4.1 Exponential Functions and 4.2 Graphs

Properties of Exponential Functions: What is an exponential function?

Function where the **variable** is the exponent and the **base** is a positive constant. The simplest of these are of the form: $f(x) = ab^x$, where $b > 0$

The y-intercept of f is $(0, a)$. The domain of f is all real numbers.

The range of f is all positive real numbers for $a > 0$, and negative for $a < 0$.

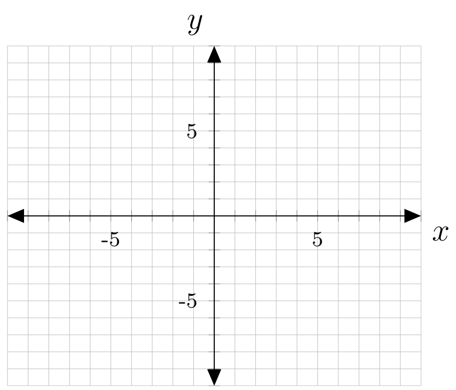
We will also consider what is arguably the most useful exponential function:

$$f(x) = e^x$$

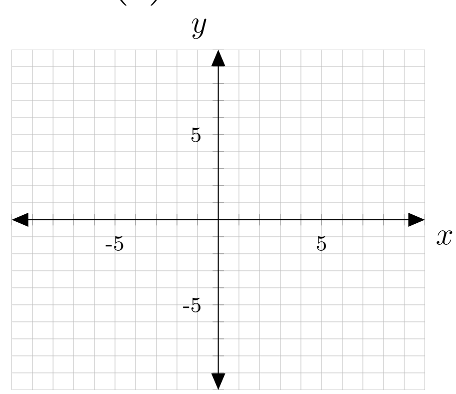
*Note: The **number e** is approximately 2.718281828459...

Example: Graph each function:

i) $f(x) = 2^x$



ii) $f(x) = \left(\frac{3}{2}\right)^x$



Definition Exponential Growth: These are both examples of exponential growth. When $b > 1$, the function $f(x) = b^x$ has the set of all real numbers as its domain, the graph has the general shape above, and the following properties:

i) The graph is above the x-axis

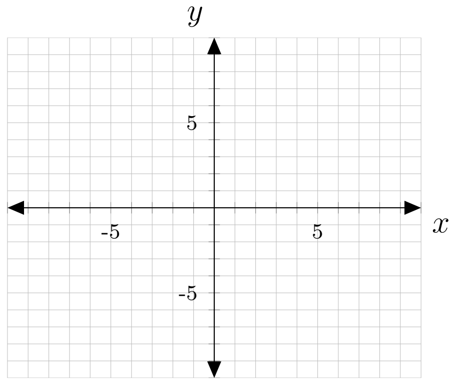
ii) The y-intercept is 1

iii) The graph climbs steeply to the right

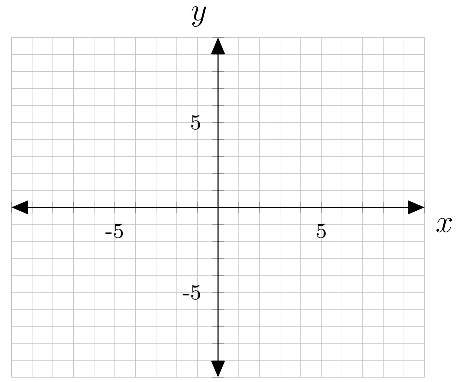
iv) The negative x-axis is a horizontal asymptote

v) The larger the base b, the more steeply the graph rises to the right

iii) $f(x) = 2^{-x}$



iv) $f(x) = \left(\frac{1}{10}\right)^x$



Definition Exponential Decay: These are both examples of exponential decay. When $0 < b < 1$, the function $f(x) = b^x$ has the set of all real numbers as its domain, the graph has the general shape above, and the following properties:

i) The graph is above the x-axis

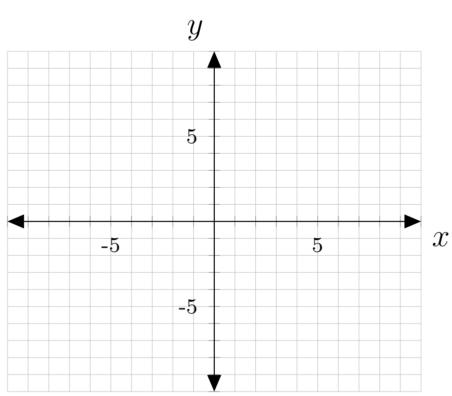
ii) The y-intercept is 1

iii) The graph falls sharply to the right

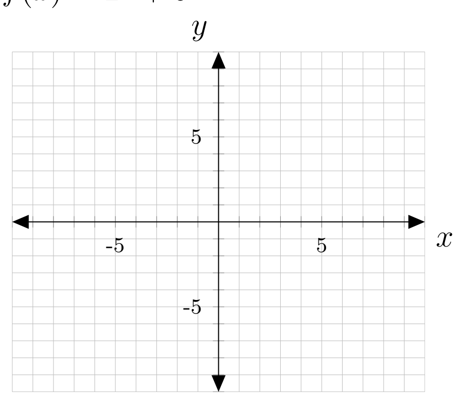
iv) The positive x-axis is a horizontal asymptote

v) The smaller the base b, the more sharply the graph falls to the right

v) $f(x) = 2^{x+6}$



vi) $f(x) = 2^x + 6$



Properties of Translations of Exponential Functions:

A translation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

Where the parent function $f(x) = b^x$, $b > 1$ is

i) Shifted horizontally c units to the left

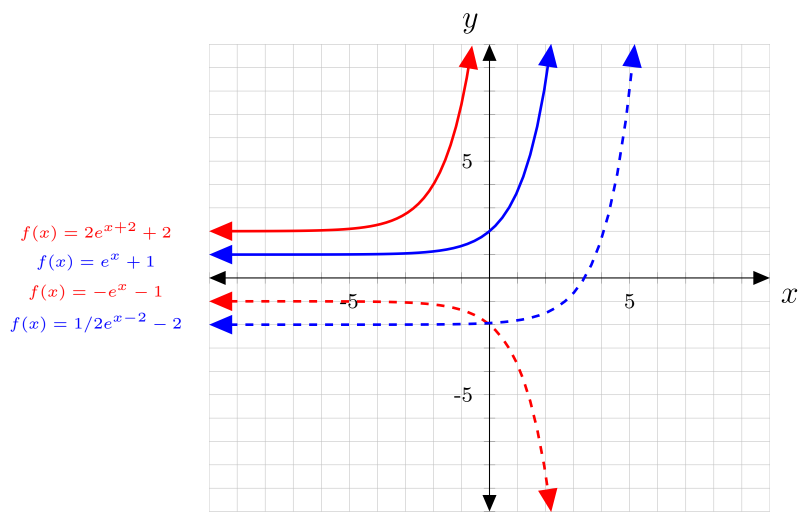
ii) Stretched vertically by a factor of $|a|$ if $|a| > 0$

iii) Compressed vertically by a factor of $|a|$ if $0 < |a| < 1$

iv) Shifted vertically d units

v) Reflected about the x-axis when $a < 0$

Example:



Example: Give a rule of the form $f(x) = a^x$ for the exponential function whose graph contains the point $(-2, 121)$

Example: Give a rule of the form $f(x) = ab^x$ for the exponential function whose graph contains the points $(0, 3)$ and $(2, 12)$

Properties of Simple and Compound Interest:

Simple Interest: earn interest each year on your original investment

To find the future value, we have the following:

$$FV = P(1 + rt)$$

P = original investment, r = interest rate, t = number of years

Compound Interest: earn interest on both your original investment and previously earned interest

To find the future value (or compound amount), we have the following:

$$FV = P(1 + i)^n$$

P = original investment, i = interest rate per period, n = number of periods

*Note: i is not simply the interest rate!

$$i = \frac{\text{interest rate}}{\text{times per year compounded}} = \frac{r}{m}$$

Comparing compound and simple interest:

Example: If $7000 is deposited into an account that pays 4% interest compounded annually, how much money is in the account after 9 years? What if it is compounded quarterly?

Continuous Compounding: informally, interest compounded as frequently as possible

Lets say we invest \$1 for one year at an annual interest rate of 100%, compounded n times per year. What happens?

$$FV = P(1+i)^n = 1\left(1+\frac{1}{n}\right)^n$$

Compounded	n	Future Value
Monthly	12	
Daily	365	
Every Minute	525,600	
Every Second	31,536,000	

Continuous Growth/Decay:

$$f(t) = ae^{rt}$$

$a =$ initial value, $r =$ continuous growth rate per unit time, $t =$ time elapsed

Continuous Compound Interest:

Let $a = P =$ principal amount, $r =$ interest rate per unit time, $t =$ time invested

Example: Suppose that to settle a debt, you will be paid \$10,000 at the end of 10 years.

i) If you can get an interest rate of 4% compounded continuously, what is the present value of the \$10,000?

ii) If the person offers you \$8000 immediately to settle his/her debt, should you take it?

4.1 Problems: 4-6, 9, 10, 15, 17, 23, 26, 39, 41, 58, 59, 61-64

4.2 Problems: 3, 8, 13-23, 26-28, 51-54

4.3 Logarithmic Functions and 4.4 Graphs

The Logarithm Function

If $x = a^y$, then we say that y is the logarithm base a of x. So

$$y = \log_a(x) \text{ if and only if } x = a^y$$

If $a = 10$, then we write $y = \log(x)$. If $a = e$, then we write $y = \ln(x)$.

Notice that since $a^y > 0$ for all y, we have that $x > 0$ with $x = e^y$.

Thus, the domain is all positive real numbers and the range is all real numbers.

Properties of Logarithms

Let x and a be positive real numbers, $a \neq 1$, and r any real number. Then we have:

i) $\log_a(1) = 0$ and $\ln(1) = 0$ since $a^0 = 1$

ii) $\log_a(a) = 1$ and $\ln(e) = 1$ since $a^1 = a$

iii) $\log_a(a^r) = r$ and $\ln(e^r) = r$ since $(a)^r = (a^r)$

iv) $a^{\log_a(x)} = x$ and $e^{\ln(x)} = x$

Example: Evaluate the logarithmic expression without using a calculator:

i) $\log_2 16$ ii) $\ln \frac{1}{e^2}$ iii) $\log 0.01$ iv) $\log_2 \sqrt{2}$

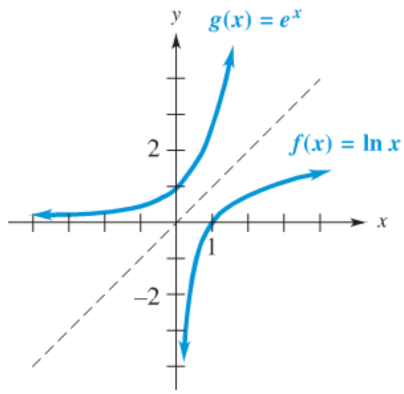
Example: Convert to a logarithmic equation or exponential equation:

i) $3^3 = 27$ ii) $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$ iii) $\log_6 36 = 2$ iv) $\log_{\sqrt{3}} 81 = 8$

Example: Find the domain of the following:

i) $f(x) = \log(5x + 10) + 3$ ii) $g(x) = \ln(-x) + 8$

*Note: Exponential and Logarithmic functions are actually inverses and their graphs are reflected over the line $y = x$. This means that we know basic **points** and **asymptotes**.



Properties of Translations of Logarithmic Functions:

A translation of a logarithmic function has the form

$$f(x) = a\log_b(x + c) + d$$

Where the parent function $f(x) = \log_b(x)$, $b > 1$ is

i) Shifted horizontally c units to the left

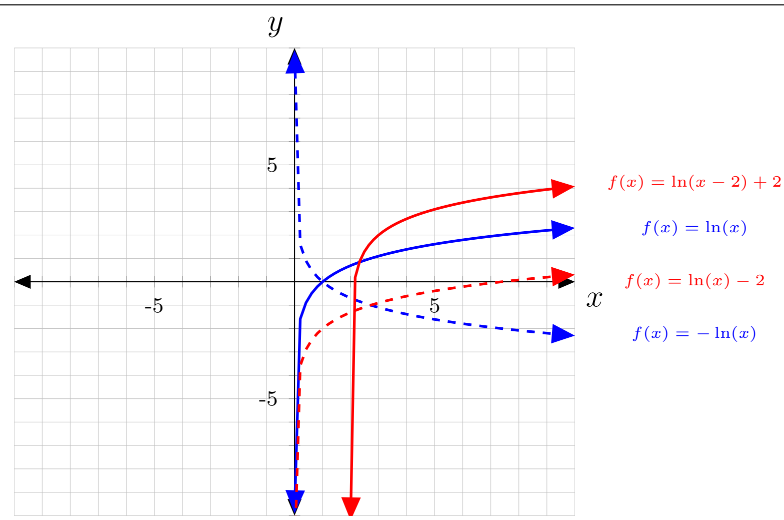
ii) Stretched vertically by a factor of $|a|$ if $|a| > 0$

iii) Compressed vertically by a factor of $|a|$ if $0 < |a| < 1$

iv) Shifted vertically d units

v) Reflected about the x-axis when $a < 0$

Example:



4.3 Problems: 9, 14-16, 24, 25, 27, 31, 34-37, 39, 40, 42, 43, 46, 47, 50, 51, 59-60

4.4 Problems: 21, 23, 38-40, 56, 60

4.5 Logarithmic Properties

Properties of Logarithms

The first four properties below are from the previous sections.

Let x, y, and a be positive real numbers, $a \neq 1$, and r any real number. Then we have:

i) $\log_a(1) = 0$ and $\ln(1) = 0$ since $a^0 = 1$

ii) $\log_a(a) = 1$ and $\ln(e) = 1$ since $a^1 = a$

iii) $\log_a(a^r) = r$ and $\ln(e^r) = r$ since $(a)^r = (a^r)$

iv) $a^{\log_a(x)} = x$ and $e^{\ln(x)} = x$

v) $\log_a(xy) = \log_a(x) + \log_a(y)$

vi) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

vii) $\log_a(x^r) = r \cdot \log_a(x)$

Example: Write the expression as a single logarithm with coefficient of 1:

i) $\log_a x + \log_a y - \log_a m$

ii) $2\log_m a - 3\log_m b^2$

iii) $3\ln 3 - \dfrac{1}{2}\ln 36$

iv) $\dfrac{1}{3}\log 8 + \log 5 - 3\log 2$

Example: Write each expression as a sum and/or difference of logarithms, with all variables to the first degree:

i) $\log_2 \dfrac{6x}{y}$

ii) $\log_5 \dfrac{5\sqrt{7}}{3}$

iii) $\log_m \sqrt{\dfrac{5r^3}{z^5}}$

iv) $\log_2 \dfrac{ab}{cd}$

4.5 Problems: 9, 11, 13, 14, 38, 39

4.6 Logarithmic and Exponential Equations and 4.7 Models

*Note: If $\log_a u = \log_a v$ then $u = v$, and if $a^u = a^v$ then $u = v$

Example: Solve each logarithmic equation. Express solutions as an exact answer.

i) $\log_2(x+9) - \log_2 x = log_2(x+1)$

ii) $\log_5(4x) = \log_5(x+3) + log_5(x-1)$

iii) $\log_6(2x+4) = 2$

iv) $\log(m+25) = 1 + \log(2m-7)$

v) $\ln(x+2) - 4 = -\ln(x-2)$

vi) $\ln(e^y) - 2\ln(e) = -\ln(e^4)$

Example: Solve each exponential equation. Express solutions as an exact answer.

i) $2^x = 7$

ii) $10^{3y-9} = 7$

iii) $3e^{x^2} = 1200$

iv) $5^{6x-3} = 2^{4x+1}$

Example: According to projections by the US Census Bureau, the world population (in billions) is approximated by the function $f(x) = 4.834(1.011)^x$, where $x = 4$ corresponds to the year 1984. When will the population reach 7 billion?

Example: In the central Sierra Nevada mountains of California, the percent of moisture that falls as snow rather than rain is approximated reasonably well by $p(h) = 86.3 \ln h - 680$, where p is the percent of moisture as snow at an altitude of h feet (with $3000 \leq h < 8500$). At what altitude is 50 percent of the moisture snow?

Applications in Radioactive Decay

Suppose we are given the half-life, t_1, of an isotope and we want to find the (continuous) decay rate. We can do so by the following steps:

i) We write the exponential function $A(t) = A_0 e^{kt}$.

ii) We then solve for k by letting $A(t_1) = \frac{1}{2}A_0$ as follows

This k is the decay rate of our given isotope.

Example: The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, t.

Example: If we start with 200 mg, how much of the isotope would remain after 2 years?

4.6 Problems: 5, 7, 12, 14, 17, 24, 45, 46, 48, 53, 55, 67

4.7 Problems: 25, 29, 23, 34