4.1 Linear Approximation and Applications

In this section, we will learn how to approximate how small changes in an input to a function affect the output of the function.

For example, we know that the square root of 16 is 4, but how much larger would the square root of 16.2 be?

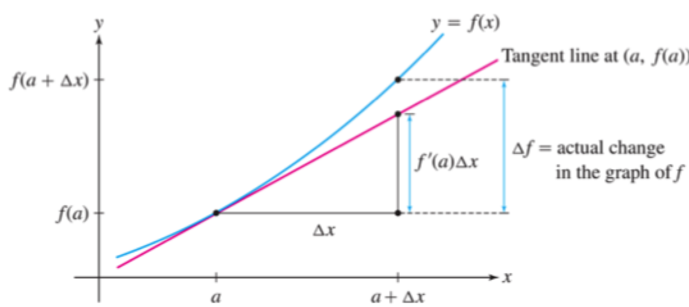
If we have a function $f(x)$, we are interested in estimating the change

$$\Delta f = f(a + \Delta x) - f(a)$$

where Δx is small. Note that Δf is exact.

The **Linear Approximation** uses the derivative to estimate Δf *without* computing the change exactly. Because the function f and its tangent line are very close when Δx is "small enough," we can just use the instantaneous rate of change of f, i.e. the slope of the tangent line at a, to predict the function's actual change. So

$$\Delta f \approx f'(a) \cdot \Delta x$$



The **Linear Approximation Error**: Error=|Actual - Approximation|=$|\Delta f - f'(a)\Delta x|$

The **Linear Approximation *Percent* Error**: Percentage Error=$\left|\frac{\text{Error}}{\text{Actual}}\right| \times 100\%$

Linearization:
To approximate $f(x)$, we can use the **Linearization of** f. If f is differentiable at a and x is close to a, then

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

Note: These methods are desirable because linear functions are usually easier to use and compute than nonlinear ones.

Example 1: Estimate $\Delta f = f(4.02) - f(4)$ given $f(x) = x^3$. Then find the actual difference.

Example 2: Find the linearization at $x = a$ and then use it to approximate $f(b)$.

(a) $f(x) = 1/x$, $a = 2$, $b = 2.02$

(b) $f(x) = e^x \ln x$, $a = 1$, $b = 1.02$

4.2 Extreme Values

Recall:

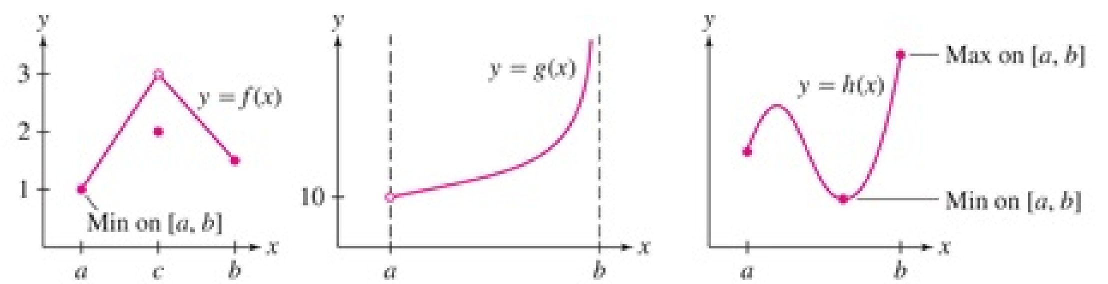
- If $f'(x) > 0$ at $x = a$, then the tangent line to f at a has ______________ slope.
- If $f'(x) < 0$ at $x = a$, then the tangent line to f at a has ______________ slope.
- If $f'(x) = 0$ at $x = a$, then the tangent line to f at a is ______________.

We refer to the maximum and minimum values as **extreme values** or **extrema**. The process of finding these extreme values is called **optimization**.

Extreme Values on an Interval:
Let f be a function on an interval I and let $a \in I$. We say that $f(a)$ is the

- **Absolute minimum** of f if $f(a) \leq f(x)$ for all $x \in I$.
- **Absolute maximum** of f on I if $f(a) \geq f(x)$ for all $x \in I$.

Note: Not every function will have a minimum and/or maximum value!



(A) Discontinuous function with no max on [a, b], and a min at x = a.

(B) Continuous function with no min or max on the open interval (a, b).

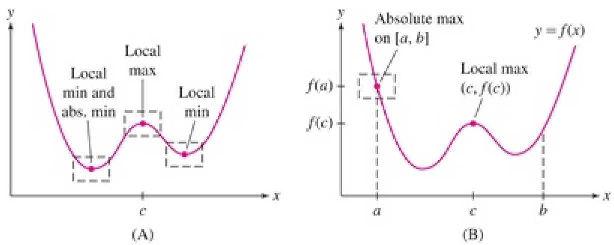
(C) Every continuous function on a closed interval [a, b] has both a min and a max on [a, b].

Theorem: Existence of Extrema on a Closed Interval
A continuous function f on a closed (bounded) interval $I = [a, b]$ takes on both a minimum and a maximum value on I.

Local Extrema:

We say that $f(c)$ is a

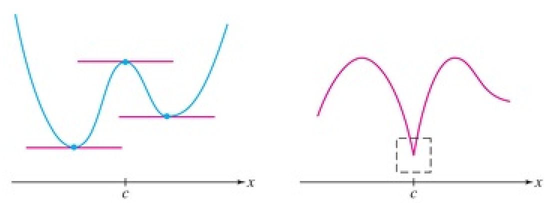
- **Local minimum** occurring at $x = c$ if $f(c)$ is the minimum value of f on some open interval (in the domain of f) containing c.
- **Local maximum** occurring at $x = c$ if $f(c)$ is the maximum value of f on some open interval (in the domain of f) containing c.



We find these local extrema (and possibly the absolute extrema) by locating **critical points** or **critical values**.

Critical Points:

A number c in the domain of f is called a **critical point** if either $f'(c) = 0$ or $f'(c)$ does not exist.



(A) Tangent line is horizontal at the local extrema.

(B) This local minimum occurs at a point where the function is not differentiable.

Example 1: Find the critical points of the function:

(a) $y = x^3 + x^2 - x$

(b) $y = \dfrac{1}{x^2}$

(c) $y = |x|$

Theorem: Fermat's Theorem on Local Extrema
If $f(c)$ is a local minimum or maximum, then c is a critical point of f.

Proof.

BUT: the other way around is **not** always true! Just because c is a critical point of f does not mean that $f(c)$ is necessarily a maximum or minimum (sometimes this point is a "point of inflection," more on this later).

Optimizing on a Closed Interval

Theorem: Extreme Values on a Closed Interval
Assume that f is continuous on $[a, b]$ and let $f(c)$ be the minimum or maximum value on $[a, b]$. Then c is either a critical point or one of the endpoints a or b.

Strategy for Finding Extreme Values on a Closed Interval [a,b]:

1. Find the critical points on $[a, b]$.
2. Find the function value at each critical point and at the two endpoints.
3. The largest function value from Step 2 is the maximum and the smallest function value from Step 2 is the minimum value. (Min/max values are y-values and they occur at the corresponding x-values.)

Example 2: Find the extrema of $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0, 2]$.

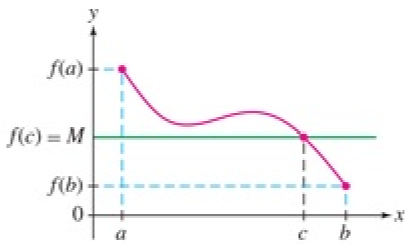
Example 3: Find the extrema of $y = x^2 - 8\ln x$ on $[1, 4]$.

Example 4: Find the extrema of $y = \theta - 2\sin\theta$ on $[0, 2\pi]$.

Recall: Intermediate Value Theorem (IVT; 2.8)

Intuition: If f is continuous, then it won't skip y-values!

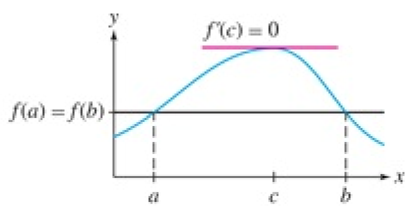
If f is continuous on a closed interval $[a, b]$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$ such that $f(c) = M$.



Rolle's Theorem:

Intuition: If f is continuous, its derivatives exist between $x = a$ and $x = b$, and $f(a) = f(b)$, then f had to change direction between a and b and at that point, the derivative is zero.

Assume that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c between a and b such that $f'(c) = 0$.



Example 5: Show that $f(x) = x^3 + 9x - 4$ has precisely one real root.

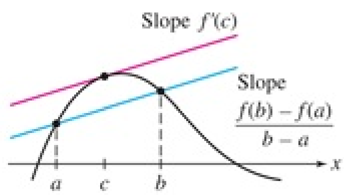
4.3 The Mean Value Theorem and Monotonicity

The Mean Value Theorem:

Intuition: If f is continuous and its derivatives exist between $x = a$ and $x = b$, then there is a point between a and b where the slope of the tangent line is the same as the slope of the secant line through a and b.

Assume that f is continuous on the closed interval $[a, b]$ and differentiable on (a, b). Then there exists at least one value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Note: Rolle's Theorem is a special case of the MVT where $f(a) = f(b)$.

Corollary:

If f is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b). In other words, $f(x) = C$ for some constant C.

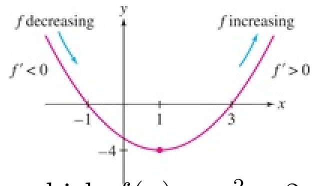
Example 1: Find a point c satisfying the conclusion of the MVT for $y = \sqrt{x}$ on the interval $[1, 9]$.

Theorem: The Sign of the Derivative

Let f be a differentiable function on an open interval (a, b).

- If $f'(x) > 0$ for $x \in (a, b)$, then f is *increasing* on (a, b).
- If $f'(x) < 0$ for $x \in (a, b)$, then f is *decreasing* on (a, b).

We say that f is **monotonic** on (a, b) if it is either increasing or decreasing on (a, b).

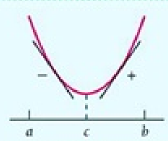
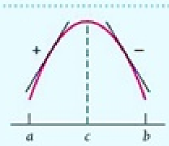
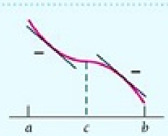
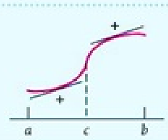


Example 2: Find the intervals on which $f(x) = x^2 - 2x - 3$ is monotonic.

Theorem: First Derivative Test for Critical Points

Let c be a critical point of f. Then

- $f'(x)$ changes from + to - at $c \implies f(c)$ is a local maximum.
- $f'(x)$ changes from - to + at $c \implies f(c)$ is a local minimum.

Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	−	+	Decreasing on (a, c); increasing on (c, b)
	Relative maximum	+	−	Increasing on (a, c); decreasing on (c, b)
	No relative maxima or minima	−	−	Decreasing on (a, b)
	No relative maxima or minima	+	+	Increasing on (a, b)

Using the First Derivative Test to Find Local Extrema:

Step 1 Find the critical points.

(a) These are the x-values where $f'(x) = 0$ or $f'(x)$ is undefined (but $f(x)$ is defined).

Step 2 Divide the x-axis into intervals using the critical values and any x-values for which f is undefined (Note: the latter will *not* produce extrema.)

Step 3 Find the sign of $f'(x)$ on the intervals between the critical points.

(a) To do this, pick a test point inside each interval (so don't pick a critical value).

(b) Plug this test point into $f'(x)$ (not $f(x)$.)

Step 4 Use the First Derivative Test at each critical point to determine where the local extrema occur.

(a) The extreme values are the function values, not the x-values where they occur.

Example 3: Find the critical points and the intervals on which the function is increasing or decreasing. Use the First Derivative Test to determine any local extrema.

(a) $f(x) = x^3 - 12x$

(b) $y = x^3 - 6x^2 + 12x$

(c) $g(\theta) = \theta - 2\cos\theta$ on $[0, 2\pi]$

(d) $y = 1 - (x-1)^{2/3}$

(e) $y = x^2 e^x$

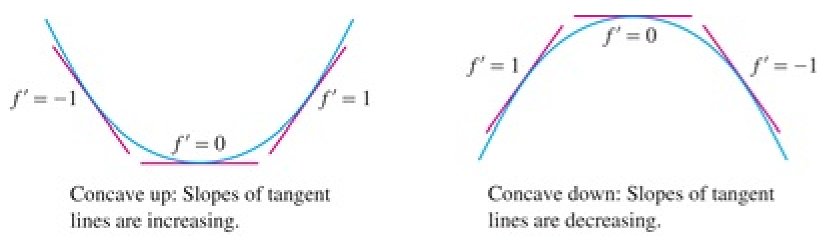
4.4 The Second Derivative and Concavity

In this section, we learn about concavity, which refers to the way a graph bends. Informally, a curve is *concave up* if it bends up and *concave down* if it bends down.

Concavity:

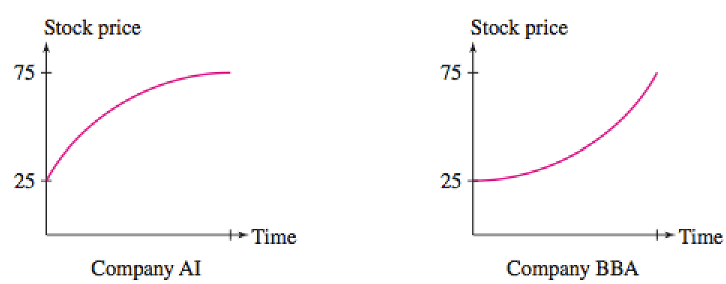
Let f be a differentiable function on an open interval (a, b). Then

- f is **concave up** on (a, b) if f' is increasing on (a, b)
  (i.e. the slopes of its tangent lines are increasing)
- f is **concave down** on (a, b) if f' is decreasing on (a, b)
  (i.e. the slopes of its tangent lines are decreasing)



Note: A function can be decreasing even as its derivative is increasing.

Example 1: The stocks of two companies, Arenot Industries (AI) and Blurbenthal Business Associates (BBA), went up in value. Both currently sell for \$25. However, one is clearly a better investment than the other. Explain in terms of concavity.



Test for Concavity

Assume that $f''(x)$ exists for all $x \in (a,b)$.

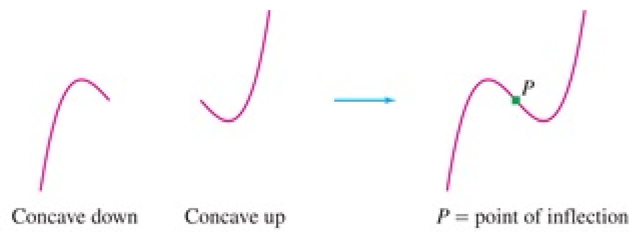
- If $f''(x) > 0$ for all $x \in (a,b)$, then f is concave up on (a,b).
- If $f''(x) < 0$ for all $x \in (a,b)$, then f is concave down on (a,b).

Point of Inflection:

We say that $P = (c, f(C))$ is a **point of inflection** of f if the concavity changes from up to down or from down to up at $x = c$.

These occur at **hypercritical values**, where $f''(x)$ is undefined or where $f''(x) = 0$.

However, not every hypercritical value is a point of inflection!



Using the Second Derivative to Find Points of Inflection:

Step 1 Find the hypercritical values.

(a) These are the x-values where $f''(x) = 0$ or $f''(x)$ is undefined (but $f(x)$ is defined).

Step 2 Divide the x-axis into intervals using the hypercritical values and any x-values for which f is undefined (Note: the latter will *not* produce points of inflection.)

Step 3 Find the sign of $f''(x)$ on the intervals between the hypercritical points.

(a) To do this, pick a test point inside each interval (so don't pick a hypercritical value).

(b) Plug this test point into $f''(x)$ (not $f(x)$ or $f'(x)$.)

Step 4 Use the information in step 3 to determine where $f(x)$ is concave up or concave down over each interval and to label any points of inflection.

(a) Make sure to find the value of $f(x)$ for any hypercritical value that yields a point of inflection.

Example 2: Find all points of inflection and determine the intervals on which the function is concave up and concave down.

a) $y = \cos x$ on $[0, 2\pi]$

b) $y = x^{5/3}$

c) $y = \dfrac{1}{12}x^4 + 4x + 9$

The Second Derivative Test

Let c be a critical point of $f(x)$. If $f''(c)$ exists, then

- $f''(c) > 0$ implies that $f(c)$ is a local minimum
- $f''(c) < 0$ implies that $f(c)$ is a local maximum
- $f''(c) = 0$ means that the test is inconclusive; $f(c)$ may be a local max, a local min, or neither *and we need to go back to the First Derivative Test to find out!*

Example 3: Analyze the critical points of $g(x) = x^5 - \frac{5}{4}x^4$ using the Second Derivative Test.

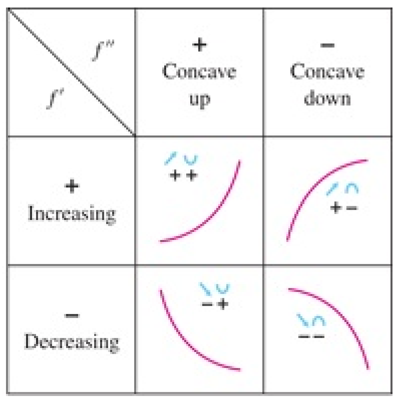
Example 4: Find all critical points, points of inflection, local extrema, intervals on which the function is increasing/decreasing, intervals on which the function is concave up/concave down.

$$f(x) = x^{3/2} - 4x^{-1/2} \qquad (x > 0)$$

4.6 Analyzing and Sketching Graphs of Functions

Important Things to Consider when Graphing:

- *The Four Basic Shapes of a Graph*: Most graphs are made up of smaller arcs (or sections of a curve) that have one of four basic shapes, corresponding to the four possible sign combinations of f' and f''.
- *Transition Points*: locations where the basic shape changes due to a sign change in either f' (local extrema) or f'' (point of inflection) - see diagram.
- *Asymptotic Behavior*: the behavior of $f(x)$ as x approaches either $\pm\infty$.
- *Miscellaneous*: be sure any intercepts and asymptotes are clearly indicated where appropriate.



Using the First and Second Derivative for Graphing a Function:

1. Check the domain of the function. We can find where vertical asymptotes will be here.

2. Find all transition points.

   (a) First find the critical points. Draw a sign chart for f'. Classify all local extrema. Determine the intervals on which f is increasing and decreasing.

   (b) Find the points of inflection. Draw a sign chart for f''. Determine the intervals on which f is concave up and concave down.

3. Use the information from Step 2 to determine the sign combination for each interval.

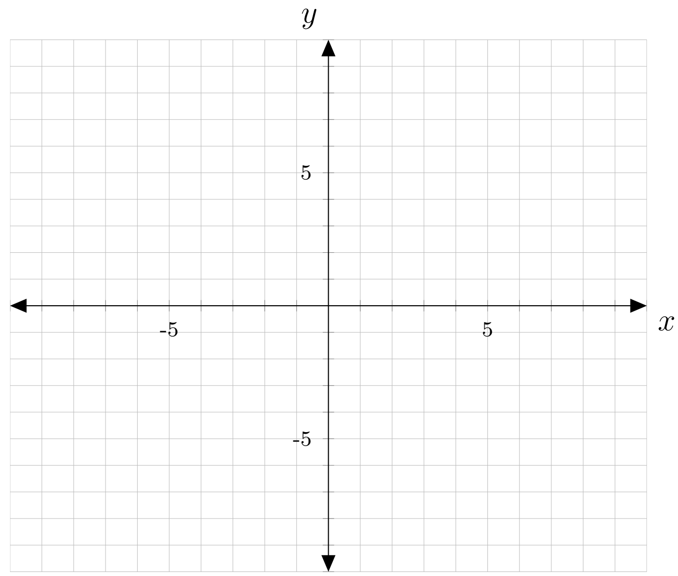
4. Determine the asymptotic behavior of f. Also determine any useful points–usually intercepts.

5. Plot all relevant points and asymptotes. Sketch the graph using the above information.

Example 1: Sketch the following graphs.

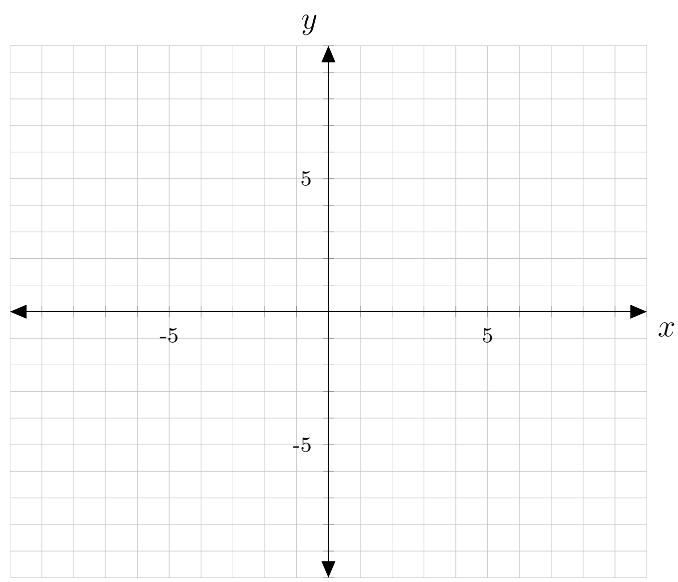
a) Sketch the graph of a function satisfying all of the given conditions:

- $h(x)$ is increasing and concave down on $(-\infty, -2)$
- $h(x)$ is increasing and concave up on $(-2, \infty)$



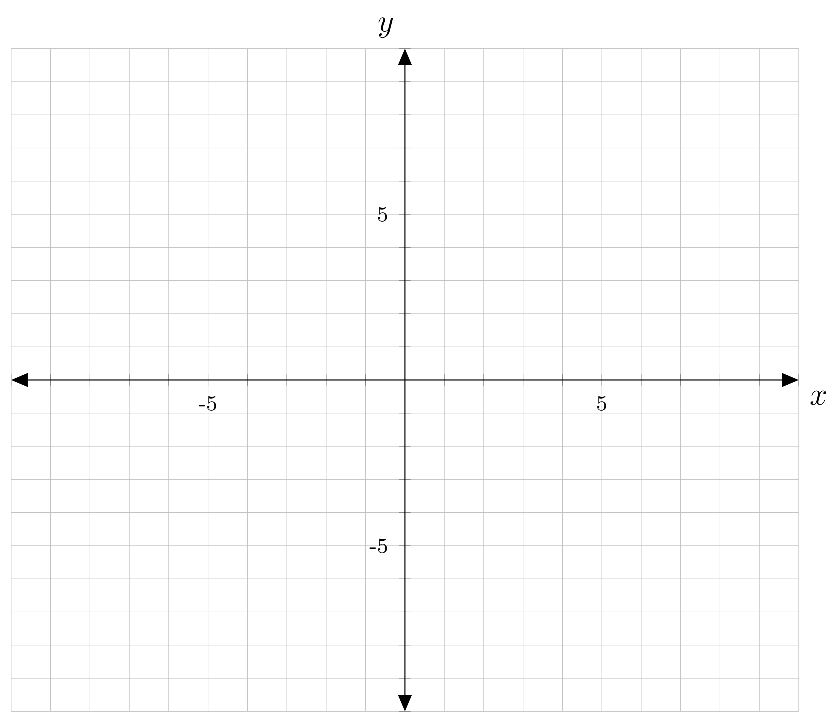
b) Sketch the graph of a function satisfying all of the given conditions:

- $f(-3) = -5, f(1) = 3, f(5) = 7$
- $f'(x) < 0$ on $(-\infty, -3)$ and $(5, \infty)$; $f'(x) > 0$ on $(-3, 5)$; $f'(-3) = 0$ and $f'(5) = 0$
- $f''(x) > 0$ on $(-\infty, 1)$; $f''(x) < 0$ on $(1, \infty)$

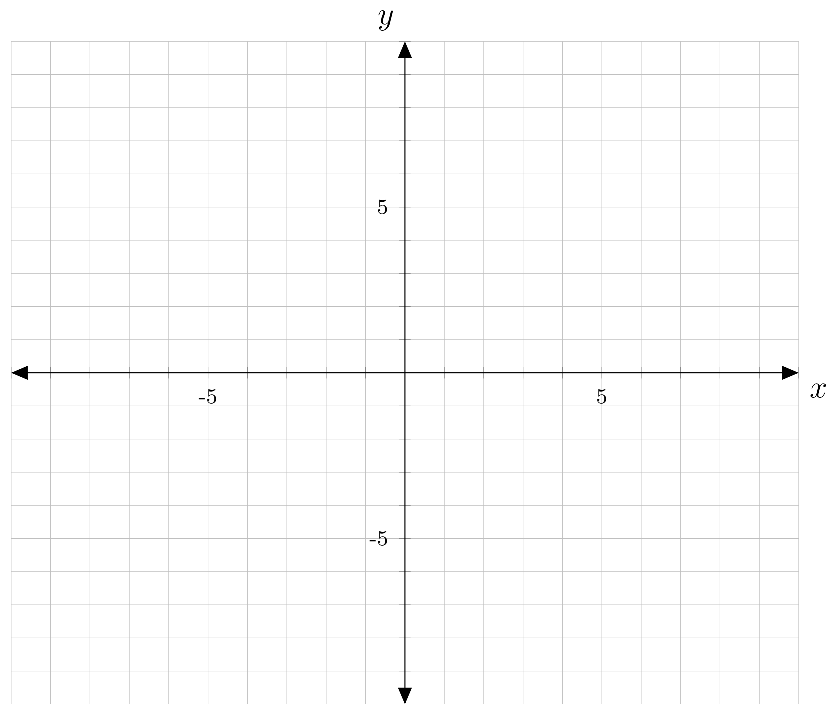


Example 2: Find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

(a) $y = x^3 - 3x + 5$



(b) $f(x) = \dfrac{3x+2}{2x-4}$



4.7 Applied Optimization

In applied optimization problems, we need to find the function whose minimum or maximum we need. This is called the **objective function**.

We also might need an equation that relates two or more variables in an optimization problem, called the **constraint equation**.

We also need to identify which interval we are working on and if it is open or closed.

Solving Applied Optimization Problems:

1. Choose variables.
   (a) Draw a picture, if applicable.
   (b) Determine which quantities are relevant.
   (c) Assign appropriate variables.
2. Find the objective function and the interval.
   (a) Restate as an optimization problem for a function over an interval.
   (b) If the function depends on more that one variable, use a constraint equation to write it as a function of just one variable.
3. Optimize the objective function.
   (a) If we are working on a closed interval, we need to compare the function values at the endpoints and at any critical points inside the interval.
   (b) If we are working on an open interval, the function doesn't necessarily take on a max or min value. If it does, these must occur at critical points within the interval. We need to analyze the behavior of the function as we approach the endpoints of the interval.

Example 1: A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

Example 2: Determine the dimensions of a rectangular solid (with a square base) of maximum volume if its surface area is 150 square inches.

Example 3: Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius 4.

Example 4: Find the point on the line $y = x$ that is closest to the point $(0, -1)$.

Example 5: Enclose a rectangular pasture with area 16000 m^2 with parallel sides made of chain link fence costing \$20/m and wooden fence costing \$50/m. Minimize the cost.

Example 6: Your task is to build a road joining a ranch to a highway that enables drivers to reach the city in the shortest time. How should this be done if the speed limit is 60 km/h on the road and 110 km/h on the highway? The perpendicular distance from the ranch to the highway is 30 km and the city is 50 km down the highway.