p. 52].

From Lemma 3.12 (ii) follows that D in the Lemma is uniquely determined up to isomorphism. The module D (and its maps γ, δ) is the **pushout** of the given diagram of maps.
We will also use the explicit construction of D and its maps, that is $D := (A \times B)/H$, where $H := \{(c\alpha, -c\beta) | c \in C\}$, $\gamma : a \mapsto (a, 0) + H$ and $\delta : b \mapsto (0, b) + H$. See [19, Chapter 10, p. 51 ff] for further details on pushouts.

Remark 3.13 *In the proof of Lemma 3.14 we will use a similar construction:*
For any partial automorphism μ of M we define $M' := (M \times M)/H$, with
$H := \{(x\mu^{\mathfrak{r}}, -x) | x \in \operatorname{Dom} \mu^{\mathfrak{r}}\} \subseteq M \times M$. This is a natural generalization of the

pushout construction replacing the homomorphisms α and β in Lemma 3.12 by partial homomorphisms. This "partial pushout" can be derived from the ordinary pushout as in Lemma 3.12, i.e. the "total pushout" can be rediscovered inside M'. We have the following pushout diagram,

$$\begin{array}{ccc} M \supseteq \operatorname{Dom}\mu & \xrightarrow{\ \mu\ } & \operatorname{Im}\mu \subseteq M \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \pi_0 \\ \operatorname{Dom}\mu & \xrightarrow[\ \pi_1\ ]{} & (\operatorname{Im}\mu \times \operatorname{Dom}\mu)/H \subseteq M' = (M \times M)/H \end{array}$$

where $\pi_0 : M \to (M \times 0 + H)/H,\ x \mapsto (x, 0) + H$ and $\pi_1 : M \to (0 \times M + H)/H$, $x \mapsto (0, x) + H$ are the canonical injections.
However, we will not use this in the proof of Lemma 3.14 to give priority to more fundamental facts.

The following lemma is a central result for proving the main theorem. Its proof is not complicated but needs lengthy bookkeeping.

Lemma 3.14 (The Pushout-Construction)
Let $\mathfrak{x} = (M^{\mathfrak{x}}, \mathfrak{F}^{\mathfrak{x}}, \pi^{\mathfrak{x}}) \in \mathfrak{K}$ $(\mathfrak{K}^*)$ with $\mathfrak{F} = \{\varphi_i | i \in J\}$, and choose $t \in J$. Then the following holds:
(a) There is $\mathfrak{x} \sqsubseteq \mathfrak{y} := (M^{\mathfrak{y}}, \mathfrak{F}^{\mathfrak{y}}, \pi^{\mathfrak{y}})$, where $M^{\mathfrak{y}}$ is a pushout using $M^{\mathfrak{x}}$ and $\mathfrak{F}^{\mathfrak{y}} = \mathfrak{F}^{\mathfrak{x}}$, $\varphi_i^{\mathfrak{y}} = \varphi_i^{\mathfrak{x}}$ for all $i \neq t$. Moreover, $\operatorname{Dom}\varphi_t^{\mathfrak{y}} = M^{\mathfrak{x}}$, $\operatorname{Im}\varphi_t^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\varphi_t^{\mathfrak{x}}$ and $\operatorname{Dom}\varphi_t^{\mathfrak{y}} + \operatorname{Im}\varphi_t^{\mathfrak{y}} = M^{\mathfrak{y}}$. We call $M^{\mathfrak{y}}$ the **(Dom-pushout of φ_t)**.
(b) There is $\mathfrak{x} \sqsubseteq \mathfrak{y} := (M^{\mathfrak{y}}, \mathfrak{F}^{\mathfrak{y}}, \pi^{\mathfrak{y}})$, where $M^{\mathfrak{y}}$ is a pushout using $M^{\mathfrak{x}}$ and $\mathfrak{F}^{\mathfrak{y}} = \mathfrak{F}^{\mathfrak{x}}$, $\varphi_i^{\mathfrak{y}} = \varphi_i^{\mathfrak{x}}$ for all $i \neq t$. Moreover, $\operatorname{Im}\varphi_t^{\mathfrak{y}} = M^{\mathfrak{x}}$, $\operatorname{Dom}\varphi_t^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Dom}\varphi_t^{\mathfrak{x}}$ and $\operatorname{Dom}\varphi_t^{\mathfrak{y}} + \operatorname{Im}\varphi_t^{\mathfrak{y}} = M^{\mathfrak{y}}$. We call $M^{\mathfrak{y}}$ the **(Im-pushout of φ_t)**.

Proof: Let $M := M^{\mathfrak{x}}$, $M' := M^{\mathfrak{y}}$, $\mathfrak{F} := \mathfrak{F}^{\mathfrak{y}} := \mathfrak{F}^{\mathfrak{x}}$ and $\mu := \varphi_t$.
(a): Define the pushout $M' := (M \times M)/H$, with

$$H := \{(x\mu^{\mathfrak{x}}, -x) | x \in \operatorname{Dom}\mu^{\mathfrak{x}}\} \subseteq M \times M.$$

If $U \subseteq M$ then let $U_0 := (U \times 0 + H)/H \subseteq M'$ and $U_1 := (0 \times U + H)/H \subseteq M'$ and define the injections $\pi_0 : M \to M_0$, $\pi_1 : M \to M_1$ by $x\pi_0 = (x, 0) + H$ and

$x\pi_1 = (0, x) + H$ for all $x \in M$. First we claim

$M' = M_0 + M_1$, $D := M_0 \cap M_1 = (\operatorname{Im} \mu^{\mathfrak{x}})_0 = (\operatorname{Dom} \mu^{\mathfrak{x}})_1$ *and* $M \cong M_0 \cong M_1$. **(1)**

Proof: Obviously, $M' = M_0 + M_1$ holds because $(x, y) + H = [(x, 0) + H] + [(0, y) + H]$ for any $(x, y) + H \in M'$.

If $m \in D$, then $m \in M_0 \cap M_1$ and $m = (x, 0) + H = (0, y) + H$, hence $(x, -y) \in H$ and $y\mu^{\mathfrak{x}} = x$. It follows that $y \in \operatorname{Dom} \mu^{\mathfrak{x}}$, $x \in \operatorname{Im} \mu^{\mathfrak{x}}$ and $m = (x, 0) + H \in (\operatorname{Im} \mu^{\mathfrak{x}})_0$, $m = (0, y) + H \in (\operatorname{Dom} \mu^{\mathfrak{x}})_1$, thus $D \subseteq (\operatorname{Im} \mu^{\mathfrak{x}})_0 \wedge D \subseteq (\operatorname{Dom} \mu^{\mathfrak{x}})_1$.

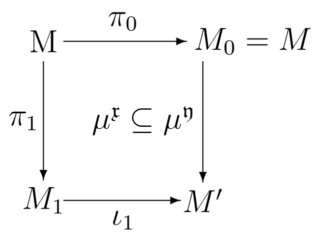
The reverse inclusion "$\supseteq$" can be shown similarly; hence $D = (\operatorname{Im} \mu^{\mathfrak{x}})_0 = (\operatorname{Dom} \mu^{\mathfrak{x}})_1$ holds.

The homomorphism $\pi_0 : M \to M_0 \subseteq M'$ is obviously surjective. From $x_1\pi_0 = x_2\pi_0$ follows $(x_1, 0) + H = (x_2, 0) + H$, hence $(x_1 - x_2, 0) \in H$, and $x_1 - x_2 = 0\mu^{\mathfrak{x}} = 0$ implies $x_1 = x_2$. Thus π_0 is an isomorphism and $M \cong M_0$ follows. Similarly $M \cong M_1$ holds. This proofs (1).

Using π_0 we identify $M = M_0 \subseteq M'$. We set $\varphi_i^{\mathfrak{y}} := \varphi_i^{\mathfrak{x}}$ for $i \neq t$. If $i = t$ we let

$$\varphi_t^{\mathfrak{y}} = \mu^{\mathfrak{y}} : M = M_0 \to M_1, (x, 0) + H \mapsto (0, x) + H,$$

which is an isomorphism. For $x \in \operatorname{Dom} \mu^{\mathfrak{x}}$ we get
$x\mu^{\mathfrak{y}} = [(x, 0) + H]\mu^{\mathfrak{y}} = (0, x) + H = (x\mu^{\mathfrak{x}}, 0) + H = x\mu^{\mathfrak{x}}$. Thus $\mu^{\mathfrak{x}} \subseteq \mu^{\mathfrak{y}}$, and from (1) we conclude $\operatorname{Im} \mu^{\mathfrak{y}} \cap M = M_0 \cap M_1 = \operatorname{Im} \mu^{\mathfrak{x}}$. Let ι_1 be the canonical injection from M_1 into M'. We summarize our results in the following commuting diagram:



Next we claim that

M' and M'/M are free and $\varphi_i^{\mathfrak{y}} \in \operatorname{pAut} M'$ holds for all $i \in J$. **(2)**

Proof: First note that

$$M'/D = ((M \times M)/H)/D \cong ((M \times M)/H)/((\operatorname{Im} \mu^{\mathfrak{x}})_0 + (\operatorname{Dom} \mu^{\mathfrak{x}})_1) =$$

$$(M \times M)/(\operatorname{Im}\mu^{\mathfrak{x}} \times \operatorname{Dom}\mu^{\mathfrak{x}}) \cong (M/\operatorname{Im}\mu^{\mathfrak{x}}) \times (M/\operatorname{Dom}\mu^{\mathfrak{x}})$$

is free, because $\mu^{\mathfrak{x}} \in \operatorname{pAut} M$ implies that $M/\operatorname{Im}\mu^{\mathfrak{x}}$ and $M/\operatorname{Dom}\mu^{\mathfrak{x}}$ are free.

It follows, that $M'/M_0 = (M_0 + M_1)/M_0 \cong M_1/D \subseteq M'/D$ and

$M'/M_1 = (M_0 + M_1)/M_1 \cong M_0/D \subseteq M'/D$ are free and hence $\varphi_t^{\mathfrak{y}} = \mu^{\mathfrak{y}} \in \operatorname{pAut} M'$.

Moreover, $M'/M = M'/M_0$ is free; and since M is free also M' is free.

We have $\varphi_i^{\mathfrak{y}} = \varphi_i^{\mathfrak{x}}$ for $i \neq t$. From $\varphi_i^{\mathfrak{y}} \in \operatorname{pAut} M$ follows that $M/\operatorname{Dom}\varphi_i^{\mathfrak{y}}$ and $M/\operatorname{Im}\varphi_i^{\mathfrak{y}}$ are free, thus using freeness of M'/M also $M'/\operatorname{Dom}\varphi_i^{\mathfrak{y}}$ and $M'/\operatorname{Im}\varphi_i^{\mathfrak{y}}$ are free. Therefore $\varphi_i^{\mathfrak{y}} \in \operatorname{pAut} M'$ for all $i \neq t$. This proves (2).

If $\varphi = r \cdot \xi_1 \mu^{\varepsilon_1} \xi_2 \cdots \mu^{\varepsilon_{k-1}} \xi_k$ with $r \in R^*$ $(0 \neq \varepsilon_j \in \mathbb{Z})$ and all $\xi_j \in \langle \varphi_i | i \in J \wedge i \neq t \rangle$ are monomials in their reduced representation, then for all $z \in M$ the following holds:

If $z \in \operatorname{Dom}\varphi^{\mathfrak{y}}$ and $z\varphi^{\mathfrak{y}} \in M$, then $z \in \operatorname{Dom}\varphi^{\mathfrak{x}}$. **(3)**

Proof: Again we consider the **path** $[z]$ of the element z for $\varphi^{\mathfrak{y}}$, consisting of

$$z_0 := z,\ z_1 := rz,\ z_2 := z_1\xi_1^{\mathfrak{y}},\ z_3 := z_2(\mu^{\mathfrak{y}})^{\varepsilon_1},\ \cdots,\ z_{2k} := z_{2k-1}\xi_k^{\mathfrak{y}}.$$

We want to show that $z \in \operatorname{Dom}\varphi^{\mathfrak{x}}$, replacing in $[z]$ step by step the $\mathfrak{y}$'s by $\mathfrak{x}$'s.

Since $\xi_i \neq 1$ for $1 < i < k$, we have $\xi_i^{\mathfrak{y}} = \xi_i^{\mathfrak{x}}$. In particular, $z_{2i-1} \in \operatorname{Dom}\xi_i^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}$ and $z_{2i} \in \operatorname{Im}\xi_i^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}$ for $1 < i < k$.

Observe that $\xi_i = 1$ may occur for $i \in \{1, k\}$. In this case $\xi_i^{\mathfrak{x}} = \operatorname{id}_M$ and $\xi_i^{\mathfrak{y}} = \operatorname{id}_{M'}$, in particular $\xi_i^{\mathfrak{y}} \neq \xi_i^{\mathfrak{x}}$. Thus we must look onto the case $i \in \{1, k\}$ more carefully.

By assumption of (3) we have $z_0 = z \in M^{\mathfrak{x}}$ hence $z_1 = rz \in M^{\mathfrak{x}}$ and $z_{2k} = z\varphi^{\mathfrak{y}} \in M^{\mathfrak{x}}$.

If $\xi_1 \neq 1$, then $\xi_1^{\mathfrak{y}} = \xi_1^{\mathfrak{x}}$ and $z_2 \in \operatorname{Im}\xi_1^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}$. If $\xi_1 = 1$, then $z_2 = z_1\xi_1^{\mathfrak{y}} = z_1 \in M^{\mathfrak{x}}$ and $\xi_1^{\mathfrak{y}} = 1^{\mathfrak{y}}$ can be replaced $\xi_1^{\mathfrak{x}} = 1^{\mathfrak{x}}$. Similar arguments hold for ξ_k and z_{2k-1}.

It follows that $[z]$ belongs to $M^{\mathfrak{x}}$.

Thus we can replace all $\xi_i^{\mathfrak{y}}$'s by $\xi_i^{\mathfrak{x}}$'s. **(4)**

Next we claim that

$m_1(\mu^{\mathfrak{y}})^{\varepsilon} = m_2$ *for* $m_1, m_2 \in M^{\mathfrak{x}}$, $\varepsilon \in \mathbb{Z}$ *implies* $m_1(\mu^{\mathfrak{x}})^{\varepsilon} = m_2$. **(5)**

We prove (5) by induction.

For $\varepsilon = 0$ the statement is obvious, because $(\mu^{\mathfrak{y}})^0 = 1^{\mathfrak{y}}$ and $(\mu^{\mathfrak{x}})^0 = 1^{\mathfrak{x}}$.

If $\varepsilon < 0$, then we exchange m_1 and m_2. Hence we may assume that $\varepsilon > 0$, hence $\varepsilon \geq 1$. Let $m_1, m_2 \in M^{\mathfrak{x}}$ with $m_1(\mu^{\mathfrak{y}})^{\varepsilon+1} = m_2$. If $m_3 := m_1(\mu^{\mathfrak{y}})^{\varepsilon}$, then $m_3\mu^{\mathfrak{y}} = m_2 \in \operatorname{Im}\mu^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\mu^{\mathfrak{x}}$, hence $m_3 \in \operatorname{Dom}\mu^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}$. Therefore $m_1(\mu^{\mathfrak{x}})^{\varepsilon} = m_3$ by assumption (5) and $m_3\mu^{\mathfrak{x}} = m_2$ holds; thus $m_1(\mu^{\mathfrak{x}})^{\varepsilon+1} = m_2$ and the statement also holds for $\varepsilon + 1$.

By (5) we can replace all maps $(\mu^{\mathfrak{y}})^{\varepsilon_j}$ by $(\mu^{\mathfrak{x}})^{\varepsilon_j}$ without changing the path of z. Together with (4) we conclude $z\varphi^{\mathfrak{x}} = z\varphi^{\mathfrak{y}}$ and therefore $z \in \operatorname{Dom}\varphi^{\mathfrak{x}}$.

We now claim that

$\pi^{\mathfrak{y}}$ ***satisfies condition* (ii) *of the U-property for* $\mathrm{n} > 1$.**

$\mathfrak{y}$ is obtained from $\mathfrak{x}$ by a Dom-pushout of the partial automorphism $\mu^{\mathfrak{x}}$ with $\mu \in \mathfrak{F}^{\mathfrak{x}}$. We know already that $\mathfrak{F}^{\mathfrak{y}} = \mathfrak{F}^{\mathfrak{x}}$, $M^{\mathfrak{x}} \subseteq_* M^{\mathfrak{y}}$, $\operatorname{Dom}\mu^{\mathfrak{y}} = M^{\mathfrak{x}}$ and $\operatorname{Im}\mu^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\mu^{\mathfrak{x}}$. Let $f = \sum_{j=1}^{n} r_j\mu_j$ be a polynomial and $\{b_i | 1 \leq i \leq m\}$ be a set of independent elements in $M^{\mathfrak{y}}$. Let $b_i f^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}}$ for all i and some $\varphi \in \mathfrak{F}_{\pm}$. Moreover suppose that there are $1 \leq j_i \leq n$ with $b_i\mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}}$ for all i. We must show $m < n$.
We will distinguish three cases depending on the relation between $\operatorname{Dom} f^{\mathfrak{y}}$ and $\{b_i | 1 \leq i \leq m\}$.

Case A: Suppose $b_i \in \operatorname{Dom}\mu_j^{\mathfrak{x}}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. This in particular implies $b_i \in M^{\mathfrak{x}}$.
a) If $\varphi \notin \{\mu, \mu^{-1}\}$, then $\varphi^{\mathfrak{y}} = \varphi^{\mathfrak{x}}$ and therefore $b_i \in M^{\mathfrak{x}}$ with $b_i\mu_{j_i}^{\mathfrak{x}} = b_i\mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}} = \operatorname{Dom}\varphi^{\mathfrak{x}}$ and $b_i f^{\mathfrak{x}} = b_i f^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}} = \operatorname{Dom}\varphi^{\mathfrak{x}}$ for all $1 \leq i \leq m$. Now the U-property of $\pi^{\mathfrak{x}}$ yields $m < n$.
b) If $\varphi = \mu$, then $b_i\mu_{j_i}^{\mathfrak{y}} = b_i\mu_{j_i}^{\mathfrak{x}} \in M^{\mathfrak{x}} = \operatorname{Dom}\varphi^{\mathfrak{y}}$ for all $1 \leq i \leq m$. This contradicts our assumption $b_i\mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}}$.
c) If $\varphi = \mu^{-1}$, we have $b_i \in M^{\mathfrak{x}}$ with $b_i\mu_{j_i}^{\mathfrak{x}} = b_i\mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{x}} \subseteq \operatorname{Dom}\varphi^{\mathfrak{y}}$ and $b_i f^{\mathfrak{x}} = b_i f^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\mu^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\mu^{\mathfrak{x}} = \operatorname{Dom}\varphi^{\mathfrak{x}}$ for all $1 \leq i \leq m$. Now the U-property of $\pi^{\mathfrak{x}}$ yields $m < n$.

Case B: Let $b_i \in M^{\mathfrak{x}}$ for all $1 \leq i \leq m$ and $b_{i'} \notin \operatorname{Dom}\mu_{j'}^{\mathfrak{x}}$ for some $1 \leq i' \leq m$,

$1 \leq j' \leq n$.

Then $b_i \notin \operatorname{Dom} \mu^{\mathfrak{x}}_{j'}$ *without loss of generality for all* $1 \leq i \leq m$. **(6)**

Otherwise we can construct from $\{b_i | 1 \leq i \leq m\}$ and linear combination with $b_{i'}$ another independent set $\{b'_i | 1 \leq i \leq m\} \subseteq M^{\mathfrak{x}}$ with $b'_i f^{\mathfrak{y}} \in \operatorname{Dom} \varphi^{\mathfrak{y}}$,
$b'_i \mu^{\mathfrak{y}}_{j_i} \notin \operatorname{Dom} \varphi^{\mathfrak{y}} \subseteq_* M^{\mathfrak{y}}$ and $b'_i \notin \operatorname{Dom} \mu^{\mathfrak{x}}_{j'} \subseteq_* M^{\mathfrak{x}}$ for all i.
(In case that $b_i \in \operatorname{Dom} \mu^{\mathfrak{x}}_{j'}$ try $b'_i := b_i + r_i b_{i'}$ and choose $0 \neq r_i \in R$ such that in particular $b'_i \mu^{\mathfrak{y}}_{j_i} \notin \operatorname{Dom} \varphi^{\mathfrak{y}}$. This is possible, because $b_i \mu^{\mathfrak{y}}_{j_i} \notin \operatorname{Dom} \varphi^{\mathfrak{y}} \subseteq_* M'$.)
Thus we can exchange the two sets and rename the new family as $\{b_i | 1 \leq i \leq m\}$.

Next we put $\mathfrak{J} := \{1 \leq j \leq n | \ \exists 1 \leq i \leq m : b_i \notin \operatorname{Dom} \mu^{\mathfrak{x}}_j\} \neq \emptyset$, because $j' \in \mathfrak{J}$.

Thus for any $j \in \mathfrak{J}$ *there exists an element* $b^j \in \{b_i | 1 \leq i \leq m\} \subseteq M^{\mathfrak{x}}$ *with* $b^j \notin \operatorname{Dom} \mu^{\mathfrak{x}}_j$.
(7)

Let $C(\mathfrak{J}) := \{j \in \mathbb{Z} | 1 \leq j \leq n\} \setminus \mathfrak{J}$, and write $\mu_j = \xi_1 \mu^{\varepsilon_1} \xi_2 \cdots \mu^{\varepsilon_{k-1}} \xi_k$ ($0 \neq \varepsilon_l \in \mathbb{Z}$, $\xi_l \in \langle \varphi_i | i \in J \wedge i \neq t \rangle$) as reduced form of μ_j for any $1 \leq j \leq n$. Here ξ_i and ε_i are dependent on j, but in the sequel we will omit this index without causing further misunderstanding. First we note:

If $j \in \mathfrak{J}$ *then* $b^j \mu^{\mathfrak{y}}_j \notin M^{\mathfrak{x}}$. **(8)**

Proof: If $b^j \mu^{\mathfrak{y}}_j \in M^{\mathfrak{x}}$, then $b^j, b^j \mu^{\mathfrak{y}}_j \in M^{\mathfrak{x}}$; together with (3) follows $b^j \in \operatorname{Dom} \mu^{\mathfrak{x}}_j$ contradicting (7). Next we show:

For $j \in \mathfrak{J}$ *follows* $\xi_k = 1$ *and* $\varepsilon_{k-1} > 0$ *in the reduced form* $\xi_1 \mu^{\varepsilon_1} \xi_2 \cdots \mu^{\varepsilon_{k-1}} \xi_k$ *of* μ_j. **(9)**

Proof: Assume that $\xi_k \neq 1$, then $\xi^{\mathfrak{y}}_k = \xi^{\mathfrak{x}}_k$ implies $b^j \mu^{\mathfrak{y}}_j \in \operatorname{Im} \xi^{\mathfrak{y}}_k = \operatorname{Im} \xi^{\mathfrak{x}}_k \subseteq M^{\mathfrak{x}}$ contradicting (8). Moreover, if $\varepsilon_{k-1} < 0$, then $b^j \mu^{\mathfrak{y}}_j \in \operatorname{Im} (\mu^{\mathfrak{y}})^{-1} = \operatorname{Dom} \mu^{\mathfrak{y}} = M^{\mathfrak{x}}$ contradicts (8).
Finally, if $\varepsilon_{k-1} = 0$, then μ does not appear in the representation of μ_j, hence the representation of μ_j reduces to $\mu_j = \xi_1$. From $\xi_1 \neq 1$ follows $\xi^{\mathfrak{y}}_1 = \xi^{\mathfrak{x}}_1$, hence $b^j \mu^{\mathfrak{y}}_j \in \operatorname{Im} \xi^{\mathfrak{y}}_1 = \operatorname{Im} \xi^{\mathfrak{x}}_1 \subseteq M^{\mathfrak{x}}$ contradicts (8). If $\xi_1 = 1$, then $b^j \mu^{\mathfrak{y}}_j = b^j \in M^{\mathfrak{x}}$ also contradicts (8), and thus (9) follows.
Next we must distinguish cases depending on μ and φ.

Subcase B.1: $\varphi \neq \mu^{-1}$.

Then $\operatorname{Dom}\varphi^{\mathfrak{y}} \subseteq M^{\mathfrak{x}}$, **(10)**

since $\operatorname{Dom}\varphi^{\mathfrak{y}} = \operatorname{Dom}\varphi^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}$ for $\varphi \notin \{\mu, \mu^{-1}\}$ and $\operatorname{Dom}\varphi^{\mathfrak{y}} = M^{\mathfrak{x}}$ for $\varphi = \mu$.

We write $\mu'_j := \xi_1\mu^{\varepsilon_1}\xi_2\cdots\mu^{\varepsilon_{k-1}-1}$ *for all* $j \in \mathfrak{J}$, *hence* $\mu_j = \mu'_j\mu$ *is reduced by* (9). **(11)**
This gives rise to a new polynomial $f' := \sum_{j\in\mathfrak{J}} r_j\mu'_j \in R\langle\mathfrak{F}\rangle$. From $\mathfrak{J} \neq \emptyset$ follows $f' \neq 0$. We now claim:

$b_i \in \operatorname{Dom}\mu'^{\mathfrak{x}}_j$ $(j \in \mathfrak{J})$ *and* $b_i\mu'^{\mathfrak{x}}_{j'} \notin \operatorname{Dom}\mu^{\mathfrak{x}}$ (*see also* (6)) *for all* $1 \leq i \leq m$. **(12)**
Proof: If $1 \leq i \leq m$ and $j \in \mathfrak{J}$, then (11) shows $b_i \in \operatorname{Dom}\mu_j^{\mathfrak{y}} = \operatorname{Dom}\mu'^{\mathfrak{y}}_j\mu^{\mathfrak{y}}$, hence $b_i\mu'^{\mathfrak{y}}_j \in \operatorname{Dom}\mu^{\mathfrak{y}} = M^{\mathfrak{x}}$. Since $b_i, b_i\mu'^{\mathfrak{y}}_j \in M^{\mathfrak{x}}$ we obtain $b_i \in \operatorname{Dom}\mu'^{\mathfrak{x}}_j$ using (3). This is the first claim in (12).
If $b_i\mu'^{\mathfrak{x}}_{j'} \in \operatorname{Dom}\mu^{\mathfrak{x}}$, then $b_i \in \operatorname{Dom}\mu'^{\mathfrak{x}}_{j'}\mu^{\mathfrak{x}} = \operatorname{Dom}\mu^{\mathfrak{x}}_{j'}$ contradicting (6). This proves the second claim in (12). Next we show:

$b_i f'^{\mathfrak{x}} \in \operatorname{Dom}\mu^{\mathfrak{x}}$ *for all* $1 \leq i \leq m$. **(13)**
Proof: From (10), the definition of $\mathfrak{J}$ and our assumption follows

$$\sum_{j\in C(\mathfrak{J})} r_j b_i\mu_j^{\mathfrak{x}} + \sum_{j\in\mathfrak{J}} r_j b_i\mu_j^{\mathfrak{y}} = \sum_{j\in C(\mathfrak{J})} r_j b_i\mu_j^{\mathfrak{y}} + \sum_{j\in\mathfrak{J}} r_j b_i\mu_j^{\mathfrak{y}} = b_i f^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}} \subseteq M^{\mathfrak{x}}.$$

Moreover $\sum_{j\in C(\mathfrak{J})} r_j b_i\mu_j^{\mathfrak{y}} \in M^{\mathfrak{x}}$, thus $b_i f'^{\mathfrak{x}}\mu^{\mathfrak{y}} = \sum_{j\in\mathfrak{J}} r_j b_i\mu_j^{\mathfrak{y}} \in \operatorname{Im}\mu^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\mu^{\mathfrak{x}}$, hence $b_i f'^{\mathfrak{x}} \in \operatorname{Dom}\mu^{\mathfrak{x}}$.

We summarize our last claims:
$\{b_i | 1 \leq i \leq m\} \subseteq M^{\mathfrak{x}}$ is a set of independent elements with $b_i f'^{\mathfrak{x}} \in \operatorname{Dom}\varphi'^{\mathfrak{x}}$ for all i and some $\varphi' \in \mathfrak{F}_{\pm}$, and there are $j'_i \in \mathfrak{J}$ with $b_i\mu'^{\mathfrak{x}}_{j'_i} \notin \operatorname{Dom}\varphi'^{\mathfrak{x}}$ for all i, if we set $\varphi' := \mu$ and $j'_i := j'$. In particular $f' = \sum_{j\in\mathfrak{J}} r_j\mu'_j$ is a reduced polynomial of rank $|\mathfrak{J}| > 1$.
($|\mathfrak{J}| > 1$ follows from $b_i f'^{\mathfrak{x}} \in \operatorname{Dom}\varphi'^{\mathfrak{x}}$ and $b_i\mu'^{\mathfrak{x}}_{j'_i} \notin \operatorname{Dom}\varphi'^{\mathfrak{x}}$.)
The U-property of $\pi^{\mathfrak{x}}$ now yields $m < |\mathfrak{J}| \leq n$.

Subcase B.2: $\varphi = \mu^{-1}$.

This time we define $f'' := \sum_{j\in C(\mathfrak{J})} r_j\mu_j \in R\langle\mathfrak{F}\rangle$ and claim:

$b_i \in \operatorname{Dom}\mu_j^{\mathfrak{x}}$ *and* $b_i\mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}}$ *with* $j_i \in C(\mathfrak{J})$ *for any* $1 \leq i \leq m$, $j \in C(\mathfrak{J})$. **(14)**
Proof: Suppose $b_i \notin \operatorname{Dom}\mu_j^{\mathfrak{x}}$. Then the definition of $\mathfrak{J}$ yields $j \in \mathfrak{J}$ contradicting $j \in C(\mathfrak{J})$. This is the first claim of (14).

The second claim $b_i\mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}}$ follows by the assumptions.

If $j_i \in \mathfrak{J}$, then $b_i\mu_{j_i}^{\mathfrak{y}} \in \operatorname{Im}\mu^{\mathfrak{y}} = \operatorname{Dom}\varphi^{\mathfrak{y}}$ by (9), contradicting $b_i\mu_{j_i}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}}$. Hence (14) follows. Now we show:

$b_i f''^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}}$ *for all* $1 \le i \le m.$ **(15)**

Proof: We have

$$\sum_{j\in C(\mathfrak{J})} r_j b_i \mu_j^{\mathfrak{y}} + \sum_{j\in\mathfrak{J}} r_j b_i \mu_j^{\mathfrak{y}} = b_i f^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}}.$$

By (9) follows $\sum_{j\in\mathfrak{J}} r_j b_i \mu_j^{\mathfrak{y}} \in \operatorname{Im}\mu^{\mathfrak{y}} = \operatorname{Dom}\varphi^{\mathfrak{y}}$, thus also $b_i f''^{\mathfrak{y}} = \sum_{j\in C(\mathfrak{J})} r_j b_i \mu_j^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}}$.

Again we summarize the last two claims:

$\{b_i | 1 \le i \le m\} \subseteq M^{\mathfrak{x}}$ is a set of independent elements with $b_i f''^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}}$ for all i and some $\varphi \in \mathfrak{F}_\pm$ and there are $j_i \in C(\mathfrak{J})$ with $b_i\mu_{j_i}{}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}}$ for all i. In particular $f'' = \sum_{j\in C(\mathfrak{J})} r_j\mu_j$ is a reduced polynomial of rank $|C(\mathfrak{J})| > 1$.

($|C(\mathfrak{J})| > 1$ follows from $b_i f''^{\mathfrak{y}} \in \operatorname{Dom}\varphi^{\mathfrak{y}}$ and $b_i\mu_{j_i}{}^{\mathfrak{y}} \notin \operatorname{Dom}\varphi^{\mathfrak{y}}$.)

By case A now follows $m < |C(\mathfrak{J})| \le n$.

Case C: Let $b_{i'} \notin M^{\mathfrak{x}}$ for some $1 \le i' \le m$.

If $\mu_j = \xi_1\mu^{\varepsilon_1}\xi_2\cdots\mu^{\varepsilon_{k-1}}\xi_k$ denotes again the reduced representation of μ_j, then we claim:

$\xi_1 = 1$ *and* $\varepsilon_1 \le 0$ *in the reduced form of* μ_j $(1 \le j \le n)$. **(16)**

Proof: If $\xi_1 \ne 1$, then $\xi_1^{\mathfrak{y}} = \xi_1^{\mathfrak{x}}$, hence $b_{i'} \in \operatorname{Dom}\xi_1^{\mathfrak{y}} = \operatorname{Dom}\xi_1^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}$ contradicting $b_{i'} \notin M^{\mathfrak{x}}$. Thus $\xi_1 = 1$.

If $\varepsilon_1 > 0$, then $b_{i'} \in \operatorname{Dom}\mu^{\mathfrak{y}} = M^{\mathfrak{x}}$ contradicting $b_{i'} \notin M^{\mathfrak{x}}$, thus (16) follows.

Next we show:

$b_i \in \operatorname{Dom}(\mu^{\mathfrak{y}})^{-1}$ *for* $1 \le i \le m.$ **(17)**

Proof: By (16) either $\mu_j = 1$ or $\mu_j = \mu^{\varepsilon_1}\xi_2\cdots\mu^{\varepsilon_{k-1}}\xi_k$ with $\varepsilon_1 < 0$. The polynomial f has at least 2 summands, thus $\mu_j = \mu^{\varepsilon_1}\xi_2\cdots\mu^{\varepsilon_{k-1}}\xi_k$ and $\varepsilon_1 < 0$ for at least one j follows. Therefore (17) holds.

Now $b_i' := b_i(\mu^{\mathfrak{y}})^{-1} \in \operatorname{Dom}\mu^{\mathfrak{y}} = M^{\mathfrak{x}}$ for any $1 \le i \le m$ is well defined by (17) and together with f also $f' := \mu f = \sum_{j=1}^n r_j\mu\mu_j = \sum_{j=1}^n r_j\mu_j' \in R\langle\mathfrak{F}\rangle$ with $\mu_j' := \mu\mu_j$ is a polynomial in reduced form.

Thus $f' = \sum_{j=1}^{n} r_j \mu'_j \in R\langle\mathfrak{F}\rangle$, $\varphi \in \mathfrak{F}_\pm$ and a set $\{b'_i | 1 \leq i \leq m\} \subseteq M^{\mathfrak{x}}$ of independent elements with $b'_i f'^{\mathfrak{y}} \in \operatorname{Dom} \varphi^{\mathfrak{y}}$ for all i exist. Moreover, there are $1 \leq j_i \leq n$ with $b'_i \mu'^{\mathfrak{y}}_{j_i} \notin \operatorname{Dom} \varphi^{\mathfrak{y}}$ for all i.

Now we apply cases A and B and $m < n$ follows in this final case.

Thus $\pi^{\mathfrak{y}}$ satisfies condition (ii) of the U-property for $n > 1$.

$\pi^{\mathfrak{y}}$ *satisfies condition* (i) *of the U-property for* $\mathbf{n} > 1$.

Let $f = \sum_{j=1}^{n} r_j \mu_j$ be a polynomial and $\{b_i | 1 \leq i \leq m\}$ be a set of independent elements in $M^{\mathfrak{y}}$, such that $b_i f^{\mathfrak{y}} = 0$ for all i. We must show $m < n - 1$.

The proof of condition (i) is similar to the proof of (ii). Thus we only point out the significant changes.

Case A: Let $b_i \in \operatorname{Dom} \mu_j^{\mathfrak{x}}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. In particular $b_i \in M^{\mathfrak{x}}$.

It follows that $b_i \in M^{\mathfrak{x}}$ with $b_i f^{\mathfrak{x}} = b_i f^{\mathfrak{y}} = 0$ for all $1 \leq i \leq m$. The U-property for $\pi^{\mathfrak{x}}$ yields $m < n - 1$.

Case B: Let $b_i \in M^{\mathfrak{x}}$ for all $1 \leq i \leq m$, but there are $1 \leq i' \leq m$, $1 \leq j' \leq n$ with $b_{i'} \notin \operatorname{Dom} \mu_{j'}^{\mathfrak{x}}$.

Then $b_i \notin \operatorname{Dom} \mu_{j'}^{\mathfrak{x}}$ without loss of generality for all $1 \leq i \leq m$.

Set $\mathfrak{J} := \{1 \leq j \leq n | \; \exists 1 \leq i \leq m : b_i \notin \operatorname{Dom} \mu_j^{\mathfrak{x}}\}$, where $j' \in \mathfrak{J} \neq \emptyset$. For any $j \in \mathfrak{J}$ there is some $b^j \in \{b_i | 1 \leq i \leq m\} \subseteq M^{\mathfrak{x}}$ with $b^j \notin \operatorname{Dom} \mu_j^{\mathfrak{x}}$.

Write $\mu_j = \xi_1 \mu^{\varepsilon_1} \xi_2 \cdots \mu^{\varepsilon_{k-1}} \xi_k$ $(0 \neq \varepsilon_l \in \mathbb{Z}$, $\xi_l \in \langle \varphi_i | i \in J \wedge i \neq t\rangle)$ in reduced form $(1 \leq j \leq n)$.

Following the proof of (ii) we can show the next claims.

If $j \in \mathfrak{J}$, then $b^j \mu_j^{\mathfrak{y}} \notin M^{\mathfrak{x}}$. (see (8))

If $j \in \mathfrak{J}$, then $\xi_k = 1$ and $\varepsilon_{k-1} > 0$ in the reduced form of μ_j. (see (9))

Set $\mu'_j := \xi_1 \mu^{\varepsilon_1} \xi_2 \cdots \mu^{\varepsilon_{k-1} - 1}$ and $f' := \sum_{j \in \mathfrak{J}} r_j \mu'_j \in R\langle\mathfrak{F}\rangle$. (see (11))

$b_i \in \operatorname{Dom} \mu'^{\mathfrak{x}}_j$ $(j \in \mathfrak{J})$ and $b_i \mu'^{\mathfrak{x}}_{j'} \notin \operatorname{Dom} \mu^{\mathfrak{x}}$ (see also (6)) for all $1 \leq i \leq m$. (see (12))

$b_i f'^{\mathfrak{x}} \in \operatorname{Dom} \mu^{\mathfrak{x}}$ *for all* $1 \leq i \leq m$. *Moreover,* $b_i f'^{\mathfrak{x}} = 0$ *(*$1 \leq i \leq m$*) if* $C(\mathfrak{J}) = \emptyset$*,* which requires a proof. By definition of $\mathfrak{J}$ and our preliminaries we have

$$\sum_{j\in C(\mathfrak{J})} r_j b_i \mu_j^{\mathfrak{x}} + \sum_{j\in\mathfrak{J}} r_j b_i \mu_j^{\mathfrak{y}} = \sum_{j\in C(\mathfrak{J})} r_j b_i \mu_j^{\mathfrak{y}} + \sum_{j\in\mathfrak{J}} r_j b_i \mu_j^{\mathfrak{y}} = b_i f^{\mathfrak{y}} = 0.$$

Also from $\sum_{j\in C(\mathfrak{J})} r_j b_i \mu_j^{\mathfrak{y}} \in M^{\mathfrak{x}}$ follows $b_i f'^{\mathfrak{x}} \mu^{\mathfrak{y}} = \sum_{j\in\mathfrak{J}} r_j b_i \mu_j^{\mathfrak{y}} \in \operatorname{Im}\mu^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\mu^{\mathfrak{x}}$, hence $b_i f'^{\mathfrak{x}} \in \operatorname{Dom}\mu^{\mathfrak{x}}$.

If $C(\mathfrak{J}) = \emptyset$, then $b_i f'^{\mathfrak{x}} \mu^{\mathfrak{y}} = \sum_{j\in\mathfrak{J}} r_j b_i \mu_j^{\mathfrak{y}} = b_i f^{\mathfrak{y}} = 0$, hence $b_i f'^{\mathfrak{x}} = 0$ for all i.

Subcase B.1: If $C(\mathfrak{J}) \neq \emptyset$, then $|\mathfrak{J}| \leq n-1$.

We summarize some of the claims: $\{b_i | 1 \leq i \leq m\} \subseteq M^{\mathfrak{x}}$ is a set of independent elements with $b_i f'^{\mathfrak{x}} \in \operatorname{Dom}\varphi^{\mathfrak{x}}$ for all i and some fixed $\varphi \in \mathfrak{F}_{\pm}$, and there exist $j_i' \in \mathfrak{J}$ with $b_i {\mu'_{j_i'}}^{\mathfrak{x}} \notin \operatorname{Dom}\varphi^{\mathfrak{x}}$ for all i. Set $\varphi := \mu$ and $j_i' := j'$. Thus $f' = \sum_{j\in\mathfrak{J}} r_j \mu_j'$ is a reduced polynomial of rank $|\mathfrak{J}| > 1$. The U-property of $\pi^{\mathfrak{x}}$ now yields $m < |\mathfrak{J}| \leq n-1$.

Subcase B.2: If $C(\mathfrak{J}) = \emptyset$, then $|\mathfrak{J}| = n$.

Again we can summarize some of the claims: $\{b_i | 1 \leq i \leq m\} \subseteq M^{\mathfrak{x}}$ is a set of independent elements with $b_i f'^{\mathfrak{x}} = 0$ for all i, and $f' = \sum_{j\in\mathfrak{J}} r_j \mu_j'$ is a reduced polynomial of rank $|\mathfrak{J}| > 1$. The U-property of $\pi^{\mathfrak{x}}$ now yields $m < |\mathfrak{J}| - 1 = n-1$.

Case C: Let $b_{i'} \notin M^{\mathfrak{x}}$ for some $1 \leq i' \leq m$.

By case A and B in the proof of (ii) we already showed $m < n-1$.

Thus $m < n-1$ in all possible cases and all other conditions of the Definitions 3.8 and 3.10 are obvious, hence the free and $\aleph_1$-free case of (a) are simultaneously shown.

(b): This is an immediate consequence of (a).

To see this we define $\mathfrak{x}' := (M^{\mathfrak{x}}, \mathfrak{F}^{\mathfrak{x}}, \pi^{\mathfrak{x}'}) \in \mathfrak{K}$, where $\pi^{\mathfrak{x}'} := [\pi^{\mathfrak{x}} \cup (\varphi_t, (\varphi_t^{\mathfrak{x}})^{-1})] \setminus (\varphi_t, \varphi_t^{\mathfrak{x}})$. Thus $\varphi_t^{\mathfrak{x}'} = (\varphi_t^{\mathfrak{x}})^{-1}$ and by (a) there is $\mathfrak{x}' \sqsubseteq \mathfrak{y}' := (M^{\mathfrak{y}'}, \mathfrak{F}^{\mathfrak{y}'}, \pi^{\mathfrak{y}'})$, where $M^{\mathfrak{y}'}$ is a pushout of $M^{\mathfrak{x}}$, with $\mathfrak{F}^{\mathfrak{y}'} = \mathfrak{F}^{\mathfrak{x}}$, $\varphi_i^{\mathfrak{y}'} = \varphi_i^{\mathfrak{x}}$ for $i \neq t$, $\operatorname{Dom}\varphi_t^{\mathfrak{y}'} = M^{\mathfrak{x}}$, $\operatorname{Im}\varphi_t^{\mathfrak{y}'} \cap M^{\mathfrak{x}} = \operatorname{Im}(\varphi_t^{\mathfrak{x}})^{-1}$ and $\operatorname{Dom}\varphi_t^{\mathfrak{y}'} + \operatorname{Im}\varphi_t^{\mathfrak{y}'} = M^{\mathfrak{y}'}$.

Now let $\mathfrak{y} := (M^{\mathfrak{y}'}, \mathfrak{F}^{\mathfrak{y}'}, \pi^{\mathfrak{y}}) \in \mathfrak{K}$ with $\pi^{\mathfrak{y}} := [\pi^{\mathfrak{y}'} \cup (\varphi_t, (\varphi_t^{\mathfrak{y}'})^{-1})] \setminus (\varphi_t, \varphi_t^{\mathfrak{y}'})$. Hence $\mathfrak{x} \sqsubseteq \mathfrak{y}$ with $M^{\mathfrak{y}'}$ from above and $\mathfrak{F}^{\mathfrak{y}'} = \mathfrak{F}^{\mathfrak{x}}$, $\varphi_i^{\mathfrak{y}} = \varphi_i^{\mathfrak{x}}$ for $i \neq t$, $\operatorname{Im}\varphi_t^{\mathfrak{y}} = \operatorname{Dom}\varphi_t^{\mathfrak{y}'} = M^{\mathfrak{x}}$, $\operatorname{Dom}\varphi_t^{\mathfrak{y}} \cap M^{\mathfrak{x}} = \operatorname{Im}\varphi_t^{\mathfrak{y}'} \cap M^{\mathfrak{x}} = \operatorname{Im}(\varphi_t^{\mathfrak{x}})^{-1} = \operatorname{Dom}\varphi_t^{\mathfrak{x}}$ and $\operatorname{Dom}\varphi_t^{\mathfrak{y}} + \operatorname{Im}\varphi_t^{\mathfrak{y}} = M^{\mathfrak{y}'}$. □

We would like to note that extensions $(M', \mathfrak{F}', \pi')$ of tripels $(M, \mathfrak{F}, \pi) \in \mathfrak{K}$ $(\mathfrak{K}^*)$ based on Dom- respectively Im-pushout do not change the cardinality of the first component: $|M| = |M'|$. Moreover, for Dom-pushouts with $\operatorname{Dom} \varphi_t \neq M$ (Im-pushouts with $\operatorname{Im} \varphi_t \neq M$) we obtain $M \neq M'$ and $|M' \setminus M| = |M|$.

3.4 The Free UT-Construction

We want to apply Lemma 3.11 and Lemma 3.14 inductively, which will lead to triples $\mathfrak{x} \in \mathfrak{K}$ $(\mathfrak{K}^*)$. We begin with some elementary facts.

Let α be a limit ordinal and $\mathfrak{x}^j = (M^j, \mathfrak{F}^j, \pi^j) \in \mathfrak{K}$ $(j \in \alpha)$ be a family of triples. We also let $\mathfrak{F}^j = \{\varphi_i | i \in J^j\}$ for $j \in \alpha$.
This family $(\mathfrak{x}^j)_{j\in\alpha}$ is a **chain** in $(\mathfrak{K}, \subseteq)$ (respectively in $(\mathfrak{K}, \sqsubseteq)$), if $\mathfrak{x}^\beta \subseteq \mathfrak{x}^\gamma$ (respectively $\mathfrak{x}^\beta \sqsubseteq \mathfrak{x}^\gamma$) holds for all $\beta \leq \gamma < \alpha$, and the chain is **continuous** if in addition $\mathfrak{x}^\beta = \bigcup_{j\in\beta} \mathfrak{x}^j$ for all limit ordinals $\beta \in \alpha$. The **supremum** $\bigcup_{j\in\beta} \mathfrak{x}^j = (M, \mathfrak{F}, \pi)$ is defined componentwise as $M := \bigcup_{j\in\beta} M^j$, $\mathfrak{F} := \bigcup_{j\in\beta} \mathfrak{F}^j$ with indexing set $J := \bigcup_{j\in\beta} J^j$ and $\pi(\varphi_t) := \bigcup_{i(t)<j<\beta} \pi^j(\varphi_t)$ for all $t \in J$, where we have to chose an $i(t) \in \beta$ with $t \in J^{i(t)}$. Note that $\pi(\varphi_t)$ is well-defined since π^j extends $\pi^{j'}$ if $j' < j$.
The definition of chains in $\mathfrak{K}^*$ is obvious.

Lemma 3.15 (Taking Suprema)
(a) Any continuous chain $(\mathfrak{x}^j)_{j\in\alpha} = (M^j, \mathfrak{F}^j, \pi^j)_{j\in\alpha}$ *in* $(\mathfrak{K}, \sqsubseteq)$ $(\alpha \in \mathrm{LORD})$, *which is obtained by iterated addition of baby-automorphisms and application of pushout-constructions, has a supremum* $\mathfrak{x} := \bigcup_{j\in\alpha} \mathfrak{x}^j \in \mathfrak{K}$, *and* $\mathfrak{x}^j \sqsubseteq \mathfrak{x}$ *holds for all* $j \in \alpha$.
(*The same holds, when replacing* $\mathfrak{K}$ *by* $\mathfrak{K}^*$.)
(b) Any continuous chain $(\mathfrak{x}^j)_{j\in\alpha} = (M^j, \mathfrak{F}^j, \pi^j)_{j\in\alpha}$ *in* $(\mathfrak{K}, \sqsubseteq)$ $(\alpha \in \mathrm{LORD})$ *with* $\pi^j(\mathfrak{F}^j) \subseteq \operatorname{Aut} M^j$ *for all* $j \in \alpha$ *satisfies* $\mathfrak{x} := \bigcup_{j\in\alpha} \mathfrak{x}^j \in \mathfrak{K}$ *and* $\mathfrak{x}^j \sqsubseteq \mathfrak{x}$ *for all* $j \in \alpha$.
(*The same holds, when replacing* $\mathfrak{K}$ *by* $\mathfrak{K}^*$.)

Proof:
(a): Define $\mathfrak{x} = \bigcup_{j\in\alpha} \mathfrak{x}^j = (M, \mathfrak{F}, \pi)$ and set $\varphi^j := \pi^j(\varphi)$ for all $\varphi \in R^* \times \langle \mathfrak{F}^j \rangle$, $j \in \alpha$.
First we claim:

M and M/M^β are free for all $\beta < \alpha$. **(1)**

Proof: If $j \in \alpha$ then $\mathfrak{x}^j \in \mathfrak{K}$, $\mathfrak{x}^j \sqsubseteq \mathfrak{x}^{j+1}$ and $M^j \sqsubseteq M^{j+1}$ are free. Thus $M^{j+1} = M^j \oplus D^j$ decomposes and $M = \bigcup_{j\in\alpha} M^j = M^0 \oplus \bigoplus_{j\in\alpha} D^j$, $M/M^\beta = \bigoplus_{\beta\leq j<\alpha} D^j$ are free.

The $\aleph_1$-free case is also immediate (see Corollary 2.11):

M and M/M^β are $\aleph_1$-free for all $\beta < \alpha$. **(1)***

Next we show:

$\varphi_t^{\mathfrak{x}} \in \mathrm{pAut}\, M$ *($\mathrm{pAut}^* M$) for all $t \in J$.* **(2)**

Proof: The supremum $\varphi_t^{\mathfrak{x}} := \bigcup_{i(t)<j<\alpha} \varphi_t^j$ is defined as the union of graphs φ_t^j.

Case 1: If $\varphi_t^{\mathfrak{x}} = \varphi_t^\beta$ for some $\beta < \alpha$, then M/M^β is free by (1) and $M^\beta/\mathrm{Dom}\,\varphi_t^\beta$ is free because $\varphi_t^\beta \in \mathrm{pAut}\,(M^\beta)$.

Thus $M/\mathrm{Dom}\,\varphi_t^{\mathfrak{x}} = M/\mathrm{Dom}\,\varphi_t^\beta$ is free.

Case 2: The chain $(\varphi_t^j)_{i(t)<j<\alpha}$ is strictly increasing.

Passing to a cofinal subchain (if necessary), we may assume that $\varphi_t^j \neq \varphi_t^{j+1}$ for all $j \in \alpha$. This chain is obtained by pushout-constructions of φ_t only.

Using Lemma 3.14 recursively $N^j := M^j \cap \mathrm{Dom}\,\varphi_t^{\mathfrak{x}} \in \{M^j, \mathrm{Dom}\,\varphi_t^j\}$ is free. **(3)**

Note that $N^j = \mathrm{Dom}\,\varphi_t^j \neq M^j$ holds only, if all subsequent pushouts after $\mathfrak{x}^j$ are Im-pushouts of φ_t. With (3) follows, that M^j/N^j is free for all $j \in \alpha$.

If $\mathfrak{x}^{j+1}$ is constructed from $\mathfrak{x}^j$ by Dom-pushout of φ_t, then $M^j + \mathrm{Dom}\,\varphi_t^{j+1} = M^j \sqsubseteq M^{j+1}$ follows. If $\mathfrak{x}^{j+1}$ is constructed from $\mathfrak{x}^j$ by an Im-pushout of φ_t, then $M^j + \mathrm{Dom}\,\varphi_t^{j+1} = \mathrm{Im}\,\varphi_t^{j+1} + \mathrm{Dom}\,\varphi_t^{j+1} = M^{j+1}$ by Lemma 3.14. Using (3) we derive:

$M^{j+1}/(M^j + N^{j+1}) \in \{M^{j+1}/M^{j+1}, M^{j+1}/(M^j + \mathrm{Dom}\,\varphi_t^{j+1})\}$ *is free.* **(4)**

By freeness of M^{j+1}/M^j and $M^j/\mathrm{Dom}\,\varphi_t^j$ it follows, that $\mathrm{Dom}\,\varphi_t^{j+1}/\mathrm{Dom}\,\varphi_t^j \subseteq M^{j+1}/\mathrm{Dom}\,\varphi_t^j$ and $\mathrm{Dom}\,\varphi_t^{j+1}/M^j \subseteq M^{j+1}/M^j$ are free. Using (3) we derive:

$N^{j+1}/N^j \in \{M^{j+1}/M^j, \mathrm{Dom}\,\varphi_t^{j+1}/M^j, M^{j+1}/\mathrm{Dom}\,\varphi_t^j, \mathrm{Dom}\,\varphi_t^{j+1}/\mathrm{Dom}\,\varphi_t^j\}$ *is free.* **(5)**

Thus the chain $(N^j)_{i(t)<j<\alpha}$ with $N^j = M^j \cap \mathrm{Dom}\,\varphi_t^{\mathfrak{x}}$ satisfies $N^j \sqsubseteq M^j$, $M^j \cap N^{j+1} = N^j$, $M^j + N^{j+1} \sqsubseteq M^{j+1}$ and $N^j \sqsubseteq N^{j+1}$ are free. We define recursively a basis

$B := \bigcup_{i(t)<j<\alpha} B^j$ of the free R-module M such that $\langle B^j \rangle = M^j$ and $\langle N^j \cap B^j \rangle = N^j$. In particular $N = \langle N \cap B \rangle$ for $N := \bigcup_{i(t)<j<\alpha} N^j = \operatorname{Dom} \varphi_t^{\mathfrak{x}}$, and $M/\operatorname{Dom} \varphi_t^{\mathfrak{x}} \cong \langle (M \setminus \operatorname{Dom} \varphi_t^{\mathfrak{x}}) \cap B \rangle$ is free.
From the freeness of $M/\operatorname{Dom} \varphi_t^{\mathfrak{x}}$ follows, that $M/\operatorname{Im} \varphi_t^{\mathfrak{x}}$ is free, substituting φ_t by φ_t^{-1}.

Next we consider the case $\mathfrak{K}^*$ concerning $\aleph_1$-freeness. The proof is similar to the free case $\mathfrak{K}$: For the chain $(N^j)_{i(t)<j<\alpha}$ with $N^j = M^j \cap \operatorname{Dom} \varphi_t^{\mathfrak{x}}$ the factors M^j/N^j, $M^{j+1}/(M^j + N^{j+1})$, N^{j+1}/N^j are $\aleph_1$-free and $M^j \cap N^{j+1} = N^j$ holds.
If $j, k < \alpha$, then $(M^{j+1} + N^k)/(M^j + N^k) = ((M^j + N^k) + M^{j+1})/(M^j + N^k)$
$\cong M^{j+1}/((M^j + N^k) \cap M^{j+1}) = M^{j+1}/(M^j + N^k \cap M^{j+1})$ by the modular law.
Now the definition $N^k = M^k \cap \operatorname{Dom} \varphi_t^{\mathfrak{x}}$ applies and
$(M^{j+1} + N^k)/(M^j + N^k) \cong M^{j+1}/(M^j + N^k \cap M^{j+1}) = M^{j+1}/(M^j + N^{j+1})$ is $\aleph_1$-free for $j < k$. Thus $(M^j + N^k) \subseteq_* (M^{j+1} + N^k)$ for all $j < k \leq \alpha$; in particular $M^j + \operatorname{Dom} \varphi_t^{\mathfrak{x}} = \bigcup_{k<\alpha} (M^j + N^k) \subseteq_* \bigcup_{k<\alpha} (M^{j+1} + N^k) = M^{j+1} + \operatorname{Dom} \varphi_t^{\mathfrak{x}}$ for these two continuous chains.
Now purity of $(M^j + \operatorname{Dom} \varphi_t^{\mathfrak{x}})/\operatorname{Dom} \varphi_t^{\mathfrak{x}} \subseteq_* (M^{j+1} + \operatorname{Dom} \varphi_t^{\mathfrak{x}})/\operatorname{Dom} \varphi_t^{\mathfrak{x}}$ follows for the continuous chain $((M^j + \operatorname{Dom} \varphi_t^{\mathfrak{x}})/\operatorname{Dom} \varphi_t^{\mathfrak{x}})_{j<\alpha}$, where the modules $(M^j + \operatorname{Dom} \varphi_t^{\mathfrak{x}})/\operatorname{Dom} \varphi_t^{\mathfrak{x}} \cong M^j/(M^j \cap \operatorname{Dom} \varphi_t^{\mathfrak{x}}) = M^j/N^j$ are $\aleph_1$-free. Thus finally also $\bigcup_{j<\alpha} (M^j + \operatorname{Dom} \varphi_t^{\mathfrak{x}})/\operatorname{Dom} \varphi_t^{\mathfrak{x}} = M/\operatorname{Dom} \varphi_t^{\mathfrak{x}}$ is $\aleph_1$-free.
From $\aleph_1$-freeness of $M/\operatorname{Dom} \varphi_t^{\mathfrak{x}}$ follows $\aleph_1$-freeness of $M/\operatorname{Im} \varphi_t^{\mathfrak{x}}$, substituting φ_t by φ_t^{-1}.

We also note that

π satisfies the U-property for M,

because the U-property is of finite character. All other conditions of the Definitions 3.8 and 3.10 are obvious, thus $\mathfrak{x} \in \mathfrak{K}$ and clearly $\mathfrak{x}^j \sqsubseteq \mathfrak{x}$, and (a) is shown.

(b): Let $\mathfrak{x} = \bigcup_{j\in\alpha} \mathfrak{x}^j = (M, \mathfrak{F}, \pi)$ and set $\varphi^j := \pi^j(\varphi)$ for all $\varphi \in R^* \times \langle \mathfrak{F}^j \rangle$, $j \in \alpha$. First we claim:

$\varphi_t^{\mathfrak{x}} \in \operatorname{Aut} M$ holds for all $t \in J$.
Proof: From $\varphi_t^{\mathfrak{x}} := \bigcup_{i(t)<j<\alpha} \varphi_t^j$ and $\varphi_t^j \in \operatorname{Aut} M^j$ it follows $\operatorname{Dom} \varphi_t^{\mathfrak{x}} = \bigcup_{i(t)<j<\alpha} \operatorname{Dom} \varphi_t^j =$

$\bigcup_{i(t)<j<\alpha} M^j = M$. Similarly $\operatorname{Im}\varphi_t^{\mathfrak{x}} = M$ holds and thus $\varphi_t^{\mathfrak{x}} \in \operatorname{Aut} M$ is an automorphism of M.

The remaining conditions of Definitions 3.8 and 3.10 are obvious. $\square$

Observe, that for a continuous chain $(\mathfrak{x}^j)_{j\in\alpha} = (M^j, \mathfrak{F}^j, \pi^j)_{j\in\alpha}$ in $(\mathfrak{K}, \subseteq)$ respectively $(\mathfrak{K}^*, \subseteq)$ the supremum $\mathfrak{x} = \bigcup_{j\in\alpha} \mathfrak{x}^j = (M, \mathfrak{F}, \pi)$ need not belong to $\mathfrak{K}$. In particular M may not be free ($\aleph_1$-free).

Now we can prove the main claim of this chapter.

Theorem 3.16 (Free UT-Construction)
If $\mathfrak{x} = (M, \mathfrak{F}, \pi) \in \mathfrak{K}$ ($\mathfrak{K}^*$), then there is $\mathfrak{x} \sqsubseteq \mathfrak{x}' = (M', \mathfrak{F}', \pi') \in \mathfrak{K}$ ($\mathfrak{K}^*$) with the following properties:

(i) $\pi'(\mathfrak{F}') \subseteq \operatorname{Aut} M'$.

(ii) $\pi'(R^* \times \langle \mathfrak{F}' \rangle)$ *acts uniquely transitive on* $\mathfrak{p}M'$.

(iii) $|M'| = |M|$.

Proof: If $M = 0$ there is nothing to show. If $M \neq 0$, then we construct recursively a chain $(\mathfrak{x}_n)_{n\in\omega}$ in $(\mathfrak{K}, \sqsubseteq)$ starting with $\mathfrak{x}_0 := \mathfrak{x}$.

Step 1: Recursively adding baby-automorphisms we construct a continuous chain $(\mathfrak{x}_0^i)_{i\in|M|\cdot|M|}$ in $(\mathfrak{K}, \sqsubseteq)$ such that $\mathfrak{x}_0^0 := \mathfrak{x}_0$ and $\pi_1(R^* \times \langle \mathfrak{F}_1 \rangle) \subseteq \operatorname{pAut} M_1$ acts transitively on $\mathfrak{p}M_1$ for $\mathfrak{x}_0 \sqsubseteq \mathfrak{x}_1 := \bigcup_{i\in|M|\cdot|M|} \mathfrak{x}_0^i = (M_1, \mathfrak{F}_1, \pi_1) \in \mathfrak{K}$. In particular, $M_1 = M_0 = M$, hence $|M_1| = |M|$.

Step 2: Recursively applying Dom- and Im-pushouts we construct a chain $(\mathfrak{x}_1^i)_{i\in|M|\cdot\omega}$ in $(\mathfrak{K}, \sqsubseteq)$ such that $\mathfrak{x}_1^0 := \mathfrak{x}_1$ and $\pi_2(\mathfrak{F}_2) \subseteq \operatorname{Aut} M_2$ for $\mathfrak{x}_1 \sqsubseteq \mathfrak{x}_2 := \bigcup_{i\in|M|\cdot\omega} \mathfrak{x}_1^i = (M_2, \mathfrak{F}_2, \pi_2) \in \mathfrak{K}$. In particular, $\mathfrak{F}_2 = \mathfrak{F}_1$ and $|M| \leq |M_2| \leq \aleph_0 \cdot |M|^2 = |M|$, hence $|M_2| = |M|$.

The chain $(\mathfrak{x}_n)_{n\in\omega}$ in $(\mathfrak{K}, \sqsubseteq)$ now is built up by alternated use of Step 1 and Step 2. Set $\mathfrak{x}' := \bigcup_{n\in\omega} \mathfrak{x}_n = (M', \mathfrak{F}', \pi') \in \mathfrak{K}$. Property (i) of the theorem follows by step 2. By step 1 and the U-property of π' the uniqueness of the transitivity of $\pi'(R^* \times \langle \mathfrak{F}' \rangle)$ on

$\mathfrak{p}M'$ follows.

Obviously $\mathfrak{x} \sqsubseteq \mathfrak{x}'$ and $|M| \leq |M'| \leq \aleph_0 \cdot |M| = |M|$, hence $|M'| = |M|$. $\square$

Thus the following definition is reasonable.

Definition 3.17 *Let* $\mathfrak{A}$ $(\mathfrak{A}^*)$ *be the family of all triples* $\mathfrak{x} = (M, \mathfrak{F}, \pi) \in \mathfrak{K}$ $(\mathfrak{K}^*)$ *with* $\pi(\mathfrak{F}) \subseteq \operatorname{Aut} M$.

The supremum $\bigcup_{j\in\alpha} \mathfrak{x}^j$ of any continuous chain $(\mathfrak{x}^j)_{j\in\alpha}$ in $(\mathfrak{A}, \sqsubseteq)$ (in $(\mathfrak{A}^*, \sqsubseteq)$) belongs to $\mathfrak{A}$ $(\mathfrak{A}^*)$, cf. Lemma 3.15. Furthermore the following Corollary holds.

Corollary 3.18 (Taking Suprema)
Let $(\mathfrak{x}^j)_{j\in\alpha} = (M^j, \mathfrak{F}^j, \pi^j)_{j\in\alpha}$ *a continuous chain in* $(\mathfrak{A}^*, \subseteq)$ $(\alpha \in \mathrm{LORD})$ *with pure ascending chain* $\bigcup_{j\in\alpha} M^j$ *and* $\mathfrak{y} \in \mathfrak{A}^*$ *with* $\mathfrak{y} \sqsubseteq \mathfrak{x}^j$ *for all* $j \in \alpha$. *Then* $\mathfrak{y} \sqsubseteq \mathfrak{x} \in \mathfrak{A}^*$ *holds for* $\mathfrak{x} := \bigcup_{j\in\alpha} \mathfrak{x}^j$.

Proof: See Lemma 3.15 (b).

If $\mathfrak{x} = (M, \mathfrak{F}, \pi) \in \mathfrak{A}$, then $\pi : R^* \times F \to \operatorname{Aut} M$ extends uniquely to a ring homomorphism:
$\pi : RF \to \operatorname{End} M$, $(f = \sum_{i=1}^n r_i\mu_i \longmapsto \pi(f) = \sum_{i=1}^n r_i\mu_i^{\mathfrak{x}})$ with $r_i \in R$, $\mu_i \in F$, $n \in \omega$.

4 The Step-Lemma

With Theorem 3.16 we can construct for any $\mathfrak{x} = (M, \mathfrak{F}, \pi) \in \mathfrak{K}$ $(\mathfrak{K}^*)$ some $\mathfrak{x} \sqsubseteq \mathfrak{x}' = (M', \mathfrak{F}', \pi') \in \mathfrak{K}$ $(\mathfrak{K}^*)$ such that the subgroup $\pi'(R^* \times \langle \mathfrak{F}' \rangle)$ acts uniquely transitive on $\mathfrak{p}M'$. If we achieve $\pi'(R^* \times \langle \mathfrak{F}' \rangle) = \operatorname{Aut} M'$, then M' will be a UT-module. For this purpose Step-Lemmas are used to control or eliminate unwanted automorphisms or endomorphisms. Theorem 4.12 and Theorem 4.13 are such Step-Lemmas.

4.1 Algebraic Part of the Free Case

Throughout this section let $(\mathfrak{x}_i)_{i\in\omega}$ be a chain in $(\mathfrak{A}, \sqsubseteq)$ with $\mathfrak{x}_i = (M_i, \mathfrak{F}_i, \pi_i)$ and $M_i \neq 0$, $\mathfrak{F}_i \neq \emptyset$, $M_{i+1} = M_i \oplus D_i$ for all $i \in \omega$. It will help to view $M_i \oplus D_i$ as an external direct sum, because we want $\widehat{M_i \oplus D_i} = \widehat{M_i} \oplus \widehat{D_i}$ for the completion discussed below. This is in contrast to the following example: Let $\pi \in \widehat{R}$ be algebraically independent over R, then $R + \pi R = R \bigoplus \pi R$ but $\widehat{R + \pi R} = \widehat{R}$. This is consequence of the fact that π, although algebraically independent over R, fails to be algebraically independent over $\widehat{R}$, a hidden dependence.
In particular, $M_i \oplus D_i$ has to be a direct sum, which is a pure submodule of its completion. We also require:

For any $i \in \omega$ there is an $e_i \in D_i$ with $R\langle \mathfrak{F}_i \rangle \cong e_i R\langle \mathfrak{F}_i \rangle \sqsubseteq D_i$; (*)
and $\pi_{i+1}(\varphi_t) \restriction e_i R\langle \mathfrak{F}_i \rangle = \varphi_t$ holds for all $\varphi_t \in \mathfrak{F}_i$.

Furthermore, let $\mathfrak{x} := \bigcup_{i\in\omega} \mathfrak{x}_i = (M^{\mathfrak{x}}, \mathfrak{F}, \pi^{\mathfrak{x}})$ and fix a prime element p of the PID R with $R \neq \widehat{R}_p$. For any R-module M let $\widehat{M}$ be its p-adic completion; also we write $\widehat{R_p} = \widehat{R}$. Clearly $M^{\mathfrak{x}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$, which will be used several times.

To simplify notation, let $RF := R\langle \mathfrak{F} \rangle$ and $RF_i := R\langle \mathfrak{F}_i \rangle$.
For any $f \in RF$ the unique extension of $f^{\mathfrak{x}}$ in $\operatorname{End} \widehat{M^{\mathfrak{x}}}$ will be denoted by $f^{\mathfrak{x}}$ as well; $RF^{\mathfrak{x}}$ will denote the subring $\{f^{\mathfrak{x}} | f \in RF\} \subseteq \operatorname{End} \widehat{M^{\mathfrak{x}}}$ and $\widehat{M^{\mathfrak{x}}}$ becomes a **right** $RF^{\mathfrak{x}}$-module.
From $M^{\mathfrak{x}} = M_j \oplus \bigoplus_{j\leq i<\omega} D_i$ follows $\widehat{M^{\mathfrak{x}}} \subseteq \widehat{M_j} \oplus \prod_{j\leq i<\omega} \widehat{D_i}$ for all $j \in \omega$, and from

$e_iRF_i \sqsubseteq D_i$ also $e_i\widehat{RF}_i \sqsubseteq \widehat{D}_i$. Thus $m \in \widehat{M^{\mathfrak{x}}}$ has a unique $\widehat{M}_i$-component, $\widehat{D}_i$-component and $\widehat{RF}_i$-component.

Definition 4.1 *For $m \in M^{\mathfrak{x}}$ and $\widehat{r} := \sum_{i\in\omega} p^i r_i \in \widehat{R}$ consider a* **branch** *of $\widehat{M^{\mathfrak{x}}}$ defined as $y := \widehat{r}m + \sum_{i\in\omega} p^i e_i = \sum_{i\in\omega} p^i(r_i m + e_i) \in \widehat{M^{\mathfrak{x}}}$.*
Moreover, if $\varphi \in \langle\mathfrak{F}\rangle$, $k \in \omega$ set
$y^k_\varphi := \sum_{k\leq i<\omega} p^{i-k}(r_i m + e_i)\varphi^{\mathfrak{x}} = (\sum_{k\leq i<\omega} p^{i-k} r_i) m\varphi^{\mathfrak{x}} + \sum_{k\leq i<\omega} p^{i-k} e_i \varphi^{\mathfrak{x}} \in \widehat{M^{\mathfrak{x}}}$.

We will keep $\widehat{r}$ and y fixed below.
We now summarize:

Lemma 4.2
(a) If $\varphi_t \in \mathfrak{F}$, then $\varphi^{\mathfrak{x}}_t \in \operatorname{Aut} \widehat{M^{\mathfrak{x}}}$.
(b) If $y \in \widehat{M^{\mathfrak{x}}}$ is a branch and $f \in RF$ satisfies $yf^{\mathfrak{x}} \in M^{\mathfrak{x}}$, then $f = 0$.

Proof:
(a): If $\widehat{m} \in \widehat{M^{\mathfrak{x}}}$ and $\varphi_t \in \mathfrak{F}$, then let $(m_i)_{i\in\omega}$ be a Cauchy-sequence in $M^{\mathfrak{x}}$ converging to $\widehat{m}$. From $\varphi^{\mathfrak{x}}_t \in \operatorname{Aut} M^{\mathfrak{x}}$ follows that $(m_i(\varphi^{\mathfrak{x}}_t)^{-1})_{i\in\omega}$ is also a Cauchy-sequence in $M^{\mathfrak{x}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$ converging to $\widehat{m'} \in \widehat{M^{\mathfrak{x}}}$. It follows $\widehat{m'}\varphi^{\mathfrak{x}}_t = \widehat{m}$. So $\varphi^{\mathfrak{x}}_t$ is surjective.
From $\widehat{m}\varphi^{\mathfrak{x}}_t = 0$ follows that $(m_i\varphi^{\mathfrak{x}}_t)_{i\in\omega}$ is a zero sequence. Without loss of generality let $p^i | m_i\varphi^{\mathfrak{x}}_t$ for all $i \in \omega$, so we have $p^i | m_i$ for all $i \in \omega$; now $(m_i)_{i\in\omega}$ is a zero sequence and $\widehat{m} = 0$. Thus $\varphi^{\mathfrak{x}}_t$ is also injective.
(b): Let $y = \widehat{r}m + \sum_{i\in\omega} p^i e_i \in \widehat{M^{\mathfrak{x}}}$ and $f \in RF$ with $yf^{\mathfrak{x}} \in M^{\mathfrak{x}}$ as in (b).
We can choose $j \in \omega$ large enough such that $m \in M_j$, $yf^{\mathfrak{x}} \in M_j$, $f \in RF_j$. Consider the $\widehat{RF_j}$-component of the equation $\widehat{r}mf^{\mathfrak{x}} + \sum_{i\in\omega} p^i e_i f^{\mathfrak{x}} = yf^{\mathfrak{x}}$:
From $m \in M_j$ and $f \in RF_j$ follows $\widehat{r}mf^{\mathfrak{x}} \in \widehat{M_j}$, $yf^{\mathfrak{x}} \in M_j$, $p^i e_i f^{\mathfrak{x}} \in M_j$ for $i < j$ and $p^i e_i f^{\mathfrak{x}} = p^i e_i f \in e_i RF_i$ for $i \geq j$. Thus $p^j e_j f = 0$ and hence $e_j f = 0$, which implies $f = 0$. □

For a submodule U of a torsion-free R-module M and a prime element p of R let $M_{*p} := \{m \in M \mid \exists\, n \in \omega : p^n m \in U\}$ be the **p-purification** of U in M.
Given a branch y of $\widehat{M^{\mathfrak{x}}}$ the module $\langle M^{\mathfrak{x}}, y\rangle_{RF^{\mathfrak{x}}}$ is an R-module as

$R \subseteq RF^{\mathfrak{x}} \subseteq \text{End}\, \widehat{M^{\mathfrak{x}}}$. Hence $(\langle M^{\mathfrak{x}}, y\rangle_{RF^{\mathfrak{x}}})_{*p} = \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}}\rangle_{*p}$ is an R-module and we let $M^{\mathfrak{y}} := \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}}\rangle_{*p}$.

If $y = \widehat{r}m + \sum_{i\in\omega} p^i e_i$ and $\varphi \in \langle\mathfrak{F}\rangle$, then $m\varphi^{\mathfrak{x}} \in M^{\mathfrak{x}}$ implies $m\varphi^{\mathfrak{x}} \in M_j$ and $\widehat{r}m\varphi^{\mathfrak{x}} \in \widehat{M_j}$ for a suitable $j = j(\varphi) \in \omega$. Moreover, if $\varphi \in \langle\mathfrak{F}\rangle$, then $\varphi \in \langle\mathfrak{F}_{k(\varphi)}\rangle$ for some $j \leq k(\varphi)$. If $k(\varphi) \leq k$, then $y^k_\varphi = (\sum_{k\leq i<\omega} p^{i-k} r_i)m\varphi^{\mathfrak{x}} + \sum_{k\leq i<\omega} p^{i-k} e_i\varphi$, with $(\sum_{k\leq i<\omega} p^{i-k} r_i)m\varphi^{\mathfrak{x}}$ the $\widehat{M_{k(\varphi)}}$-component of y^k_φ and $p^{i-k}e_i\varphi$ the $\widehat{RF_i}$-component of y^k_φ for any $k(\varphi) \leq k \leq i$ respectively.
Now let $k(\varphi)$ be fixed as above.

In our Step-Lemma we will concentrate on modules $M^{\mathfrak{y}}$ defined by suitable branches y. Next we want to show that these $M^{\mathfrak{y}}$ are free.

Lemma 4.3
(a) $M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, y^k_\varphi | \varphi \in \langle\mathfrak{F}\rangle, k \in \omega\rangle_R$.
(b) $M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, y^k_\varphi | \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\rangle_R$.

Proof:
(a):
" $\subseteq$ ": If $m \in M^{\mathfrak{y}}$, then there are $k \in \omega$, $m' \in M^{\mathfrak{x}}$ and $f = \sum_{i=1}^n s_i\mu_i \in RF$ with $p^k m = m' + yf^{\mathfrak{x}}$. Thus $p^k m = m' + yf^{\mathfrak{x}} = m' + \sum_{i=1}^n s_i y\mu_i^{\mathfrak{x}} = m' + \sum_{i=1}^n s_i y^0_{\mu_i} = m' + \sum_{i=1}^n s_i(m'_i + p^k y^k_{\mu_i}) = m' + \sum_{i=1}^n s_i m'_i + p^k \sum_{i=1}^n s_i y^k_{\mu_i}$, hence $p^k m = m'' + p^k \sum_{i=1}^n s_i y^k_{\mu_i}$ for suitable $m'_i, m'' \in M^{\mathfrak{x}}$. In particular $m'' \in M^{\mathfrak{x}} \cap p^k\widehat{M^{\mathfrak{x}}} = p^k M^{\mathfrak{x}}$, thus $m'' = p^k m'''$ with $m''' \in M^{\mathfrak{x}}$ since $M^{\mathfrak{x}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$. It follows that $m = m''' + \sum_{i=1}^n s_i y^k_{\mu_i} \in \langle M^{\mathfrak{x}}, y^k_\varphi | \varphi \in \langle\mathfrak{F}\rangle, k \in \omega\rangle_R$.
" $\supseteq$ ": This follows from $p^k y^k_\varphi = y\varphi^{\mathfrak{x}} - \sum_{i=0}^{k-1} p^i(r_i m + e_i)\varphi^{\mathfrak{x}} \in \langle M^{\mathfrak{x}}, y\rangle_{RF^{\mathfrak{x}}} \subseteq M^{\mathfrak{y}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$.
(b): We will show $\langle M^{\mathfrak{x}}, y^k_\varphi | \varphi \in \langle\mathfrak{F}\rangle, k \in \omega\rangle_R = \langle M^{\mathfrak{x}}, y^k_\varphi | 0 \neq \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\rangle_R$.
From $y^k_\varphi = p^l y^{k+l}_\varphi + \sum_{i=k}^{k+l-1} p^{i-k}(r_i m + e_i)\varphi^{\mathfrak{x}}$ follows "$\subseteq$". The reverse inclusion is trivial. □

Lemma 4.4 *If $\varphi \in \langle \mathfrak{F} \rangle$, then the following holds:*

(a) $\{y_\varphi^k | \ k(\varphi) \le k < \omega\}$ is a set of independent elements.

(b) $\langle y_\varphi^k | \ k(\varphi) \le k < \omega \rangle_R \cap M^\mathfrak{x} = \langle (r_k m + e_k)\varphi^\mathfrak{x} | k(\varphi) \le k < \omega \rangle_R$.

Proof:

(a): Let $\sum_{k(\varphi)\le k} \lambda_k y_\varphi^k = 0$ be a linear combination of the elements y_φ^k, where almost all λ_k are 0. Suppose that some $\lambda_{k'} \neq 0$ and choose k' to be minimal. The $\widehat{RF_{k'}}$-component of the equation $\sum_{k(\varphi)\le k} \lambda_k y_\varphi^k = 0$ forces $\lambda_{k'} e_{k'} \varphi = 0$, thus $\lambda_{k'} = 0$, a contradiction. Therefore (a) holds.

(b):

" $\subseteq$ ": If $m' := \sum_{k(\varphi)\le k} \lambda_k y_\varphi^k \in M^\mathfrak{x}$, then $\sum_{k(\varphi)\le k} \lambda_k y_\varphi^k \in M_{k'}$ follows for a suitable $k' \in \omega$. Choose $l \ge \max\{k(\varphi), k'\}$. The $\widehat{RF_l}$-component of $\sum_{k(\varphi)\le k} \lambda_k y_\varphi^k \in M_{k'}$ must be zero:

$$\sum_{k(\varphi)\le k\le l} \lambda_k p^{l-k} e_l \varphi = 0, \text{ hence } \sum_{k(\varphi)\le k\le l} \lambda_k p^{l-k} = 0. \qquad \textbf{(1)}$$

Using (1) we write $m' = \sum_{k(\varphi)\le k} \lambda_k y_\varphi^k = \sum_{k(\varphi)\le k} \lambda_k \sum_{k\le i} p^{i-k}(r_i m + e_i)\varphi^\mathfrak{x} =$
$\sum_{i\in\omega}(\sum_{k(\varphi)\le k\le i} \lambda_k p^{i-k})(r_i m + e_i)\varphi^\mathfrak{x} = \sum_{0\le i<k''}(\sum_{k(\varphi)\le k\le i} \lambda_k p^{i-k})(r_i m + e_i)\varphi^\mathfrak{x}$.
Hence $m' \in \langle (r_k m + e_k)\varphi^\mathfrak{x} | k(\varphi) \le k < \omega \rangle_R$.

" $\supseteq$ ": The reverse inclusion is immediate since $y_\varphi^k - p y_\varphi^{k+1} = (r_k m + e_k)\varphi^\mathfrak{x}$. □

Lemma 4.5 $\{(r_k m + e_k)\varphi^\mathfrak{x} | \ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \le k < \omega\}$ *is a set of independent elements.*

Proof: Let $\sum_{\varphi\in\langle\mathfrak{F}\rangle, k(\varphi)\le k} \lambda_\varphi^k (r_k m + e_k)\varphi^\mathfrak{x} = 0$ with $\lambda_\varphi^k = 0$ for almost all k, φ. If there is $\lambda_\varphi^k \neq 0$, then let $k' := \max\ \{k | \ \exists \varphi \in \langle \mathfrak{F} \rangle : \ k(\varphi) \le k < \omega \wedge \lambda_\varphi^k \neq 0\}$.
The $\widehat{RF_{k'}}$-component of the equation $\sum_{\varphi\in\langle\mathfrak{F}\rangle, k(\varphi)\le k} \lambda_\varphi^k (r_k m + e_k)\varphi^\mathfrak{x} = 0$ is:
$\sum_{\varphi\in\langle\mathfrak{F}\rangle, k(\varphi)\le k'} \lambda_\varphi^{k'} e_{k'} \varphi = 0$, thus $\sum_{\varphi\in\langle\mathfrak{F}\rangle} \lambda_\varphi^{k'} \varphi = 0$ and $\lambda_\varphi^{k'} = 0$ for all $\varphi \in \langle \mathfrak{F} \rangle$, contradicting $\lambda_{\varphi'}^{k'} \neq 0$ for a suitable $\varphi' \in \langle \mathfrak{F} \rangle$. The lemma follows. □

Lemma 4.6

(a) $\{y_\varphi^k | \ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \le k < \omega\}$ is a set of independent elements.

(b) $\langle y_\varphi^k | \ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \le k < \omega \rangle_R \cap M^\mathfrak{x} = \langle (r_k m + e_k)\varphi^\mathfrak{x} | \ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \le k < \omega \rangle_R$.

Proof:

(b): Let $m' := \sum_{\varphi\in\langle\mathfrak{F}\rangle, k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k \in M^\mathfrak{r}$ with $\lambda_\varphi^k = 0$ for almost all coefficients k, φ. For $\varphi \in \langle\mathfrak{F}\rangle$ we can write

$$\sum_{k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k = m_\varphi + r_\varphi y_\varphi^{k'(\varphi)} \qquad \textbf{(1)}$$

with suitable $m_\varphi \in M^\mathfrak{r}$, $r_\varphi \in R$, $k'(\varphi) := \max\{k(\varphi) \le k < \omega | \ \lambda_\varphi^k \ne 0\}$ and $m_\varphi = 0$, $r_\varphi = 0$ for almost all $\varphi \in \langle\mathfrak{F}\rangle$. Hence $m' = \sum_{\varphi\in\langle\mathfrak{F}\rangle, k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k = \sum_{\varphi\in\langle\mathfrak{F}\rangle}(\sum_{k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k)$ $= \sum_{\varphi\in\langle\mathfrak{F}\rangle} m_\varphi + \sum_{\varphi\in\langle\mathfrak{F}\rangle} r_\varphi y_\varphi^{k'(\varphi)} \in M^\mathfrak{r}$. Also $\sum_{\varphi\in\langle\mathfrak{F}\rangle} m_\varphi \in M^\mathfrak{r}$, thus $\sum_{\varphi\in\langle\mathfrak{F}\rangle} r_\varphi y_\varphi^{k'(\varphi)} \in M^\mathfrak{r}$ and $\sum_{\varphi\in\langle\mathfrak{F}\rangle} r_\varphi y_\varphi^{k'(\varphi)} \in M_{k'}$ follows for a large enough $k' \in \omega$.

If l is an upper bound of $\max\{k'(\varphi), k' | \ \varphi \in \langle\mathfrak{F}\rangle \wedge r_\varphi \ne 0\}$, then the $\widehat{RF_l}$-component of $\sum_{\varphi\in\langle\mathfrak{F}\rangle} r_\varphi y_\varphi^{k'(\varphi)} \in M_{k'}$ vanishes. It follows that $\sum_{\varphi\in\langle\mathfrak{F}\rangle} r_\varphi p^{l-k'(\varphi)} e_l \varphi = 0$ and hence $\sum_{\varphi\in\langle\mathfrak{F}\rangle} r_\varphi p^{l-k'(\varphi)} \varphi = 0$, which implies $\forall \ \varphi \in \langle\mathfrak{F}\rangle : r_\varphi = 0$.

Now (1) implies

$$\sum_{k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k = m_\varphi \in M^\mathfrak{r} \text{ for all } \varphi \in \langle\mathfrak{F}\rangle. \qquad \textbf{(2)}$$

Using (2) and Lemma 4.4 we determine (b):

$\langle y_\varphi^k | \ \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \le k < \omega\rangle_R \cap M^\mathfrak{r} = \sum_{\varphi\in\langle\mathfrak{F}\rangle}\langle y_\varphi^k | \ k(\varphi) \le k < \omega\rangle_R \cap M^\mathfrak{r} = \sum_{\varphi\in\langle\mathfrak{F}\rangle}\langle (r_k m + e_k)\varphi^\mathfrak{r} | k(\varphi) \le k < \omega\rangle_R = \langle (r_k m + e_k)\varphi^\mathfrak{r} | \ \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \le k < \omega\rangle_R$.

(a): Let $m' = \sum_{\varphi\in\langle\mathfrak{F}\rangle, k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k = 0$ be as above and suppose $m' = 0$. From $m' = 0 \in M^\mathfrak{r}$ and (2) follows $\sum_{k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k \in M^\mathfrak{r}$ for all $\varphi \in \langle\mathfrak{F}\rangle$ and Lemma 4.4 (b), Lemma 4.5 apply. We get

$$X := \bigoplus_{\varphi\in\langle\mathfrak{F}\rangle} \langle (r_k m + e_k)\varphi^\mathfrak{r} | k(\varphi) \le k < \omega\rangle_R$$

and

$$\sum_{k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k \in \langle (r_k m + e_k)\varphi^\mathfrak{r} | k(\varphi) \le k < \omega\rangle_R.$$

Now $m' = 0$ implies $\sum_{k(\varphi)\le k} \lambda_\varphi^k y_\varphi^k = 0$ for all $\varphi \in \langle\mathfrak{F}\rangle$ and by Lemma 4.4 (a) also $\lambda_\varphi^k = 0$ for all $\varphi \in \langle\mathfrak{F}\rangle$, $k(\varphi) \le k < \omega$. Hence all coefficients of m' are trivial, and (a) follows. □

4.2 Algebraic Part of the $\aleph_1$-Free Case

We will use the algebraic preliminaries from Section 4.1 and consider a chain $(\mathfrak{x}_i)_{i\in\omega}$ in $(\mathfrak{A}^*, \sqsubseteq)$ with $\mathfrak{x}_i = (M_i, \mathfrak{F}_i, \pi_i)$ and $M_i \neq 0$, $\mathfrak{F}_i \neq \emptyset$ for all $i \in \omega$. Then $\mathfrak{x}_\omega := \bigcup_{i\in\omega} \mathfrak{x}_i = (M_\omega, \mathfrak{F}, \pi_\omega) \in \mathfrak{A}^*$ by Lemma 3.15 (b). Let $(B_i)_{i\in\omega}$ be a chain of free R-modules B_i with $B_i \subseteq_{*p} M_i \subseteq_{*p} \widehat{B_i}$ and $B_{i+1} = B_i \oplus D_i$ for all $i \in \omega$. Hence $B := \bigcup_{i\in\omega} B_i$ is a free R-module. Moreover let $\mathfrak{x}_\omega \subseteq \mathfrak{x} := (M^\mathfrak{x}, \mathfrak{F}, \pi^\mathfrak{x})$ with $B \subseteq_{*p} M_\omega \subseteq_* M^\mathfrak{x} \subseteq_{*p} \widehat{B}$ and $\mathfrak{x}_i \sqsubseteq \mathfrak{x}$ for all $i \in \omega$. Note that $\widehat{B} = \widehat{M^\mathfrak{x}}$. As in Section 4.1 (*) we assume:

For any $i \in \omega$ there is an $e_i \in D_i$ with $R\langle\mathfrak{F}_i\rangle \cong e_i R\langle\mathfrak{F}_i\rangle \sqsubseteq D_i$; **(*)**
and $\pi_{i+1}(\varphi_t) \restriction e_i R\langle\mathfrak{F}_i\rangle = \varphi_t$ holds for all $\varphi_t \in \mathfrak{F}_i$.

Let $RF := R\langle\mathfrak{F}\rangle$ and $RF_i := R\langle\mathfrak{F}_i\rangle$.
Again $B = B_j \oplus \bigoplus_{j\le i<\omega} D_i$ implies $\widehat{M^\mathfrak{x}} = \widehat{B} \subseteq \widehat{B_j} \oplus \prod_{j\le i<\omega} \widehat{D_i}$ for all $j \in \omega$, and any $m \in \widehat{M^\mathfrak{x}}$ has unique $\widehat{B_i}$-components, $\widehat{D_i}$-components and $\widehat{RF_i}$-components for each $i \in \omega$. Thus we assign a support $[m] \subseteq \omega$ to m:

$$[m] := \{i \in \omega \mid \textit{The } RF_i\textit{-component of } m \textit{ is not } 0.\}$$

With these notions we can formulate our last condition on the chain $(\mathfrak{x}_i)_{i\in\omega}$:

Any $m \in M^\mathfrak{x}$ has finite support $[m]$.

Also recall the Definition 4.1 of $\widehat{r}$ and the branch (element) y from Section 4.1.
The next lemma is a modification of Lemma 4.2. Also its proof is very similar.

Lemma 4.7
(a) If $\varphi_t \in \mathfrak{F}$, then $\varphi_t^\mathfrak{x} \in \operatorname{Aut} \widehat{M^\mathfrak{x}}$.
(b) If $y \in \widehat{M^\mathfrak{x}}$ is a branch and $f \in RF$ satisfies $yf^\mathfrak{x} \in M^\mathfrak{x}$, then $f = 0$.

Proof:
(a): See the proof of Lemma 4.2 (a).
(b): Let $y = \widehat{r}m + \sum_{i\in\omega} p^i e_i \in \widehat{M^\mathfrak{x}}$ and $f \in RF$ with $yf^\mathfrak{x} \in M^\mathfrak{x}$ as in (b). Thus the supports $[m]$ and $[yf^\mathfrak{x}]$ are finite and we can choose $j \in \omega$ large enough such that $[m] \subseteq j$, $[yf^\mathfrak{x}] \subseteq j$, $f \in RF_j$. The $\widehat{RF_j}$-component of the equation $\widehat{r}mf^\mathfrak{x} + \sum_{i\in\omega} p^i e_i f^\mathfrak{x} = yf^\mathfrak{x}$

forces $p^j e_j f = 0$, thus $e_j f = 0$, and $f = 0$. $\square$

We now define $M^{\mathfrak{y}} := \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}} \rangle_{*p}$.
For $y = \widehat{r}m + \sum_{i\in\omega} p^i e_i$ and $\varphi \in \langle \mathfrak{F} \rangle$ with $m\varphi^{\mathfrak{x}} \in M^{\mathfrak{x}}$ follows $[\widehat{r}m\varphi^{\mathfrak{x}}] \subseteq [m\varphi^{\mathfrak{x}}]$, which is a finite support, thus $[\widehat{r}m\varphi^{\mathfrak{x}}] \subseteq j(\varphi)$ for some $j(\varphi) \in \omega$, and the $\widehat{RF_k}$-components of $\widehat{r}m\varphi^{\mathfrak{x}}$ vanish for $k \geq j(\varphi)$. Now for every $\varphi \in \langle \mathfrak{F} \rangle$ there exists $k(\varphi) \geq j(\varphi)$ with $\varphi \in \langle \mathfrak{F}_{k(\varphi)} \rangle$, and if $k \geq k(\varphi)$, then $y^k_\varphi = (\sum_{k\leq i<\omega} p^{i-k} r_i) m\varphi^{\mathfrak{x}} + \sum_{k\leq i<\omega} p^{i-k} e_i \varphi$ has $\widehat{RF_i}$-component $p^{i-k} e_i \varphi$ $(i \geq k)$.
We associate with φ, y the element $k(\varphi) \in \omega$.

The next lemmata are variants of Lemma 4.3 to Lemma 4.6; they are needed to determine the structure of $M^{\mathfrak{y}}$.

Lemma 4.8
(a) $M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, y^k_\varphi | \varphi \in \langle \mathfrak{F} \rangle, k \in \omega \rangle_R$.
(b) $M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, y^k_\varphi | \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega \rangle_R$.

Proof: See Lemma 4.3.

Lemma 4.9 *If* $\varphi \in \langle \mathfrak{F} \rangle$*, then:*
(a) $\{y^k_\varphi |\ k(\varphi) \leq k < \omega\}$ *is a set of independent elements.*
(b) $\langle y^k_\varphi |\ k(\varphi) \leq k < \omega \rangle_R \cap M^{\mathfrak{x}} = \langle (r_k m + e_k)\varphi^{\mathfrak{x}} | k(\varphi) \leq k < \omega \rangle_R$.

Proof: See Lemma 4.4.

Lemma 4.10 $\{(r_k m + e_k)\varphi^{\mathfrak{x}} |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega\}$ *is a set of independent elements.*

Proof: See Lemma 4.5.

Lemma 4.11
(a) $\{y^k_\varphi |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega\}$ *is a set of independent elements.*
(b) $\langle y^k_\varphi |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega \rangle_R \cap M^{\mathfrak{x}} = \langle (r_k m + e_k)\varphi^{\mathfrak{x}} |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega \rangle_R$.

Proof: See Lemma 4.6.

4.3 The two Step-Lemmas

We continue to apply the algebraic preliminaries discussed in Section 4.1 and want to prove the Step-Lemmas. They are used to get rid of endomorphisms as discussed in (d) below.

One of the first to develop this idea was Brendan Goldsmith, see [27].

The chain $\mathfrak{x}_i = (M_i, \mathfrak{F}_i, \pi_i) \in \mathfrak{A}$ $(i \in \omega)$ is **proper** if $0 \neq M_i \subset M_{i+1}$ and $\emptyset \neq \mathfrak{F}_i \subset \mathfrak{F}_{i+1}$ for all $i \in \omega$.

Theorem 4.12 (Step-Lemma, Free Case)

Let $(\mathfrak{x}_i)_{i\in\omega}$ be a proper chain in $(\mathfrak{A}, \sqsubseteq)$ with $\mathfrak{x}_i = (M_i, \mathfrak{F}_i, \pi_i)$ and $M_{i+1} = M_i \oplus D_i$ for all $i \in \omega$. There are $e_i \in D_i$ such that $RF_i \cong e_iRF_i \sqsubseteq D_i$ hold with $\pi_{i+1}(\varphi_t) \restriction e_iRF_i = \varphi_t$ for all $\varphi_t \in \mathfrak{F}_i$, $i \in \omega$. Let $\mathfrak{x} = \bigcup_{i\in\omega} \mathfrak{x}_i = (M^{\mathfrak{x}}, \mathfrak{F}, \pi^{\mathfrak{x}}) \in \mathfrak{A}$ and $y = \widehat{r}m + \sum_{i\in\omega} p^i e_i$ be a branch of $\widehat{M^{\mathfrak{x}}}$.

If $M^{\mathfrak{y}} := \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}}\rangle_{*p}$, $\pi^{\mathfrak{y}}(\varphi_t) = \varphi_t^{\mathfrak{y}} := \varphi_t^{\mathfrak{x}} \restriction M^{\mathfrak{y}}$ for all $\varphi_t \in \mathfrak{F}$ and $\mathfrak{y} := (M^{\mathfrak{y}}, \mathfrak{F}, \pi^{\mathfrak{y}})$, then the following holds:

(a) $\mathfrak{x} \subseteq \mathfrak{y} \in \mathfrak{A}$.

(b) $|M^{\mathfrak{y}}| = |M^{\mathfrak{y}} \setminus M^{\mathfrak{x}}| = |M^{\mathfrak{x}}|$ and $\mathfrak{x}_n \sqsubseteq \mathfrak{y}$ for all $n \in \omega$.

(c) $M^{\mathfrak{y}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$ and $M^{\mathfrak{y}}/M^{\mathfrak{x}} \neq 0$ is p-divisible, in particular $\mathfrak{x} \not\sqsubseteq \mathfrak{y}$.

(d) If $\eta^{\mathfrak{x}} \in \operatorname{End} M^{\mathfrak{x}} \setminus \pi^{\mathfrak{x}}(RF)$ (also $\eta^{\mathfrak{x}} \in \operatorname{End} \widehat{M^{\mathfrak{x}}}$), then there are a branch $y \in \widehat{M^{\mathfrak{x}}}$, $M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}}\rangle_{*p}$ and $\eta^{\mathfrak{y}} := \eta^{\mathfrak{x}} \restriction M^{\mathfrak{y}}$ with $y\eta^{\mathfrak{y}} \notin M^{\mathfrak{y}}$, hence $\eta^{\mathfrak{x}}$ does not extend to an endomorphism of $M^{\mathfrak{y}}$.

Proof:

(a): Obviously $M^{\mathfrak{x}} \subseteq M^{\mathfrak{y}}$ and $\pi^{\mathfrak{x}} \subseteq \pi^{\mathfrak{y}}$ by continuity.

Set $C := M^{\mathfrak{x}}$ and $D := \langle y_\varphi^k |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega \rangle_R$.

Now C is free, and D is free by Lemma 4.6 (a). By Lemma 4.3 (b) follows $C + D = M^{\mathfrak{y}}$ and $C \cap D = \langle (r_k m + e_k)\varphi^{\mathfrak{x}} |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega \rangle_R$ by Lemma 4.6 (b).

Clearly there is a basis $B := \bigcup_{i\in\omega} B_i$ of $M^{\mathfrak{x}}$ such that $\langle B_i \rangle = M_i$, $\langle B_{i+1} \setminus B_i \rangle = D_i$ and $\{e_i\varphi | \varphi \in \langle \mathfrak{F}_i \rangle\} \subseteq B_{i+1}$ for all $i \in \omega$, hence $B \cap e_iRF_i$ is a basis of e_iRF_i. If

$$B_i' := (B_i \setminus \{e_k\varphi |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < i\}) \cup \{(r_k m + e_k)\varphi^{\mathfrak{x}} |\ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < i\},$$

then

B'_i is a basis of M_i for all $i \in \omega$. **(1)**

We prove (1) using induction. $B'_0 = B_0$ is a basis of M_0. Now assume that B'_i is a basis of M_i for some $i \geq 0$.

a) Any $m' \in M_{i+1}$ can be expressed as

$$m' = \sum_{j=1}^{n_1} \lambda_j b_j + \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} e_{k_j} \varphi_j$$

with $b_j \in B_{i+1} \cap B'_{i+1}$, $e_{k_j}\varphi_j \in B_{i+1} \setminus B'_{i+1}$. Thus
$m' = \sum_{j=1}^{n_1} \lambda_j b_j + \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} (r_{k_j} m + e_{k_j}) \varphi_j^{\mathfrak{x}} - \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} r_{k_j} m \varphi_j^{\mathfrak{x}}$ using basis elements from B'_{i+1}, where $\sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} r_{k_j} m \varphi_j^{\mathfrak{x}} \in M_i \in \langle B'_i \rangle$.
Thus $M_{i+1} \subseteq \langle B'_{i+1} \rangle$ and obviously $\langle B'_{i+1} \rangle \subseteq M_{i+1}$. It follows that
$\langle B'_{i+1} \rangle = M_{i+1}$. **(2)**

b) Next we show that B'_{i+1} is a set of independent elements; then (1) will follow by induction. Let $\sum_{j=1}^{n_1} \lambda_j b_j + \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} (r_{k_j} m + e_{k_j}) \varphi_j^{\mathfrak{x}} = 0$ with $b_j \in B'_{i+1} \cap B_{i+1}$, $(r_{k_j} m + e_{k_j}) \varphi_j^{\mathfrak{x}} \in B'_{i+1} \setminus B_{i+1}$ and only non-trivial coefficients, and suppose that this sum is not degenerated. Then

$$\sum_{j=1}^{n_1} \lambda_j b_j + \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} e_{k_j} \varphi_j = - \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} r_{k_j} m \varphi_j^{\mathfrak{x}} \in M_i$$

is a linear combination of distinct elements in B_{i+1}. Thus $\sum_{j=1}^{n_1} \lambda_j b_j + \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} e_{k_j} \varphi_j$ is a linear combination in B_i and $\sum_{j=1}^{n_1} \lambda_j b_j + \sum_{j=1}^{n_2} \lambda_{\varphi_j}^{k_j} (r_{k_j} m + e_{k_j}) \varphi_j^{\mathfrak{x}} = 0$ is a non-trivial linear combination of basis elements in B'_i, a contradiction. Next we show:

$M^{\mathfrak{y}}$ is free. **(3)**

Proof: The set $B' := \bigcup_{i \in \omega} B'_i$ is a basis of $M^{\mathfrak{x}}$ by (1). Moreover
$\{(r_k m + e_k)\varphi^{\mathfrak{x}} | \ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega\} \subseteq B'$, hence
$C \cap D = \langle (r_k m + e_k)\varphi^{\mathfrak{x}} | \ \varphi \in \langle \mathfrak{F} \rangle, k(\varphi) \leq k < \omega \rangle_R \sqsubseteq \langle B' \rangle = C$.
In particular $C/(C \cap D) \cong (C + D)/D$ is free, and since D is free as we have seen above, together with $(C + D)/D$ also $C + D = M^{\mathfrak{y}}$ is free as well.

$\varphi_t^{\mathfrak{y}} \in \operatorname{Aut} M^{\mathfrak{y}}$ is well defined for all $\varphi_t \in \mathfrak{F}$. **(4)**

Proof: First we note that $\varphi_t^{\mathfrak{x}} \in \operatorname{Aut} \widehat{M^{\mathfrak{x}}}$ for all $\varphi_t \in \mathfrak{F}$ by Lemma 4.2 (a). Also note that $M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}}\rangle_{*p}$ is an $RF^{\mathfrak{x}}$-module as p-purification of an $RF^{\mathfrak{x}}$-module, thus $\varphi_t^{\mathfrak{y}} = \varphi_t^{\mathfrak{x}} \restriction M^{\mathfrak{y}} \in \operatorname{Aut} M^{\mathfrak{y}}$. Now we want to show:

$\pi^{\mathfrak{y}}$ *satisfies the U-property on* $M^{\mathfrak{y}}$. **(5)**

Proof: Let $z \in M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}}\rangle_{*p}$ and $0 \neq f \in RF$ with $zf^{\mathfrak{y}} = 0$. Thus $p^k z = z' + yg^{\mathfrak{x}}$ for some k and $z' \in M^{\mathfrak{x}}$, $g \in RF$. From $zf^{\mathfrak{y}} = 0$ follows $p^k zf^{\mathfrak{y}} = 0$, hence $(z' + yg^{\mathfrak{x}})f^{\mathfrak{y}} = 0$, $yg^{\mathfrak{x}}f^{\mathfrak{x}} = -z'f^{\mathfrak{x}} \in M^{\mathfrak{x}}$ and $gf = 0$, $g = 0$ by Lemma 4.2 (b). In particular $p^k z = z' \in M^{\mathfrak{x}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$, thus $z \in M^{\mathfrak{x}}$. We have seen, that

$$0 \neq f \in RF \text{ implies } \ker f^{\mathfrak{y}} = \ker f^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}.$$

Thus Definition 3.5 (i) of the U-property for $\pi^{\mathfrak{x}}$ induces the U-property Definition 3.5 (i) for $\pi^{\mathfrak{y}}$. Condition (ii) of the U-property is trivial because $\varphi_t^{\mathfrak{y}} \in \operatorname{Aut} M^{\mathfrak{y}}$ for all $\varphi_t \in \mathfrak{F}$ and $\operatorname{Dom} f^{\mathfrak{y}} = M^{\mathfrak{y}}$ for all $f \in RF$. It follows that $\mathfrak{x} \subseteq \mathfrak{y} \in \mathfrak{A}$ and (a) is shown.

Next we verify **(b)** of the theorem and begin with

$\mathfrak{x}_n \sqsubseteq \mathfrak{y}$ *holds for all* $n \in \omega$. **(6)**

Proof: If $n \in \omega$ then $\mathfrak{x}_n \subseteq \mathfrak{x} \subseteq \mathfrak{y}$, thus by (a) it is enough to prove that $M^{\mathfrak{y}}/M_n$ is free. Recall that we associate with φ, y an element $k(\varphi) \in \omega$. We also may assume that $n \leq k(\varphi)$. Thus

$C := M^{\mathfrak{x}}$ and $D := \langle y_\varphi^k |\ \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\rangle_R$ are free, **(7)**

and

$C + D = M^{\mathfrak{y}}$, $C \cap D = \langle (r_k m + e_k)\varphi^{\mathfrak{x}} |\ \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\rangle_R$ **(8)**

as in the proof of (a). Also,

$$B' := (B \setminus \{e_k\varphi | \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\}) \cup \{(r_k m + e_k)\varphi^{\mathfrak{x}} | \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\}$$

is a basis of $M^{\mathfrak{x}}$ with $B_n \dot\cup \{(r_k m + e_k)\varphi^{\mathfrak{x}} |\ \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\} \subseteq B'$ and

$(C \cap D) \oplus M_n \sqsubseteq C$. **(9)**

From (9) follows $D \cap M_n = (C \cap D) \cap M_n = 0$, hence $D + M_n = D \oplus M_n$ and $C/M_n \cap (D + M_n)/M_n = [C \cap (D + M_n)]/M_n = [(C \cap D) + M_n]/M_n \sqsubseteq C/M_n$. Thus $C' \cap D' \sqsubseteq C'$ with $C' := C/M_n = M^{\mathfrak{x}}/M_n$ is free and

$D' := (D + M_n)/M_n = (D \oplus M_n)/M_n \cong D$ is free as well.

As in (a) we finish the proof of (b) showing that $C' + D' = (C + D)/M_n = M^{\mathfrak{y}}/M_n$ is free. Next we claim

$|M^{\mathfrak{y}}| = |M^{\mathfrak{y}} \setminus M^{\mathfrak{x}}| = |M^{\mathfrak{x}}|.$ **(10)**

Proof: Clearly $|M^{\mathfrak{x}}| \leq |M^{\mathfrak{y}}| = |\langle M^{\mathfrak{x}}, y^k_\varphi | \varphi \in \langle \mathfrak{F} \rangle, k \in \omega \rangle_R| \leq |M^{\mathfrak{x}}| \cdot |R| \cdot |\langle \mathfrak{F} \rangle| = |M^{\mathfrak{x}}|$, thus $|M^{\mathfrak{y}}| = |M^{\mathfrak{x}}|$ follows by Lemma 4.3 (a) and $\aleph_0 \leq |R| \leq |M^{\mathfrak{x}}|$, $|\mathfrak{F}| \leq |M^{\mathfrak{x}}|$.
Also $y = y1^{\mathfrak{x}} \notin M^{\mathfrak{x}}$ follows from Lemma 4.2 (b), and thus $|M^{\mathfrak{y}} \setminus M^{\mathfrak{x}}| = |M^{\mathfrak{x}}|$.

Condition **(c)** of the theorem is obvious.

In order to show **(d)** we first try $y = \sum_{i \in \omega} p^i e_i$ and let $M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}} \rangle_{*p}$.
If $y\eta^{\mathfrak{y}} \notin M^{\mathfrak{y}}$, then the proof is finished. Thus assume $y\eta^{\mathfrak{y}} \in M^{\mathfrak{y}}$. In particular $y\eta^{\mathfrak{x}} \in M^{\mathfrak{y}} = \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}} \rangle_{*p}$, hence $p^n y\eta^{\mathfrak{x}} = m' + yf^{\mathfrak{x}}$ holds for some $n \in \omega$, $m' \in M^{\mathfrak{x}}$ and $f \in RF$. Next we claim:

There is $m \in M^{\mathfrak{x}}$ with $p^n m\eta^{\mathfrak{x}} \neq mf^{\mathfrak{x}}$. **(11)**

Proof: Suppose $p^n m\eta^{\mathfrak{x}} = mf^{\mathfrak{x}}$ for all $m \in M^{\mathfrak{x}}$.
If $k \in \omega$, choose $m = e_k \in M^{\mathfrak{x}}$ and $f \in RF_k$ with $p^n e_k\eta^{\mathfrak{x}} = e_k f^{\mathfrak{x}} = e_k f$. Hence $p^n | e_k f$ and $p^n | f$. Thus $f = p^n g$ for some $g \in RF$, and for all $m \in M^{\mathfrak{x}}$ follows $p^n m\eta^{\mathfrak{x}} = mf^{\mathfrak{x}} = p^n mg^{\mathfrak{x}}$. It follows that $m\eta^{\mathfrak{x}} = mg^{\mathfrak{x}}$ and $\eta^{\mathfrak{x}} = g^{\mathfrak{x}} \in \pi^{\mathfrak{x}}(RF)$ contradicting the choice of $\eta^{\mathfrak{x}}$. Thus (11) holds.

There is $\widehat{r} \in \widehat{R}$ with $\widehat{r}(p^n m\eta^{\mathfrak{x}} - mf^{\mathfrak{x}}) \notin M^{\mathfrak{x}}$. **(12)**

Proof: By (11) we conclude $\lambda := p^n m\eta^{\mathfrak{x}} - mf^{\mathfrak{x}} \neq 0$. Suppose $\widehat{r}\lambda \in M^{\mathfrak{x}}$ for all $\widehat{r} \in \widehat{R}$.
Choose an $r_i \in R$ $(i \in \omega)$ with $\widehat{r} - r_i \in p^i\widehat{R}$. Thus $r_i\lambda = (r_i - \widehat{r})\lambda \in p^i\widehat{M^{\mathfrak{x}}} \cap M^{\mathfrak{x}}$ and $r_i\lambda \in p^i M^{\mathfrak{x}}$ by purity. Note that $0 \neq \lambda \in \widehat{M^{\mathfrak{x}}}$ has finite p-height, but $r_i \in p^k R$ $(i \in \omega)$ for any $k \in \omega$ large enough, thus $\widehat{r} = 0$.
It follows that the map $\widehat{R} \to M^{\mathfrak{x}}$ $(\widehat{r} \mapsto \widehat{r}\lambda)$ is a monomorphism. **(13)**

The module $M^{\mathfrak{x}}$ is free and R is p-cotorsion-free by assumption. Thus (13) is impossible and there is $\widehat{r} \in \widehat{R}$ with $\widehat{r}\lambda \notin M^{\mathfrak{x}}$.

We now try another branch for $M^{\mathfrak{y}}$ and will succeed.

Let $y' := \widehat{r}m + y = \widehat{r}m + \sum_{i\in\omega} p^i e_i$ with m, $\widehat{r}$ as above and let $M^{\mathfrak{y}'} := \langle M^{\mathfrak{x}}, y'RF^{\mathfrak{x}}\rangle_{*p}$, $\eta^{\mathfrak{y}'} := \eta^{\mathfrak{x}} \restriction M^{\mathfrak{y}'}$. We now claim:

$$y'\eta^{\mathfrak{y}'} \notin M^{\mathfrak{y}'}. \tag{14}$$

Proof: If $y'\eta^{\mathfrak{y}'} \in M^{\mathfrak{y}'}$, then $p^{n'}y'\eta^{\mathfrak{x}} = m'' + y'g^{\mathfrak{x}}$ for some $n' \in \omega$, $m'' \in M^{\mathfrak{x}}$ and $g \in RF$. The two equations $p^n y\eta^{\mathfrak{x}} = m' + yf^{\mathfrak{x}}$, $p^{n'}y'\eta^{\mathfrak{x}} = m'' + y'g^{\mathfrak{x}}$ connected by $y' = \widehat{r}m + y$ give rise to $p^{n+n'}\widehat{r}\cdot m\eta^{\mathfrak{x}} = p^{n+n'}(\widehat{r}m)\eta^{\mathfrak{x}} = p^{n+n'}[y'\eta^{\mathfrak{x}} - y\eta^{\mathfrak{x}}] = p^n(m'' + y'g^{\mathfrak{x}}) - p^{n'}(m' + yf^{\mathfrak{x}}) = y(p^n g^{\mathfrak{x}} - p^{n'}f^{\mathfrak{x}}) + p^n\widehat{r}\cdot mg^{\mathfrak{x}} + p^n m'' - p^{n'}m'$, hence

$$y(p^n g^{\mathfrak{x}} - p^{n'}f^{\mathfrak{x}}) = p^{n+n'}\widehat{r}\cdot m\eta^{\mathfrak{x}} - p^n\widehat{r}\cdot mg^{\mathfrak{x}} - p^n m'' + p^{n'}m' \in M^{\mathfrak{y}} \cap \widehat{M_k}, \tag{15}$$

where $k \in \omega$ is chosen with $m\eta^{\mathfrak{x}}$, $mg^{\mathfrak{x}}$, m', $m'' \in M_k$.

The module $M^{\mathfrak{y}}/M_k$ is free by (6), hence M_k is p-adically closed in $M^{\mathfrak{y}}$ and $M^{\mathfrak{y}} \cap \widehat{M_k} = M_k$ is immediate. With (15) and Lemma 4.2 (b) we obtain $y(p^n g^{\mathfrak{x}} - p^{n'}f^{\mathfrak{x}}) \in M_k \subseteq M^{\mathfrak{x}}$ and $p^n g - p^{n'}f = 0$.

Substituting $p^n g = p^{n'}f$ into (15) gives

$p^{n+n'}\widehat{r}\cdot m\eta^{\mathfrak{x}} = y(p^n g^{\mathfrak{x}} - p^{n'}f^{\mathfrak{x}}) + p^n\widehat{r}\cdot mg^{\mathfrak{x}} + p^n m'' - p^{n'}m' = p^{n'}\widehat{r}\cdot mf^{\mathfrak{x}} + p^n m'' - p^{n'}m'$

Therefore $p^{n'}\widehat{r}(p^n m\eta^{\mathfrak{x}} - mf^{\mathfrak{x}}) = p^n m'' - p^{n'}m' \in M^{\mathfrak{x}}$, and $\widehat{r}(p^n m\eta^{\mathfrak{x}} - mf^{\mathfrak{x}}) \in M^{\mathfrak{x}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$ contradicts (12). Thus (14) and (d) follow. □

We continue to assume the algebraic preliminaries from Section 4.2.

Theorem 4.13 (Step-Lemma, $\aleph_1$-Free Case)

Let $(\mathfrak{x}_i)_{i\in\omega}$ be a proper chain in $(\mathfrak{A}^*, \sqsubseteq)$ with $\mathfrak{x}_i = (M_i, \mathfrak{F}_i, \pi_i)$ and $\mathfrak{x}_\omega = \bigcup_{i\in\omega}\mathfrak{x}_i = (M_\omega, \mathfrak{F}, \pi_\omega) \in \mathfrak{A}^*$. Moreover let $(B_i)_{i\in\omega}$ be a chain of free R-modules with $B_i \subseteq_{*p} M_i \subseteq_{*p} \widehat{B_i}$ and $B_{i+1} = B_i \oplus D_i$ for all $i \in \omega$. Set $B := \bigcup_{i\in\omega} B_i$ and let $\mathfrak{x}_\omega \subseteq \mathfrak{x} := (M^{\mathfrak{x}}, \mathfrak{F}, \pi^{\mathfrak{x}}) \in \mathfrak{A}^*$ with $B \subseteq_{*p} M_\omega \subseteq_* M^{\mathfrak{x}} \subseteq_{*p} \widehat{B}$ and $\mathfrak{x}_i \sqsubseteq \mathfrak{x}$ for all $i \in \omega$. Also let $RF_i \cong e_iRF_i \sqsubseteq D_i$ for all $i \in \omega$ with $\pi_{i+1}(\varphi_t) \restriction e_iRF_i = \varphi_t$ for all $\varphi_t \in \mathfrak{F}_i$.

If $y = \widehat{r}m + \sum_{i\in\omega} p^i e_i$ is a branch of $\widehat{M^{\mathfrak{x}}}$ such that

support $[m]$ is finite for all $m \in M^{\mathfrak{x}}$, **(+)**

set $M^{\mathfrak{y}} := \langle M^{\mathfrak{x}}, yRF^{\mathfrak{x}}\rangle_{*p}$ and $\pi^{\mathfrak{y}}(\varphi_t) = \varphi_t^{\mathfrak{y}} := \varphi_t^{\mathfrak{x}} \restriction M^{\mathfrak{y}}$ for all $\varphi_t \in \mathfrak{F}$.

Then $\mathfrak{y} := (M^{\mathfrak{y}}, \mathfrak{F}, \pi^{\mathfrak{y}})$ satisfies the following conditions:

(a) $\mathfrak{x} \subseteq \mathfrak{y} \in \mathfrak{A}^*$.

(b) $|M^{\mathfrak{y}}| = |M^{\mathfrak{y}} \setminus M^{\mathfrak{x}}| = |M^{\mathfrak{x}}|$ *and* $\mathfrak{x}_n \sqsubseteq \mathfrak{y}$ *for all* $n \in \omega$.

(c) $M^{\mathfrak{y}} \subseteq_{*p} \widehat{M^{\mathfrak{x}}}$ *and* $M^{\mathfrak{y}}/M^{\mathfrak{x}} \neq 0$ *is p-divisible, in particular* $\mathfrak{x} \not\sqsubseteq \mathfrak{y}$.

(d) If $\eta^{\mathfrak{x}} : U \to M^{\mathfrak{x}}$ *is a homomorphism with* $P \subseteq U \subseteq M^{\mathfrak{x}}$, $P := \bigoplus_{i\in\omega} Re_i$, $\eta^{\mathfrak{x}} \neq f^{\mathfrak{x}} \restriction U$ *for all* $f \in RF$, *and* $\eta^{\mathfrak{y}} := \widehat{\eta^{\mathfrak{x}}} \restriction M^{\mathfrak{y}} \cap \widehat{U}$, *then we can choose* $y \in \widehat{M^{\mathfrak{x}}}$ *with* $y\eta^{\mathfrak{y}} \notin M^{\mathfrak{y}}$, *i.e.* $\eta^{\mathfrak{x}}$ *does not extend to an endomorphism of* $M^{\mathfrak{y}}$.

Proof: The proof is similar to the last one, thus we will be brief on it.

(a): Obviously $M^{\mathfrak{x}} \subseteq M^{\mathfrak{y}}$ and $\pi^{\mathfrak{x}} \subseteq \pi^{\mathfrak{y}}$.

Let $C_i := \{m \in M^{\mathfrak{x}} | \ [m] \subseteq i\} \subseteq M^{\mathfrak{y}}$, $D_i := \langle y^i_\varphi | \ \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq i\rangle_R$ for all $i \in \omega$ and define $H_i := C_i + D_i \subseteq \widehat{M^{\mathfrak{x}}}$. First we show:

$(H_i)_{i\in\omega}$ *is a pure, ascending chain of* $\aleph_1$*-free submodules with* $\bigcup_{i\in\omega} H_i = M^{\mathfrak{y}}$, **(1)**

starting with

$H_i = C_i \oplus D_i \ (i \in \omega)$. **(2)**

Proof: If $m = \sum_{\varphi\in\langle\mathfrak{F}\rangle,\ k(\varphi)\leq i} \lambda_\varphi y^i_\varphi \in C_i \cap D_i$, then the RF_i-component of m is $\sum_{\varphi\in\langle\mathfrak{F}\rangle,\ k(\varphi)\leq i} \lambda_\varphi e_i\varphi = 0$, thus $\lambda_\varphi = 0$ for all $i \in \omega$ and $m = \sum_{\varphi\in\langle\mathfrak{F}\rangle,\ k(\varphi)\leq i} \lambda_\varphi y^i_\varphi = 0$. Claim (2) follows.

$H_i \subseteq H_{i+1}$ *for all* $i \in \omega$ *and* $\bigcup_{i\in\omega} H_i = M^{\mathfrak{y}}$. **(3)**

Proof: $H_i = C_i + D_i \subseteq C_{i+1} + D_{i+1} = H_{i+1}$ follows from $C_i \subseteq C_{i+1}$ and the fact that $y^i_\varphi = (r_i m + e_i)\varphi^{\mathfrak{x}} + p y^{i+1}_\varphi \in H_{i+1}$ with $r_i m\varphi^{\mathfrak{x}} + e_i\varphi \in C_{i+1}$, $y^{i+1}_\varphi \in D_{i+1}$ for all $y^i_\varphi \in D_i$. Thus $\bigcup_{i\in\omega} H_i = \bigcup_{i\in\omega} C_i + \bigcup_{i\in\omega} D_i = M^{\mathfrak{x}} + \langle y^k_\varphi | \ \varphi \in \langle\mathfrak{F}\rangle, k(\varphi) \leq k < \omega\rangle$, which is $M^{\mathfrak{y}}$ by Lemma 4.8 (b). Now we show the purity of the chain:

$H_i \subseteq_* H_{i+1} \ (i \in \omega)$. **(4)**

Proof: Let q be a prime and $h \in H_{i+1}$, $h' \in H_i$ with $qh = h'$.

Then $h = c + e_i f + \sum_{\varphi\in\langle\mathfrak{F}\rangle,\ k(\varphi)\leq i+1} r_\varphi y^{i+1}_\varphi = c + e_i f + y^{i+1}_1 g^{\mathfrak{x}}$

and $h = c' + \sum_{\varphi\in\langle\mathfrak{F}\rangle,\ k(\varphi)\leq i} r'_\varphi y^i_\varphi = c' + y^i_1 g'^{\mathfrak{x}}$ for suitable $c' \in C_i$, $c + e_i f \in C_{i+1}$ with $c \in \widehat{M^{\mathfrak{x}}}$, $[c] \subseteq i$ and $f \in \widehat{RF_i}$, $r_\varphi, r'_\varphi \in R$, $g' \in RF_i$ and $g \in RF_{i+1}$. We therefore have

$q(c + e_i f + y^{i+1}_1 g^{\mathfrak{x}}) = c' + y^i_1 g'^{\mathfrak{x}} = c' + r_i m g'^{\mathfrak{x}} + e_i g' + p y^{i+1}_1 g'^{\mathfrak{x}}$. **(5)**

The $\widehat{RF_i}$-component of (5) gives $qf = g'$, while the $\widehat{RF_{i+1}}$-component of (5) is $qg = pg'$. Thus $qg = pqf$ and $pf = g \in p\widehat{RF_i} \cap RF_{i+1} = pRF_i$.

Hence $f \in RF_i$ with $g' = qf$ and $g = pf$ holds, and $c = (c + e_i f) - e_i f \in C_i$ follows. Combining this, we get purity

$$h = c + e_i f + y_1^{i+1} g^{\mathfrak{x}} = c + (e_i + p y_1^{i+1}) f^{\mathfrak{x}} = c - r_i m f^{\mathfrak{x}} + (r_i m + e_i + p y_1^{i+1}) f^{\mathfrak{x}}$$
$$= c - r_i m f^{\mathfrak{x}} + y_1^i f^{\mathfrak{x}} = c - r_i m \frac{g'^{\mathfrak{x}}}{q} + y_1^i \frac{g'^{\mathfrak{x}}}{q} \in H_i.$$

We finish the proof of (1) with

H_i/M_j and H_i are $\aleph_1$-free for $j \leq i$. **(6)**

Proof: If $j \leq i$, then $M_j \subseteq C_j \subseteq C_i \subseteq H_i$ and therefore

$$H_i/M_j = (C_i \oplus D_i)/M_j \cong C_i/M_j \oplus D_i.$$

Now $C_i/M_j \subseteq M^{\mathfrak{x}}/M_j$ is $\aleph_1$-free by the preliminaries, while D_i is free using Lemma 4.11 (a), hence H_i/M_j is $\aleph_1$-free. This shows half of (6).

Moreover M_j is $\aleph_1$-free by assumption on $\mathfrak{x}_j$, thus H_i is $\aleph_1$-free as well.

$M^{\mathfrak{y}}$ is $\aleph_1$-free. **(7)**

This follows trivially from (1) and Pontryagin's Theorem.

From the proof of Theorem 4.12 we also see, that for all $\varphi_t \in \mathfrak{F}$ the map $\varphi_t^{\mathfrak{y}} \in \operatorname{Aut} M^{\mathfrak{y}}$ is well defined, and $\pi^{\mathfrak{y}}$ satisfies the U-property for $M^{\mathfrak{y}}$. Thus (a) of the theorem is shown.

(b): In order to show $\mathfrak{x}_n \sqsubseteq \mathfrak{y}$ $(n \in \omega)$, we note that $\mathfrak{x}_n \subseteq \mathfrak{x} \subseteq \mathfrak{y}$ and by (a) it is enough to prove that $M^{\mathfrak{y}}/M_n$ is $\aleph_1$-free:

From (1) and (6) follows that $(H_i/M_n)_{i \in \omega,\ i \geq n}$ is a pure, ascending chain of $\aleph_1$-free modules with $M^{\mathfrak{y}}/M_n$. Thus also $M^{\mathfrak{y}}/M_n$ is $\aleph_1$-free, as required.

$|M^{\mathfrak{y}}| = |M^{\mathfrak{y}} \setminus M^{\mathfrak{x}}| = |M^{\mathfrak{x}}|$ follows by the same easy argument as in Theorem 4.12.

(c): See Theorem 4.12.

(d): From branch $\widehat{r}m + y = \widehat{r}m + \sum_{i\in\omega} p^i e_i \in \widehat{P}$ follows that $y\widehat{\eta^{\mathfrak{x}}}$ is well defined.

As in Theorem 4.12 the following claims are immediate:

There exists an $m \in U$ *with* $p^n m\eta^{\mathfrak{x}} \neq mf^{\mathfrak{x}}$. **(11)**

There exists an $\widehat{r} \in \widehat{R}$ *with* $\widehat{r}(p^n m\eta^{\mathfrak{x}} - mf^{\mathfrak{x}}) \notin M^{\mathfrak{x}}$, **(12)**

and we show

$y'\eta^{\mathfrak{y}'} \notin M^{\mathfrak{y}'}$. **(14)**

Again we compare y and y':

$y(p^n g^{\mathfrak{x}} - p^{n'} f^{\mathfrak{x}}) = p^{n+n'}\widehat{r} \cdot m\eta^{\mathfrak{x}} - p^n \widehat{r} \cdot mg^{\mathfrak{x}} - p^n m'' + p^{n'} m'$

Observe that the support

$[p^{n+n'}\widehat{r} \cdot m\eta^{\mathfrak{x}} - p^n \widehat{r} \cdot mg^{\mathfrak{x}} - p^n m'' + p^{n'} m'] \subseteq [m\eta^{\mathfrak{x}}] \cup [mg^{\mathfrak{x}}] \cup [m''] \cup [m']$ is finite.

Thus also $[y(p^n g^{\mathfrak{x}} - p^{n'} f^{\mathfrak{x}})]$ is finite and an easy support argument shows $p^n g - p^{n'} f = 0$.

Then we can follow the proof of Theorem 4.12 to get (14). □

5 The Main Construction

In this section we adjoin various bits needed for Theorem 2.28.

The next lemma will ensure that automorphisms coming from $\mathfrak{F}$ also act on many copies of $R\langle\mathfrak{F}\rangle \subseteq M$ naturally by scalar multiplication.

Lemma 5.1 (RF-Construction)
Let be $\mathfrak{x} = (M^{\mathfrak{x}}, \mathfrak{F}, \pi^{\mathfrak{x}}) \in \mathfrak{A}$ $(\mathfrak{A}^*)$ *with* $\mathfrak{F} = \{\varphi_t | t \in J\}$. *For any* $t \in J$ *define* $\varphi_t^{\mathfrak{y}} := \varphi_t^{\mathfrak{x}} \times \varphi_t \in \operatorname{Aut} M^{\mathfrak{x}} \times \operatorname{Aut} R\langle\mathfrak{F}\rangle \subseteq \operatorname{Aut} M^{\mathfrak{y}}$. *Set* $M^{\mathfrak{y}} := M^{\mathfrak{x}} \times R\langle\mathfrak{F}\rangle$ *and define* $\pi^{\mathfrak{y}} := \bigcup_{t \in J} (\varphi_t, \varphi_t^{\mathfrak{y}})$ *as graph. Then* $\mathfrak{x} \sqsubseteq \mathfrak{y} := (M^{\mathfrak{y}}, \mathfrak{F}, \pi^{\mathfrak{y}}) \in \mathfrak{K}$ *holds, where we identify* $M^{\mathfrak{x}}$ *with* $(M^{\mathfrak{x}}, 0) \subseteq M^{\mathfrak{y}}$.

In this case we write $M^{\mathfrak{y}} = M^{\mathfrak{x}} \oplus eRF$, let $e = (0, 1)$ and note that $e \cdot r = r \cdot e$ commutes for all $r \in R$, thus eRF is a left R-module and $\varphi_t^{\mathfrak{y}} \restriction eRF = \varphi_t$ holds.

Proof of Lemma 5.1:

$M^{\mathfrak{y}}$ and $M^{\mathfrak{y}}/M^{\mathfrak{x}}$ are free. **(1)**

Proof: The R-module $M^{\mathfrak{y}}/M^{\mathfrak{x}} \cong R\langle\mathfrak{F}\rangle \cong \bigoplus_{m \in \langle\mathfrak{F}\rangle} R \cdot m$ is obviously free.
With $M^{\mathfrak{x}}$ and $M^{\mathfrak{y}}/M^{\mathfrak{x}}$ also $M^{\mathfrak{y}}$ is free. We also have:

$\pi^{\mathfrak{y}}$ satisfies the U-property on $M^{\mathfrak{y}}$. **(2)**

Proof: Let $z = (z_1, z_2) \in M^{\mathfrak{y}}$ and $0 \neq f \in R\langle\mathfrak{F}\rangle$ with $z_1 \in M^{\mathfrak{x}}$, $z_2 \in R\langle\mathfrak{F}\rangle$, $zf^{\mathfrak{y}} = 0$ be given. From $zf^{\mathfrak{y}} = 0$ follows $z_1 f^{\mathfrak{x}} = 0$ and $z_2 f = 0$, thus $z_2 = 0$. We conclude $z = (z_1, z_2) = (z_1, 0) \in M^{\mathfrak{x}}$.
Thus we have shown for any $0 \neq f \in R\langle\mathfrak{F}\rangle$ that $\ker f^{\mathfrak{y}} = \ker f^{\mathfrak{x}} \subseteq M^{\mathfrak{x}}$.
Now Definition 3.5 (i) of the U-property for $\pi^{\mathfrak{x}}$ carries over to $\pi^{\mathfrak{y}}$. Definition 3.5 (ii) holds automatically for automorphisms. Thus (2) follows, and the remaining assertions in Lemma 5.1 are obvious. $\square$

The following corollary provides an easy but useful topological argument.

Corollary 5.2 *Let $M \subseteq M'$ be two R-modules such that M' and M'/M are p-reduced. Then M is p-adically closed in M'.*

Proof: Let m_i $(i \in \omega)$ be a p-adic Cauchy sequence in M converging to $m \in M'$.
We may assume $(m - m_i) \in p^i M'$ for all $i \in \omega$. Thus we have
$m + M = (m - m_i) + M \in p^i(M'/M)$ for all $i \in \omega$ and $m + M = M$ because M'/M is p-reduced, hence $m \in M$. $\square$

Now we present the **Constructions** of the modules for Theorem 2.28.

5.1 The Free Case

Let $\kappa > \aleph_0$ be a regular, non-reflecting cardinal and R be a PID with $|R| < \kappa$ and $R \neq \widehat{R}_p$ for some prime element $p \in R$. Choose a set M of cardinality $|M| = \kappa$. Also choose a κ-filtration $M = \bigcup_{\alpha<\kappa} M_\alpha$ with

$$|M_0| = |M_1 \setminus M_0| = |R| \text{ and } |M_\alpha| = |M_{\alpha+1} \setminus M_\alpha| = |R||\alpha| \text{ for all } 0 < \alpha < \kappa \;. \qquad (*)$$

Let $E \subseteq \kappa^o = \{\alpha \in \kappa | \operatorname{cf}(\alpha) = \aleph_0\}$ be a non-reflecting, stationary set and $\{\Phi_\alpha : M_\alpha \to M_\alpha | \alpha \in E\}$ be a system of Jensen-functions for the κ-filtration $M = \bigcup_{\alpha<\kappa} M_\alpha$.

We want to assign inductively a module structure to the sets M_α such that we get a continuous chain $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ in $(\mathfrak{A}, \subseteq)$ with $\mathfrak{x}_\alpha = (M_\alpha, \mathfrak{F}_\alpha, \pi_\alpha)$. Thus M_α will be a left R-module and a right $\operatorname{End} M_\alpha$-module.
The union $M := \bigcup_{\alpha<\kappa} M_\alpha$ of this continuous chain will satisfy condition (a) of Theorem 2.28 as left R-module.
We will carry out the following steps inductively.

Let $\mathfrak{x}_0 := (M_0, \mathfrak{F}_0, \pi_0) \in \mathfrak{A}$ with $M_0 := R \oplus R$, $\mathfrak{F}_0 := \emptyset$ and $\pi_0 := \emptyset$.

Suppose that the structure on M_β $(\beta < \alpha)$ is defined.
Case 1: $\alpha = \beta + 1,\ \beta \notin E$.

First construct $\mathfrak{x}_\beta \sqsubseteq \mathfrak{y}_\beta := (M'_\beta, \mathfrak{F}_\beta, \pi'_\beta) \in \mathfrak{A}$ using the RF-construction and then $\mathfrak{y}_\beta \sqsubseteq \mathfrak{x}_\alpha \in \mathfrak{A}$ using the UT-construction.

Case 2: $\alpha = \beta + 1,\ \beta \in E$.

- If the Jensen-function $\Phi_\beta : M_\beta \to M_\beta$ is an R-homomorphism from $\operatorname{End} M_\beta \backslash \pi_\beta(R\langle\mathfrak{F}_\beta\rangle)$ construct $\mathfrak{x}_\beta \subseteq \mathfrak{x}_\alpha$ using the Step-Lemma 4.12 such that $y\Phi_\alpha \notin M_\alpha$ for some $y \in M_\alpha$, where Φ_α is the uniquely defined extension of Φ_β to M_α.
  For the identifications needed here see also the proof of Lemma 5.3, Case 2.

- Otherwise use the construction described in Case 1.

Case 3: $\alpha \in \text{LORD} \cap \kappa$.

Set $\mathfrak{x}_\alpha := \bigcup_{\beta<\alpha} \mathfrak{x}_\beta$.

We deduce some easy facts about the constructed chain $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$.

Lemma 5.3 *Let $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ be constructed as above.*
(a) $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ is a well-defined continuous chain in $(\mathfrak{A}, \subseteq)$.
(b) If $\beta \le \alpha < \kappa$, $\beta \notin E$, then $\mathfrak{x}_\beta \sqsubseteq \mathfrak{x}_\alpha$.
(c) $|M_0| = |M_1 \setminus M_0| = |R|$ and $|M_\alpha| = |M_{\alpha+1} \setminus M_\alpha| = |R||\alpha|$ holds for all $0 < \alpha < \kappa$, i.e. $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ satisfies (*) of the κ-filtration $M = \bigcup_{\alpha<\kappa} M_\alpha$.

Proof: We prove (a), (b) and (c) simultaneously by transfinite induction on $\alpha < \kappa$.
For $\alpha = 0$ properties (a), (b) and (c) are obvious.
Thus let $0 < \alpha < \kappa$ and suppose that the properties (a), (b) and (c) hold for all ordinals less than α. We will distinguish three different cases as above.
Case 1: $\alpha = \beta + 1,\ \beta \notin E$.
From $\mathfrak{x}_\beta \sqsubseteq \mathfrak{y}_\beta \sqsubseteq \mathfrak{x}_\alpha \in \mathfrak{A}$ and the induction hypothesis follows $\mathfrak{x}_\gamma \subseteq \mathfrak{x}_\alpha$ for all $\gamma \le \alpha$ and $\mathfrak{x}_\delta \sqsubseteq \mathfrak{x}_\alpha$ for all $\delta \le \alpha$, $\delta \notin E$.
For the RF-construction $M'_\beta = M_\beta \oplus R\langle \mathfrak{F}_\beta \rangle$ follows $|M_\beta| \le |M'_\beta| = |M_\beta| \cdot |R| \cdot |\langle \mathfrak{F}_\beta \rangle| = |M_\beta| \cdot |\mathfrak{F}_\beta| \le |M_\beta|^2 = |M_\beta|$, $|M'_\beta| = |M_\beta|$ (observe that $\aleph_0 \le |R| \le |M_\beta|$, $|\mathfrak{F}_\beta| \le |M_\beta|$).
For the next UT-construction follows $|M_\alpha| = |M'_\beta| = |M_\beta| = |R||\beta| = |R||\alpha|$.
By UT-construction baby-automorphisms will be extended to total automorphisms via Dom- and Im-pushouts. Thus $|M_\alpha \setminus M_\beta| = |M_\beta| = |R||\beta|$.
Case 2: $\alpha = \beta + 1,\ \beta \in E$.
If $\Phi_\beta \notin \operatorname{End} M_\beta \setminus \pi_\beta(R\langle \mathfrak{F}_\beta \rangle)$, the claims (a), (b) and (c) follow similar to Case 1. Therefore suppose $\Phi_\beta \in \operatorname{End} M_\beta \setminus \pi_\beta(R\langle \mathfrak{F}_\beta \rangle)$.
From $\beta \in E \subseteq \kappa^o$ we obtain an ascending chain $(\beta_i)_{i\in\omega}$ of ordinals $0 < \beta_i \notin E \subseteq \text{LORD}$ with $\bigcup_{i\in\omega} \beta_i = \beta$. By induction hypothesis we defined for the chain $(\beta_i)_{i\in\omega}$ a sequence $(\mathfrak{x}_{\beta_i})_{i\in\omega}$ of triples in $(\mathfrak{A}, \sqsubseteq)$ with $\bigcup_{i\in\omega} \mathfrak{x}_{\beta_i} = \mathfrak{x}_\beta$. In particular $\mathfrak{x}_{\beta_i} \sqsubseteq \mathfrak{y}_{\beta_i} \sqsubseteq \mathfrak{x}_{\beta_i+1} \sqsubseteq \mathfrak{x}_{\beta_{(i+1)}}$ for all $i \in \omega$, where $M'_{\beta_i} = M_{\beta_i} \oplus e_i R\langle \mathfrak{F}_{\beta_i} \rangle \sqsubseteq M_{\beta_{(i+1)}}$ with $\pi_{\beta_{(i+1)}}(\varphi_t) \restriction e_i R\langle \mathfrak{F}_{\beta_i} \rangle = \varphi_t$

for all $\varphi_t \in \mathfrak{F}_{\beta_i}$ using the RF-construction. Thus for all $i \in \omega$ there is a representation of $M_{\beta_{(i+1)}} = M_{\beta_i} \oplus D_i$ with $R\langle\mathfrak{F}_{\beta_i}\rangle \cong e_i R\langle\mathfrak{F}_{\beta_i}\rangle \sqsubseteq D_i$.

Thus $(\mathfrak{x}_{\beta_i})_{i\in\omega}$ and the endomorphism Φ_β satisfy the preliminaries of the Step-Lemma 4.12, which now applies: There is a suitable $\mathfrak{x}_\beta \subseteq \mathfrak{x}_\alpha$ by this lemma.

It follows that $\mathfrak{x}_\gamma \subseteq \mathfrak{x}_\alpha$ for all $\gamma \leq \alpha$. If $\delta < \alpha$, $\delta \notin E$, then there is an $i \in \omega$ with $\delta < \beta_i$ and $\mathfrak{x}_\delta \sqsubseteq \mathfrak{x}_{\beta_i}$ by induction hypothesis. Using Step-Lemma 4.12 (b) we deduce $\mathfrak{x}_{\beta_i} \sqsubseteq \mathfrak{x}_\alpha$, thus $\mathfrak{x}_\delta \sqsubseteq \mathfrak{x}_\alpha$.

Also Lemma 4.12 (b) implies $|M_\alpha| = |M_\alpha \setminus M_\beta| = |M_\beta| = |R||\beta| = |R||\alpha|$.

Case 3: Let $\alpha < \kappa$ be a limit cardinal.

By continuity $\mathfrak{x}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{x}_\beta$. We can write $\alpha = \bigcup_{i<\rho} \beta_i$ with $\rho = \mathrm{cf}\,(\alpha)$ and $\beta_i \in \alpha$. We also may assume $\beta_i \notin E$. If $\rho = \aleph_0$, then replace β_i by $\beta_i + 1$ if necessary, and if $\rho > \aleph_0$, then apply that E is not reflexive. There is a corresponding chain $(\mathfrak{x}_{\beta_i})_{i\in\lambda}$ in $(\mathfrak{A}, \sqsubseteq)$ with $\bigcup_{i\in\lambda} \mathfrak{x}_{\beta_i} = \mathfrak{x}_\alpha$. Hence $\mathfrak{x}_{\beta_i} \sqsubseteq \mathfrak{x}_\alpha \in \mathfrak{A}$ for all $i \in \lambda$ by Lemma 3.15, and M_α is a free R-module. Now Lemma 5.3 (b) is immediate.

Finally $|M_\alpha| = |M_0| + \sum_{\beta<\alpha} |M_{\beta+1} \setminus M_\beta| = |R| + \sum_{\beta<\alpha} |R||\beta| = |R||\alpha|$ holds. □

We want to prove some basic properties of the R-module $M := \bigcup_{\alpha<\kappa} M_\alpha$.

Theorem 5.4 *If $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ is the chain as above and $\mathfrak{x} = \bigcup_{\alpha<\kappa} \mathfrak{x}_\alpha = (M, \mathfrak{F}, \pi)$, then the following holds.*

(a) $|M| = \kappa$ and $\pi(R^* \times F) \subseteq \mathrm{Aut}\, M$ acts uniquely transitive on $\mathfrak{p}M$.

(b) M is strongly κ-free.

(c) If $\Phi \in \mathrm{End}\, M$, then there are a stationary set $E_\Phi \subseteq \kappa$ and a sequence $(f_\alpha)_{\alpha\in E_\Phi}$ with $f_\alpha \in R\langle\mathfrak{F}_\alpha\rangle$ such that $\Phi \restriction M_\alpha = \pi_\alpha(f_\alpha)$ holds for all $\alpha \in E_\Phi$.

In particular $M_\alpha \pi_\beta(f_\alpha - f_\beta) = 0$ holds for all $\alpha \leq \beta < \kappa$.

Proof:

(a): $|M| = \kappa$ and $\pi(R^* \times F) \subseteq \mathrm{Aut}\, M$ is obvious. Thus $\mathfrak{p}M = \bigcup_{\alpha<\kappa} \mathfrak{p}M_\alpha$ by Lemma 5.3 (b). If $\alpha < \kappa$, $\alpha \notin E$, then $\pi_{\alpha+1}(R^* \times \langle\mathfrak{F}_{\alpha+1}\rangle) \subseteq \mathrm{Aut}\, M_{\alpha+1}$ acts uniquely transitive on $\mathfrak{p}M_{\alpha+1}$, thus $\pi(R^* \times F) \subseteq \mathrm{Aut}\, M$ acts uniquely transitive on $\mathfrak{p}M$.

(b): Let $U \subseteq M/M_{\alpha+1}$ a submodule with $|U| < \kappa$ for all $\alpha < \kappa$. There is $\alpha < \beta < \kappa$, with $U \subseteq M_\beta/M_{\alpha+1}$, because κ is regular. By Lemma 5.3 (b) follows $\mathfrak{x}_{\alpha+1} \sqsubseteq \mathfrak{x}_\beta$ and $U \subseteq M_\beta/M_{\alpha+1}$ is free.

By Lemma 2.12 (b) and the filtration $M := \bigcup_{\alpha\in\kappa} M_\alpha$ the module M must be κ-free.

(c): If $\Phi \in \operatorname{End} M$, then $\diamondsuit_\kappa(E)$ provides a stationary set $E_\Phi := \{\alpha \in E \mid \Phi \restriction M_\alpha = \Phi_\alpha\}$ (Φ_α is the Jensen-function for α). We now claim

$\Phi \restriction M_\alpha \in \pi_\alpha(R\langle\mathfrak{F}_\alpha\rangle)$ *holds for all* $\alpha \in E_\Phi$. **(1)**

Proof: If $\alpha \in E_\Phi \subseteq E$ with $\Phi_\alpha = \Phi \restriction M_\alpha \in \operatorname{End} M_\alpha \setminus \pi_\alpha(R\langle\mathfrak{F}_\alpha\rangle)$, then $\mathfrak{x}_{\alpha+1}$ is constructed from $\mathfrak{x}_\alpha$ with the Step-Lemma, thus $M_{\alpha+1} \subseteq_{*p} (\widehat{M_\alpha})_p$ holds.

We can write any $m \in M_{\alpha+1}$ as limit of a sequence $(m_i)_{i\in\omega} \subseteq M_\alpha$ in the p-adic topology on $M_{\alpha+1} \subseteq_{*p} (\widehat{M_\alpha})_p$. Thus $(\Phi_\alpha(m_i))_{i\in\omega} \subseteq M_\alpha$ converges to $\Phi(m)$ by continuity. By Theorem 5.4 (b) $M_{\alpha+1} \subseteq M$ and $M/M_{\alpha+1}$ are κ-free and $M_{\alpha+1}$ is p-adically closed in M, thus $\Phi(m) \in M_{\alpha+1}$ (see Corollary 5.2). Hence $\Phi(M_{\alpha+1}) \subseteq M_{\alpha+1}$; thus $\Phi_\alpha = \Phi \restriction M_\alpha$ and Φ_α extends to $\Phi \restriction M_{\alpha+1} \in \operatorname{End}(M_{\alpha+1})$, which contradicts Lemma 4.12 (d).

Thus $\Phi \restriction M_\alpha = \pi_\alpha(f_\alpha) \in \pi_\alpha(R\langle\mathfrak{F}_\alpha\rangle)$ for some $f_\alpha \in R\langle\mathfrak{F}_\alpha\rangle$ and (1) follows. Finally, we claim

$M_\alpha\pi_\beta(f_\alpha - f_\beta) = 0$ *for all* $\alpha \leq \beta < \kappa$. **(2)**

Proof: Observe that π_β is a weak extension of π_α, thus

$M_\alpha\pi_\beta(f_\alpha - f_\beta) = M_\alpha(\pi_\beta(f_\alpha) - \pi_\beta(f_\beta)) = M_\alpha(\pi_\alpha(f_\alpha) - \pi_\beta(f_\beta)) =$

$M_\alpha(\Phi \restriction M_\alpha - \Phi \restriction M_\beta) = M_\alpha(\Phi - \Phi) = 0.$ □

We are ready to prove Theorem 2.28 (a).

Proof: The module M is strongly κ-free of cardinality κ by the last lemma and the following inclusion is obvious

$\pi(R\langle\mathfrak{F}\rangle) \subseteq \operatorname{End} M.$ **(1)**

If $\Phi \in \operatorname{End} M$ then we apply Theorem 5.4 (c). There is a family $(f_\alpha)_{\alpha\in E_\Phi}$ with $f_\alpha \in R\langle\mathfrak{F}_\alpha\rangle$ such that $\Phi \restriction M_\alpha = \pi_\alpha(f_\alpha)$ for all $\alpha \in E_\Phi$, hence $M_\alpha\pi_\beta(f_\alpha - f_\beta) = 0$ for all $\alpha \leq \beta < \kappa$. If $\aleph_0 \leq \alpha$, then M_α has infinite rank, thus Definition 3.5 (i) of the U-property of π_β gives $f_\alpha - f_\beta = 0$. The sequence $(f_\alpha)_{\alpha\in E_\Phi}$ becomes stationary with $f_\alpha = f$ for some fixed $f \in RF$ and all $\alpha \in E_\Phi$ large enough. We get

$\Phi = \bigcup_{\alpha\in E_\Phi} \pi_\alpha(f_\alpha) = \bigcup_{\alpha\in E_\Phi} \pi_\alpha(f) = \pi(f)$, and with (1) also $\operatorname{End} M = \pi(R\langle\mathfrak{F}\rangle)$ follows.

By the same arguments π is a ring monomorphism, thus $\pi(R\langle\mathfrak{F}\rangle) \cong R\langle\mathfrak{F}\rangle$.
From $\operatorname{End} M = \pi(R\langle\mathfrak{F}\rangle) \cong R\langle\mathfrak{F}\rangle$ follows $\operatorname{Aut} M = \pi(R^* \times F)$ and M is a UT-module by Theorem 5.4 (a). □

5.2 The $\aleph_1$-Free Case

Let $\kappa > \aleph_0$ be a regular cardinal with $\kappa^{\aleph_0} = \kappa$ and R be a PID with $|R| \leq \kappa$ and $R \neq \widehat{R}_p$ for some prime element $p \in R$.
We adopt the notions of the General Black Box 2.14:
Let $T := T_{\kappa\times\kappa\times\aleph_0} = T_\kappa \times T_\kappa \times T_{\aleph_0}$ be the canonical tree. A **norm** $||\cdot||$ is defined on this tree with respect to the first coordinate.
In the following let $B := \bigoplus_{\tau\in T} B_\tau$ with all $B_\tau := R^{(\kappa)} := \bigoplus_{\lambda\in\kappa} R$, which will be our basic module of consideration for the given PID R. For a fixed prime element $p \in R$ the p-adic completion of B will be called $\widehat{B}$. We also set $T_\alpha := T_{\alpha\times\kappa\times\aleph_0}$ and $B_\alpha := \bigoplus_{\tau\in T_\alpha} B_\tau$.
For any element $b = \sum_{\tau\in T} b_\tau \in \widehat{B}$ let $[b] := \{\tau | b_\tau \neq 0\}$ be the **support** of b. For any subset $X \subseteq \widehat{B}$ set $||X|| := ||[X]||$. Also reserve some elements $e_\tau \in B_\tau$.
Finally let $E \subseteq \kappa^o$ be a stationary set. We decompose $E = \dot{\bigcup}_{\gamma\in\kappa} E_\gamma$ with E_γ stationary, see [32, Theorem 85, p. 433], and assign a set $E_X := E_\gamma$ to every countable set $X \subseteq \widehat{B'}$. (Recall that $\kappa^{\aleph_0} = \kappa$!)
Let $p^X_\beta = (f^X_\beta, \varphi^X_\beta)$ be a list of traps for the R-module B and stationary set E_X given by the General Black Box, Theorem 2.14.

We want to construct inductively a continuous chain $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ in $(\mathfrak{A}^*, \subseteq)$ with $\mathfrak{x}_\alpha = (M_\alpha, \mathfrak{F}_\alpha, \pi_\alpha)$ such that M_α is sandwiched between B_α and $\widehat{B_\alpha}$:

$$B_\alpha \subseteq M_\alpha \subseteq \widehat{B_\alpha} \text{ for all } 0 \neq \alpha \in \kappa.$$

The union $M := \bigcup_{\alpha<\kappa} M_\alpha$ of this continuous chain will satisfy condition (b) of Theorem

2.28 as left R-module.

We will carry out the following steps inductively.

First Steps of the Induction

Define the continuous chain $(\mathfrak{x}'_\alpha)_{\alpha\in\kappa}$ with $\mathfrak{x}'_\alpha = (M'_\alpha, \mathfrak{F}'_\alpha, \pi'_\alpha)$ in $\mathfrak{A}^*$:

- Let $\mathfrak{x}'_0 := (M'_0, \mathfrak{F}'_0, \pi'_0) \in \mathfrak{A}^*$ with $M'_0 := R$, $\mathfrak{F}'_0 := \emptyset$ and $\pi'_0 := \emptyset$.
- First construct $\mathfrak{x}'_\alpha \sqsubseteq \mathfrak{y}'_\alpha := (M''_\alpha, \mathfrak{F}'_\alpha, \pi''_\alpha) \in \mathfrak{A}^*$ using the RF-construction and then $\mathfrak{y}'_\alpha \sqsubseteq \mathfrak{x}'_{\alpha+1} \in \mathfrak{A}^*$ using the UT-construction.
- Set $\mathfrak{x}'_\alpha := \bigcup_{\beta<\alpha} \mathfrak{x}'_\beta$ at limit points.

We define $\mathfrak{x}_0 := \bigcup_{\alpha<\kappa} \mathfrak{x}'_\alpha$.

Suppose that the structure on M_β $(\beta < \alpha)$ is defined.

Case 1: $\alpha = \beta + 1,\ \beta \notin E$.

Define the continuous chain $(\mathfrak{x}_{\beta\gamma})_{\gamma\in\kappa}$ with $\mathfrak{x}_{\beta\gamma} = (M_{\beta\gamma}, \mathfrak{F}_{\beta\gamma}, \pi_{\beta\gamma})$ in $\mathfrak{A}^*$:

- $\mathfrak{x}_{\beta 0} := \mathfrak{x}_\beta$.
- First construct $\mathfrak{x}_{\beta\gamma} \sqsubseteq \mathfrak{y}_{\beta\gamma} := (M'_{\beta\gamma}, \mathfrak{F}_{\beta\gamma}, \pi'_{\beta\gamma}) \in \mathfrak{A}^*$ using the RF-construction and then $\mathfrak{y}_{\beta\gamma} \sqsubseteq \mathfrak{x}_{\beta(\gamma+1)} \in \mathfrak{A}^*$ using the UT-construction.
- Set $\mathfrak{x}_{\beta\gamma} := \bigcup_{\delta<\gamma} \mathfrak{x}_{\beta\delta}$ at limit points.

We define $\mathfrak{x}_\alpha := \bigcup_{\gamma<\kappa} \mathfrak{x}_{\beta\gamma}$.

Case 2: $\alpha = \beta + 1,\ \beta \in E_X$.

Let $(p^X_\varepsilon)_{\gamma\le\varepsilon<\delta}$ for suitable ordinals γ, δ denote the sublist of $p^X_\varepsilon = (f^X_\varepsilon, \varphi^X_\varepsilon)$ consisting of all traps with $||p^X_\varepsilon|| = \beta$.

We recursively define a set $I(\beta) \subseteq \{\varepsilon|\ \gamma \le \varepsilon < \delta\}$ and a continuous chain $(\mathfrak{x}_{\beta\varepsilon})_{\gamma\le\varepsilon\le\delta}$ with $\mathfrak{x}_{\beta\varepsilon} = (M_{\beta\varepsilon}, \mathfrak{F}_\beta, \pi_{\beta\varepsilon})$ in $\mathfrak{A}^*$:

- $\mathfrak{x}_{\beta\gamma} := \mathfrak{x}_\beta$.
- If $X \subseteq M_\beta \cap \operatorname{Dom}\varphi^X_\varepsilon$, $\operatorname{Im}(\varphi^X_\varepsilon \restriction M_\beta) \subseteq M_\beta$ and $(\varphi^X_\varepsilon \neq f^{\mathfrak{x}_\beta}) \restriction M_\beta$ for all $f \in R\langle\mathfrak{F}_\beta\rangle$, define $I_{\varepsilon+1}(\beta) := I_\varepsilon(\beta) \cup \{\varepsilon\}$. Let $f'^X_\varepsilon : \omega \to \kappa \times \kappa \times \aleph_0$

setting $f'^X_\varepsilon(i) := (f^X_\varepsilon(i), g^X_\varepsilon(i))$ for all $i \in \omega$ and some function $g^X_\varepsilon : \omega \to \aleph_0$. Set

$$y^X_\varepsilon = \widehat{r}^X_\varepsilon m^X_\varepsilon + \sum_{i\in\omega} p^i e_{f'^X_\varepsilon \restriction (i+1)}.$$

Define

$M_{\beta(\varepsilon+1)} := \langle M_{\beta\varepsilon}, y^X_\varepsilon RF^{\mathfrak{x}_\beta}\rangle_{*p}$ and $\pi_{\beta(\varepsilon+1)}(\varphi_t) = \varphi_t^{\mathfrak{x}_\beta} \restriction M_{\beta(\varepsilon+1)}$ $(\varphi_t \in \mathfrak{F}_\beta)$.

Using Step-Lemma 4.13 (d) construct $\mathfrak{x}_{\beta(\varepsilon+1)}$ from $\mathfrak{x}_{\beta\varepsilon}$ and choose $\widehat{r}^X_\varepsilon m^X_\varepsilon \in \widehat{M_\beta} \cap \widehat{\mathrm{Dom}\,\varphi^X_\varepsilon}$ such that

$$y^X_\varepsilon \varphi^X_\varepsilon \notin M_{\beta(\varepsilon+1)}.$$

- Otherwise set $I_{\varepsilon+1}(\beta) := I_\varepsilon(\beta)$ and $\mathfrak{x}_{\beta(\varepsilon+1)} := \mathfrak{x}_{\beta\varepsilon}$.
- Set $\mathfrak{x}_{\beta\varepsilon} := \bigcup_{\gamma\leq\varepsilon'<\varepsilon} \mathfrak{x}_{\beta\varepsilon'}$ at limit points.

Finally define $I(\beta) := I_\delta(\beta)$ and $\mathfrak{y}_\beta := \mathfrak{x}_{\beta\delta}$ as suprema of ascending chains and $\{y^X_\varepsilon | \varepsilon \in I(\beta)\}$ as family of branches. Furthermore $\{y^X_\varepsilon | \varepsilon \in I(\beta)\}$ will be constructed such that condition (+) of Step-Lemma 4.13 and

$$y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \notin M_{\beta\varepsilon} \textit{ for } \gamma \leq \varepsilon' < \varepsilon \leq \delta,\ \varepsilon' \in I(\beta) \qquad (\star)$$

holds. The consistence proof of (+) and ($\star$) is part of Lemma 8.4.

Construct $\mathfrak{x}_\alpha$ from $\mathfrak{y}_\beta$ as in Case 1.

Case 3: $\alpha \in \mathrm{LORD} \cap \kappa$.

Set $\mathfrak{x}_\alpha := \bigcup_{\beta<\alpha} \mathfrak{x}_\beta$.

We deduce some easy facts about the constructed chain $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$.
See also Lemma 5.3 and Theorem 5.4.

Lemma 5.5 *Let* $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ *be constructed as above.*
(a) $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ *is a well-defined continuous chain in* $(\mathfrak{A}^*, \subseteq)$.
(b) If $\alpha \leq \beta < \kappa$ *with* α *a successor ordinal, then* $\mathfrak{x}_\alpha \sqsubseteq \mathfrak{x}_\beta$. *Furthermore* $\mathfrak{y}_\alpha \sqsubseteq \mathfrak{x}_{\alpha+1}$ *for all* $\alpha \in E$.
(c) $B_\alpha \subseteq_* S_\alpha \subseteq \widehat{B_\alpha}$ *for all* $0 \neq \alpha \in \kappa$.

Proof: This is mainly an easy transfinite induction on $\alpha < \kappa$ using Theorem 3.16, Theorem 4.13 and Lemma 5.1. We therefore concentrate on the more interesting arguments only.

(a) *Step-Lemma 4.13, condition* (+) *and* ($\star$) *are consistent with the construction.*
Let $\beta \in E_X$ and $\varepsilon \in I(\beta)$. Suppose that the branches $y^X_{\varepsilon'}$ ($\varepsilon' < \varepsilon,\ \varepsilon' \in I(\beta)$) and triples $\mathfrak{x}_{\beta\varepsilon'}$ ($\varepsilon' \leq \varepsilon$) are defined. Observe that

$M_{\beta\varepsilon'} := \langle M_\beta, y^X_{\varepsilon''} RF^{\mathfrak{x}_\beta} | \varepsilon'' < \varepsilon', \varepsilon'' \in I(\beta)\rangle_{*p}$ and $\pi_{\beta\varepsilon'}(\varphi_t) = \varphi_t^{\mathfrak{x}_\beta} \restriction M_{\beta\varepsilon'}$ $(\varphi_t \in \mathfrak{F}_\beta)$. **(1)**

Furthermore let (+) hold and

$y^X_{\varepsilon''}\varphi^X_{\varepsilon''} \notin M_{\beta\varepsilon'} \quad (\varepsilon'' < \varepsilon' < \varepsilon,\ \varepsilon'' \in I(\beta)).$ **(2)**

As basis for Step-Lemma 4.13 serves the proper ascending chain $\mathfrak{x}_\beta = \bigcup_{i\in\omega} \mathfrak{x}_{f^X_{\varepsilon 1}(i)}$ in $(\mathfrak{A}^*, \sqsubseteq)$ of cofinality $\mathrm{cf}\,(\beta) = \aleph_0$, where $f^X_\varepsilon : \omega \to \kappa \times \kappa$ with $f^X_\varepsilon(i) = (f^X_{\varepsilon 1}(i), f^X_{\varepsilon 2}(i))$ is the stretched branch of trap p^X_ε. We furthermore set $\mathfrak{x} := \mathfrak{x}_{\beta\varepsilon}$.
Define $f'^X_\varepsilon : \omega \to \kappa \times \kappa \times \aleph_0$ setting $f'^X_\varepsilon(i) := (f^X_\varepsilon(i), g^X_\varepsilon(i))$ for all $i \in \omega$ and some function $g^X_\varepsilon : \omega \to \aleph_0$. Set

$$y^X_\varepsilon := \widehat{r}^X_\varepsilon m^X_\varepsilon + \sum_{i\in\omega} p^i e_{f'^X_\varepsilon \restriction (i+1)}.$$

Observe that $||f'^X_\varepsilon \restriction (i+1)|| < ||p^X_\varepsilon|| = \beta$, thus $f'^X_\varepsilon \restriction (i+1) \in T_\beta$ and $e_{f'^X_e \restriction (i+1)}$ is already identified as free generator in M_β, and $y^X_\varepsilon \in \widehat{M_\beta}$.
The General Black Box, Theorem 2.14 (iii), gives $\mathrm{Br}\,(f^X_\varepsilon \times T_{\aleph_0}) \cap \mathrm{Br}\,(f^X_{\varepsilon'} \times T_{\aleph_0}) = \emptyset$ for $\varepsilon' < \varepsilon$. Observing that the supports $[y^X_\varepsilon]_B$ and $[y^X_{\varepsilon'}]_B$ of the branches y^X_ε and $y^X_{\varepsilon'}$ with respect to the basis of B are determined by $f'^X_\varepsilon \in \mathrm{Br}\,(f^X_\varepsilon \times T_{\aleph_0})$ and $f'^X_{\varepsilon'} \in \mathrm{Br}\,(f^X_{\varepsilon'} \times T_{\aleph_0})$, an easy support argument leads to

$$[y^X_\varepsilon]_B \cap [y^X_{\varepsilon'}]_B \textit{ is finite for all } \varepsilon' < \varepsilon.$$

Together with (1) follows (+) for y^X_ε independent of the choice of g^X_ε.
Set $\eta^{\mathfrak{x}_\beta} := \varphi^X_\varepsilon \restriction M_\beta$ as test-function for Step-Lemma 4.13 (d) and remind that

$$(\varphi^X_\varepsilon \neq f^{\mathfrak{x}_\beta}) \restriction M_\beta \textit{ for all } f \in R\langle \mathfrak{F}_\beta \rangle$$

from $\varepsilon \in I(\beta)$. Independent of the choice of g^X_ε holds $f'^X_\varepsilon \restriction (i+1) \in f^X_\varepsilon \times T_{\aleph_0}$, thus $e_{f'^X_\varepsilon \restriction (i+1)} \in \mathrm{Dom}\,\varphi^X_\varepsilon$ by Definition 2.13. Furthermore $X \subseteq M_\beta \cap \mathrm{Dom}\,\varphi^X_\varepsilon$, thus

$e_{f'^X_\varepsilon \restriction (i+1)} \in \operatorname{Dom}\mu$, $X \subseteq \operatorname{Dom}\mu$ and Step-Lemma 4.13 (d) applies to choose $\widehat{r}^X_\varepsilon m^X_\varepsilon \in \widehat{M_\beta} \cap \widehat{\operatorname{Dom}\varphi^X_\varepsilon}$ such that $y^X_\varepsilon \eta = y^X_\varepsilon \varphi^X_\varepsilon \notin M_{\beta(\varepsilon+1)}$.

To gain also $y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \notin M_{\beta(\varepsilon+1)}$ for $\varepsilon' < \varepsilon, \varepsilon' \in I(\beta)$ we distinguish the following two cases.

Case 1: $\varepsilon' + 2^{\aleph_0} \leq \varepsilon$.

Here we can combine Theorem 2.14 (iv) and Definition 2.13 (iii) to gain

$$\operatorname{Br}(f^X_\varepsilon \times T_{\aleph_0}) \cap \operatorname{Br}[P^X_{\varepsilon'}] = \emptyset, \text{ where } y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \in \widehat{\operatorname{Im}(\varphi^X_{\varepsilon'})} \subseteq \widehat{P^X_{\varepsilon'}}.$$

In particular $\operatorname{Br}(f^X_\varepsilon \times T_{\aleph_0}) \cap \operatorname{Br}[y^X_{\varepsilon'} \varphi^X_{\varepsilon'}] = \emptyset$ and together with Definition 2.13 (iv) follows

$[y^X_\varepsilon]_B \cap [y^X_{\varepsilon'} \varphi^X_{\varepsilon'}]_B$ *is finite* **(3)**

for the supports in B. Now suppose, there is $\varepsilon' < \varepsilon, \varepsilon' \in I(\beta)$ with $y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \in M_{\beta(\varepsilon+1)}$. Then using support arguments $y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \in M_{\beta\varepsilon} = \bigcup_{\varepsilon'' < \varepsilon} M_{\beta\varepsilon''}$ follows from (1) and (3) contradicting (2).

Case 2: $\varepsilon' < \varepsilon < \varepsilon' + 2^{\aleph_0}$.

First observe that

there exists at most one function $g^X_\varepsilon : \omega \longrightarrow \aleph_0$ *with* $y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \in M_{\beta(\varepsilon+1)}$. **(4)**

Assume that $y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \in M_{\beta(\varepsilon+1)}$ for two functions $g^X_{\varepsilon 1}$ and $g^X_{\varepsilon 2}$, then using support arguments $y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \in M_{\beta\varepsilon} = \bigcup_{\varepsilon'' < \varepsilon} M_{\beta\varepsilon''}$ follows contradicting (2).

Now observe that there are less than $2^{\aleph_0}$ ordinals ε' with $\varepsilon' < \varepsilon < \varepsilon' + 2^{\aleph_0}$, while there are $2^{\aleph_0}$ functions $g^X_\varepsilon : \omega \longrightarrow \aleph_0$. Thus there is always some function g^X_ε such that $y^X_{\varepsilon'} \varphi^X_{\varepsilon'} \notin M_{\beta(\varepsilon+1)}$ holds simultaneously for all ε' with $\varepsilon' < \varepsilon < \varepsilon' + 2^{\aleph_0}$.

This g^X_ε is sufficient for $(\star)$.

(c) For this we refer to [23]. □

Proving $B_\alpha \subseteq_* S_\alpha \subseteq \widehat{B_\alpha}$ is essential for connecting our combinatorial arguments with our algebraical arguments and we have to spare no efforts to show the existence of a suitable algebraic independent basis. We will carry this out in detail in Section 8.2.

Next we state some basic properties of the R-module $M := \bigcup_{\alpha < \kappa} M_\alpha$.

Theorem 5.6 *If* $(\mathfrak{x}_\alpha)_{\alpha<\kappa}$ *is the chain as above and* $\mathfrak{x} = \bigcup_{\alpha<\kappa} \mathfrak{x}_\alpha = (M, \mathfrak{F}, \pi)$*, then the following holds.*

(a) $\mathfrak{x} \in \mathfrak{A}^*$.

(b) $|M| = \kappa$ *and* $\pi(R^* \times F) \subseteq \operatorname{Aut} M$ *acts uniquely transitive on* $\mathfrak{p}M$.

(c) M *and* M/M_α *are* $\aleph_1$*-free for all successor ordinal* $\alpha < \kappa$*. Furthermore* $M/M^{\mathfrak{y}_\alpha}$ *is* $\aleph_1$*-free for all* $\alpha \in E$.

(d) $B \subseteq_* M \subseteq \widehat{B}$.

(e) If $\Psi \in \operatorname{End} M$ *and* $X \subseteq M$ *a submodule of countable rank, then there exists an* $f \in R\langle\mathfrak{F}\rangle$ *with* $(\Psi = f^{\mathfrak{x}}) \restriction X$.

Proof: (a), (b), (c) and (d) are an immediate consequence of Definition 3.5, Lemma 3.15, Corollary 3.18 and Lemma 5.5.

(e): Let $\Psi \in \operatorname{End} M$. Then $\Psi : M \to M \subseteq \widehat{B}$ lifts uniquely to an endomorphism $\Psi : \widehat{B} \to \widehat{B}$. Set

$$C := \{\alpha \in \kappa \mid X \subseteq M_\alpha\}$$

as cub. Applying the General Black Box, Theorem 2.14, for X countable, C and $\varphi := \Psi$ there exists a trap $p_\alpha^X = (f_\alpha^X, \varphi_\alpha^X)$, that catches X, C and φ. Thus the following holds:

(a) $X \subseteq P_\alpha^X := \operatorname{Dom} \varphi_\alpha^X \subseteq \widehat{B}$ and $\varphi_\alpha^X \in \operatorname{End}(P_\alpha^X)$.

(b) $||X|| < ||p_\alpha^X|| = \beta \in C \cap E_X \subseteq \kappa^o$ and $||x|| < \beta$ for all $x \in P_\alpha^X$.

(c) $\varphi \restriction P_\alpha^X = \varphi_\alpha^X$.

In particular $\Psi \restriction P_\alpha^X = \varphi_\alpha^X$ is a partial endomorphism of $\widehat{B_\beta}$ with $X \subseteq M_\beta \cap P_\alpha^X$. Furthermore

$$(\Psi = \varphi_\alpha^X) \restriction (M_\beta \cap P_\alpha^X)$$

is a partial endomorphism of M_β: For every $x \in M_\beta \cap P_\alpha^X$ holds $x\varphi_\alpha^X = x\Psi \in M \cap \widehat{M_\gamma}$ with (a) and (b), where $\gamma < \beta \in \kappa^o$ is some successor ordinal. From M, M/M_γ $\aleph_1$-free and Corollary 5.2 now follows $x\varphi_\alpha^X \in M_\gamma \subseteq M_\beta$.

Lets assume $\alpha \in I(\beta)$. Thus $y_\alpha^X \in M$ and the continuity of Ψ gives

$$y_\alpha^X \Psi = y_\alpha^X \varphi_\alpha^X \in \widehat{M_\beta} \cap M.$$

Furthermore from M, $M/M^{\mathfrak{y}_\beta}$ $\aleph_1$-free and Corollary 5.2 follows $y_\alpha^X \varphi_\alpha^X \in M^{\mathfrak{y}_\beta} = M_{\beta\delta}$ contradicting ($\star$). Thus $\alpha \notin I(\beta)$. In particular, there exists some $f \in R\langle\mathfrak{F}\rangle$ with $(\Psi = f^{\mathfrak{x}_\beta}) \restriction (M_\beta \cap P_\alpha^X)$, hence $(\Psi = f^{\mathfrak{x}}) \restriction X$. ◻

Now the proof of Theorem 2.28 (b) is similar to the proof of Theorem 2.28 (a).

Proof: The module M is $\aleph_1$-free of cardinality κ by the last lemma and the following inclusion is obvious

$\pi(R\langle\mathfrak{F}\rangle) \subseteq \operatorname{End} M$. **(1)**

If $\Psi \in \operatorname{End} M$ then we apply Theorem 5.6 (e). For every submodule $X \subseteq M$ of countable rank exists some $f_X \in R\langle\mathfrak{F}\rangle$ such that $(\Psi = f_X^{\mathfrak{x}}) \restriction X$. In particular

$$(\Psi = f_{A+B}^{\mathfrak{x}} = f_{A+C}^{\mathfrak{x}}) \restriction A$$

for arbitrary submodules $A, B, C \subseteq M$ of countable rank, hence

$$A\pi(f_{A+B} - f_{A+C}) = 0$$

Now A has infinite rank, thus Definition 3.5 (i) of the U-property of π gives

$$f_{A+B} = f_{A+C}.$$

In particular there exists some $f \in R\langle\mathfrak{F}\rangle$ with $f_X = f$ for all submodules $X \subseteq M$ of countable rank. We get $\Psi = f^{\mathfrak{x}}$, and with (1) also $\operatorname{End} M = \pi(R\langle\mathfrak{F}\rangle)$ follows.

By the same arguments π is a ring monomorphism, thus $\pi(R\langle\mathfrak{F}\rangle) \cong R\langle\mathfrak{F}\rangle$.
From $\operatorname{End} M = \pi(R\langle\mathfrak{F}\rangle) \cong R\langle\mathfrak{F}\rangle$ follows $\operatorname{Aut} M = \pi(R^* \times F)$ and M is a UT-module by Theorem 5.6 (b). ◻

Part II

UT-Modules with PIDs as Endomorphism Rings

6 Some Algebraic Tools

We will need some easy algebraic observations for a special class $\mathfrak{K}$ of R-algebras, and a theorem on ring localizations, which helps to shorten arguments on UT-modules.

6.1 Unit-Free Algebras

We introduce particular almost free R-algebras:
Again, let R be a PID with p-adic completion $\widehat{R} := \widehat{R_p}$ for some prime element p, such that $\widehat{R}$ has two transcendental elements π and π' over R. Recall that π, π' are **transcendental** over R, if 0 is the only polynomial $f(x, x') \in R[x, x']$ such that $f(\pi, \pi') = 0$ (viewed in $\widehat{R}$).

Definition 6.1
(a) We call a commutative R-algebra S **unit-free** *if ${}_RS$ is $\aleph_1$-free and $\mathfrak{p}_RS = S^*$.*
(b) Let $\mathfrak{K} := \mathfrak{K}_R$ be the family of all unit-free R-algebras S such that π, π' from above are two transcendental elements over S.

Lemma 6.2 *Let S be a unit-free R-algebra. Then S is a PID having the same ideal structure as R. Also $A \subseteq_{*S} B$ if and only if $A \subseteq_{*R} B$ holds for arbitrary S-modules A and B.*

Proof: The cyclic R-module $1R \subseteq S$ is isomorphic to R, because ${}_RS$ is $\aleph_1$-free; we identify $R = 1R \subseteq S$.

Next we apply that ${}_RS$ is an $\aleph_1$-free R-module:
If $s, s' \in S$, there are $e, e' \in \mathfrak{p}_RS = S^*$ with $s = r \cdot e$ and $s' = r' \cdot e'$ for suitable $r, r' \in R$.

If also $ss' = 0$, then $rr' = 0$, hence either $s = 0$ or $s' = 0$ and S is a domain.

From $s = r \cdot e$ also follows

$sS = reS = rS$. **(1)**

Now let $J = \langle s_i S | i \in I \rangle$ be any ideal of S. By (1) there are suitable elements $r_i \in R$ with $J = \langle r_i S | i \in I \rangle$. Let $r := \gcd\{r_i | i \in I\}$ be the greatest common divisor in R.

Hence $rR = \langle r_i R | i \in I \rangle$. **(2)**

We will show that $J = rS$.

If $x \in J$, then there are $t_i \in S$ with $x = \sum_{i \in I} r_i t_i = r \cdot \sum_{i \in I} \frac{r_i}{r} t_i \in rS$. Thus $J \subseteq rS$.

Conversely, if $rs \in rS$, then by (2) follows $r = \sum_{i \in I} r_i t_i \in (r_i | i \in I)$ for some $t_i \in R$.

Thus $rs \in J$ and also $rS \subseteq J$.

In particular all ideals of S are of the form rS for some $r \in R$.

The first claim of Lemma 6.2 follows.

Now suppose $A \subseteq_{*S} B$ as in Lemma 6.2. From $R \subseteq S$ follows obviously $A \subseteq_{*R} B$. Conversely, let $A \subseteq_{*R} B$ and consider $sb = a$ for $s = re$ with $s \in S$, $r \in R$, $e \in S^*$, $a \in A$ and $b \in B$. From $sb = reb = a$ follows $rb = e^{-1}a \in A$ and by R-purity follows $ra' = e^{-1}a$ for some $a' \in A$. Thus $sa' = rea' = a$ and $A \subseteq_{*S} B$ follows. □

Corollary 6.3 *$\mathfrak{K}$ is closed under taking unions of pure ascending continuous chains.*

Proof: Being a unit-free R-algebra is a property of finite character, hence Corollary 6.3 is immediate. □

Lemma 6.4 *Let $T \in \mathfrak{K}_S$ and $S \in \mathfrak{K}_R$. Then $T \in \mathfrak{K}_R$.*

Proof: Let $T \in \mathfrak{K}_S$ and $S \in \mathfrak{K}_R$. Thus $\mathfrak{p}_R T = \mathfrak{p}_S T = T^*$ (see Lemma 6.2).

Let $n \in \omega$ and elements $t_i \in T$ $(i \in n)$ be arbitrary. Then

$$\langle t_i | i \in n \rangle_{*R} \subseteq \langle t_i | i \in n \rangle_{*S} \subseteq T,$$

where $\langle t_i | i \in n \rangle_{*S}$ is S-free and an $\aleph_1$-free R-module. In particular $\langle t_i | i \in n \rangle_{*R}$ is R-free and T is an $\aleph_1$-free R-module by Pontryagin's Theorem. Thus $T \in \mathfrak{K}_R$. □

6.2 Ring Localizations

Next we want to show that $\mathfrak{K}$ is closed under special localizations.

A subset $T \subseteq S$ of a PID S is **multiplicatively closed** if $1 \in T$, $0 \notin T$, $t_1 \cdot t_2 \in T$ for all $t_1, t_2 \in T$.

Definition 6.5 *Let T be a multiplicatively closed subset of the PID S.*
Then $S_T := \{\frac{s}{t} | s \in S, t \in T\}$ is the **localization of S at T**.

Clearly (see [1, Chapter 5.8, p. 159]) we have the

Corollary 6.6 *Let S, T be as above. The localization S_T is a canonical subring of the quotient ring $Q(S)$.*

Next we will show the existence of many unit-free R-algebras used for the construction of UT-modules.

Lemma 6.7 *If S is a PID and μ an ordinal, then*

$$T := \mathfrak{p}_S S[x_\alpha | \alpha \in \mu] := \mathfrak{p}_S(S[x_\alpha | \alpha \in \mu])$$

is a multiplicatively closed subset of the polynomial ring $S[x_\alpha|\alpha \in \mu]$ and $S[x_\alpha|\alpha \in \mu]_T$ is an $\aleph_1$-free S-module.

Proof: The polynomial ring $S[x_\alpha|\alpha \in \mu]$ is $S[x_\alpha|\alpha \in \mu] = \bigoplus_{\mathfrak{m} \in \mathfrak{M}_\mu} S\mathfrak{m}$ as S-module, where $\mathfrak{M}_\mu := \langle x_\alpha | \alpha \in \mu \rangle$ is a multiplicative group of monomials. We impose a lexicographic ordering on $\mathfrak{M}_\mu$ by setting $x_\alpha < x_\beta$ for $\alpha < \beta$ and comparing the exponents occurring in the monomials. This ordering is preserved under multiplication.

Let $f = \sum_{i=0}^m a_i \mathfrak{m}_i, g = \sum_{j=0}^n b_j \mathfrak{m}'_j \in \mathfrak{p}_S S[x_\alpha|\alpha \in \mu]$ and assume $fg \notin \mathfrak{p}_S S[x_\alpha|\alpha \in \mu]$. There is a prime element $q \in S$ with $q|fg$. Without loss of generality let $q \nmid a_i, b_j$ for all coefficients of f and g, and $\mathfrak{m}_k < \mathfrak{m}_l$, $\mathfrak{m}'_k < \mathfrak{m}'_l$ for $k < l$, thus the monomials are indexed in correct order.

The $\mathfrak{m}_0\mathfrak{m}'_0$-component of fg gives

$$q|fg \Longrightarrow q|a_0 b_0 \Longrightarrow q|a_0 \vee q|b_0$$

contradicting the choice of f and g. Therefore

$\mathfrak{p}_S S[x_\alpha|\alpha \in \mu]$ *is multiplicatively closed in* $S[x_\alpha|\alpha \in \mu]$ **(1)**

and the localization $S[x_\alpha|\alpha \in \mu]_T$ is defined.

$S[x_\alpha|\alpha \in \mu]_T$ *is an* $\aleph_1$*-free* S*-module.* **(2)**

Proof: Let $n \in \omega$ and elements $\frac{f_i}{g_i} \in S[x_\alpha|\alpha \in \mu]_T$ $(i \in n)$ be arbitrary. By Pontryagin's Theorem it is sufficient, to show that $\left\langle \frac{f_i}{g_i}|i \in n\right\rangle_* \subseteq S[x_\alpha|\alpha \in \mu]_T$ is S-free.
By (1) we know that with $g_i \in T$ for all $i \in n$ also $G := \prod_{i\in n} g_i \in T$, and
$\Phi: \ S[x]_T \to S[x]_T,\ \frac{f}{g} \longrightarrow \frac{fG}{g}$ is an S-module isomorphism. Therefore it is sufficient to show that $\Phi(\left\langle \frac{f_i}{g_i}|i \in n\right\rangle_*) \cong \left\langle \Phi(\frac{f_i}{g_i})|i \in n\right\rangle_* \subseteq S[x_\alpha|\alpha \in \mu] \subseteq_* S[x_\alpha|\alpha \in \mu]_T$ is S-free.
But this is obvious due to the fact, that $S[x_\alpha|\alpha \in \mu]$ is an S-free module. □

Theorem 6.8 *Let* λ *an ordinal of cofinality* $\operatorname{cf}(\lambda) = \aleph_0$ *and* $S = \bigcup_{\alpha\in\lambda} S_\alpha \in \mathfrak{K}$ *an* R*-pure ascending chain in the class* $\mathfrak{K}$ *of rings.*
Further let $V = \bigcup_{k\in\omega} S[y_{\alpha k}|\alpha \in \mu]$ *be an ascending chain of rings and set*
$V(I) := \bigcup_{k\in\omega} S[y_{\alpha k}|\alpha \in I]$ *for any finite* $I \subseteq \mu$.
Depending on I *there are ascending chains* $\bigcup_{k\in\omega} K(\alpha, k) = \omega$ $(\alpha \in I)$, $\bigcup_{k\in\omega} \nu'(k) = \lambda$ *and some* $N \in \omega$ *with* $V(I) = \bigcup_{k\in\omega,k>N} S_{\nu'(k)}[y_{\alpha K(\alpha,k)}|\alpha \in I] \subseteq_* V$ *an* R*-pure ascending chain of rings, where* $S_{\nu'(k)}[y_{\alpha K(\alpha,k)}|\alpha \in I] \cong S_{\nu'(k)}[x_\alpha|\alpha \in I]$.
Let the pair $\{\pi, \pi'\}$ *of transcendentals over* S *also be transcendental over* V.
Then $V_T \in \mathfrak{K}$, *where* $T = \mathfrak{p}_R V$.

Proof: Obviously, V as the union of an ascending chain of commutative rings is again a commutative ring.

V *is an* $\aleph_1$*-free* R*-module.* **(1)**

Proof: Let $n \in \omega$ and elements $s_i \in V$ $(i \in n)$ be arbitrary.
Then $s_i \in S[y_{\alpha k(i)}|\alpha \in I(i)]$ holds for suitable $k(i) \in \omega$ and finite $I(i) \subseteq \mu$.
Thus $s_i \in V(I)$ $(i \in n)$ for $I := \bigcup_{i\in n} I(i)$ finite. Now for large $k > N$ we have $\langle s_i|i \in n\rangle_{R*} \subseteq S_{\nu'(k)}[y_{\alpha K(\alpha,k)}|\alpha \in I] \subseteq_* V$, where $S_{\nu'(k)}[y_{\alpha K(\alpha,k)}|\alpha \in I] \cong$
$S_{\nu'(k)}[x_\alpha|\alpha \in I]$ is $\aleph_1$-free. Hence $\langle s_i|i \in n\rangle_{R*}$ is R-free and V is $\aleph_1$-free by Pontryagin's

Theorem.

The set T is multiplicatively closed. **(2)**

Proof: Let t_1, $t_2 \in T$ with $t_1 t_2 \notin T$. Then t_1, t_2 are pure in V while $t_1 t_2$ is not. As above there are some $k \in \omega$, $I \subseteq \mu$ finite, such that t_1, t_2 are pure in $S_{\nu'(k)}[y_{\alpha K(\alpha,k)}|\alpha \in I]$ while $t_1 t_2$ is not. This contradicts $S_{\nu'(k)}[y_{\alpha K(\alpha,k)}|\alpha \in I] \cong S_{\nu'(k)}[x_\alpha|\alpha \in I]$ and Lemma 6.7, where we use $\mathfrak{p}_R S_{\nu'(k)}[x_\alpha|\alpha \in I] = \mathfrak{p}_{S'_\nu(k)} S_{\nu'(k)}[x_\alpha|\alpha \in I]$.

Hence the localization $V_T \in \mathfrak{K}$ is defined.

The set $\{\pi, \pi'\}$ is transcendental over V_T. **(3)**

Proof: Let $0 \neq f(x,y) \in V_T(x,y)$ with $f(\pi,\pi') = 0$. Multiplying f with the common denominator of its coefficients gives some polynomial $0 \neq F(x,y) \in V(x,y)$ with $F(\pi,\pi') = 0$. This is a contradiction to $\{\pi,\pi'\}$ transcendental over V.

$\mathfrak{p}_R V_T = \{\frac{s}{t} \mid s \in \mathfrak{p}_R V = T, t \in T\} = (V_T)^*$ *and* $V \subseteq_* V_T$. **(4)**

Proof: We start with the first claim.

"$\subseteq$" is obvious. For "$\supseteq$" let $q \in R$ be a prime, $s, s' \in V$ and $t, t' \in T$ with $q \cdot \frac{s'}{t'} = \frac{s}{t}$. Again there are some $k \in \omega$, $I \subseteq \mu$ finite, such that $s, s', t, t' \in S_{\nu'(k)}[y_{\alpha K(\alpha,k)}|\alpha \in I] \cong S_{\nu'(k)}[x_\alpha|\alpha \in I]$. Thus $q|s$ and $\frac{s}{t} \notin \{\frac{s}{t} \mid s \in \mathfrak{p}_R V = T, t \in T\}$ follows.

The second part of (4) follows similarly.

V_T is an $\aleph_1$-free R-module. **(5)**

Proof: Let $n \in \omega$ and elements $\frac{s_i}{t_i} \in V_T$ $(i \in n)$ be arbitrary. By Pontryagin's Theorem it is sufficient, to show that $\left\langle \frac{s_i}{t_i} | i \in n \right\rangle_* \subseteq V_T$ is R-free.

By (2) we know that with $t_i \in T$ for all $i \in n$ also $t := \prod_{i \in n} t_i \in T$, and $\Phi: V_T \to V_T$, $\frac{s'}{t'} \longrightarrow \frac{s't}{t'}$ is an R-module isomorphism. Therefore it is sufficient to show that $\Phi(\left\langle \frac{s_i}{t_i} | i \in n \right\rangle_*) \cong \left\langle \Phi(\frac{s_i}{t_i}) | i \in n \right\rangle_* \subseteq V \subseteq_* V_T$ is R-free. But this follows directly from (1). $\square$

Next we have two easy applications of Theorem 6.8.

Corollary 6.9 (x-Localization)

If $S \in \mathfrak{K}$, then $S[x]_T \in \mathfrak{K}$, where $T := \mathfrak{p}_R S[x]$.

Furthermore $S[x]_T/S$ is an $\aleph_1$-free R-module, $S[x]_T \in \mathfrak{K}_S$, $S[x] \subseteq S[x]_T$ as rings and $S \oplus \bigoplus_{k\in\omega} Sx^{k+1} \subseteq_* S[x]_T$ as R-modules holds.

Proof: $S[x]_T \in \mathfrak{K}$ is a special case of Theorem 6.8 for $\lambda := \omega$, $S_\alpha := S$, $\mu := 1$, $y_{\alpha k} := x$, $K(\alpha, k) := k$ and $\nu'(k) := k$. Obviously $\{\pi, \pi'\}$ is transcendental over $S[x]$.
$S[x] \subseteq S[x]_T$ and $S[x] = S \oplus \bigoplus_{k\in\omega} Sx^{k+1} \subseteq S[x]_T$ is obvious. If $f, g \in S[x]$, $q \in R$ prime and $h \in T$ with $f = q \cdot \frac{g}{h}$, then $qg = fh$ and $q|f$. Thus $\frac{g}{h} = \frac{f}{q} \in S[x]$ and $S[x] \subseteq_* S[x]_T$ follows.
From $S \subseteq_* S[x] \subseteq_* S[x]_T$ it follows that $S[x]_T/S$ is $\aleph_1$-free.
Furthermore from $S \in \mathfrak{K}_S$ we conclude $S[x]_T \in \mathfrak{K}_S$ by Corollary 6.9, where $T = \mathfrak{p}_R S[x] = \mathfrak{p}_S S[x]$ by Lemma 6.2. □

Corollary 6.10 (px-Localization)
If $S \in \mathfrak{K}$ and $p \in R$ a prime, then $S[px]_T \in \mathfrak{K}$ and $S[px]_T \subseteq \widehat{S[x]}_p$, where $T := \mathfrak{p}_R S[px]$. Furthermore the following holds: $S[px]_T/S$ is an $\aleph_1$-free R-module, $S[px]_T \in \mathfrak{K}_S$, we have inclusions $S[px] \subseteq S[px]_T$ as rings and $S \oplus \bigoplus_{k\in\omega} Sx^{k+1} \subseteq_* S[px]_T$ as R-modules.

Proof: Together with x also px is transcendental over S and Corollary 6.10 is a special case of Corollary 6.9. It remains to show that $S[px]_T \subseteq \widehat{S[x]}_p$.
Let $f = \sum_{i=0}^n a_i(px)^i \in T$, in particular $p \nmid a_0 \in S$. Thus a_0 is invertible in $\widehat{S}_p$ and f is invertible in $\widehat{S[x]}_p$, giving $S[px]_T \subseteq \widehat{S[x]}_p$. □

Another important application of Theorem 6.8 will be the following Step-Lemma 7.7.

7 The Step-Lemma

Let λ be an ordinal of cofinality $\mathrm{cf}(\lambda) = \aleph_0$ and $(S_\alpha)_{\alpha\in\lambda}$ be an R-pure ascending chain in $\mathfrak{K}$ with S_β/S_α $\aleph_1$-free as R-module and $S_\beta \in \mathfrak{K}_{S_\alpha}$ for all $\alpha \le \beta \in \lambda$ and successor ordinals $\alpha \in \lambda$. Then $S_\lambda := \bigcup_{\alpha\in\lambda} S_\alpha \in \mathfrak{K}$ by Corollary 6.3. Again we demand $S_\lambda \subseteq_{*p} \widehat{S_\lambda}$. Suppose there is a transcendental element $e_\alpha \in S_{\alpha+1}$ over S_α, hence

$S_\alpha[e_\alpha] \subseteq S_{\alpha+1}$ as rings and $S_\alpha \oplus \bigoplus_{k\in\omega} S_\alpha e_\alpha^{k+1} \subseteq_* S_{\alpha+1}$ $(\star)$
as R-modules for all successor ordinals $\alpha \in \lambda$.

Iterated use of $(\star)$ gives $M_\alpha := \bigoplus_{\mathfrak{m}\in\mathfrak{M}_\alpha} S_\alpha\mathfrak{m} \subseteq_* S_\lambda$ as (external) direct sum with $\widehat{M_\alpha} \subseteq \prod_{\mathfrak{m}\in\mathfrak{M}_\alpha} \widehat{S_\alpha}\mathfrak{m}$ and $\widehat{M_\alpha} \subseteq_* \widehat{S_\lambda}$ for all successor ordinals $\alpha \in \lambda$, and $S_\lambda = \bigcup_{\alpha\in\lambda} M_\alpha$ is an ascending chain of $\aleph_1$-free modules, where $\mathfrak{M}_\alpha := \langle e_\beta | \alpha \le \beta \in \lambda \setminus \mathrm{LORD}\rangle$ is the multiplicative group of monomials. Thus every element of $\widehat{M_\alpha}$ can be expressed as a sum of at most countably many $\mathfrak{m}$-components (coming from $\widehat{S_\alpha}\mathfrak{m}$); i.e. we can assign to $m \in \widehat{M_\alpha}$ a support $[m]_\alpha \subseteq \mathfrak{M}_\alpha$ consisting of all $\mathfrak{m}$ for which the $\mathfrak{m}$-component of m is not zero. Further define $\sup\,[m]_\alpha$ as minimal ordinal γ with $[m]_\alpha \subseteq \langle e_\beta | \beta \in \gamma\rangle$
Next follows the notion of branches on S_λ.

Definition 7.1 *If $m \in S_\lambda$, $\widehat{r} := \sum_{i\in\omega} p^i r_i \in \widehat{R} \subseteq \widehat{S_\lambda}$ and $(\nu(i))_{i\in\omega}$ is a strictly ascending sequence of successor ordinals with $\bigcup_{i\in\omega} \nu(i) = \lambda$, then*
*$y := \widehat{r}m + \sum_{i\in\omega} p^i e_{\nu(i)} = \sum_{i\in\omega} p^i(r_i m + e_{\nu(i)}) \in \widehat{S_\lambda}$ is a **branch** of $\widehat{S_\lambda}$.*
For all $k \in \omega$ set
$y_k := \sum_{k\le i<\omega} p^{i-k}(r_i m + e_{\nu(i)}) = (\sum_{k\le i<\omega} p^{i-k} r_i)m + \sum_{k\le i<\omega} p^{i-k} e_{\nu(i)} \in \widehat{S_\lambda}$.

Let $\{y_\alpha | \alpha \le \mu\}$ be a family of branches of $\widehat{S_\lambda}$ for some ordinal μ.
With this notions we now can formulate our last condition on the chain $(S_\alpha)_{\alpha\in\lambda}$:

If $\alpha \in \lambda \setminus LORD, \beta \ne \gamma \le \mu$ with $y_\beta, y_\gamma \in \widehat{M_\alpha}$, then $[y_\beta]_\alpha \cap [y_\gamma]_\alpha$ is finite. $(+)$

Observe that in Definition 7.1 the initial segments y_k themselves are branches.
In the following $y, \widehat{r}$ and y_k will be as in Definition 7.1. For the additional index α elements y_α, $\widehat{r}_\alpha$ and $y_{\alpha k}$ are defined similarly.
Recall that π, π' are transcendental over $S_\lambda \in \mathfrak{K}$; we will apply $\widehat{r}_\alpha \in \{\pi, \pi'\}$.

Set $V := S_\lambda[y_\alpha|\alpha \leq \mu] \subseteq \widehat{S_\lambda}$ as induced ring extension. Thus $\widehat{V} = \widehat{S_\lambda}$. In the following V will always be viewed as R-subalgebra of the $\widehat{R}$-algebra $\widehat{S_\lambda}$.

Our goal will be the ring localization $S := (V_{*p})_T$, where $T := \mathfrak{p}_R(V_{*p})$ has to be multiplicatively closed. As a first step towards this goal we state some fundamental facts about V_{*p}. As for the previous step lemmas support arguments will play a crucial role. Therefore we will often give the basic ideas of the proofs only, thus omitting technical index arguments. For further exercise see Sections 4.1 and 4.3.

Observation 7.2 *The following facts can be easily derived using support arguments: Let $\alpha \in \lambda$ be a successor ordinal. Then for any $m \in \widehat{M_\alpha}$ the support $[m]_\alpha$ is finite or countable. For all $m \in \widehat{M_\alpha} \cap S_\lambda$ the support $[m]_\alpha$, finite or not, is bounded, in particular* $\sup [m]_\alpha < \lambda$. *In contrast the support $[y]_\alpha$ of a branch $y \in \widehat{M_\alpha}$ is always countable and unbounded, i.e.* $\sup [m]_\alpha = \lambda$.

Let $\{y_k|k \in n\}$ be a finite set of branches with property $(+)$ and $\mathfrak{M}' := \langle y_k|k \in n\rangle$ the induced multiplicative group of monomials. Then $(+)$ says that for all $i \neq j \in n$ the branches y_i and y_j have only finitely many generators e_α in common.

Adding these joint generators to S_α, $[y_i]_\alpha \cap [y_j]_\alpha = \{1\}$ holds for all $i \neq j \in n$ and large successor ordinals $\alpha \in \lambda$. Furthermore $[\mathfrak{m}]_\alpha \cap [\mathfrak{m}']_\alpha = \{1\}$ for all $\mathfrak{m} \neq \mathfrak{m}' \in \mathfrak{M}'$ and $[\mathfrak{m}]_\alpha$ is infinite unbounded for all $1 \neq \mathfrak{m} \in \mathfrak{M}'$. Thus the support $[f]_\alpha$ of every polynomial f in the variables $\{y_k|k \in n\}$ splits into disjoint parts, each belonging to a non-constant monomial of f.

In the next lemma we employ arguments using the polynomial representation of elements in $V = S_\lambda[y_\alpha|\alpha \leq \mu]$. Define $\mathfrak{M} := \langle x_\alpha|\alpha \leq \mu\rangle$ as multiplicative group of freely generated monomials and $\mathfrak{M}' := \langle y_\alpha|\alpha \leq \mu\rangle$ as multiplicative group of the branches y_α. Equip $\mathfrak{M}$ and $\mathfrak{M}'$ each with a lexicographic ordering defined as in Lemma 6.7.

Lemma 7.3 $S_\lambda[y_\alpha|\alpha \leq \mu] \cong S_\lambda[x_\alpha|\alpha \leq \mu]$ *holds as ring isomorphism, where the x_α are free generators. The set $\{\pi, \pi', y_\alpha|\alpha \leq \mu\}$ is transcendental over S_λ.*

Proof:

The set $\{\pi, \pi', y_\alpha|\alpha \leq \mu\}$ is transcendental over S_λ. **(1)**

Proof: Assume that there is a polynomial $f(x, x', x_\alpha|\alpha \le \mu) \neq 0$ over the coefficient ring S_λ with $f(\pi, \pi', y_\alpha|\alpha \le \mu) = 0$. Write $f(x, x', x_\alpha|\alpha \le \mu) = \sum_{i=0}^{n} f_i(x, x')\,\mathfrak{m}_i$ with distinct monomials $\mathfrak{m}_i \in \langle x_\alpha|\alpha \le \mu\rangle$. Thus $\sum_{i=0}^{n} f_i(\pi, \pi')\,\mathfrak{m}'_i = 0$ holds, where $\mathfrak{m}'_i \in \mathfrak{M}'$ is the by $\mathfrak{m}_i$ induced monomial and $f_i(x, x') \neq 0$ for all $0 \le i \le n$. Thus for large successor ordinals $\alpha \in \lambda$ the support $[\sum_{i=0}^{n} f_i(\pi, \pi')\,\mathfrak{m}'_i]_\alpha$ is bounded, while by Observation 7.2 it splits into disjoint unbounded parts for every non-constant monomial $\mathfrak{m}_i$ occuring in f. Therefore $f_i(\pi, \pi') = 0$ must hold for all monomials $\mathfrak{m}_i \neq 1$. Finally the transcendence of $\{\pi, \pi'\}$ over S_λ gives $f_i = 0$ for all monomials $\mathfrak{m}_i \neq 1$.
Thus the polynomial $f(x, x', x_\alpha|\alpha \le \mu) \neq 0$ reduces to $f(x, x') \neq 0$ with $f(\pi, \pi') = 0$. From $f(\pi, \pi') = 0$ follows $f = 0$ contradicting our assumption $f \neq 0$, therefore $\{\pi, \pi', y_\alpha|\alpha \le \mu\}$ is transcendent over S_λ.

$S_\lambda[y_\alpha|\alpha \le \mu] \cong S_\lambda[x_\alpha|\alpha \le \mu]$. **(2)**

Proof: This follows directly from (1). The canonical isomorphism
$S_\lambda[x_\alpha|\alpha \le \mu] \overset{\Phi}{\longleftrightarrow} S_\lambda[y_\alpha|\alpha \le \mu]$ is well defined by $\Phi(x_\alpha) = y_\alpha$. □

In the following Lemmas let $\nu' = (\nu'(k))_{k\in\omega} \subseteq \lambda$ be another ascending unbounded sequence of ordinals.
To make the definition of initial segments y_k consistent with ν' we introduce the useful abbreviation $y_{\nu' k} := y_{K(\nu',k)}$ with $K(\nu', k) := \min\{i \in \omega|\nu'(k) \le \nu(i)\}$ for all $k \in \omega$, where ν is the ascending sequence related to the branch y. Keep in mind, that the $y_{\nu' k}$ are special initial segments.

Lemma 7.4 *$V_{*p} = \bigcup_{k\in\omega} S_\lambda[y_{\alpha k}|\alpha \le \mu] = \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu' k}|\alpha \le \mu]$ is an ascending chain of rings for every ascending unbounded sequence ν'.*

Proof: We only show that $V_{*p} = \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu' k}|\alpha \le \mu]$, the proof of
$V_{*p} = \bigcup_{k\in\omega} S_\lambda[y_{\alpha k}|\alpha \le \mu]$ is similar.
$V_{*p} \supseteq \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu' k}|\alpha \le \mu]$ is obviously an ascending chain using that

$y_k = p^l y_{k+l} + \sum_{i=0}^{l-1} p^i (r_{k+i} m + e_{\nu(k+i)})$ **(*)**

for all initial segments y_k, where $k, l \in \omega$ and $y \in \{y_\alpha|\alpha \le \mu\}$, while $(K(\nu', k))_{k\in\omega}$ is an ascending unbounded sequence.

To see, that also "$\subseteq$" holds, let $v \in V_{*p}$ be given. Thus $p^i v \in V = S_\lambda[y_\alpha|\alpha \leq \mu]$ holds for some $i \in \omega$. Now only finitely many branches y_α ($\alpha \in I \subseteq \mu$, $|I| < \aleph_0$) occur in the representation of $p^i v$. For any $\alpha \in I$ choose $j(\alpha) \in \omega$ with $K(\nu', j(\alpha)) \geq i$. Then $v \in S_\lambda[y_{\alpha\nu'k'}|\alpha \leq \mu]$ for $k' := \max\{j(\alpha)|\alpha \in I\}$, where we use (*) to split off from $p^i v$ some p^i-divisible polynomial of initial segments and $S_\lambda \subseteq_{*p} \widehat{S_\lambda}$ for the p^i-divisibility of the remaining term. $\square$

Lemma 7.4 clearly implies, that V_{*p} is a subring of $\widehat{S_\lambda}$.
Furthermore Lemma 7.3 and 7.4 hold also under restriction of the index set μ, in particular for every finite $I \subseteq \mu$. We define $V(I) := \bigcup_{k\in\omega} S_\lambda[y_{\alpha k}|\alpha \in I]$.

Lemma 7.5 *The pair $\{\pi, \pi'\}$ is transcendental over V_{*p} and*
$V(I) = \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu'k}|\alpha \in I] \subseteq_* V$ *for every finite* $I \subseteq \mu$.

Proof: From Lemma 7.3 follows directly:
The set $\{\pi, \pi'\}$ is transcendental over $V = S_\lambda[y_\alpha|\alpha \leq \mu]$. **(1)**
From (1) it takes only a small step to:

The set $\{\pi, \pi'\}$ is transcendental over V_{*p}. **(2)**
Proof: Let $0 \neq f(x,y) \in V_{*p}(x,y)$ with $f(\pi, \pi') = 0$. Multiplying f with a suitable p^n ($n \in \omega$) gives some $0 \neq F(x,y) \in V(x,y)$ with $F(\pi, \pi') = 0$. This contradicts (1).

Similar to Lemma 7.4 follows
$V(I) = \bigcup_{k\in\omega} S_\lambda[y_{\alpha k}|\alpha \in I] = \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu'k}|\alpha \in I]$ **(3)**
as ascending chains. Thus for the second claim it remains to show, that

$V(I) \subseteq_* V_{*p}$. **(4)**
Proof: If $\alpha \in \lambda \setminus \mathrm{LORD}, \beta \neq \gamma \leq \mu$ with $y_\beta, y_\gamma \in \widehat{M_\alpha}$, then from $[y_\beta]_\alpha \cap [y_\gamma]_\alpha$ being finite it follows that $[y_{\beta\nu'k}]_\alpha \cap [y_{\gamma\nu'k}]_\alpha \subseteq [y_\beta]_\alpha \cap [y_\gamma]_\alpha$ is finite. Thus also family $\{y_{\alpha\nu'k}|\alpha \leq \mu\}$ has property (+) and from Lemma 7.3 follows $S_\lambda[y_{\alpha\nu'k}|\alpha \leq \mu] \cong S_\lambda[x_\alpha|\alpha \leq \mu]$ as canonical isomorphism. Now $S_\lambda[x_\alpha|\alpha \in I] \subseteq_* S_\lambda[x_\alpha|\alpha \leq \mu]$ gives $S_\lambda[y_{\alpha\nu'k}|\alpha \in I] \subseteq_* S_\lambda[y_{\alpha\nu'k}|\alpha \leq \mu]$ and $V(I) = \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu'k}|\alpha \in I] \subseteq_* \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu'k}|\alpha \leq \mu] = V_{*p}$. $\square$

In the following Lemmas we need to take a closer look at the polynomial representations of elements $f \in S_{\nu'(k)}[y_{\alpha\nu'k}|\alpha \le \mu]$. We therefore define $\mathfrak{M}'_{\nu'k} := \langle y_{\alpha\nu'k}|\alpha \le \mu\rangle$. Equip $\mathfrak{M}'_{\nu'k}$ with a lexicographic ordering as in Lemma 6.7.

Lemma 7.6 *For any finite $I \subseteq \mu$ there exists an $N \in \omega$ with*
$V(I) = \bigcup_{k\in\omega,k>N} S_{\nu'(k)}[y_{\alpha\nu'k}|\alpha \in I]$ an R-pure ascending chain of rings.

Proof: First we show

$\bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu'k}|\alpha \in I] = \bigcup_{k\in\omega} S_{\nu'(k)}[y_{\alpha\nu'k}|\alpha \in I]$. **(1)**

Proof: "$\supseteq$" is clear and the right side of the equation is obviously an ascending chain. To prove "$\subseteq$", let $v \in S_\lambda[y_{\alpha\nu'k}|\alpha \in I]$ be given. Choose $k' \ge k$ such, that all finitely many coefficients of the polynomial representation of v are elements of $S_{\nu'(k')}$. Then $v \in S_{\nu'(k')}[y_{\alpha\nu'k'}|\alpha \le \mu]$.
Thus $V(I) = \bigcup_{k\in\omega} S_\lambda[y_{\alpha\nu'k}|\alpha \in I] = \bigcup_{k\in\omega} S_{\nu'(k)}[y_{\alpha\nu'k}|\alpha \in I]$.

$S_{\nu'(k)}[y_{\alpha\nu'k}|\alpha \in I] \subseteq_* S_{\nu'(k+1)}[y_{\alpha\nu'k+1}|\alpha \in I]$ for all $N < k \in \omega$ and some $N \in \omega$. **(2)**

Proof: Let q be a prime element and elements $f = \sum_{i=0}^m s_i\mathfrak{m}_i \in S_{\nu'(k)}[y_{\alpha\nu'k}|\alpha \in I]$ and $g = \sum_{j=0}^n s'_j\mathfrak{m}''_j \in S_{\nu'(k+1)}[y_{\alpha\nu'k+1}|\alpha \in I]$ in their polynomial representations with $s_i, s'_j \in S_\lambda$, $\mathfrak{m}_i \in \mathfrak{M}'_{\nu'k}$, $\mathfrak{m}''_j \in \mathfrak{M}'_{\nu'k+1}$ and $qg = f$ be given. Without loss of generality let $q \nmid s_i$ for all coefficients of f and $\mathfrak{m}_i < \mathfrak{m}_{i+1}$ for all $0 \le i < m$.
We now take advantage of equation (*) to gain a representation of f in the ring $S_{\nu'(k+1)}[y_{\alpha\nu'k+1}|\alpha \le \mu]$. We then can compare coefficients in the equation $qg = f$.
Note that the monomial of maximal order in the representation of f in
$S_{\nu'(k+1)}[y_{\alpha\nu'k+1}|\alpha \le \mu]$ is $\mathfrak{m}'_m$, where we get $\mathfrak{m}'_m$ by replacing all $y_{\alpha\nu'k}$ in $\mathfrak{m}_m$ by $y_{\alpha\nu'k+1}$. The coefficient of the $\mathfrak{m}'_m$-component is $p^k s_m$ for suitable $k \in \omega$. Thus $q|p^k s_m$, and
$q = p$. **(3)**

For large $N \in \omega$ the supports of the branches $y_{\alpha\nu'k}$ $(\alpha \in I)$ and thus of the $\mathfrak{m}_i$ are totally disjoint. Using (*) again to split off some p-divisible component of $f = \sum_{i=0}^m s_i\mathfrak{m}_i$ in $S_{\nu'(k+1)}[y_{\alpha\nu'k+1}|\alpha \in I]$ we can look on the remainder. The disjointness of supports now gives $p|s_i$ from $p|\sum_{i=0}^m s_i\mathfrak{m}_i$. This contradicts $p = q \nmid s_i$ by (3). ◻

Theorem 7.7 (Step-Lemma)

Let λ be an ordinal of cofinality $\mathrm{cf}(\lambda) = \aleph_0$ *and let $(S_\alpha)_{\alpha\in\lambda}$ be an R-pure ascending chain in $\mathfrak{K}$ with supremum $S_\lambda := \bigcup_{\alpha\in\lambda} S_\alpha \in \mathfrak{K}$ such that S_β/S_α is an $\aleph_1$-free R-module and $S_\beta \in \mathfrak{K}_{S_\alpha}$ for all $\alpha \leq \beta \in \lambda$ and successor ordinals $\alpha \in \lambda$. There shall also exist absolute free ring variables e_α with $S_\alpha[e_\alpha] \subseteq S_{\alpha+1}$ as rings and $S_\alpha \oplus \bigoplus_{k\in\omega} S_\alpha e_\alpha^{k+1} \subseteq_* S_{\alpha+1}$ as R-modules for all successor ordinals $\alpha \in \lambda$. Let a family $\{y_\alpha | \alpha \leq \mu\}$ of branches $y_\alpha := \widehat{r}_\alpha m_\alpha + \sum_{i\in\omega} p^i e_{\nu_\alpha(i)}$ of $\widehat{S_\lambda}$ be given for some ordinal μ, such that*

for all $\alpha \in \lambda \setminus LORD, \beta \neq \gamma \leq \mu$ with $y_\beta, y_\gamma \in \widehat{M_\alpha}$ the set $[y_\beta]_\alpha \cap [y_\gamma]_\alpha$ is finite. (+)

Set $V := S_\lambda[y_\alpha|\alpha \leq \mu] \subseteq \widehat{S_\lambda}$ and $S := (V_{*p})_T$, where $T := \mathfrak{p}_R(V_{*p})$. Then:

(a) $S \in \mathfrak{K}$ and $S \in \mathfrak{K}_{S_\alpha}$ for all successor ordinals $\alpha < \lambda$.

(b) S/S_α is an $\aleph_1$-free R-module for all successor ordinals $\alpha < \lambda$.

(c) $S \subseteq_{*p} \widehat{S_\lambda}$ and $S/S_\lambda \neq 0$ is p-divisible, in particular S/S_λ is not $\aleph_1$-free.

(d) If $\eta : U \to S_\lambda$ is a homomorphism with $P \subseteq U \subseteq S_\lambda$, $P := \bigoplus_{i\in\omega} Re_{\nu_\mu(i)}$, and $a, b \in U \cap \mathfrak{p}S_\lambda$ with $(\eta \neq s) \upharpoonright Ra \oplus Rb$ for all $s \in S_\lambda$, and $\eta := \widehat{\eta} \upharpoonright S \cap \widehat{U}$, then we can choose $y_\mu \in \widehat{S_\lambda}$ with $y_\mu\eta \notin S$, i.e. η does not extend to an endomorphism of S.

Proof:

(a): $S \in \mathfrak{K}$ follows directly from Lemma 7.5, Lemma 7.6 and Theorem 6.8, where we set $S := S_\lambda$, $V := V_{*p}$ and $K(\alpha, k) := K(\nu', k)$.

From our preliminaries follows that $(S_\beta)_{\alpha\leq\beta\in\lambda}$ is an S_α-pure ascending chain in $\mathfrak{K}_{S_\alpha}$ for all successor ordinals $\alpha < \lambda$. Thus we may replace the basic PID R by S_α, and also $S \in \mathfrak{K}_{S_\alpha}$ holds.

(b): For all successor ordinals $\alpha < \lambda$ we have $S_\alpha \subseteq_{*S_\alpha} S$ and $S \in \mathfrak{K}_{S_\alpha}$ by (a). Thus S/S_α is an $\aleph_1$-free S_α-module, in particular an $\aleph_1$-free R-module.

(c): For PID S_λ we have $\widehat{S_\lambda} = (\ \widehat{S_\lambda}\ )^* \,\dot\cup\, p\widehat{S_\lambda}$. Thus for the quotient ring of $\widehat{S_\lambda}$ holds the identity

$$Q(\widehat{S_\lambda}) = \widehat{S_\lambda}\,[\tfrac{1}{p}]. \qquad (++)$$

Now $V_{*p} \subseteq_{*p} \widehat{S_\lambda}$ leads to $T = \mathfrak{p}_R(V_{*p}) \subseteq (\ \widehat{S_\lambda}\ )^*$ and $S = (V_{*p})_T \subseteq_{*p} \widehat{S_\lambda}$. Obviously $S \neq S_\lambda$, thus $0 \neq S/S_\lambda \subseteq \widehat{S_\lambda}/S_\lambda$ is p-divisible.

(d): In order to show **(d)** we first try $y_\mu := y = -\pi a + \sum_{i\in\omega} p^i e_{\nu_\mu(i)}$, which gives a ring $S^0 := (V_{*p})_T$. This does not affect (+).

If $y\eta \notin S^0$, then the proof is finished. Thus $y\eta \in S^0$. In particular $y\eta \in S^0 = (V_{*p})_T$, thus

$$y\eta = \frac{f(y)}{g(y)} \qquad \textbf{(1)}$$

holds for some polynomials $f, g \in S_\lambda[y_\alpha, y|\alpha < \mu]$.

We now try another branch for y_μ and will succeed.

Let $y'_\mu := y' = \pi' b + \sum_{i\in\omega} p^i e_{\nu_\mu(i)}$ and set $S^1 := (V'_{*p})_{T'}$. We claim:

$$y'\eta \notin S^1. \qquad \textbf{(2)}$$

Proof: If $y'\eta \in S^1$, then

$$y'\eta = \frac{f'(y')}{g'(y')} \qquad \textbf{(3)}$$

holds for some polynomials $f', g' \in S_\lambda[y_\alpha, y'|\alpha < \mu]$.

The equations (1) and (3) connected by $y' = \pi a + \pi' b + y$ give rise to

$(\pi a + \pi' b)\eta = \dfrac{f'(y')}{g'(y')} - \dfrac{f(y)}{g(y)} = \dfrac{A(y,y')}{B(y,y')}$, thus

$(\pi a + \pi' b)\eta \cdot B(y, \pi a + \pi' b + y) = A(y, \pi a + \pi' b + y)$ and

$$(\pi \cdot a\eta + \pi' \cdot b\eta)\ B(y, \pi a + \pi' b + y) = A(y, \pi a + \pi' b + y), \qquad \textbf{(4)}$$

where $A, 0 \neq B \in S_\lambda[y_\alpha, y, y'|\alpha < \mu]$ and $a\eta, b\eta \in S_\lambda$.

Now Lemma 7.3 again applies to compare coefficients. We take a look at the polynomial $B(y, y')$ and choose first the exponent m and then the exponent n as large as possible, such that the $y^m y'^n$-component of $B(y, y') \neq 0$ is non-trivial. Thus the $\pi^{n+1} y^m$-component of (4) reads $(\pi a)^n y^m \pi \cdot a\eta \cdot u(y_\alpha) = (\pi a)^{n+1} y^m \cdot v(y_\alpha)$, giving

$$a\eta \cdot u(y_\alpha) = a \cdot v(y_\alpha), \qquad \textbf{(5)}$$

where $0 \neq u, v \in S_\lambda[y_\alpha|\alpha < \mu]$. Similarly the $\pi'^{n+1} y^m$-component of (4) reads $(\pi' b)^n y^m \pi' \cdot b\eta \cdot u(y_\alpha) = (\pi' b)^{n+1} y^m \cdot v(y_\alpha)$ giving

$$b\eta \cdot u(y_\alpha) = b \cdot v(y_\alpha) \qquad \textbf{(6)}$$

with polynomials u, v as in (5) by symmetry of πa and $\pi' b$ in (4).

Next comparing coefficients in (5) and (6) we have

$$a\eta \cdot s = a \cdot t \quad \wedge \quad b\eta \cdot s = b \cdot t$$

for suitable $0 \neq s, t \in S_\lambda$. Thus

$$a\eta = \frac{t}{s}\, a \quad \wedge \quad b\eta = \frac{t}{s}\, b \quad \text{and} \quad (\ \eta = \frac{t}{s}\ ) \upharpoonright Ra \oplus Rb$$

with $\frac{t}{s} \in S_\lambda$, contradicting the preliminaries of Step-Lemma 7.7 (d). $\square$

Observation 7.8 *By Step-Lemma 7.7 (c) we do not leave the appreciated ring* $\widehat{S_\lambda}$ *though using ring localizations. This is an important aspect for the practicability of the Step-Lemma construction.*
Observe $|S_\lambda| \leq |S| \leq |V_{*p}| \cdot |T| \leq |V|^2 = |V| = |S_\lambda| \cdot |\mu|$, *thus* $|S| = |S_\lambda|$ *for* $|\mu| \leq |S_\lambda|$.

Observation 7.9 *Step-Lemma 7.7 (d) can be reformulated in the following way.*

(d) For any $a, b \in \mathfrak{p}S_\lambda$ *there exist two extensions* S^0 *and* S^1 *of* S_λ *only depending on the bounded part of branch* y_μ, a *and* b, *such that for any homomorphism* $\eta : U \to S_\lambda$ *with* $P \subseteq U \subseteq S_\lambda$, $P := \bigoplus_{i\in\omega} Re_{\nu_\mu(i)}$, *and* $a, b \in U$ *the following holds:*
If η *extends to both* S^0 *and* S^1, *then* $(\eta = s) \upharpoonright Ra \oplus Rb$ *for some* $s \in S_\lambda$.

Thus (d) in first line is independent of η. *This is an important fact for the weak diamond construction.*

8 The Main Construction

In this section we adjoin various bits needed for Theorem 2.29.

8.1 The Weak Diamond Case

Let $\kappa > \aleph_0$ be a successor cardinal and R be a PID with $|R| < \kappa$ and $\{\pi, \pi'\} \subseteq \widehat{R}_p$ transcendental over R for some prime element $p \in R$. Choose a set S of cardinality $|S| = \kappa$. Also choose a κ-filtration $S = \bigcup_{\alpha<\kappa} S_\alpha$ with

$$|S_0| = |S_1 \setminus S_0| = |R| \text{ and } |S_\alpha| = |S_{\alpha+1} \setminus S_\alpha| = |R||\alpha| \text{ for all } 0 < \alpha < \kappa \qquad (*)$$

and an element $e_\alpha \in S_{\alpha+1} \setminus S_\alpha$ for all successor ordinals $\alpha \in \kappa$.
Let $E \subseteq \kappa^o = \{\alpha \in \kappa |\ \mathrm{cf}\,(\alpha) = \aleph_0\}$ be a stationary set with $\Phi_\kappa(E)$. We decompose $E = \dot{\bigcup}_{\gamma\in\kappa} E_\gamma$ with E_γ stationary and $\Phi_\kappa(E_\gamma)$, see [20, Theorem 2.1.16, p. 56], and assign a set $E_{ab} := E_\gamma$ to every pair $(a, b) \in S \times S$.
For every $\alpha \in E$ choose a strictly ascending sequence of successor ordinals $\bigcup_{i\in\omega} \nu_\alpha(i) = \alpha$ and let y_α be the induced branch.

We want to assign inductively a ring structure to the sets S_α such that we get a continuous chain $(S_\alpha)_{\alpha<\kappa}$ in $\mathfrak{K}$. In particular S_α will be a left R-module and a right $\mathrm{End}\,_R S_\alpha$-module.
The union $S := \bigcup_{\alpha<\kappa} S_\alpha$ of this continuous chain will satisfy condition (a) of Theorem 2.29 as left R-module.

Next we have to define a suitable weak diamond function $F : E \to 2$.
The ring structure of S_α can be viewed as a certain subset of the set S_α^6, and maps $U \subseteq S_\alpha \to S_\alpha$ are subsets of S_α^2, thus ring structure and partial endomorphisms can be viewed as subsets of $\mathrm{Rel}_\alpha := S_\alpha^6 \cup S_\alpha^2$, where the S_α^6-part describes the ring S_α. Next we will for simplicity suppress the algebraic structure of S_α and restrict ourselves to $\mathfrak{P}(\mathrm{Rel}_\alpha)$. We will now define partition functions

$$p_\alpha^{ab} : \mathfrak{P}(\mathrm{Rel}_\alpha) \to 2 \quad \text{for all} \quad \alpha \in E_{ab}.$$

We let $p_\alpha^{ab}(X) = 0$ for $X \in \mathfrak{P}(\mathrm{Rel}_\alpha)$, if the following holds.

1. $S_\alpha = \bigcup_{\beta \in \alpha} S_\beta$ is an R-pure ascending chain in $\mathfrak{K}$ with the ring structure induced by X.

2. S_β / S_γ is an $\aleph_1$-free R-module and $S_\beta \in \mathfrak{K}_{S_\gamma}$ for all $\gamma \leq \beta \in \alpha$ and successor ordinals $\gamma \in \alpha$.

3. $S_\gamma[e_\gamma] \subseteq S_{\gamma+1}$ as rings and $S_\gamma \oplus \bigoplus_{k \in \omega} S_\gamma e_\gamma^{k+1} \subseteq_* S_{\gamma+1}$ as R-modules for all successor ordinals $\gamma \in \alpha$.

4. The map $\eta : U \to S_\alpha$ induced by X is a partial endomorphism with $a, b \in U \cap \mathfrak{p} S_\alpha$.

5. If S_α^0 and S_α^1 are fixed extensions of S_α given by S^0 and S^1 in Step-Lemma 7.9 (d) for $\mu := 0$ and $y_0 := \widehat{r}_\alpha m_\alpha + \sum_{i \in \omega} p^i e_{\nu_\alpha(i)}$, then η does not extend to S_α^0.

Otherwise we put $p_\alpha^{ab}(X) = 1$.
By $\Phi_\kappa(E_{ab})$ there is a global partition function $F^{ab} : E_{ab} \to 2$ such that for all $X \in \mathfrak{P}(\mathrm{Rel}) = \mathfrak{P}(S^6 \cup S^2)$ the set

$$\{\alpha \in E_{ab} | p_\alpha^{ab}(X \cap \mathrm{Rel}_\alpha) = F^{ab}(\alpha)\} \text{ is stationary in } \kappa. \qquad (+)$$

We will carry out the following construction steps inductively.

Let $S_0 := R$.

Suppose that the structure on S_β ($\beta < \alpha$) is defined.

Case 1: $\alpha = \beta + 1,\ \beta \notin E$.

Construct S_α from S_β using x-Localization and identify $e_\beta := x$, thus

$$S_\alpha := S_\beta[e_\beta]_{\mathfrak{p} S_\beta[e_\beta]}.$$

Case 2: $\alpha = \beta + 1,\ \beta \in E_{ab}$.

- If $a, b \in \mathfrak{p}S_\beta$, let S_β^0 and S_β^1 the fixed extensions of S_β given by S^0 and S^1 in Step-Lemma 7.9 (d) for $\mu := 0$ and $y_0 := \widehat{r}_\beta m_\beta + \sum_{i\in\omega} p^i e_{\nu_\beta(i)}$. Then we define

$$S_\alpha := S_\beta^{F^{ab}(\beta)}.$$

- Otherwise use the construction described in Case 1.

Case 3: $\alpha \in \mathrm{LORD} \cap \kappa$.

Set $S_\alpha := \bigcup_{\beta<\alpha} S_\beta$.

We deduce some easy facts about the constructed chain $(S_\alpha)_{\alpha<\kappa}$.
See also Lemma 5.3 and Theorem 5.4.

Lemma 8.1 *Let $(S_\alpha)_{\alpha<\kappa}$ be constructed as above.*
(a) $(S_\alpha)_{\alpha<\kappa}$ is a well-defined pure continuous chain in $\mathfrak{K}$.
(b) If $\alpha \leq \beta < \kappa$ with α a successor ordinal, then S_β/S_α is an $\aleph_1$-free R-module and $S_\beta \in \mathfrak{K}_{S_\alpha}$.
(c) $|S_0| = |S_1 \setminus S_0| = |R|$ and $|S_\alpha| = |S_{\alpha+1} \setminus S_\alpha| = |R||\alpha|$ holds for all $0 < \alpha < \kappa$, i.e. $(S_\alpha)_{\alpha<\kappa}$ satisfies (*) of the κ-filtration $S = \bigcup_{\alpha<\kappa} S_\alpha$.

Proof: This is an easy transfinite induction on $\alpha < \kappa$ using Lemma 6.4, Corollary 6.9 and Theorem 7.7.

We state some basic properties of the R-algebra $S := \bigcup_{\alpha<\kappa} S_\alpha$.

Theorem 8.2 *If $(S_\alpha)_{\alpha<\kappa}$ is the chain as above and $S = \bigcup_{\alpha<\kappa} S_\alpha$, then the following holds.*
(a) $S \in \mathfrak{K}$ is an $\aleph_1$-free R-module and $|S| = \kappa$.
(b) $S \in \mathfrak{K}_{S_\alpha}$ and S/S_α is $\aleph_1$-free for all successor ordinals $\alpha < \kappa$.
(c) If $\Psi \in \mathrm{End}_R S$ and $a, b \in \mathfrak{p}S$, then there exists an $s \in S$ with $(\Psi = s) \restriction Ra \oplus Rb$.

Proof: (a) and (b) are an immediate consequence of Corollary 6.3 and Lemma 8.1.

(c): For any $\Psi \in \operatorname{End}_R S$, let $X \in \mathfrak{P}(\mathrm{Rel})$ be the by the ring structure of S and endomorphism Ψ induced set. With (+) the set

$E := \{\alpha \in E_{ab} \mid p_\alpha^{ab}(X \cap \ \mathrm{Rel}_\alpha) = F^{ab}(\alpha)\}$ **(1)**

is stationary in κ. Furthermore

$$C := \{\alpha \in \kappa \mid S_\alpha \Psi \subseteq S_\alpha\}$$

is a cub. In particular $\Psi \restriction S_\alpha \in \operatorname{End} S_\alpha$ holds for all $\alpha \in C$. For all $\alpha \in C$ let $X_\alpha \in \mathfrak{P}(\mathrm{Rel}_\alpha)$ be the by the ring structure of S_α and endomorphism $\Psi \restriction S_\alpha$ induced set. Then $X_\alpha = X \cap \ \mathrm{Rel}_\alpha$ holds. **(2)**

The set $E \cap C$ is stationary. Choose $\beta \in E \cap C \neq \emptyset$ large such that $a, b \in S_\beta$. In particular $\beta \in E_{ab}$ and $S_{\beta+1}$ is constructed from S_β with the Step-Lemma, thus $S_{\beta+1} \subseteq_{*p} \widehat{(S_\beta)}_p$ holds.

We can write any $m \in S_{\beta+1}$ as limit of a sequence $(m_i)_{i\in\omega} \subseteq S_\beta$ in the p-adic topology on $S_{\beta+1} \subseteq_{*p} \widehat{(S_\beta)}_p$. Thus $(\Psi_\beta(m_i))_{i\in\omega} \subseteq S_\beta$ converges to $\Psi(m)$ by continuity. By Lemma 8.1 and Lemma 8.2 $S_{\beta+1} \subseteq S$ and $S/S_{\beta+1}$ are $\aleph_1$-free, thus $S_{\beta+1}$ is p-adically closed in S and $\Psi(m) \in S_{\beta+1}$ (see Corollary 5.2). Hence $\Psi(S_{\alpha+1}) \subseteq S_{\alpha+1}$; thus

$\Psi \restriction S_\alpha$ *extends to* $\Psi \restriction S_{\alpha+1} \in \operatorname{End}(S_{\alpha+1})$. **(3)**

Now assume that no $s \in S_\beta$ exists with $(\Psi = s) \restriction Ra \oplus Rb \subseteq S_\beta$.

Then $\Psi \restriction S_\alpha$ does not lift to both S_β^0 and S_β^1 by Step-Lemma 7.9 (d). In particular $\Psi \restriction S_\alpha$ does not lift to $S_\beta^{p_\beta^{ab}(X_\beta)}$ by definition of the partition function p_β^{ab}. But for $\beta \in E \cap C$ holds

$$p_\beta^{ab}(X_\beta) \overset{(2)}{=} p_\beta^{ab}(X \cap \ \mathrm{Rel}_\beta) \overset{(1)}{=} F^{ab}(\beta) \text{ and } S_\beta^{p_\beta^{ab}(X_\beta)} = S_\beta^{F^{ab}(\beta)} = S_{\beta+1}$$

since Case 2 of our construction. Thus $\Psi \restriction S_\alpha$ does not lift to $S_{\beta+1}$ contradicting (3). □

We are ready to prove Theorem 2.29 (a).

Proof: Ring $S \in \mathfrak{K}$ is a PID and an $\aleph_1$-free R-module by the last lemma, and the following inclusion is obvious

$S \subseteq \operatorname{End}_R S.$ **(1)**

If $\Psi \in \operatorname{End}_R S$ then we apply Theorem 8.2 (c). There is a family $(s_{ab})_{a,b \in \mathfrak{p}S}$ of elements $s_{ab} \in S$ such that $(\Psi = s_{ab}) \restriction Ra \oplus Rb \subseteq S$ for all $a, b \in \mathfrak{p}S$. In particular $(\Psi = s_{ab} = s_{ab'}) \restriction Ra$, hence $a(s_{ab} - s_{ab'}) = 0$ and $s_{ab} = s_{ab'}$ holds for all $a, b, b' \in \mathfrak{p}S$. Thus there exists a universal constant $s \in S$ with $s_{ab} = s$ for all $a, b \in \mathfrak{p}S$, and $(\Psi = s) \restriction \sum_{a \in \mathfrak{p}S} Ra = S$ holds. This yields $\Psi = s \in S$, hence

$\operatorname{End}_R S = S$ and $\operatorname{Aut}_R S = S^*$. **(2)**

In particular S is an E(R)-algebra.

$\operatorname{Aut}_R S = S^*$ *acts uniquely transitive on* S. **(3)**

Proof: $\operatorname{Aut}_R S$ acts transitively on S, because the automorphism $\frac{a}{b}$ maps a to b for arbitrary $a, b \in \mathfrak{p}S = S^*$. On the other hand follows $s_1 = s_2$ from $as_1 = as_2$ for arbitrary $s_1, s_2 \in \operatorname{Aut}_R S$ and $a \in \mathfrak{p}S$. Thus $\operatorname{Aut}_R S$ acts also uniquely on S and S is a UT-module. □

8.2 The Black Box Case

Let $\kappa > \aleph_0$ be a regular cardinal with $\kappa^{\aleph_0} = \kappa$ and R be a PID with $|R| \leq \kappa$ and $\{\pi, \pi'\} \subseteq \widehat{R}_p$ transcendental over R for some prime element $p \in R$.

We adopt the notions of the General Black Box 2.14:

Let $T := T_{\kappa \times \kappa \times \aleph_0} = T_\kappa \times T_\kappa \times T_{\aleph_0}$ be the canonical tree. A **norm** $|| \cdot ||$ is defined on this tree with respect to the first coordinate.

In the following let $B := \bigoplus_{\tau \in T} B_\tau$ with all $B_\tau := R^{(\kappa)} := \bigoplus_{\lambda \in \kappa} R$, which will be our basic module of consideration for the given PID R. For our fixed prime element $p \in R$ the p-adic completion of B will be called $\widehat{B}$. We also set $T_\alpha := T_{\alpha \times \kappa \times \aleph_0}$ and $B_\alpha := \bigoplus_{\tau \in T_\alpha} B_\tau$. In particular $B_0 = T_0 = \emptyset$ and $B = \bigcup_{\alpha \in \kappa} B_\alpha$.

For any element $b = \sum_{\tau \in T} b_\tau \in \widehat{B}$ let $[b] := \{\tau | b_\tau \neq 0\}$ be the **support** of b. For any subset $X \subseteq \widehat{B}$ set $||X|| := ||[X]||$.

Next we define the R-modules B_τ more precisely:

Reserve free generators e_α $(\alpha \in \kappa)$ and e_τ $(\tau \in T)$. Moreover choose bijections $\pi_\alpha : \kappa \to T_{\alpha+1} \setminus T_\alpha$ for all $\alpha \in \kappa$. Define

$$\mathfrak{M}_\alpha := \langle e_\gamma | \gamma < \alpha \rangle \text{ and } \mathfrak{M}_{\alpha\beta} := \langle e_\gamma, e_\tau | \gamma < \kappa,\ ||\tau|| < \alpha \text{ or } \tau = \pi_\alpha(\gamma'),\ \gamma' < \beta \rangle$$

as multiplicative groups of freely generated monomials. Thus $(\mathfrak{M}_\alpha)_{\alpha\in\kappa}$ is a continuous ascending chain with $\bigcup_{\alpha\in\kappa} \mathfrak{M}_\alpha = \mathfrak{M}_{00}$ and $(\mathfrak{M}_{\alpha\beta})_{(\alpha,\beta)\in\kappa\times\kappa}$ is a continuous ascending chain with supremum $\mathfrak{M}_{\kappa,\kappa}$ respecting the lexicographic order of $\kappa \times \kappa$. For any $\mathfrak{m} \in \mathfrak{M}_{\kappa,\kappa}$ let $|\mathfrak{m}|$ be the sum of all exponents occurring in the monomial $\mathfrak{m}$.
Let be

$$B_{\pi_0(0)} := \bigoplus_{\mathfrak{m}\in\mathfrak{M}_{01}} R\mathfrak{m} \text{ and } B_{\pi_\alpha(\beta)} := \bigoplus_{\mathfrak{m}\in\mathfrak{M}_{\alpha\beta}, n\in\omega} R\mathfrak{m}e^{n+1}_{\pi_\alpha(\beta)} \text{ for } (\alpha,\beta) \neq (0,0).$$

Furthermore set $e'_\alpha := p \cdot e_\alpha$ $(\alpha \in \kappa)$ and $e'_\tau := p \cdot e_\tau$ $(\tau \in T)$. We define the notions B', B'_τ, B'_α, $\mathfrak{M}'_\alpha$ and $\mathfrak{M}'_{\alpha\beta}$ similar to B, B_τ, B_α, $\mathfrak{M}_\alpha$ and $\mathfrak{M}_{\alpha\beta}$ using the free generators e'_α, e'_τ $(\alpha \in \kappa, \tau \in T)$ and the bijections π_α $(\alpha \in \kappa)$ from above. Especially $B' \subseteq B$ and $\widehat{B'} \subseteq \widehat{B}$ holds, but $\widehat{B'} \not\subseteq_{*p} \widehat{B}$.
Let the R-module endomorphism $\Gamma : \widehat{B} \to \widehat{B'}$ be defined by $\Gamma(\mathfrak{m}) = p^{|\mathfrak{m}|}\mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}_{\kappa\kappa}$. In particular Γ is the ring endomorphism induced by $\Gamma(e_\alpha) = e'_\alpha$ and $\Gamma(e_\tau) = e'_\tau$ $(\alpha \in \kappa, \tau \in T)$.

Corollary 8.3 *The map $\Gamma : \widehat{B} \to \widehat{B'} \subseteq \widehat{B}$ is a ring isomorphism.*

Proof: Obviously Γ is surjective, and

$$\widehat{B} \subseteq \bigoplus_{\tau\in T} \widehat{B_\tau},\ \widehat{B_{\pi_0(0)}} \subseteq \bigoplus_{\mathfrak{m}\in\mathfrak{M}_{01}} \widehat{R}\mathfrak{m},\ \widehat{B_{\pi_\alpha(\beta)}} \subseteq \bigoplus_{\mathfrak{m}\in\mathfrak{M}_{\alpha\beta}, n\in\omega} \widehat{R}\mathfrak{m}e^{n+1}_{\pi_\alpha(\beta)}$$

holds for $(\alpha,\beta) \neq (0,0)$. In particular every $b \in \widehat{B}$ can be expressed as sum of at most countably many $\mathfrak{m}$-components (coming from $\widehat{R}\mathfrak{m}$, $\mathfrak{m} \in \mathfrak{M}_{\kappa\kappa}$). For any $0 \neq b \in \widehat{B}$ choose some $\mathfrak{m} \in \mathfrak{M}_{\kappa\kappa}$ with non-trivial $\mathfrak{m}$-component $0 \neq \widehat{r} \in \widehat{R}$. Then $0 \neq p^{|\mathfrak{m}|}\widehat{r}$ is the $\mathfrak{m}$-component of $\Gamma(b)$ in $\widehat{B}$, thus $\Gamma(b) \neq 0$. □

Finally let $E \subseteq \kappa^o$ be a stationary set. We decompose $E = \dot{\bigcup}_{\gamma\in\kappa} E_\gamma$ with E_γ stationary, see [32, Theorem 85, p. 433], and assign a set $E_{ab} := E_\gamma$ to every pair $(a,b) \in \widehat{B'} \times \widehat{B'}$.

(Recall $\kappa^{\aleph_0} = \kappa$!)
Let $p_\beta^{ab} = (f_\beta^{ab}, \varphi_\beta^{ab})$ be a list of traps for the R-module B and stationary set E_{ab} given by the General Black Box, Theorem 2.14.

We want to construct inductively a continuous chain of rings $(S_\alpha)_{\alpha\in\kappa}$ in $\mathfrak{K}$, such that as R-module S_α is sandwiched between B'_α and $\widehat{B_\alpha}$:

$$B'_\alpha \subseteq S_\alpha \subseteq \widehat{B_\alpha} \text{ for all } 0 \neq \alpha \in \kappa.$$

The union $S := \bigcup_{\alpha<\kappa} S_\alpha$ of this continuous chain will satisfy condition (b) of Theorem 2.29 as left R-module.
We will carry out the following steps inductively.

First Steps of the Induction

Define the continuous chain $(R_\alpha)_{\alpha\in\kappa}$ in $\mathfrak{K}$:

- $R_0 := R$.
- Construct $R_{\alpha+1}$ from R_α using x-Localization and identify $e'_\alpha := x$, thus
$$R_{\alpha+1} := R_\alpha[e'_\alpha]_{\mathfrak{p}R_\alpha[e'_\alpha]}.$$
- Set $R_\alpha := \bigcup_{\beta<\alpha} R_\beta$ at limit points.

We define $S_0 := \bigcup_{\alpha<\kappa} R_\alpha$.

Suppose that the structure on S_β $(\beta < \alpha)$ is defined.
Case 1: $\alpha = \beta + 1,\ \beta \notin E$. Define the continuous chain $(S_{\beta\gamma})_{\gamma\in\kappa}$ in $\mathfrak{K}$:

- $S_{\beta 0} := S_\beta$.
- Construct $S_{\beta(\gamma+1)}$ from $S_{\beta\gamma}$ using x-Localization and identify $e'_{\pi_\beta(\gamma)} := x$, thus
$$S_{\beta(\gamma+1)} := (S_{\beta\gamma}[e'_{\pi_\beta(\gamma)}])_{\mathfrak{p}S_{\beta\gamma}[e'_{\pi_\beta(\gamma)}]}.$$
- Set $S_{\beta\gamma} := \bigcup_{\delta<\gamma} S_{\beta\delta}$ at limit points.

We define $S_\alpha := \bigcup_{\gamma<\kappa} S_{\beta\gamma}$.

Case 2: $\alpha = \beta + 1,\ \beta \in E_{ab}$.

Let $(p^{ab}_\varepsilon)_{\gamma\le\varepsilon<\delta}$ for suitable ordinals γ, δ denote the sublist of $p^{ab}_\varepsilon = (f^{ab}_\varepsilon, \varphi^{ab}_\varepsilon)$ consisting of all traps with $||p^{ab}_\varepsilon|| = \beta$. We recursively define a set $I(\beta) \subseteq \{\varepsilon |\ \gamma \le \varepsilon < \delta\}$ and a family of branches $\{y^{ab}_\varepsilon | \varepsilon \in I(\beta)\}$.

Suppose that $I_{\varepsilon'}(\beta) \subseteq \{\varepsilon'' |\ \gamma \le \varepsilon'' \le \varepsilon'\}$ and branches $y^{ab}_{\varepsilon'}$ are already defined for $\varepsilon' < \varepsilon < \delta$.

- If $a, b \in \mathfrak{p}S_\beta,\ a\Gamma^{-1}, b\Gamma^{-1} \in \operatorname{Dom}\varphi^{ab}_\varepsilon,\ \operatorname{Im}(\Gamma^{-1}\varphi^{ab}_\varepsilon \restriction S_\beta) \subseteq S_\beta$ and $(\Gamma^{-1}\varphi^{ab}_\varepsilon \ne s) \restriction Ra \oplus Rb$ for all $s \in S_\beta$, define $I_\varepsilon(\beta) := \bigcup_{\varepsilon'<\varepsilon} I_{\varepsilon'}(\beta) \cup \{\varepsilon\}$. Let $f'^{ab}_\varepsilon : \omega \to \kappa \times \kappa \times \aleph_0$ setting $f'^{ab}_\varepsilon(i) := (f^{ab}_\varepsilon(i), g^{ab}_\varepsilon(i))$ for all $i \in \omega$ and some function $g^{ab}_\varepsilon : \omega \to \aleph_0$. Set

  $$y^{ab}_\varepsilon := \widehat{r}^{ab}_\varepsilon m^{ab}_\varepsilon + \sum_{i\in\omega} p^i e'_{f'^{ab}_\varepsilon \restriction (i+1)}.$$

  Define

  $$V^\varepsilon := S_\beta[y^{ab}_{\varepsilon'} | \varepsilon' \le \varepsilon, \varepsilon' \in I(\beta)] \text{ and } S^\varepsilon := (V^\varepsilon_{*p})_T,\ \text{ where } T := \mathfrak{p}_R(V^\varepsilon_{*p}).$$

  Using Step-Lemma 7.7 (d) choose $\widehat{r}^{ab}_\varepsilon m^{ab}_\varepsilon \in \{-\pi a, \pi' b\}$ such that

  $$y^{ab}_\varepsilon \Gamma^{-1} \varphi^{ab}_\varepsilon \notin S^\varepsilon.$$

- Otherwise set $I_\varepsilon(\beta) := \bigcup_{\varepsilon'<\varepsilon} I_{\varepsilon'}(\beta)$. Thus $\varepsilon \notin I_\varepsilon(\beta)$.

Finally define $I(\beta) := \bigcup_{\gamma\le\varepsilon<\delta} I_\varepsilon(\beta)$ as supremum of an ascending chain and $\{y^{ab}_\varepsilon | \varepsilon \in I(\beta)\}$ as family of branches.
Furthermore $\{y^{ab}_\varepsilon | \varepsilon \in I(\beta)\}$ will be constructed such that condition (+) of Step-Lemma 7.7 and

$$y^{ab}_{\varepsilon'} \Gamma^{-1} \varphi^{ab}_{\varepsilon'} \notin S^\varepsilon \mathit{for}\ \varepsilon' \le \varepsilon,\ \ \varepsilon, \varepsilon' \in I(\beta) \qquad (\star)$$

holds. The consistence proof of (+) and $(\star)$ is part of Lemma 8.4.

Now we are ready to construct S_α:

- If $I(\beta) \neq \emptyset$ define S'_β from S_β and $\{y^{ab}_\varepsilon | \varepsilon \in I(\beta)\}$ using the Step-Lemma 7.7 such that

  $V'_\beta := S_\beta[y^{ab}_\varepsilon | \varepsilon \in I(\beta)]$, $S'_\beta := (V'_{\beta_{*p}})_T$ for $T := \mathfrak{p}_R(V'_{\beta_{*p}})$ and
  $y^{ab}_\varepsilon \Gamma^{-1} \varphi^{ab}_\varepsilon \notin S'_\beta$ for all $\varepsilon \in I(\beta)$. **(++)**

- Otherwise set $S'_\beta := S_\beta$.

Construct S_α from S'_β as in Case 1.

Case 3: $\alpha \in \mathrm{LORD} \cap \kappa$.

Set $S_\alpha := \bigcup_{\beta<\alpha} S_\beta$.

We deduce some easy facts about the constructed chain $(S_\alpha)_{\alpha<\kappa}$.
See also Lemma 5.3 and Theorem 5.4.

Lemma 8.4 *Let $(S_\alpha)_{\alpha<\kappa}$ be constructed as above.*
(a) $(S_\alpha)_{\alpha<\kappa}$ is a well-defined pure continuous chain in $\mathfrak{K}$.
(b) If $\alpha \leq \beta < \kappa$ with α a successor ordinal, then S_β/S_α is an $\aleph_1$-free R-module and $S_\beta \in \mathfrak{K}_{S_\alpha}$. Furthermore $S_{\alpha+1}/S'_\alpha$ is $\aleph_1$-free for all $\alpha \in E$.
(c) $B'_\alpha \subseteq S_\alpha \subseteq \widehat{B_\alpha}$ for all $0 \neq \alpha \in \kappa$.

Proof: This is mainly an easy transfinite induction on $\alpha < \kappa$ using Lemma 6.4, Corollary 6.10 and Theorem 7.7. We therefore concentrate on the more interesting arguments only.

Step-Lemma 7.7, condition (+) and ($\star$) are consistent with the construction.
Let $\beta \in E_{ab}$ and $\varepsilon \in I(\beta)$. Suppose that the branches $y^{ab}_{\varepsilon'}$ ($\varepsilon' < \varepsilon$, $\varepsilon' \in I(\beta)$) are defined and condition (+) holds. Furthermore set

$$V^{\varepsilon'} := S_\beta[y^{ab}_{\varepsilon''} | \varepsilon'' \leq \varepsilon', \varepsilon'' \in I(\beta)] \text{ and } S^{\varepsilon'} := (V^{\varepsilon'}_{*p})_T, \text{ where } T := \mathfrak{p}_R(V^{\varepsilon'}_{*p}) \text{ for all } \varepsilon' \leq \varepsilon$$

and let

$$y^{ab}_{\varepsilon''}\Gamma^{-1}\varphi^{ab}_{\varepsilon''} \notin S^{\varepsilon'} \quad (\varepsilon'' \leq \varepsilon' < \varepsilon, \varepsilon', \varepsilon'' \in I(\beta)). \qquad \textbf{(1)}$$

As basis for Step-Lemma 7.7 serves the R-pure ascending chain $S_\beta = \bigcup_{(\gamma,\delta)<(\beta,0)} S_{\gamma\delta}$ in $\mathfrak{K}$ of cofinality $\mathrm{cf}\,((\beta,0)) = \mathrm{cf}\,(\beta) = \aleph_0$, where we use the lexicographic order on $\kappa \times \kappa$. The construction gives $S_{\gamma\delta}/S_{\gamma'\delta'}$ $\aleph_1$-free and $S_{\gamma\delta} \in \mathfrak{K}_{S_{\gamma'\delta'}}$ for all $(\gamma',\delta') \leq (\gamma,\delta) < (\beta,0)$ and successor elements (γ',δ'). Also

$$S_{\gamma\delta}[e_{\pi_\gamma(\delta)}] \subseteq S_{\gamma(\delta+1)} \text{ as rings and } S_{\gamma\delta} \oplus \bigoplus_{k\in\omega} S_{\gamma\delta} e^{k+1}_{\pi_\gamma(\delta)} \subseteq_* S_{\gamma(\delta+1)} \text{ as } R\text{-modules}$$

holds for all $(\gamma,\delta) < (\beta,0)$ by Corollary 6.10.

Define $f'^{ab}_\varepsilon : \omega \to \kappa \times \kappa \times \aleph_0$ setting $f'^{ab}_\varepsilon(i) := (f^{ab}_\varepsilon(i), g^{ab}_\varepsilon(i))$ for all $i \in \omega$ and some function $g^{ab}_\varepsilon : \omega \to \aleph_0$. Set

$$y^{ab}_\varepsilon := \widehat{r}^{ab}_\varepsilon m^{ab}_\varepsilon + \sum_{i\in\omega} p^i e'_{f'^{ab}_\varepsilon \restriction (i+1)}.$$

Observe that $||f'^{ab}_\varepsilon \restriction (i+1)|| < ||p^{ab}_\varepsilon|| = \beta$, thus $f'^{ab}_\varepsilon \restriction (i+1) \in T_\beta$ and $e'_{f'^{ab}_e \restriction (i+1)}$ is already identified as free generator in S_β, and $y^{ab}_\varepsilon \in \widehat{S_\beta}$.

The General Black Box, Theorem 2.14 (iii), gives $\mathrm{Br}\,(f^{ab}_\varepsilon \times T_{\aleph_0}) \cap \mathrm{Br}\,(f^{ab}_{\varepsilon'} \times T_{\aleph_0}) = \emptyset$ for $\varepsilon' < \varepsilon$. Observing that the supports in B' of the branches y^{ab}_ε and $y^{ab}_{\varepsilon'}$ are determined by $f'^{ab}_\varepsilon \in \mathrm{Br}\,(f^{ab}_\varepsilon \times T_{\aleph_0})$ and $f'^{ab}_{\varepsilon'} \in \mathrm{Br}\,(f^{ab}_{\varepsilon'} \times T_{\aleph_0})$, an easy support argument leads to

$$[y^{ab}_\varepsilon] \cap [y^{ab}_{\varepsilon'}] \textit{ is finite for all } \varepsilon' < \varepsilon,$$

and (+) holds also for y^{ab}_ε independent of the choice of g^{ab}_ε.

Set $\eta := \Gamma^{-1}\varphi^{ab}_\varepsilon \restriction S_\beta$ as test-function for Step-Lemma 7.7 (d) and remind that

$$(\Gamma^{-1}\varphi^{ab}_\varepsilon \neq s) \restriction Ra \oplus Rb$$

from $\varepsilon \in I(\beta)$. Independent of the choice of g^{ab}_ε holds $f'^{ab}_\varepsilon \restriction (i+1) \in f^{ab}_\varepsilon \times T_{\aleph_0}$, thus $e_{f'^{ab}_\varepsilon \restriction (i+1)} \in \mathrm{Dom}\,\varphi^{ab}_\varepsilon$ by Definition 2.13. Furthermore $a\Gamma^{-1}, b\Gamma^{-1} \in \mathrm{Dom}\,\varphi^{ab}_\varepsilon$, thus $a, b, e'_{f'^{ab}_\varepsilon \restriction (i+1)} \in \mathrm{Dom}\,\eta$ and Step-Lemma 7.7 (d) applies to choose $\widehat{r}^{ab}_\varepsilon m^{ab}_\varepsilon \in \{-\pi a, \pi' b\}$ such that $y^{ab}_\varepsilon \eta = y^{ab}_\varepsilon \Gamma^{-1}\varphi^{ab}_\varepsilon \notin S^\varepsilon$.

To gain also $y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \notin S^\varepsilon$ for $\varepsilon' < \varepsilon, \varepsilon' \in I(\beta)$ we distinguish the following two cases.

Case 1: $\varepsilon' + 2^{\aleph_0} \leq \varepsilon$.

Here we can combine Theorem 2.14 (iv) and Definition 2.13 (iii) to gain

$$\mathrm{Br}\,(f^{ab}_\varepsilon \times T_{\aleph_0}) \cap \mathrm{Br}\,[P^{ab}_{\varepsilon'}] = \emptyset, \text{ where } y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \in \widehat{\mathrm{Im}\,(\varphi^{ab}_{\varepsilon'})} \subseteq \widehat{P^{ab}_{\varepsilon'}}.$$

In particular $\mathrm{Br}\,(f^{ab}_{\varepsilon} \times T_{\aleph_0}) \cap \mathrm{Br}\,[y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'}] = \emptyset$ and together with Definition 2.13 (iv) follows

$[y^{ab}_{\varepsilon}] \cap [y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'}]$ *is finite.* **(2)**

Now suppose, there is $\varepsilon' < \varepsilon, \varepsilon' \in I(\beta)$ with $y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \in S^{\varepsilon}$. Then using support arguments $y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \in \bigcup_{\varepsilon''<\varepsilon} S^{\varepsilon''}$ follows from (2) contradicting (1).

Case 2: $\varepsilon' < \varepsilon < \varepsilon' + 2^{\aleph_0}$.

First observe that

there exists at most one function $g^{ab}_{\varepsilon} : \omega \to \aleph_0$ *with* $y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \in S^{\varepsilon}$. **(3)**

Assume that $y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \in S^{\varepsilon}$ for two functions $g^{ab}_{\varepsilon 1}$ and $g^{ab}_{\varepsilon 2}$, then using support arguments $y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \in \bigcup_{\varepsilon''<\varepsilon} S^{\varepsilon''}$ follows contradicting (1).

Now observe that there are less than $2^{\aleph_0}$ ordinals ε' with $\varepsilon' < \varepsilon < \varepsilon' + 2^{\aleph_0}$, while there are $2^{\aleph_0}$ functions $g^{ab}_{\varepsilon} : \omega \to \aleph_0$. Thus there is always some function g^{ab}_{ε} such that $y^{ab}_{\varepsilon'}\Gamma^{-1}\varphi^{ab}_{\varepsilon'} \notin S^{\varepsilon}$ holds simultaneously for all ε' with $\varepsilon' < \varepsilon < \varepsilon' + 2^{\aleph_0}$.

This g^{ab}_{ε} is sufficient for $(\star)$. $\square$

We state some basic properties of the R-algebra $S := \bigcup_{\alpha<\kappa} S_\alpha$.

Theorem 8.5 *If $(S_\alpha)_{\alpha<\kappa}$ is the chain as above and $S = \bigcup_{\alpha<\kappa} S_\alpha$, then the following holds.*

(a) $S \in \mathfrak{K}$ is an $\aleph_1$-free R-module and $|S| = \kappa$.

(b) $S \in \mathfrak{K}_{S_\alpha}$ and S/S_α is $\aleph_1$-free for all successor ordinals $\alpha < \kappa$. Furthermore S/S'_α is $\aleph_1$-free for all $\alpha \in E$.

(c) $B' \subseteq S \subseteq \widehat{B}$.

(d) If $\Psi \in \mathrm{End}\,_R S$ and $a, b \in \mathfrak{p}S \cap \widehat{B'}$, then there exists an $s \in S$ with $(\Psi = s) \restriction Ra \oplus Rb$.

Proof: (a), (b) and (c) are an immediate consequence of Corollary 6.3 and Lemma 8.4.

(d): Let $\Psi \in \mathrm{End}\,_R S$. Then $\Gamma\Psi : B \to \widehat{B}$ lifts uniquely to an endomorphism $\Gamma\Psi : \widehat{B} \to \widehat{B}$. Set $C := \{\alpha \in \kappa \;|a, b \in S_\alpha\}$ as cub. Applying the General Black Box, Theorem 2.14, for $X := \{a\Gamma^{-1}, b\Gamma^{-1}\}$, C and $\varphi := \Gamma\Psi$ there exists a trap $p^{ab}_{\alpha} =$

$(f_\alpha^{ab}, \varphi_\alpha^{ab})$, that catches X, C and φ. Thus the following holds:

(a) $X \subseteq P_\alpha^{ab} := \mathrm{Dom}\, \varphi_\alpha^{ab} \subseteq \widehat{B}$ and $\varphi_\alpha^{ab} \in \mathrm{End}\,(P_\alpha^{ab})$.

(b) $||X|| < ||p_\alpha^{ab}|| = \beta \in C \cap E_{ab} \subseteq \kappa^o$ and $||x|| < \beta$ for all $x \in P_\alpha^{ab}$.

(c) $\varphi \restriction P_\alpha^{ab} = \varphi_\alpha^{ab}$.

In particular $\Gamma\Psi \restriction P_\alpha^{ab} = \varphi_\alpha^{ab}$ is a partial endomorphism of $\widehat{B_\beta}$ with $\{a\Gamma^{-1}, b\Gamma^{-1}\} \subseteq \mathrm{Dom}\, \varphi_\alpha^{ab}$. Furthermore $(\Psi = \Gamma^{-1}\varphi_\alpha^{ab}) \restriction (P_\alpha^{ab}\Gamma \cap S_\beta)$ is a partial endomorphism of S_β: For every $x \in P_\alpha^{ab}\Gamma \cap S_\beta$ holds $x\Gamma^{-1}\varphi_\alpha^{ab} = x\Psi \in S \cap \widehat{S_\gamma}$ with (a) and (b), where $\gamma < \beta \in \kappa^o$ is some successor ordinal. From S, S/S_γ $\aleph_1$-free and Corollary 5.2 now follows $x\Gamma^{-1}\varphi_\alpha^{ab} \in S_\gamma \subseteq S_\beta$.
Lets assume $\alpha \in I(\beta)$. Thus $y_\alpha^{ab} \in S$ and the continuity of Ψ gives

$$y_\alpha^{ab}\Gamma^{-1}\varphi_\alpha^{ab} = y_\alpha^{ab}\Psi \in \widehat{S_\beta} \cap S.$$

Furthermore from S, S/S'_β $\aleph_1$-free and Corollary 5.2 follows $y_\alpha^{ab}\Gamma^{-1}\varphi_\alpha^{ab} \in S'_\beta$ contradicting (++). Thus $\alpha \notin I(\beta)$. In particular, there exists some $s \in S_\beta \subseteq S$ with
$(\Psi = s) \restriction Ra \oplus Rb$. □

Now the proof of Theorem 2.29 (b) is similar to the proof of Theorem 2.29 (a).

Proof: Ring $S \in \mathfrak{K}$ is a PID and an $\aleph_1$-free R-module by the last lemma, and the following inclusion is obvious

$S \subseteq \mathrm{End}\,_R S$. **(1)**

If $\Psi \in \mathrm{End}\,_R S$ then we apply Theorem 8.5 (c). There is a family $(s_{ab})_{a,b \in \mathfrak{p}S}$ of elements $s_{ab} \in S$ such that $(\Psi = s_{ab}) \restriction Ra \oplus Rb \subseteq S$ for all $a, b \in \mathfrak{p}S \cap \widehat{B'}$. In particular $(\Psi = s_{ab} = s_{ab'}) \restriction Ra$, hence $a(s_{ab} - s_{ab'}) = 0$ and $s_{ab} = s_{ab'}$ holds for all $a, b, b' \in \mathfrak{p}S \cap \widehat{B'}$. Thus there exists a universal constant $s \in S$ with $s_{ab} = s$ for all $a, b \in \mathfrak{p}S \cap \widehat{B'}$, and an easy transfinite induction shows $\Psi = s \in S$, hence

$\mathrm{End}\,_R S = S$, $\mathrm{Aut}\,_R S = S^*$ and S is an E(R)-algebra. **(2)**

$\mathrm{Aut}\,_R S = S^*$ *acts uniquely transitive on* S. **(3)**

This follows as shown in Section 8.1. Thus S is a UT-module. □

References

[1] M. Auslander, D. Buchsbaum, *Groups, Rings, Modules*, Harper & Row, New York (1974).

[2] D. Carroll, B. Goldsmith, *On transitive and fully transitive abelian p-groups*, Proc. of the Royal Irish Academy (1) **96A** (1996), 33–41.

[3] P. Cohn, *Algebra* - Vol. 1, John Wiley & Sons, Chichester (1978).

[4] A.L.S. Corner, *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3) **13** (1963), 687–710.

[5] A.L.S. Corner, *The independence of Kaplansky's notions of transitivity and full transitivity*, Quart. J. Math. Oxford (2) **27** (1976), 15–20.

[6] A.L.S. Corner, R. Göbel, *Prescribing endomorpism algebras – A unified treatment*, Proc. London Math. Soc. (3) **50** (1985), 471–483.

[7] K. Devlin, *Fundamentals of Contemporary Set Theory*, Springer Verlag, New York (1979).

[8] K. Devlin, S. Shelah, *A weak version of* $\diamondsuit$ *which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math., **29** (1978), 239–247.

[9] M. Droste, *Structure of Partially Ordered Sets with Transitive Automorphism Groups*, Memoirs of the Am. Math. Soc. **57** (1985), Number 334.

[10] M. Dugas, R. Göbel, *On endomorphism rings of primary abelian groups*, Math. Ann. **261** (1982), 359–385.

[11] M. Dugas, R. Göbel, *Every cotorsion-free algebra is an endomorphism algebra*, Math. Z. **181** (1982), 451–470.

[12] M. Dugas, R. Göbel, *Endomorphism algebras of torsion modules II*, Abelian Group Theory, Proc. Honolulu Conf. 1982/83, Lecture Notes in Math. **1006** (1983), 400–411.

[13] M. Dugas, J. Hausen, *Torsion-free E-uniserial groups of infinite rank*, Abelian Group Theory, Proc. Perth Conf. 1987, Contemp. Math. **87** (1989), 181–189.

[14] M. Dugas, A. Mader, C. Vinsonhaler, *Large E-rings exist*, J. Algebra (1) **108** (1987), 88–101.

[15] M. Dugas, S. Shelah, *E-Transitive groups in L*, Abelian Group Theory, Proc. Perth Conf. 1987, Contemp. Math. **87** (1989), 191–199.

[16] P. Eklof, A. Mekler, *Almost Free Modules*, Elsevier Science, Amsterdam (2002).

[17] T. Faticoni, *Each countable reduced torsion-free commutative ring is a pure subring of an E-ring*, Comm. Algebra **15(12)** (1987), 2545–2564.

[18] L. Fuchs, *Abelian Groups*, Hungarian Academie of Sciences, Budapest (1958).

[19] L. Fuchs, *Infinite Abelian Groups* - Vol. 1 & 2, Academic Press, New York (1970,1973).

[20] R. Göbel, J. Trlifaj, *Approximation Theory and Endomorphism Algebras*, Walter de Gruyter, Berlin (2005).

[21] R. Göbel, S. Shelah, *On the existence of rigid $\aleph_1$-free abelian groups of cardinality* $\aleph_1$, Abelian Groups and Modules, Proc. Padova Conf. (1994), 227–237.

[22] R. Göbel, S. Shelah, *Indecomposable almost free modules – the local case*, Canadian J. Math. (4), **50** (1998), 719–738.

[23] R. Göbel, S. Shelah, *Uniquely transitive torsion-free abelian groups*, Rings, Modules, Algebras, and Abelian Groups, Marcel Dekker Pure and Applied Math. **236** (2004), 271–290.

[24] R. Göbel, S. Shelah, *Generalized E-algebras via λ-calculus I*, to be submitted (2005).

[25] R. Göbel, L. Strüngmann, *Almost free E(R)-algebras and E(A,R)-modules*, Fundamenta Mathematicae, **169** (2001), 175–192.

[26] R. Göbel, S. Shelah, L. Strüngmann, *Almost free E-rings of cardinality* $\aleph_1$, Canadian J. Math. (4), **55** (2003), 750–765.

[27] B. Goldsmith, *Essentially indecomposable modules which are almost free*, Quart. J. Math. Oxford (2), **39** (1988), 213–222.

[28] J. Hausen, *On strongly irreducible torsion-free groups*, Abelian Group Theory, Proc. Third Oberwolfach Conf. on Abelian Groups 1985, Gordon and Breach, London (1987), 351–358.

[29] J. Hausen, *E-transitive torsion-free abelian groups*, J. Algebra **107** (1987), 17–27.

[30] G. Hennecke, *Transitivitätseigenschaften abelscher p-Gruppen*, PhD thesis, University of Duisburg-Essen, Campus Essen (1999).

[31] D. Herden, *Scharf transitive* κ*-freie R-Moduln*, Diploma dissertation, University of Duisburg-Essen, Campus Essen (2003).

[32] T. Jech, *Set Theory*, Academic Press ("Second corrected edition" published 1997 by Springer Verlag).

[33] R. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.

[34] I. Kaplansky, *Infinite Abelian Groups*, University of Michigan Press (1971).

[35] C. Megibben, *Large subgroups and small homomorphisms*, Michigan Math. J. **13** (1966), 153–160.

[36] R. B. Mura, A. Rhemtulla, *Orderable Groups*, Marcel Dekker, New York (1977).

[37] P. Schultz, *The endomorphism ring of the additive group of a ring*, J. Aust. Math Soc. **15** (1973), 60–69.

[38] S. K. Sehgal, *Units in Integral Group Rings*, Pitman Monographs, New York (1993).

[39] L. Strüngmann, *Almost-Free* E(R)-*Algebras over Countable Domains*, PhD thesis, University of Duisburg-Essen, Campus Essen (1998).

[40] H. Wielandt, *Unendliche Permutationsgruppen*, Tübingen 1959/60; reprinted, York University, Toronto (1967).