RANK INEQUALITY DONE BY FREE PROBABILITY AND RANDOM MATRICES

Abstract: In linear algebra it is well-known that the matrix rank is subadditive, i.e.,

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$$

for matrices A, B. Clearly this inequality become trivial if $\operatorname{rank}(A) + \operatorname{rank}(B)$ is larger than the full size. But one can improve it by

$$\operatorname{rank}(A+B) \leq \inf_{\lambda \in \mathbb{C}} \{\operatorname{rank}(A-\lambda) + \operatorname{rank}(B+\lambda)\}$$

to get a better (actually optimal) bound for $\operatorname{rank}(A+B)$. Generally, one can ask for a polynomial p, for example, commutator $p = xy - xy$ or anticommutator $p = xy + yx$, what a rank inequality $\operatorname{rank}(p(A,B)) \leq ???$ one can hope for matrices A, B? Moreover, can it be optimal in some sense?

In this talk, we will provide some answers to the above question with the help of free probability and random matrices. It is based on a recent joint work with Octavio Arizmendi, Guillaume Cébron, Roland Speicher where we discovered some universality property of freely independent random variables in free probability theory. As an application, we can provide a machinery that can yield an optimal upper bound for $\operatorname{rank}(p(A,B))$.