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Residual a posteriori error estimates for the mixed finite element method

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In this paper, residual-based *a posteriori* error bounds are derived for the mixed finite element method applied to a model second order elliptic problem. A global upper bound for the error in the scalar variable is established, as well as a local lower bound. In addition, due to the fact that the scalar and vector variables are approximated to equal order accuracy, the dual problem may be modified to give an upper bound for the vector variable. Some comments on estimating more general error quantities are also made. The estimate effectively guides adaptive refinement for a smooth problem with a boundary layer, as well as detects the need to refine near a singularity.

Keywords: a posteriori error estimates, mixed finite element method

AMS subject classification: 35Q35, 65N30, 65N15

1. Introduction

Mixed finite elements have found successful application in flow and transport problems in porous media and many other fields. In problems such as contaminant transport and miscible displacement, the pressure and velocity fields come from Darcy's law and conservation of mass. Mixed methods provide a physically relevant, optimal order, locally conservative approximation to both pressure and velocity fields [15].

Mixed methods are based on writing a second order elliptic equation as a first order system, in so-called mixed form. For Ω a domain in $\mathbb{R}^2$, consider the problem

$$\nabla \cdot u = f, \quad u = -K\nabla p, \tag{1}$$

together with the boundary conditions

$$p = g^{\mathrm{D}},\ \Gamma^{\mathrm{D}}, \qquad u \cdot n = 0,\ \Gamma^{\mathrm{N}}. \tag{2}$$

Assume that Ω is a polygonal domain with its boundary partitioned into Dirichlet and Neumann portions Γ^{D} and Γ^{N} and that Ω, K, f, and g^{D} are sufficient for $p \in H^r(\Omega)$, $r \geqslant 2$. Here, K is a diagonal, positive-definite tensor.

Raviart and Thomas [21] propose a variational formulation based on this mixed form, construct suitable approximating spaces, and prove error estimates for a finite element method. Among other results, p, u, and $\nabla \cdot u$ are all approximated to the same order of accuracy in L^2. This work has been extended in many directions. For example, a more complicated linear problem is analyzed by Douglas and Roberts [16] and a nonlinear problem by Milner [19]. In addition, richer approximating spaces have been constructed, such as those in [10] which give a higher order flux approximation than for the scalar variable.

In order to efficiently capture fine-scale details occurring in flow and transport problems, some sort of local adaptivity, based either on mesh refinement, polynomial enrichment, or both, should be incorporated in the approximation process. At the heart of any such adaptive scheme is the notion of estimating the error in the computed solution *a posteriori*. Such estimates must be computable, based on the computed solution and the data of the problem.

Error estimation for standard Galerkin methods is a richly researched field. Surveys of various approaches appear in [1,12]. One approach of particular interest in this work is explicit estimation, in which various techniques are applied to bound the error in the computed solution by a constant times the sum of local residuals. Such estimates grow out of the work of Babuska and Rheinboldt [3,4]. While these residuals are straightforward to compute, the constant often is not. Therefore, the estimate tends to function as an indicator with a tolerance parameter. Still, these estimates can be used to successfully guide adaptive computation. Johnson et al. extend these ideas to a discontinuous in time Galerkin method for parabolic problems and to a streamline diffusion method for systems of conservation laws [17,18].

In order to control error in more localized quanitities, Becker and Rannacher [6,7] solve a dual problem numerically and approximate its derivatives to give weights for the element residuals. This leads to computable estimates for quantities such as the error at a point or the average error over some set. This approach has been successfully exploited in the context of boundary element methods by Demkowicz and Walsh [14], where the (dense) matrix may be factored once and used in both the primal and dual problems.

While error estimation is well understood for Galerkin procedures for elliptic problems, the literature contains comparably few *a posteriori* estimates for mixed methods. Braess and Verfürth [8] argue that the standard explicit techniques such as used in [17] cannot give optimal order bounds for the mixed method due to the approximation properties of the Raviart–Thomas spaces on the edges. This motivates their use of certain mesh-dependent norms which do not suffer from these difficulties. Carstensen [13] avoids such mesh-dependent norms by considering a Helmholtz decomposition of the flux space, leading to upper and lower bounds in the natural $L^2 \times H(\mathrm{div})$ norm. Wohlmuth and Hoppe [23] derive and compare analytically several different estimates. An estimate similar to [13] is considered, as well as a local subproblem technique akin to [5] and recovery-based estimates like [24,25].

The present paper derives residual-based duality estimates for mixed finite element methods. After describing the variational setting and finite element discretization in section 2, the error estimates are derived in section 3. These estimates include an upper bound for the L^2 error in the scalar variable in terms of simple residuals and a projection error in the Dirichlet boundary conditions. The element residuals also serve as a lower bound for the error in a mesh-dependent perturbation of the L^2 norm. In addition, an upper bound for the flux variable is also computed in L^2. While these estimates control the error in somewhat weaker norms than do [13,23], two noteworthy points should be made. First, they show that the duality theory indeed may be applied in a mixed context when the proper interpolation operators are used. Second, the residuals are simpler than many other estimates, so substantial error control is possible even with very simple indicators. Also included is a discussion of how more general error quantities can be controlled by this approach, much in line with the work in [6,7,14]. Finally, numerical results combining the estimate with adaptive mesh refinement are presented in section 4 and some conclusions are made at the end of the paper.

2. Variational formulation and discretization

Let $\{\mathcal{T}_h\}_h$ be a family of conforming, quasiuniform triangulation of Ω and $\mathcal{E}_h$ be the union of all edges of the triangles on a given mesh. Let $\mathcal{E}_{I,h} \subset \mathcal{E}_h$ be the set of all interior edges, and let $\mathcal{E}_{D,h} \subset \mathcal{E}_h$ be the set of all edges coinciding with the Dirichlet boundary and $\mathcal{E}_{N,h}$ be the set of all Neumann edges. It is assumed here that the triangulation is such that no edge intersects both Γ^{D} and Γ^{N}. For each T, let n_T denote the unique outward normal unit vector. For each γ, let n_γ denote a fixed, unique unit vector with the assumption that n_γ points outward if $\gamma \in \mathcal{E}_{D,h} \cup \mathcal{E}_{N,h}$. The jump of some function across some interiori edge γ is defined in the normal way.

Throughout this paper, let C, C_i, etc. denote generic positive constants independent of the mesh parameter h. For each triangle T, let h_T denote its diameter, and let h_γ denote the length of edge γ. The following condition is assumed to hold between the edges and diameters

$$C_1 h_\gamma \leqslant h_T \leqslant C_2 h_\gamma \quad \forall \gamma \subset \partial T,\ \forall T \in \mathcal{T}_h.$$

Let the space $W = L^2(\Omega)$ and $V = H(\mathrm{div}; \Omega)$ be the standard spaces. Further, let V^0 be the set of functions with vanishing normal trace in the sense defined by, for example, [11]. Let $\Lambda = L^2(\Gamma^{\mathrm{D}})$.

The pairing $(\cdot,\cdot)_S$ shall denote the standard $L^2(S)$ inner product over a two-dimensional set S, with S omitted if $S = \Omega$. Similarly, let $\langle \cdot,\cdot \rangle_\gamma$ be the one-dimensional L^2 inner product. Let $\|\cdot\|_{i,S}$ denote the standard Sobolev norm with i omitted if it is zero and S omitted if it is Ω. Finally, let $\|\cdot\|_{\mathrm{div},S}$ denote the standard $H(\mathrm{div};S)$ norm.

Now, several finite-dimensional spaces are defined. First, let $\mathcal{P}_k(S)$ denote the set of polynomials of total degree less than or equal k over the open two-dimensional set S and let $\mathcal{R}_k(s)$ be the set of polynomials of degree less than or equal k over some open line segment s. Then, the set $W_h = \{w \in W\colon\ w|_T \in \mathcal{P}_k(T)\ \forall T \in \mathcal{T}_h\}$ denotes the set of possibly discontinuous piecewise polynomials over the mesh $\mathcal{T}_h$. Also, let V_h be the standard Raviart–Thomas space of order k. Although there are multiple definitions of this space, to fix ideas, let $V_h = \{v \in V\colon\ v|_T \in (\mathcal{P}_k(T))^2 + x\mathcal{P}_k(T)\}$. Let $V_h^0 = \{v \in V_h\colon\ (v\cdot n)|_{\Gamma^{\mathrm{N}}} = 0\}$ be the finite-dimensional set of functions with vanishing normal trace on Γ^{N}. Let $\Lambda_h = \{\mu \in \Lambda\colon\ \mu|_\gamma \in \mathcal{R}_k(\gamma),\ \gamma \in \mathcal{E}_{D,h}\}$. On this space, define the norm

$$|\mu|^2_{-(1/2),h} = \sum_{\gamma \in \mathcal{E}_{D,h}} h_\gamma \|\mu\|^2_\gamma. \tag{3}$$

Two projections shall play a crucial role in establishing the error estimates in this paper. First, let $\mathcal{P}_h : H^r(\Omega) \to W_h$ be defined by

$$(p - \mathcal{P}_h p, w_h) = 0, \quad w_h \in W_h, \tag{4}$$

which is the standard L^2 projection. The error in the projection satisfies

$$\|p - \mathcal{P}_h p\| \leqslant Ch^s \|p\|_s, \quad s \leqslant \min(r, k+1). \tag{5}$$

The second projection is the divergence projection into the mixed finite element space, thoroughly described in [11]. This projection $\Pi_h : (H^r(\Omega))^d \to V_h$ is defined by

$$\big(\nabla(u - \Pi_h u), w_h\big) = 0, \quad w_h \in W_h, \tag{6}$$

it commutes with the divergence operator in the sense that

$$\nabla \cdot \Pi_h v = \mathcal{P}_h \nabla \cdot v, \tag{7}$$

and it preserves the discrete normal trace

$$\big\langle (u - \Pi_h u) \cdot n, \mu_h \rangle_\gamma = 0, \quad \mu_h \in \mathcal{R}_k(\gamma),\ \gamma \in \mathcal{E}_{D,h}. \tag{8}$$

This projection has the approximation property

$$\|u - \Pi_h u\| \leqslant Ch^s \|u\|_s, \quad s \leqslant \min(r, k+1). \tag{9}$$

The solution to (1)–(2) may be phrased in a variational setting as: Find $p \in W$, $u \in V^0$ such that

$$\begin{aligned} (\nabla \cdot u, w) &= (f, w), & w &\in W, \\ \big(K^{-1}u, v\big) - (p, \nabla \cdot v) &= -\big\langle g^{\mathrm{D}}, v \cdot n\big\rangle_{\Gamma^{\mathrm{D}}}, & v &\in V^0. \end{aligned} \tag{10}$$

Similarly, this problem is discretized by seeking a solution in the finite-dimensional subspaces. That is, find $p_h \in W_h$, $u_h \in V_h^0$ such that

$$\begin{aligned} (\nabla \cdot u_h, w_h) &= (f, w_h), & w_h &\in W_h, \\ \big(K^{-1}u_h, v_h\big) - (p_h, \nabla \cdot v_h) &= -\big\langle g^{\mathrm{D}}, v_h \cdot n\big\rangle_{\Gamma^{\mathrm{D}}}, & v_h &\in V_h^0. \end{aligned} \tag{11}$$

3. Residual a posteriori error estimate

In this section, a computable (up to a multiplicative constant) *a posteriori* bound for the error quantities $\xi \equiv p - p_h$ and $\eta \equiv u - u_h$ is established. Of first importance in error estimation are the error orthogonality relations

$$\begin{aligned} (\nabla\eta, w_h) &= 0, \quad w_h \in W_h, \\ \left(K^{-1}\eta, v_h\right) - (\xi, \nabla\cdot v_h) &= 0, \quad v_h \in V_h, \end{aligned} \tag{12}$$

which follow readily by subtracting (11) from (10).

This section begins with a minor remark about computing the error in the $H(\mathrm{div})$ seminorm. Then, duality is employed to compute an upper bound for ξ. A lower bound also is derived with an additional hypothesis and the help of a lemma from [13]. The next subsection develops an estimate for the error in η with a saturation-like hypothesis. Finally, this section is concluded with some comments on error estimation in quantities defined by general linear functionals.

3.1. $H(\mathrm{div})$ *seminorm error*

It is a trivial observation that the error in the $H(\mathrm{div})$ seminorm is exactly (up to quadrature, machine roundoff, etc.) computable. Note that

$$\nabla\cdot\eta = f - \nabla\cdot u_h. \tag{13}$$

Since both quantities on the right hand side are known, the error is directly computable

3.2. *Pressure error*

The estimate for ξ proceeds by a typical duality argument. Let the functions w and v satisfy

$$\nabla\cdot v = \xi, \qquad v = -K\nabla w, \tag{14}$$

on Ω with the boundary conditions

$$w = 0,\ \Gamma^{\mathrm{D}}, \qquad v\cdot n = 0,\ \Gamma^{\mathrm{N}}. \tag{15}$$

From elliptic regularity theory, $\|w\|_2 \leqslant \|\xi\|$. These functions satisfy the variational problem

$$\begin{aligned} (\nabla\cdot v, \phi) &= (\xi, \phi), \quad \phi \in W, \\ \left(K^{-1}v, \psi\right) - (w, \nabla\psi) &= 0, \qquad \psi \in V^0. \end{aligned} \tag{16}$$

Now, let $\phi = \xi$ and $\psi = \eta$ in (16). By subtracting the second equation in (16) from the first and using the orthogonality relation (12)

$$\begin{aligned} (\xi,\xi) &= (\nabla\cdot v, \xi) - \left(K^{-1}v, \eta\right) + (w, \nabla\cdot\eta) \\ &= (\nabla\cdot\eta, w) - \left(K^{-1}\eta, v\right) + (\xi, \nabla\cdot v) \\ &= (\nabla\cdot\eta, w - \mathcal{P}_h w) - \left(K^{-1}\eta, v - \Pi_h v\right) + \left(\xi, \nabla(v - \Pi_h v)\right). \end{aligned} \tag{17}$$

The last line can then be written as a sum over triangles, and integrating each $(\xi, \nabla(v - \Pi_h v))_T$ by parts yields

$$(\xi, \xi) = \sum_{T \in \mathcal{T}_h} \left[(\nabla \cdot \eta, w - \mathcal{P}_h w)_T - \left(K^{-1}\eta + \nabla\xi, v - \Pi_h v\right)_T \right] + \sum_{\gamma \in \mathcal{E}_{I,h}} \left\langle [\xi], (v - \Pi_h v) \cdot n_\gamma \right\rangle_\gamma + \sum_{\gamma \in \mathcal{E}_{D,h}} \left\langle \xi, (v - \Pi_h v) \cdot n_\gamma \right\rangle_\gamma. \tag{18}$$

The sum over interior edges vanishes. To see this, note first that $p \in H^2$, so it is continuous and thus $[p] = 0$ on each edge interior edge. Second, $[p_h]|_\gamma \in R_k(\gamma)$ over each edge γ. Thus, $(v - \Pi_h v) \cdot n_\gamma$ is orthogonal to it by (8).

Next, let $\tilde{d}^{\mathrm{D}}$ be the best approximation to g^{D} in Λ_h. Replacing ξ with $g^{\mathrm{D}} - \tilde{g}^{\mathrm{D}}$ in (18) is permitted again because of (8). Putting this together with (1) in (18) means that the error can be represented exactly as

$$\|\xi\|^2 = \sum_{T \in \mathcal{T}_h} \left[(f - \nabla \cdot u_h, w - \mathcal{P}_h w)_T + \left(K^{-1}u_h + \nabla p_h, v - \Pi_h v\right)_T \right] + \sum_{\gamma \in \mathcal{E}_{D,h}} \left\langle g^{\mathrm{D}} - \tilde{g}^{\mathrm{D}}, (v - \Pi_h v) \cdot n_\gamma \right\rangle_\gamma. \tag{19}$$

Using the Cauchy–Schwarz inequality, the error can be bounded by

$$\|\xi\|^2 \leqslant \theta_1 + \theta_2, \tag{20}$$

where

$$\theta_1 \equiv \sum_{T \in \mathcal{T}_h} \left[\|f - \nabla \cdot u_h\|_T \, \|w - \mathcal{P}_h w\|_T + \left\|K^{-1}u_h + \nabla p_h\right\|_T \|v - \Pi_h v\|_T \right] \tag{21}$$

and

$$\theta_2 \equiv \sum_{\gamma \in \mathcal{E}_{D,h}} \left\|g^{\mathrm{D}} - \tilde{g}^{\mathrm{D}}\right\|_\gamma \|v - \Pi_h v\|_\gamma. \tag{22}$$

Let $\mu_k = \min\{k, 1\}$, where k is the degree of the approximating space. Now, θ_1 may be bounded by

$$\begin{aligned} \theta_1 &\leqslant C \sum_{T \in \mathcal{T}_h} \left(h_T^{1+\mu_k} \|f - \nabla \cdot u_h\|_T \, \|w\|_{1+\mu_k,T} + h_T \left\|K^{-1}u_h + \nabla p_h\right\| \|v\|_{1,T} \right) \\ &\leqslant C \sum_{T \in \mathcal{T}_h} \left[\left(h_T^{1+\mu_k} \|f - \nabla \cdot u_h\|_T + h_T \left\|K^{-1}u_h + \nabla p_h\right\|_T \right)^2 \right]^{1/2} \|w\|_2. \end{aligned} \tag{23}$$

Since $w \in H^2$, it may be approximated to second order. However, the lowest order space is only capable of first order approximation, and hence the μ_k power of h in this bound. Next, θ_2 may be bounded by

$$\theta_2 \leqslant C\left|g^{\mathrm{D}} - \tilde{g}^{\mathrm{D}}\right|_{-1/2,h} \|v\|_1, \tag{24}$$

where a trace theorem, the approximation properties of Π_h, and quasiuniformity of $\mathcal{T}_h$ have been applied. Now,

$$\|\xi\|^2 \leqslant C\left[\left(\sum_{T\in\mathcal{T}_h}\omega_T^2\right)^{1/2}+\left(\sum_{\gamma\in\mathcal{E}_{D,h}}\omega_\gamma^2\right)^{1/2}\right]\|w\|_2 \tag{25}$$

where

$$\omega_T \equiv \left(h_T^{1+\mu_k}\|f-\nabla\cdot u_h\|_T + h_T\left\|K^{-1}u_h+\nabla p_h\right\|_T\right) \tag{26}$$

and

$$\omega_\gamma \equiv h_\gamma^{1/2}\left\|g^{\mathrm{D}}-\tilde{g}^{\mathrm{D}}\right\|_\gamma. \tag{27}$$

Finally, using the elliptic regularity result $\|w\|_2 \leqslant \|\xi\|$, the following theorem is established.

Theorem 1. With data K, g^{D}, f and domain Ω as assumed above and with error indicators as defined above, the error in the pressure variable for (11) is bounded *a posteriori* by

$$\|\xi\| \leqslant C\left[\left(\sum_{T\in\mathcal{T}_h}\omega_T^2\right)^{1/2}+\left(\sum_{\gamma\in\mathcal{E}_{D,h}}\omega_\gamma^2\right)^{1/2}\right]. \tag{28}$$

A few comments about the estimate are in order. First, it is typically the case that the residual $h_T\,\|K^{-1}u_h-\nabla p_h\|_T$ drives the estimate. To see this, consider just the lowest order case. In this case, $\|f-\nabla\cdot u_h\|\leqslant Ch$ globally, and thus the element residual $h_T\,\|f-\nabla\cdot u_h\|$ is actually a higher order term compared to $h_T\,\|K^{-1}u_h-\nabla p_h\|_T$. The boundary term is also a higher order term. Additionally, $\nabla(p_h|_T)=0$ for each T in the lowest order case, and so the estimator can be approximated by

$$\left(\sum_{T\in\mathcal{T}_h}\omega_T^2\right)^{1/2}+\left(\sum_{\gamma\in\mathcal{E}_{D,h}}\omega_\gamma^2\right)^{1/2} \approx \left(\sum_{T\in\mathcal{T}_h}h_T^2\left\|K^{-1}u_h\right\|_T^2\right)^{1/2}. \tag{29}$$

Since $K^{-1}u_h$ is an approximation to ∇p, this estimator is essentially a gradient-based estimator. Also note that with higher degree approximating spaces, the first residual is even smaller since $\mu_k=1$ if $k>0$. More generally, the estimator is approximately

$$\left(\sum_{T\in\mathcal{T}_h}\omega_T^2\right)^{1/2}+\left(\sum_{\gamma\in\mathcal{E}_{D,h}}\omega_\gamma^2\right)^{1/2} \approx \left(\sum_{T\in\mathcal{T}_h}h_T^2\left\|K^{-1}u_h-\nabla p_h\right\|_T^2\right)^{1/2}. \tag{30}$$

3.3. *A lower bound*

The element error indicators serve as a local lower bound for a mesh-dependent perturbation of $\|\xi\|$ if $K^{-1}|_T\in\mathcal{P}_k(T)$. First, recall that

$$f-\nabla\cdot u_h=\nabla\cdot\eta. \tag{31}$$

Then, Carstensen [13] shows that if $K^{-1}|_T \in \mathcal{P}_k(T)$, then

$$h_T \left\| K^{-1} u_h + \nabla p_h \right\|_T \leqslant C\left(\|\xi\|_T + h_T \,\|\eta\|_T \right). \tag{32}$$

Putting these two things together gives the result

Theorem 2. If $K^{-1}|_T \in \mathcal{P}_k(T)$, the element error indicator ω_T satisfies

$$\|\omega_T\|_T \leqslant C\left(\|\xi\|_T + h_T \,\|\eta\|_{\mathrm{div},T} \right). \tag{33}$$

If $K^{-1}|_T \notin \mathcal{P}_k(T)$, then it is possible to derive a similar estimate which includes an additional term due to approximating K^{-1} locally by a polynomial.

3.4. *Flux error*

With an additional assumption, similar to a saturation assumption, a similar estimate applies to the error in the flux variable as well. This assumption is well-motivated by the standard *a priori* estimates for the Raviart–Thomas spaces, which say that the scalar variable, the flux variable, and the divergence of the flux variable all converge with equal order accuracy. Therefore, assume that there exists some constant C_f, independent of h, such that

$$\|\nabla \cdot \eta\| \leqslant C_f \|\eta\|. \tag{34}$$

Let w and v solve the equation

$$\nabla \cdot v = 0, \qquad v + K\nabla w = \eta, \tag{35}$$

on Ω with the homogeneous boundary conditions

$$w = 0,\ \Gamma^{\mathrm{D}}, \qquad v \cdot n = 0,\ \Gamma^{\mathrm{N}}. \tag{36}$$

Equation (35) can be rewritten as

$$\nabla \cdot (K\nabla w) = \nabla \cdot \eta \tag{37}$$

by eliminating v.

Therefore, elliptic regularity theory gives

$$\|w\|_2 \leqslant C\|\nabla \cdot \eta\|, \tag{38}$$

so by the previous assumption,

$$\|w\|_2 \leqslant C\|\eta\|. \tag{39}$$

In addition, $v \in H(\mathrm{div})(\Omega)$ and $v|_T \in H^1(T)$ and so approximation properties will hold elementwise.

The functions w, v satisfy the variational problem

$$\begin{aligned} (\nabla \cdot v, \phi) &= 0, & \phi &\in W, \\ \left(K^{-1} v, \psi\right) - (w, \nabla \cdot \psi) &= (\eta, \psi), & \psi &\in V^0. \end{aligned} \tag{40}$$

The representation of the error equations and bounding residuals works exactly as before. Thus,

$$\begin{aligned} \|\eta\|^2 &\leqslant \theta_1 + \theta_2 &(41)\\ &\leqslant C\left[\left(\sum_{T\in\mathcal{T}_h} \omega_T^2\right)^{1/2} + \left(\sum_{\gamma\in\mathcal{E}_{D,h}} \omega_\gamma^2\right)^{1/2}\right]\|w\|_2. &(42)\end{aligned}$$

Now, using the regularity result (38) gives:

Theorem 3. With the assumptions of theorem 1 and (34) holding as well, the following error bound holds

$$\|\eta\| \leqslant C\left[\left(\sum_{T\in\mathcal{T}_h} \omega_T^2\right)^{1/2} + \left(\sum_{\gamma\in\mathcal{E}_{D,h}} \omega_\gamma^2\right)^{1/2}\right]. \tag{43}$$

The remarks regarding simplification of the pressure estimate also apply to the flux extimate as well.

3.5. *Remarks on other approximating spaces*

The question arises regarding the applicability of this and other estimates, to the other mixed approximating spaces. The upper bound on the pressure used no specific information regarding the Raviart–Thomas spaces, so it holds for other valid pairs of mixed spaces. Also, Carstensen's result used to derive the lower bound was for general mixed spaces as well. On the other hand, the flux estimate assumes that the pressure and velocity converge at the same rate, and thus requires modification to apply to the BDM spaces.

3.6. *Some notes on general error quantities*

This duality approach has also been applied to error quantities derived from general linear functionals in the case of Galerkin methods and boundary element methods [6,14]. This section sketches how such estimates might be obtained in the mixed setting. Let $f_1 \in W'$ and $f_2 \in (V^0)'$ be two continuous linear functionals. Now, rather than posing the dual problem (16) or (40), let w and v satisfy the more general problem

$$\begin{aligned} (\nabla\cdot v, \phi) &= f_1(\phi), \qquad \phi \in W,\\ \left(K^{-1}v, \psi\right) - (w, \nabla\psi) &= -f_2(\psi), \quad \psi \in V^0. \end{aligned} \tag{44}$$

Of course, (16) and (40) are special cases of (44). As in (19), selecting $\phi = \xi$ and $\psi = \eta$ and using (12), integration by parts, and the duality relations yields

$$f_1(\xi) + f_2(\eta) = \sum_{T\in\mathcal{T}_h} (f - \nabla\cdot u_h, w - \widetilde{w})_T \tag{45}$$

$$+ \sum_{T \in \mathcal{T}_h} \left(K^{-1} u_h + \nabla p_h, v - \tilde{v}\right)_T$$
$$+ \sum_{\gamma \in \mathcal{E}_{D,h}} \left\langle g^{\mathrm{D}} - \tilde{g}^{\mathrm{D}}, (v - \tilde{v}) \cdot n \right\rangle_\gamma \tag{46}$$

which exactly expresses the the error as measured by the functionals f_1, f_2. Here, $\widetilde{w} \in W_h$ and $\tilde{v} \in V_h$ are possibly the projections defined above, but possibly some other convenient interpolant as well.

Rather than using elliptic regularity to bound the dual solution in terms of the error, the dual problem is solved approximately. Then, the difference between the dual solution and its approximation must either be bounded by approximation theoretic results, or computed directly if possible. The information from the dual problem then appears as local weights for the residuals in the error estimate.

Note that this approach has some limitations compared to Galerkin methods. The mixed method uses weaker spaces than the standard Galerkin method, and so the set of acceptable linear functionals to pose the dual problem is smaller. Most notably, pointwise estimates are not covered in this framework.

4. Numerical results

The *a posteriori* error estimate derived above provides the basis for an adaptive algorithm. This section presents numerical results for two problems. In the first, the solution has a boundary layer but still lies in H^2. In addition, a model from single phase

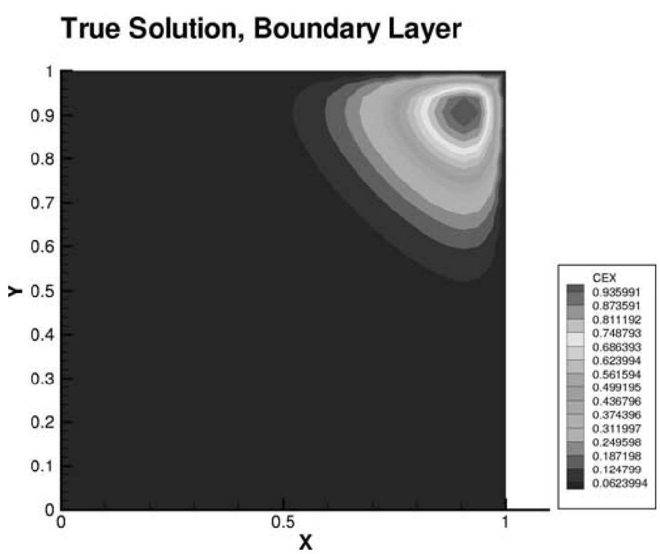

Figure 1. True solution with boundary layer in upper right corner.

flow with lower regularity is considered. All solutions are computed using the lowest order Raviart–Thomas spaces.

4.1. A smooth problem with a boundary layer

Let the permeability $K \equiv \mathcal{I}$, the identity matrix. Pose homogeneous Dirichlet conditions on the unit square and choose the source function f chosen such that

$$p(x, y) = \frac{1}{537930} x(1-x)y(1-y)\,\mathrm{e}^{10x}\mathrm{e}^{10y}. \tag{47}$$

This solution has a boundary layer in the upper right corner of the domain. It is smooth but steep in this region. The true solution appears in figure 1.

Like most adaptive mesh procedures for elliptic problems, the code begins by solving the problem on a coarse mesh. Then, if the local error indicator exceeds some user-defined tolerance, that element undergoes longest edge bisection. In order to avoid hanging nodes, which require the introduction of mortar spaces, the element sharing that longest edge is also bisected. Then, the problem is solved again on the new mesh. The code repeats this procedure a fixed number of times or until no indicators exceed the tolerance. Since the local error estimate is only proportional to the actual error, the tolerance can require tuning for each particular domain and permeability (the constant in the regularity result does not depend on the source function).

The error indicator accomplishes two important things. First, although the solution on a very coarse mesh does not reveal the boundary layer at all, the solution displayed in figure 2 begins to capture it with only three iterations of local mesh refinement. In addi-

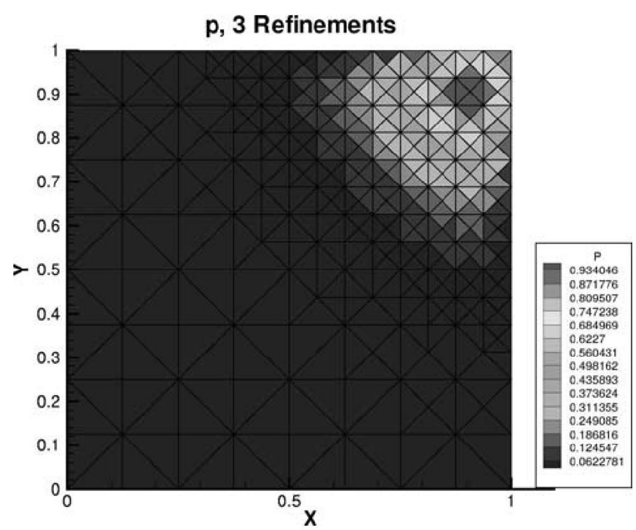

Figure 2. The boundary layer fairly refined after six refinements.

tion, the error indicator predicts the qualitative distribution of the error in both variables, as evidenced in figures 3–5.

This improvement of error in both variables per degrees of freedom is quantified in figures 6 and 7. Notice how the the adaptive refinement procedure gives a lower total error than the uniform mesh with about an eighth of the degrees of freedom as the uniform mesh.

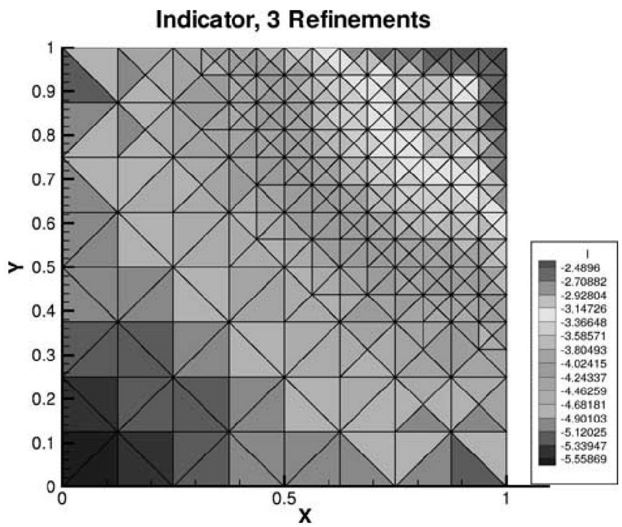

Figure 3. Large error detected in the corner.

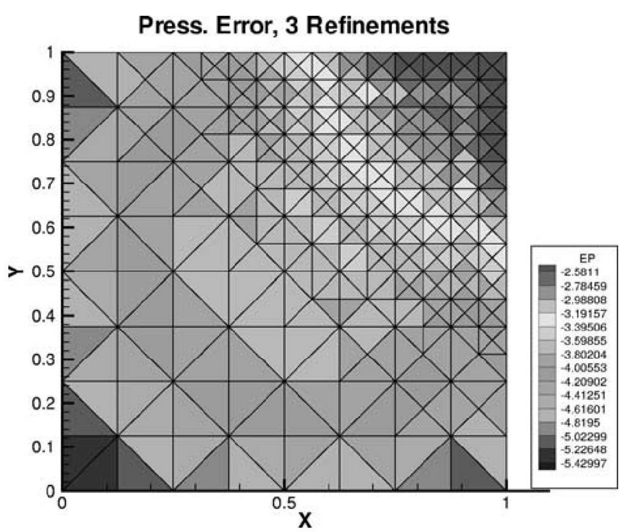

Figure 4. Logarithmic distribution of the pressure error.

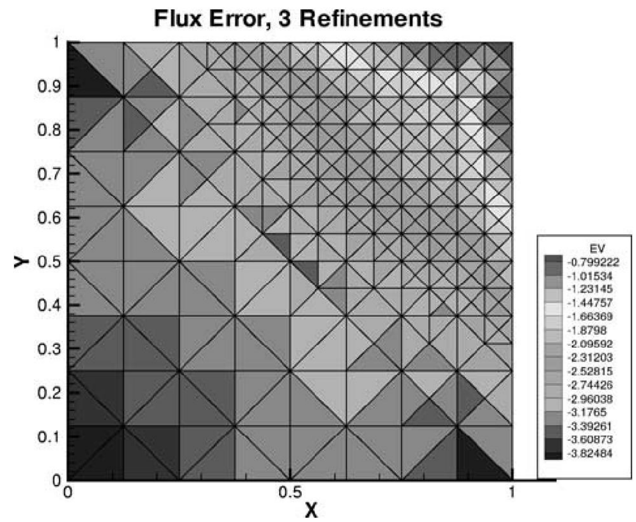

Figure 5. Logarithmic distribution of the velocity error.

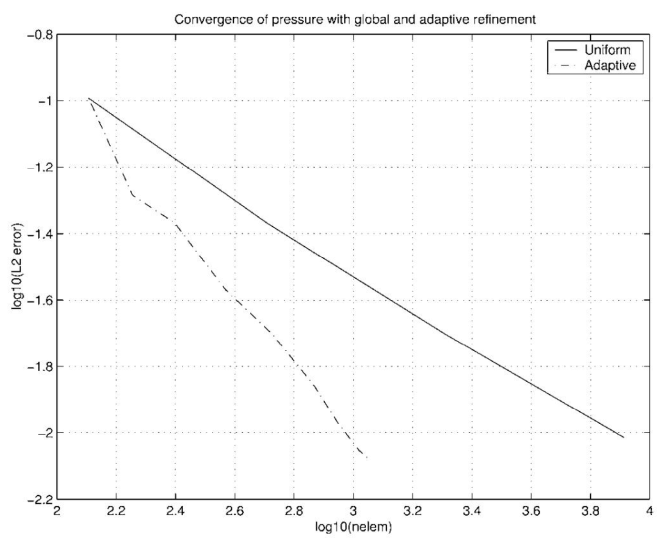

Figure 6. Convergence history for pressure.

4.2. A single-phase flow problem

Here, adaptive mesh refinement is applied to a pressure-driven, single phase flow problem. Let K be as pictured in figure 8. Let homogeneous Neumann conditions be applied along $y = 1$ and $y = 0$. Let $p = 1$ along $x = 0$ and $p = 0$ along $x = 1$. This pressure difference drives the flow from left to right, and the permeability field

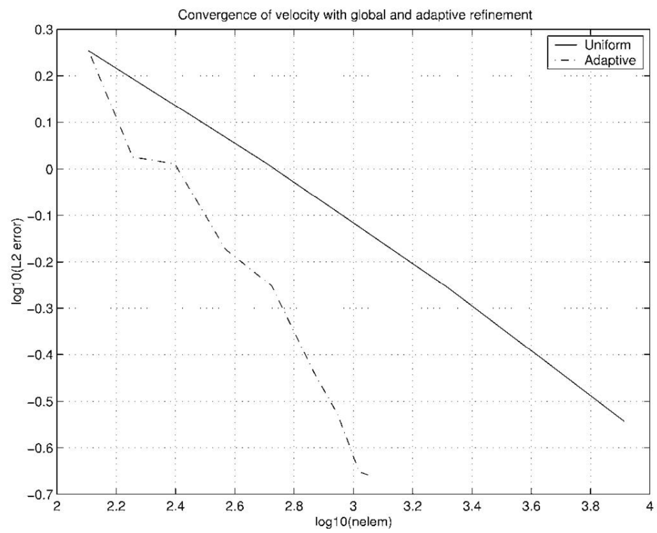

Figure 7. Convergence history for velocity.

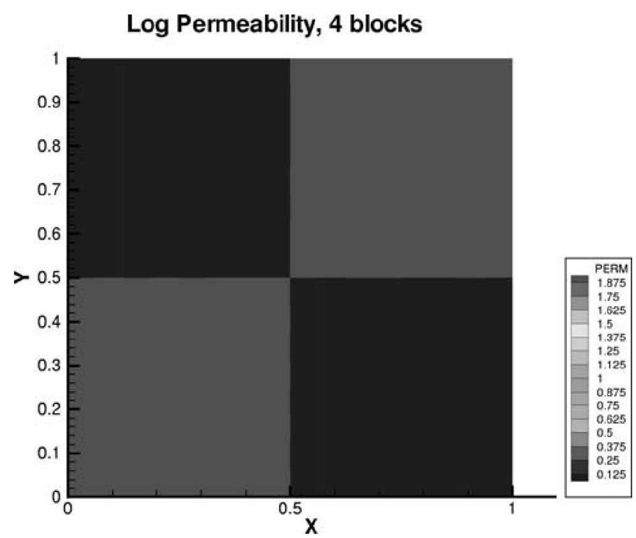

Figure 8. Block permeability field.

gives rise to singularity in the center of the domain. For a more precise discussion of the singularity, including which function space the solution lies in, see [22].

The computed pressure on a uniform mesh of 512 elements is shown in figure 9. The pressure on a uniform mesh of 4096 elements is shown in 10. Notice the under-refinement of the singularity on the coarse mesh compared to the fine. Starting from a coarse mesh, the error indicator is able to detect the need for refinement near the center

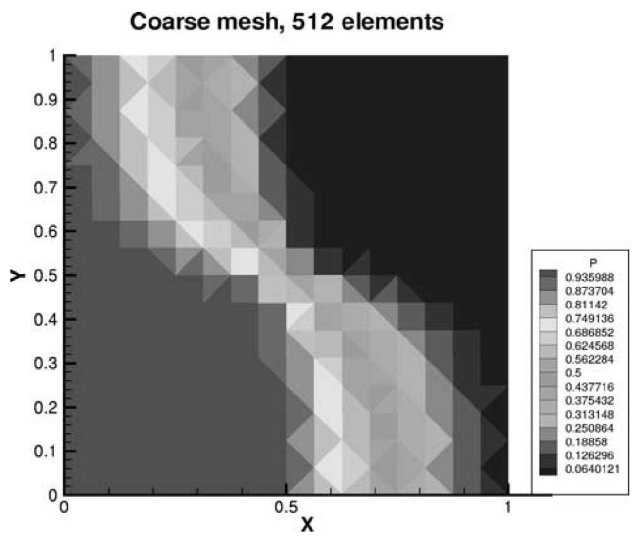

Figure 9. Coarse mesh underresolves center.

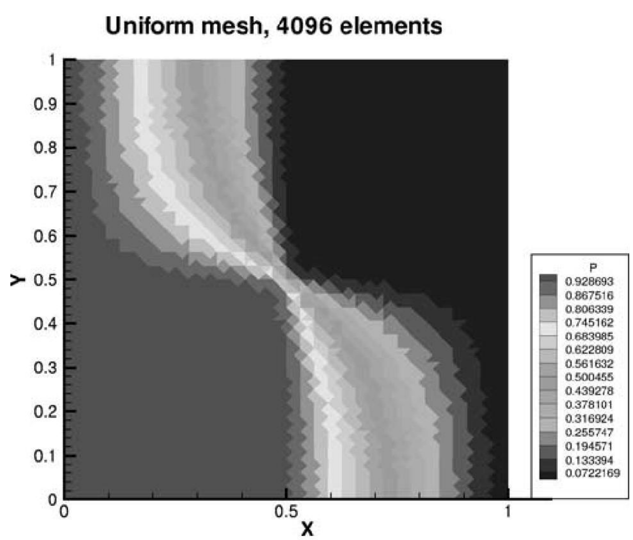

Figure 10. Pressure on a fine mesh resolves the center.

of the domain. In addition, the pressure drops are accurately modeled in the low permeability areas. Figure 11 shows the pressure on the locally refined mesh consisting of 808 elements. The mesh itself is shown in figure 12.

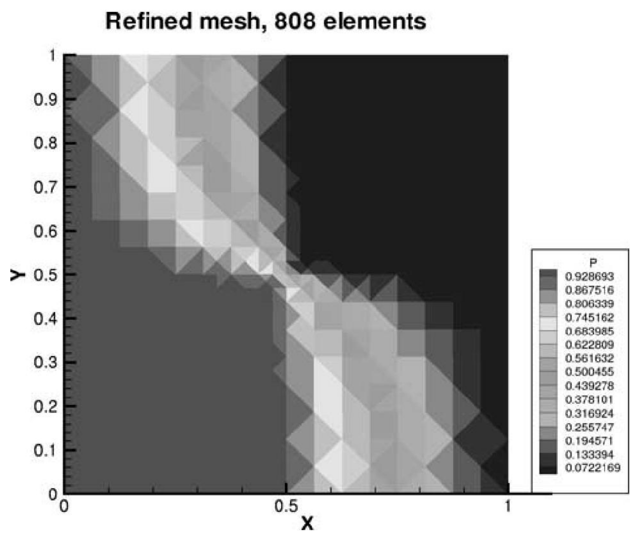

Figure 11. Local refinement captures the interesting features with only 808 elements.

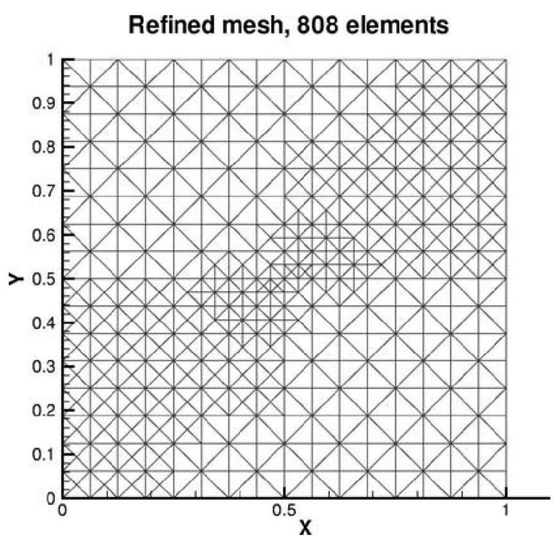

Figure 12. Local refinement concentrates the elements in the center of the domain.

5. Conclusion

Residual-based estimates have been derived for the Raviart–Thomas element. The estimates rely heavily on duality and the standard projections into the finite element spaces. Upper bounds are given for both pressure and flux variables, and a perturbed lower bound is derived for the pressure error. These estimates effectively drive adaptive mesh refinement, both for smooth problems as well as for an example problem which violates the smoothness requirements of the theory.

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