6.1 Systems of Two Linear Equations in Two Variables

Properties of Systems of Equations: What is a system of linear equations?

Example:

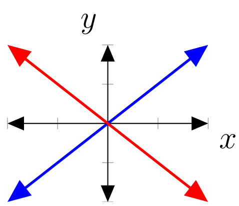
$$5x - 3y = 7$$
$$2x + 4y = 8$$

Solution of a System: a solution that satisfies all the equations of the system

Example for above: (2,1)

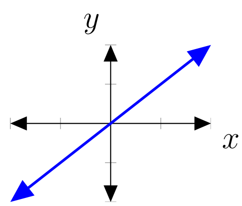
Graphs of System of Two Linear Equations of Two Variables

i) Independent System



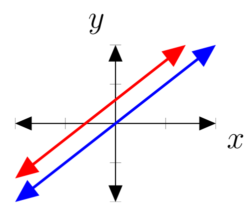
One Solution

ii) Dependent System



Infinite Solutions

iii) Inconsistent System



No Solution

We will use two different methods to solve these systems: ____________ and ____________

Example: Solve the system using substitution (Best when one variable has coefficient 1):

$2x - y = 1$

$3x + 2y = 4$

Example: Solve each system using elimination:

i) $5x + y = 4$

$3x + 2y = 1$

ii) $-4x + y = 2$

$8x - 2y = -4$

iii) $3x - 2y = 4$

$-6x + 4y = 7$

Example: A 200 seat theater charges \$8 for adults and \$5 for children. If all seats were filled and the total income was \$1435, how many adults and children were in the audience?

6.1 Problems: 2-4, 11, 14, 17, 27, 31

6.2 Larger Systems of Linear Equations

The procedure for solving large systems of equations is to transform the system into a simpler, equivalent system using three elementary operations, and then solve this system.

i) **Interchange any two equations in the system**

ii) **Multiply an equation in the system by a nonzero constant**

iii) **Replace an equation in the system by the sum of itself and a constant multiple of another**

Elimination Method Steps by using above operations:

i) Make leading coefficient of first equation 1

ii) Eliminate the leading variable of the first equation from all other equations

iii) Repeat i) and ii) for the second equation: Make its leading coefficient 1 and eliminate that variable from other equations

iv) Repeat for third equation, fourth equation, and so on.

v) Solve by back substituting for the previous variable using the variable you found.

Example: Solve the following system:

$2x + y - z = 2$

$x + 3y + 2z = 1$

$x + y + z = 2$

Matrix Methods

Matrix: The previous system can be written without listing variables as:

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Rows, Columns, and Elements of a Matrix:

Consists of horizontal rows, vertical columns, and numbers as elements.

Augmented Matrix: Draw a line to separate coefficients and constants:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

Example: Use matrices and back substitution to solve the following system:

$2x + y - z = 2$

$x + 3y + 2z = 1$

$x + y + z = 2$

Row Echelon Form of a Matrix (REF):

All rows consisting entirely of zeros (if any) are at the bottom.

The first nonzero in each row is 1 (called the leading 1).

Each leading 1 appears to the right of the leading 1 in any preceding rows.

Reduced Row Echelon Form of a Matrix (RREF):

In REF and every column with a leading 1 has zeros in all other entries.

Gauss-Jordan Method:

Similar to matrix elimination but we replace back substitution with additional elimination of variables in the matrix.

Example: Use the Gauss-Jordan Method to solve the system in the previous example.

Example: Solve the following systems of equations using any method:

i) $2x + 4y = 4$

$3x + 6y = 8$

$2x + y = 7$

ii) $2x - 3y + 4z = 6$

$x - 2y + z = 9$

$y + 2z = -12$

*Know how to input a matrix into your calculator.

Use either the REF or RREF commands to have the calculator reduce the matrix.

Be able to read the solution from the matrix given in its reduced form.

6.2 Problems: 5, 6, 11, 19, 21, 25, 27, 29, 31, 33, 44, 45, 48, 68, 71

6.3 Applications of Systems of Linear Equations

Example: An animal feed is to be made from corn, soybeans, and cottonseed. Determine how many units of each ingredient are needed to make a feed that supplies 1800g of fiber, 2800g of fat, and 2200g of protein from the information below per unit of each ingredient:

	Fiber	Fat	Protein
Corn	10g	30g	20g
Soybeans	20g	20g	40g
Cottonseed	30g	40g	25g

Example: The gross domestic product (GDP) of the United States was \$11 trillion in 2003 and is projected to be \$20 trillion in 2028 and \$30 trillion in 2044.
Let $x = 0$ correspond to 2000. Find a quadratic function $f(x) = ax^2 + bx + c$ that gives the GDP (in trillions of dollars) in year x.

6.3 Problems: 5, 14, 15, 17, 19, 21, 25, 27

6.4 Basic Matrix Operations

Properties of Matrices:

Size of a Matrix: m x n for m rows and n columns

Row Matrix (or Row Vector): 1 x n

Column Matrix (or Column Vector): m x 1

Square Matrix: $m = n$, so m x m

Position: If A is a matrix, then a_{ij} is the element in the i^{th} row and j^{th} column

Addition and Subtraction: Matrices must be same size and will be done by element

Scalar Multiplication: Product of scalar k and a matrix X is kX, where each element is k times the element of X

Example: Perform the indicated operations where possible (i.e. the matrices must be the same size):

i) $\begin{bmatrix} 2 & 5 & 7 \\ 3 & -1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 0 \\ 10 & -4 & 5 \end{bmatrix}$

ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 5 \\ 6 & 0 \end{bmatrix}$

iii) $\begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 3 & -2 \end{bmatrix}$

iv) $(-3) \begin{bmatrix} 2 & 4 & -7 \\ -6 & 2 & 1 \\ 5 & -3 & 9 \end{bmatrix}$

v) Find $2A - 3B$, when $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -4 & 2 \end{bmatrix}$

vi) Let $A = \begin{bmatrix} 1 & -2 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$. Find the matrix X satisfying $3X = A - 2B$.

Example: The shortage of organs for transplants is a continuing problem in the United States. At the end of 2000, there were 3,929 people waiting for a heart transplant, 3,514 for a lung transplant, 16,095 for a liver transplant, and 44,589 for a kidney transplant. Corresponding figures for 2003 were 3444, 3800, 16927, and 53563, respectively. In 2006, the figures were 2814, 2857, 16861, and 66961, respectively. Express this information as a 4 x 3 matrix, labeling the rows and columns appropriately.

6.4 Problems: 10, 14, 16, 21, 23, 35, 36

6.5 Matrix Products and Inverses

Matrix multiplication is **not** defined the way that addition and subtraction are, i.e. we cannot just multiply corresponding entries!

To see how we define matrix multiplication and why, consider the following example:

Example: Three warehouses in New York, Chicago, and San Francisco store sofas of type A, B, and C. The number of each type of sofa stored in each warehouse is displayed in a 3x3 matrix. The value of each sofa type is displayed in a 3x1 column matrix.

$$\begin{array}{r} \\ \textit{New York} \\ \textit{Chicago} \\ \textit{San Francisco} \end{array} \overset{\begin{array}{ccc} A & B & C \end{array}}{\begin{bmatrix} 10 & 7 & 3 \\ 5 & 9 & 6 \\ 4 & 8 & 2 \end{bmatrix}} \qquad \begin{array}{c} A \\ B \\ C \end{array} \overset{\textit{Value}}{\begin{bmatrix} 800 \\ 1000 \\ 1200 \end{bmatrix}}$$

How would we find the value of sofas of type A stored in the New York warehouse?

What about the value of sofas of type B stored in the New York warehouse?

What about the value of sofas of type C stored in the New York warehouse?

So how would we find the value of ALL sofas stored in the New York warehouse?

Multiplying Matrices: Let A be an m x n matrix and B an n x k matrix. Then AB is the m x k matrix whose entry in the i^{th} row and j^{th} column is the product of the i^{th} row of A and the j^{th} column of B.

Properties: For any matrices A, B, and C, such that all indicated products exist we have the following:

$$A(BC) = (AB)C \qquad A(B + C) = AB + AC \qquad (A + B)C = AC + BC$$

Important Note #1: When multiplying matrices, A and B, the number of columns of the first matrix, A, must be the same as the number of rows of the second matrix, B. Thus we may have that only AB exists, only BA exists, they both exist, or neither exists.

Important Note #2: If A and B are matrices such that AB and BA exist, **AB may not be equal to BA**.

Example: Perform the indicated operations where possible:

i) Find CD and DC when $C = \begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

ii) Find AB and BA when $A = \begin{bmatrix} 1 & 7 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix}$

Example: The average birth and death rates per million people for several regions of the world and the populations (in millions) of those regions are shown in the following tables:

Region	Births	Deaths
Asia	.024	.008
Latin America	.025	.007
North America	.015	.009
Europe	.011	.011

Year	Asia	Latin America	North America	Europe
1970	1996	286	226	460
1980	2440	365	252	484
1990	2906	455	277	499
2000	3683	519	310	729
2025	4723	697	364	702

i) Write the information in the 1st table as a matrix R, and the information in the second table as a matrix P.

ii) Find the product PR.

iii) Explain what PR represents.

Identity Matrix: A matrix of the form with 1 along the main diagonal and 0 elsewhere.

Identity matrices have the following property:

$$AI = IA = A \text{ for any matrix } A$$

Examples:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Convince ourselves:

Inverse Matrix: For a square matrix A, its inverse matrix (i.e.A^{-1}) must satisfy the equation: $AA^{-1} = I = A^{-1}A$

A matrix MUST be SQUARE in order to have an inverse!

However, not all square matrices have inverses (these are called singular matrices).

Note: $A^{-1} \neq \frac{1}{A}$

Example: Given the matrices A and B as follows, determine whether B is an inverse of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 1 \\ 2 & -1/2 \end{bmatrix}$$

To find the inverse of a matrix A:

i) Form the augmented matrix $\left[A \middle| I\right]$

ii) Perform row operations to get $\left[I \middle| B\right]$

iii) Then B is the inverse of A

Example: Find the inverse of $D = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Example: Find the inverse of $D = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

*Know how to find an inverse matrix on your calculator:

Enter Matrix in calculator, and then use x^{-1} key.

If Error: Singular Matrix appears, the matrix does not have an inverse!

6.5 Problems: 3, 4, 7, 8, 9, 11, 15, 17, 19, 27, 34, 35, 38, 45, 46, 51, 52, 53

6.6 Applications of Matrices

In this section, you will learn a new method for solving systems of equations.

I will not specify a required method in word problems. (i.e. use any of the methods we have discussed)

We can write a system alternatively as below:

$x + 2y + 3z = 4$

$5x + 6y + 7z = 8$

$9x + 10y + 11z = 12$

*Note- If A is invertible, we have

$$X = A^{-1} \cdot B$$

Example: Solve the following systems of equations using inverse matrices:

$x + y + z = 2x$

$2x + 3y = 5$

$x + 2y + z = -1$

6.6 Problems: 1, 5, 7, 15, 17, 19