Code for the Proof of Theorem 7

In[41]:=
```
(*Finds the coordinates of the vertices of the triangle,
taking into account that the center of mass must be 0.*)
Tx[λ_, a_] := -2 / (3 a);
Ty[λ_, a_] := (a + λ) / 3;
M1[λ_, a_] := (-a * λ) / 2;
M2[λ_, a_] := (a^2 - a * λ) / 2;
```

In[45]:=
```
(*Lines that are part of the boundary of the triangle.*)
L1[x_, λ_, a_] := λ - Ty[λ, a] + M1[λ, a] (x + 2 / a + Tx[λ, a]);
L2[x_, λ_, a_] := λ - Ty[λ, a] + M2[λ, a] (x + 2 / a + Tx[λ, a]);
```

In[47]:=
```
(*Calculates a moment of area for the triangle*)
c[x_, a_, λ_, m_, n_] :=
  Integrate[x^m * y^n, {x, -2 / a - Tx[λ, a], -Tx[λ, a]}, {y, L1[x, λ, a], L2[x, λ, a]}];
```

In[48]:=
```
(*Calculates ρ1 of the triangle*)
Rho1[x_, a_, λ_] := 4 * ((c[x, a, λ, 2, 0] * c[x, a, λ, 0, 2] - c[x, a, λ, 1, 1]^2) /
      (c[x, a, λ, 2, 0] + c[x, a, λ, 0, 2]));
```

In[49]:=
```
FullSimplify[Rho1[x, a, λ]]
```

Out[49]=
$$\frac{2\,a^2}{3\left(4+a^2\left(a^2-a\,\lambda+\lambda^2\right)\right)}$$

In[50]:=
```
D[FullSimplify[Rho1[x, a, λ]], λ]
```

Out[50]=
$$-\frac{2\,a^4\,(-a+2\,\lambda)}{3\left(4+a^2\left(a^2-a\,\lambda+\lambda^2\right)\right)^2}$$

In[51]:=
```
A[x_, a_, λ_] := 4 * (c[x, a, λ, 0, 4] × c[x, a, λ, 1, 1]^2 -
    4 c[x, a, λ, 1, 1]^4 - 2 c[x, a, λ, 0, 3] × c[x, a, λ, 1, 1] × c[x, a, λ, 1, 2] +
    c[x, a, λ, 0, 2]^3 × c[x, a, λ, 2, 0] + c[x, a, λ, 0, 3]^2 × c[x, a, λ, 2, 0] +
    4 c[x, a, λ, 1, 2]^2 × c[x, a, λ, 2, 0] - c[x, a, λ, 1, 1]^2 × c[x, a, λ, 2, 0]^2 -
    c[x, a, λ, 0, 2]^2 (c[x, a, λ, 1, 1]^2 + 2 c[x, a, λ, 2, 0]^2) -
    6 c[x, a, λ, 1, 1] × c[x, a, λ, 1, 2] × c[x, a, λ, 2, 1] -
    2 c[x, a, λ, 0, 3] × c[x, a, λ, 2, 0] × c[x, a, λ, 2, 1] +
    c[x, a, λ, 2, 0] × c[x, a, λ, 2, 1]^2 + 2 c[x, a, λ, 1, 1]^2 × c[x, a, λ, 2, 2] +
    2 c[x, a, λ, 0, 3] × c[x, a, λ, 1, 1] × c[x, a, λ, 3, 0] -
    2 c[x, a, λ, 1, 1] × c[x, a, λ, 2, 1] × c[x, a, λ, 3, 0] +
    c[x, a, λ, 0, 2] (4 c[x, a, λ, 2, 1]^2 + (c[x, a, λ, 1, 2] - c[x, a, λ, 3, 0])^2 +
       c[x, a, λ, 2, 0] (-c[x, a, λ, 0, 4] + 6 c[x, a, λ, 1, 1]^2 + c[x, a, λ, 2, 0]^2 -
          2 c[x, a, λ, 2, 2] - c[x, a, λ, 4, 0])) + c[x, a, λ, 1, 1]^2 × c[x, a, λ, 4, 0])
B[x_, a_, λ_] :=
 (c[x, a, λ, 0, 3] + c[x, a, λ, 2, 1])^2 + (c[x, a, λ, 1, 2] + c[x, a, λ, 3, 0])^2 +
  (c[x, a, λ, 0, 2] + c[x, a, λ, 2, 0]) (-c[x, a, λ, 0, 4] + 4 c[x, a, λ, 1, 1]^2 +
     (c[x, a, λ, 0, 2] - c[x, a, λ, 2, 0])^2 - 2 c[x, a, λ, 2, 2] - c[x, a, λ, 4, 0])
```

In[53]:=
```
(*Calculate ρ2 using the formula from Proposition 4.*)
Rho2[x_, a_, λ_] := A[x, a, λ] / B[x, a, λ]
```

In[54]:=
```
FullSimplify[Rho2[x, a, λ]]
```

Out[54]=
$$\left(2\,a^2\left(48+20\,a^4+3\,a^8-6\,a^3\left(4+a^4\right)\lambda+3\,a^2\left(8+3\,a^4\right)\lambda^2-6\,a^5\lambda^3+3\,a^4\lambda^4\right)\right)\Big/\left(3\left(192+224\,a^4+56\,a^8+3\,a^{12}-a^3\left(144+116\,a^4+9\,a^8\right)\lambda+8\,a^2\left(18+19\,a^4+3\,a^8\right)\lambda^2-3\,a^5\left(24+11\,a^4\right)\lambda^3+12\,a^4\left(3+2\,a^4\right)\lambda^4-9\,a^7\lambda^5+3\,a^6\lambda^6\right)\right)$$

In[55]:=
```
(*The derivative from the proof of Theorem 7.*)
FullSimplify[D[(2 a^2 (48 + 20 a^4 + 3 a^8 - 6 a^3 (4 + a^4) λ + 3 a^2 (8 + 3 a^4) λ^2 - 6 a^5 λ^3 + 3 a^4 λ^4)) /
    (3 (192 + 224 a^4 + 56 a^8 + 3 a^12 - a^3 (144 + 116 a^4 + 9 a^8) λ + 8 a^2 (18 + 19 a^4 + 3 a^8) λ^2 -
       3 a^5 (24 + 11 a^4) λ^3 + 12 a^4 (3 + 2 a^4) λ^4 - 9 a^7 λ^5 + 3 a^6 λ^6)), λ]]
```

Out[55]=
$$\left(2\,a^4\,(a-2\lambda)\left(2304+1920\,a^4+496\,a^8+120\,a^{12}+9\,a^{16}-24\,a^3\left(96+64\,a^4+20\,a^8+3\,a^{12}\right)\lambda+6\,a^2\left(384+400\,a^4+148\,a^8+27\,a^{12}\right)\lambda^2-24\,a^5\left(72+40\,a^4+9\,a^8\right)\lambda^3+3\,a^4\left(288+280\,a^4+69\,a^8\right)\lambda^4-144\,a^7\left(3+a^4\right)\lambda^5+18\,a^6\left(8+5\,a^4\right)\lambda^6-36\,a^9\lambda^7+9\,a^8\lambda^8\right)\right)\Big/\left(3\left(192+224\,a^4+56\,a^8+3\,a^{12}-a^3\left(144+116\,a^4+9\,a^8\right)\lambda+8\,a^2\left(18+19\,a^4+3\,a^8\right)\lambda^2-3\,a^5\left(24+11\,a^4\right)\lambda^3+12\,a^4\left(3+2\,a^4\right)\lambda^4-9\,a^7\lambda^5+3\,a^6\lambda^6\right)^2\right)$$

Code for the Proof of Theorem 9

In[1]:=
```
Element[a | θ, Reals]
```
Out[1]=
$$(a \mid \theta) \in \mathbb{R}$$

In[2]:=
```
(*Initialize coordinates of the triangle.*)
Tx[a_, θ_] := (-a + (2 / (a * Tan[π - θ]))) / 3
Ty[a_, θ_] := (2 / (3 a))
X1[a_, θ_] := -a - Tx[a, θ]
X2[a_, θ_] := -Tx[a, θ]
X3[a_, θ_] := (-2 / (a * Tan[θ])) - Tx[a, θ]
Y1[a_, θ_] := -Ty[a, θ]
Y2[a_, θ_] := -Ty[a, θ]
Y3[a_, θ_] := (2 / a) - Ty[a, θ]
M1[a_, θ_] := (Y3[a, θ] - Y1[a, θ]) / (X3[a, θ] - X1[a, θ])
M2[a_, θ_] := (Y3[a, θ] - Y2[a, θ]) / (X3[a, θ] - X2[a, θ])
B1[a_, θ_] := a * Sin[π - θ] - M1[a, θ] * (a * Cos[π - θ] + Tx[a, θ])
B2[a_, θ_] := a * Sin[π - θ] - M2[a, θ] * a * Cos[π - θ] - Ty[a, θ]
```

In[14]:=
```
(*Formulas for the sides of the triangle,
which will serve as limits of integration.*)
Line3[a_, θ_, x_] := M1[a, θ] * (x + a + Tx[a, θ]) - Ty[a, θ]
Line2[a_, θ_, x_] := M2[a, θ] * (x + Tx[a, θ]) - Ty[a, θ]
Line1[a_, θ_, x_] := -Ty[a, θ]
```

In[17]:=
```
(*Calculates a moment of area.*)
c[a_, θ_, m_, n_] :=
 Integrate[x^m * y^n, {x, X1[a, θ], X2[a, θ]}, {y, Y1[a, θ], Line3[a, θ, x]}] +
  Integrate[x^m * y^n, {x, X2[a, θ], X3[a, θ]}, {y, Line2[a, θ, x], Line3[a, θ, x]}]
```

In[18]:=
```
(*Formula for ρ1*)
Rho1[a_, θ_] :=
 4 * ((c[a, θ, 2, 0] × c[a, θ, 0, 2] - c[a, θ, 1, 1]^2) / (c[a, θ, 2, 0] + c[a, θ, 0, 2]))
```

In[19]:=
```
Rho1[a, θ]
```

Out[19]=
$$\left(4\left(-\left(\frac{a^6}{18\left(a^3-2\,a\,\text{Cot}[\theta]\right)^2}+\frac{8\,a^2\,\text{Cot}[\theta]^2}{9\left(a^3-2\,a\,\text{Cot}[\theta]\right)^2}-\frac{2\,\text{Cot}[\theta]\left(2\,a^4-a^2\,\text{Cot}[\theta]+4\,\text{Cot}[\theta]^2\right)}{9\left(a^3-2\,a\,\text{Cot}[\theta]\right)^2}\right)^2+\left(-\frac{4\,a^2\,\text{Cot}[\theta]}{3\left(a^2-2\,\text{Cot}[\theta]\right)^3}+\frac{8\,\text{Cot}[\theta]^2}{9\left(a^2-2\,\text{Cot}[\theta]\right)^3}-\frac{16\,\text{Cot}[\theta]^3}{9\,a^2\left(a^2-2\,\text{Cot}[\theta]\right)^3}+\frac{2\,a^7}{9\left(a^3-2\,a\,\text{Cot}[\theta]\right)^3}+\frac{16\,a^3\,\text{Cot}[\theta]^2}{9\left(a^3-2\,a\,\text{Cot}[\theta]\right)^3}\right)\left(-\frac{2\,\text{Cot}[\theta]\left(a^4+2\,\text{Cot}[\theta]^2\right)}{9\,a^2\left(a^2-2\,\text{Cot}[\theta]\right)}+\frac{a^4+8\,\text{Cot}[\theta]^2}{18\,a^2-36\,\text{Cot}[\theta]}\right)\right)\Big/\left(-\frac{4\,a^2\,\text{Cot}[\theta]}{3\left(a^2-2\,\text{Cot}[\theta]\right)^3}+\frac{8\,\text{Cot}[\theta]^2}{9\left(a^2-2\,\text{Cot}[\theta]\right)^3}-\frac{16\,\text{Cot}[\theta]^3}{9\,a^2\left(a^2-2\,\text{Cot}[\theta]\right)^3}+\frac{2\,a^7}{9\left(a^3-2\,a\,\text{Cot}[\theta]\right)^3}+\frac{16\,a^3\,\text{Cot}[\theta]^2}{9\left(a^3-2\,a\,\text{Cot}[\theta]\right)^3}-\frac{2\,\text{Cot}[\theta]\left(a^4+2\,\text{Cot}[\theta]^2\right)}{9\,a^2\left(a^2-2\,\text{Cot}[\theta]\right)}+\frac{a^4+8\,\text{Cot}[\theta]^2}{18\,a^2-36\,\text{Cot}[\theta]}\right)$$

In[20]:=
```
(*This yields equation (7) in the paper.*)
FullSimplify[
 (4 (-((a^6/(18 (a^3 - 2 a Cot[θ])^2) + (8 a^2 Cot[θ]^2)/(9 (a^3 - 2 a Cot[θ])^2) -
        (2 Cot[θ] (2 a^4 - a^2 Cot[θ] + 4 Cot[θ]^2))/(9 (a^3 - 2 a Cot[θ])^2))^2) +
    (-((4 a^2 Cot[θ])/(3 (a^2 - 2 Cot[θ])^3)) + (8 Cot[θ]^2)/(9 (a^2 - 2 Cot[θ])^3) -
     (16 Cot[θ]^3)/(9 a^2 (a^2 - 2 Cot[θ])^3) + (2 a^7)/(9 (a^3 - 2 a Cot[θ])^3) +
     (16 a^3 Cot[θ]^2)/(9 (a^3 - 2 a Cot[θ])^3)) (-((2 Cot[θ] (a^4 + 2 Cot[θ]^2))/
       (9 a^2 (a^2 - 2 Cot[θ]))) + (a^4 + 8 Cot[θ]^2)/(18 a^2 - 36 Cot[θ])))/
  (-((4 a^2 Cot[θ])/(3 (a^2 - 2 Cot[θ])^3)) + (8 Cot[θ]^2)/(9 (a^2 - 2 Cot[θ])^3) -
   (16 Cot[θ]^3)/(9 a^2 (a^2 - 2 Cot[θ])^3) + (2 a^7)/(9 (a^3 - 2 a Cot[θ])^3) +
   (16 a^3 Cot[θ]^2)/(9 (a^3 - 2 a Cot[θ])^3) - (2 Cot[θ] (a^4 + 2 Cot[θ]^2))/
    (9 a^2 (a^2 - 2 Cot[θ])) + (a^4 + 8 Cot[θ]^2)/(18 a^2 - 36 Cot[θ])))]
```

Out[20]=
$$\frac{1}{\frac{3\,a^2}{2}-3\,\text{Cot}[\theta]+\frac{6\,\text{Csc}[\theta]^2}{a^2}}$$

In[21]:=
```
Rh1[a_, θ_] := 1/(3 a^2/2 - 3 Cot[θ] + (6 Csc[θ]^2)/a^2)
```

In[22]:=
```
D[(2 a^2)/(3 a^4 - 6 a^2 Cot[θ] + 4 (Csc[θ])^2), a]
```

Out[22]=
$$-\frac{2\,a^2\left(12\,a^3-12\,a\,\text{Cot}[\theta]\right)}{\left(3\,a^4-6\,a^2\,\text{Cot}[\theta]+4\,\text{Csc}[\theta]^2\right)^2}+\frac{4\,a}{3\,a^4-6\,a^2\,\text{Cot}[\theta]+4\,\text{Csc}[\theta]^2}$$

In[23]:=
```
FullSimplify[D[Rh1[a, θ], a]]
```

Out[23]=
$$-\frac{4\,a\left(a^4-4\,\text{Csc}[\theta]^2\right)}{3\left(a^4-2\,a^2\,\text{Cot}[\theta]+4\,\text{Csc}[\theta]^2\right)^2}$$

In[24]:=
```
Solve[4 a (a^4 - 4 Csc[θ]^2) == 0, a]
```

Out[24]=
$$\left\{\{a\to 0\},\left\{a\to -\sqrt{2}\ \sqrt{\text{Csc}[\theta]}\right\},\left\{a\to -i\ \sqrt{2}\ \sqrt{\text{Csc}[\theta]}\right\},\left\{a\to i\ \sqrt{2}\ \sqrt{\text{Csc}[\theta]}\right\},\left\{a\to \sqrt{2}\ \sqrt{\text{Csc}[\theta]}\right\}\right\}$$

In[25]:=
```
(*Calculate ρ2 according to the formula in Proposition 4.*)
A[a_, θ_] :=
 4 * (c[a, θ, 0, 4] × c[a, θ, 1, 1] ^ 2 - 4 c[a, θ, 1, 1] ^ 4 - 2 c[a, θ, 0, 3] × c[a, θ, 1, 1] ×
      c[a, θ, 1, 2] + c[a, θ, 0, 2] ^ 3 × c[a, θ, 2, 0] + c[a, θ, 0, 3] ^ 2 × c[a, θ, 2, 0] +
     4 c[a, θ, 1, 2] ^ 2 × c[a, θ, 2, 0] - c[a, θ, 1, 1] ^ 2 × c[a, θ, 2, 0] ^ 2 -
     c[a, θ, 0, 2] ^ 2 (c[a, θ, 1, 1] ^ 2 + 2 c[a, θ, 2, 0] ^ 2) -
     6 c[a, θ, 1, 1] × c[a, θ, 1, 2] × c[a, θ, 2, 1] -
     2 c[a, θ, 0, 3] × c[a, θ, 2, 0] × c[a, θ, 2, 1] + c[a, θ, 2, 0] × c[a, θ, 2, 1] ^ 2 +
     2 c[a, θ, 1, 1] ^ 2 × c[a, θ, 2, 2] + 2 c[a, θ, 0, 3] × c[a, θ, 1, 1] × c[a, θ, 3, 0] -
     2 c[a, θ, 1, 1] × c[a, θ, 2, 1] × c[a, θ, 3, 0] +
     c[a, θ, 0, 2] (4 c[a, θ, 2, 1] ^ 2 + (c[a, θ, 1, 2] - c[a, θ, 3, 0]) ^ 2 +
        c[a, θ, 2, 0] (-c[a, θ, 0, 4] + 6 c[a, θ, 1, 1] ^ 2 + c[a, θ, 2, 0] ^ 2 -
           2 c[a, θ, 2, 2] - c[a, θ, 4, 0])) + c[a, θ, 1, 1] ^ 2 × c[a, θ, 4, 0])
B[a_, θ_] := (c[a, θ, 0, 3] + c[a, θ, 2, 1]) ^ 2 + (c[a, θ, 1, 2] + c[a, θ, 3, 0]) ^ 2 +
  (c[a, θ, 0, 2] + c[a, θ, 2, 0]) (-c[a, θ, 0, 4] + 4 c[a, θ, 1, 1] ^ 2 +
     (c[a, θ, 0, 2] - c[a, θ, 2, 0]) ^ 2 - 2 c[a, θ, 2, 2] - c[a, θ, 4, 0])
```

In[27]:=
```
Rho2[a_, θ_] := A[a, θ] / B[a, θ]
```

In[28]:=
```
FullSimplify[Rho2[a, θ]]
```

Out[28]=
$$\left(8\,a^2\,\text{Sin}[\theta]^2\left(-384-96\,a^4-9\,a^8+4\,a^4\left(20+3\,a^4\right)\text{Cos}[2\,\theta]+a^4\left(16-3\,a^4\right)\text{Cos}[4\,\theta]+24\,a^2\left(8+a^4\right)\text{Sin}[2\,\theta]-12\,a^6\,\text{Sin}[4\,\theta]\right)\right)/\left(3\left(-2\left(3072+2112\,a^4+376\,a^8+15\,a^{12}\right)+a^4\left(3584+936\,a^4+45\,a^8\right)\text{Cos}[2\,\theta]-2\,a^4\left(-320+9\,a^4\left(8+a^4\right)\right)\text{Cos}[4\,\theta]+a^8\left(-40+3\,a^4\right)\text{Cos}[6\,\theta]+2\,a^2\left(2304+976\,a^4+45\,a^8\right)\text{Sin}[2\,\theta]-8\,a^6\left(116+9\,a^4\right)\text{Sin}[4\,\theta]+2\,a^6\left(-16+9\,a^4\right)\text{Sin}[6\,\theta]\right)\right)$$

In[29]:=
```
(*The previous expression simplifies to this one.*)
Rh2[a_, θ_] := (-32 a^6 + 6 a^10 + 72 a^6 Csc[θ]^2 + 96 a^2 Csc[θ]^4 - 24 Cot[θ] (a^8 + 4 a^4 Csc[θ]^2)) /
   (3 (a^4 + 4 Csc[θ]^2) (-40 a^4 + 3 a^8 + 84 a^4 Csc[θ]^2 + 48 Csc[θ]^4) -
     6 a^2 Cot[θ] (-16 a^4 + 9 a^8 + 132 a^4 Csc[θ]^2 + 144 Csc[θ]^4));
```

In[32]:=
```
D[Rh2[a, θ], a]
```

Out[32]=
$$-\Big(\big(\big(-32\,a^6+6\,a^{10}+72\,a^6\,\text{Csc}[\theta]^2+96\,a^2\,\text{Csc}[\theta]^4-24\,\text{Cot}[\theta]\,\big(a^8+4\,a^4\,\text{Csc}[\theta]^2\big)\big)\big(3\,\big(a^4+4\,\text{Csc}[\theta]^2\big)\,\big(-160\,a^3+24\,a^7+336\,a^3\,\text{Csc}[\theta]^2\big)-6\,a^2\,\text{Cot}[\theta]\,\big(-64\,a^3+72\,a^7+528\,a^3\,\text{Csc}[\theta]^2\big)+12\,a^3\,\big(-40\,a^4+3\,a^8+84\,a^4\,\text{Csc}[\theta]^2+48\,\text{Csc}[\theta]^4\big)-12\,a\,\text{Cot}[\theta]\,\big(-16\,a^4+9\,a^8+132\,a^4\,\text{Csc}[\theta]^2+144\,\text{Csc}[\theta]^4\big)\big)\big)\Big/\big(3\,\big(a^4+4\,\text{Csc}[\theta]^2\big)\,\big(-40\,a^4+3\,a^8+84\,a^4\,\text{Csc}[\theta]^2+48\,\text{Csc}[\theta]^4\big)-6\,a^2\,\text{Cot}[\theta]\,\big(-16\,a^4+9\,a^8+132\,a^4\,\text{Csc}[\theta]^2+144\,\text{Csc}[\theta]^4\big)\big)^2\Big)+\big(-192\,a^5+60\,a^9+432\,a^5\,\text{Csc}[\theta]^2+192\,a\,\text{Csc}[\theta]^4-24\,\text{Cot}[\theta]\,\big(8\,a^7+16\,a^3\,\text{Csc}[\theta]^2\big)\big)\Big/\big(3\,\big(a^4+4\,\text{Csc}[\theta]^2\big)\,\big(-40\,a^4+3\,a^8+84\,a^4\,\text{Csc}[\theta]^2+48\,\text{Csc}[\theta]^4\big)-6\,a^2\,\text{Cot}[\theta]\,\big(-16\,a^4+9\,a^8+132\,a^4\,\text{Csc}[\theta]^2+144\,\text{Csc}[\theta]^4\big)\big)$$

In[33]:=
```
FullSimplify[
 -(((-32 a^6 + 6 a^10 + 72 a^6 Csc[θ]^2 + 96 a^2 Csc[θ]^4 - 24 Cot[θ] (a^8 + 4 a^4 Csc[θ]^2))
      (3 (a^4 + 4 Csc[θ]^2) (-160 a^3 + 24 a^7 + 336 a^3 Csc[θ]^2) - 6 a^2 Cot[θ] (-64 a^3 +
          72 a^7 + 528 a^3 Csc[θ]^2) + 12 a^3 (-40 a^4 + 3 a^8 + 84 a^4 Csc[θ]^2 + 48 Csc[θ]^4) -
       12 a Cot[θ] (-16 a^4 + 9 a^8 + 132 a^4 Csc[θ]^2 + 144 Csc[θ]^4)))/
    (3 (a^4 + 4 Csc[θ]^2) (-40 a^4 + 3 a^8 + 84 a^4 Csc[θ]^2 + 48 Csc[θ]^4) -
      6 a^2 Cot[θ] (-16 a^4 + 9 a^8 + 132 a^4 Csc[θ]^2 + 144 Csc[θ]^4))^2) +
  (-192 a^5 + 60 a^9 + 432 a^5 Csc[θ]^2 + 192 a Csc[θ]^4 - 24 Cot[θ] (8 a^7 + 16 a^3 Csc[θ]^2))/
   (3 (a^4 + 4 Csc[θ]^2) (-40 a^4 + 3 a^8 + 84 a^4 Csc[θ]^2 + 48 Csc[θ]^4) -
     6 a^2 Cot[θ] (-16 a^4 + 9 a^8 + 132 a^4 Csc[θ]^2 + 144 Csc[θ]^4)))]
```

Out[33]=
$$\big(4\,a\,\big(a^2-2\,\text{Csc}[\theta]\big)\,\big(a^2+2\,\text{Csc}[\theta]\big)\,\big(a^8\,\big(-256+240\,a^4-9\,a^8\big)+24\,a^2\,\text{Cot}[\theta]\,\big(a^4+4\,\text{Csc}[\theta]^2\big)\,\big(-16\,a^4+3\,a^8+12\,a^4\,\text{Csc}[\theta]^2+48\,\text{Csc}[\theta]^4\big)-48\,\text{Csc}[\theta]^2\,\big(2\,a^8\,\big(-16+3\,a^4\big)+a^4\,\big(-80+27\,a^4\big)\,\text{Csc}[\theta]^2+96\,a^4\,\text{Csc}[\theta]^4+48\,\text{Csc}[\theta]^6\big)\big)\Big/\big(3\,\big(\big(a^4+4\,\text{Csc}[\theta]^2\big)\,\big(-40\,a^4+3\,a^8+84\,a^4\,\text{Csc}[\theta]^2+48\,\text{Csc}[\theta]^4\big)-2\,a^2\,\text{Cot}[\theta]\,\big(-16\,a^4+9\,a^8+132\,a^4\,\text{Csc}[\theta]^2+144\,\text{Csc}[\theta]^4\big)\big)^2\big)$$

In[34]:=
```
(*The polynomial P_θ from the proof of Theorem 9.*)
P1[a_, θ_] := 4 a (a^2 - 2 Csc[θ]) (a^2 + 2 Csc[θ]) (a^8 (-256 + 240 a^4 - 9 a^8) +
    24 a^2 Cot[θ] (a^4 + 4 Csc[θ]^2) (-16 a^4 + 3 a^8 + 12 a^4 Csc[θ]^2 + 48 Csc[θ]^4) -
    48 Csc[θ]^2 (2 a^8 (-16 + 3 a^4) + a^4 (-80 + 27 a^4) Csc[θ]^2 + 96 a^4 Csc[θ]^4 + 48 Csc[θ]^6))
```

In[35]:=
```
S[a_, θ_] := P1[a + Sqrt[2 Csc[θ]], θ];
```

In[36]:=
```
FullSimplify[S[a, θ]]
```

Out[36]=
$$4\,a\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)\left(a+2\,\sqrt{2}\,\sqrt{\csc[\theta]}\right)\left(\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^2+2\csc[\theta]\right)$$
$$\left(\left(-256+240\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4-9\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^8\right)\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^8+\right.$$
$$24\cot[\theta]\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^2\left(\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4+4\csc[\theta]^2\right)\left(-16\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4+\right.$$
$$3\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^8+12\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4\csc[\theta]^2+48\csc[\theta]^4\Big)-48\csc[\theta]^2$$
$$\left(2\left(-16+3\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4\right)\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^8+\left(-80+27\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4\right)\right.$$
$$\left.\left.\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4\csc[\theta]^2+96\left(a+\sqrt{2}\,\sqrt{\csc[\theta]}\right)^4\csc[\theta]^4+48\csc[\theta]^6\right)\right)$$

```
(*All the coefficients of S[a_,θ_].  One can plot each of these
 functions on the interval [0,π] to verify that they are negative.*)
```

In[37]:=
```
FullSimplify[Coefficient[S[a, θ], a]]
```

Out[37]=
$$-16\,384\,(141-192\cos[\theta]+100\cos[2\theta]-24\cos[3\theta]+2\cos[4\theta])\csc[\theta]^{10}$$

In[38]:=
```
FullSimplify[Coefficient[S[a, θ], a^2]]
```

Out[38]=
$$-86\,016\,\sqrt{2}\,(141-192\cos[\theta]+100\cos[2\theta]-24\cos[3\theta]+2\cos[4\theta])\csc[\theta]^{19/2}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^3]]
```

Out[•]=
$$-4096\,(16\,737-22\,464\cos[\theta]+11\,060\cos[2\theta]-2520\cos[3\theta]+202\cos[4\theta])\csc[\theta]^{9}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^4]]
```

Out[•]=
$$-19\,456\,\sqrt{2}\,(6867-9024\cos[\theta]+4060\cos[2\theta]-840\cos[3\theta]+62\cos[4\theta])\csc[\theta]^{17/2}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^5]]
```

Out[•]=
$$-3072\,(127\,943-163\,392\cos[\theta]+65\,740\cos[2\theta]-11\,928\cos[3\theta]+774\cos[4\theta])\csc[\theta]^{8}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^6]]
```

Out[•]=
$$-208\,896\,\sqrt{2}\,(2186-2688\cos[\theta]+950\cos[2\theta]-147\cos[3\theta]+8\cos[4\theta])\csc[\theta]^{15/2}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^7]]
```

Out[•]=
$$-12\,288\,(70\,136-82\,016\cos[\theta]+25\,010\cos[2\theta]-3217\cos[3\theta]+140\cos[4\theta])\csc[\theta]^{7}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^8]]
```

Out[•]=
$$-1536\,\sqrt{2}$$
$$(438\,301-479\,232\cos[\theta]+123\,680\cos[2\theta]-12\,870\cos[3\theta]+426\cos[4\theta])\csc[\theta]^{13/2}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^9]]
```

Out[•]=
$$-1536\,(574\,243-574\,288\cos[\theta]+122\,740\cos[2\theta]-10\,010\cos[3\theta]+238\cos[4\theta])\csc[\theta]^{6}$$

In[•]:=
```
FullSimplify[Coefficient[S[a, θ], a^(10)]]
```

Out[•]=
$$-3328\,\sqrt{2}\,(146\,745-130\,680\cos[\theta]+22\,550\cos[2\theta]-1386\cos[3\theta]+22\cos[4\theta])\csc[\theta]^{11/2}$$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(11)]]**

Out[•]= $-9984\,(45\,931-35\,304\,\text{Cos}[\theta]+4770\,\text{Cos}[2\,\theta]-210\,\text{Cos}[3\,\theta]+2\,\text{Cos}[4\,\theta])\,\text{Csc}[\theta]^{5}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(12)]]**

Out[•]= $-832\,\sqrt{2}\,(219\,195-140\,376\,\text{Cos}[\theta]+14\,290\,\text{Cos}[2\,\theta]-420\,\text{Cos}[3\,\theta]+2\,\text{Cos}[4\,\theta])\,\text{Csc}[\theta]^{9/2}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(13)]]**

Out[•]= $-64\,(1\,909\,947-979\,272\,\text{Cos}[\theta]+71\,410\,\text{Cos}[2\,\theta]-1260\,\text{Cos}[3\,\theta]+2\,\text{Cos}[4\,\theta])\,\text{Csc}[\theta]^{4}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(14)]]**

Out[•]= $384\,\sqrt{2}\,(34\,914\,\text{Cos}[\theta]-340\,(262+5\,\text{Cos}[2\,\theta])+15\,\text{Cos}[3\,\theta])\,\text{Csc}[\theta]^{7/2}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(15)]]**

Out[•]= $384\,(11\,630\,\text{Cos}[\theta]-68\,(604+5\,\text{Cos}[2\,\theta])+\text{Cos}[3\,\theta])\,\text{Csc}[\theta]^{3}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(16)]]**

Out[•]= $-1632\,\sqrt{2}\,(1801-342\,\text{Cos}[\theta]+5\,\text{Cos}[2\,\theta])\,\text{Csc}[\theta]^{5/2}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(17)]]**

Out[•]= $192\left(5+513\,\text{Cot}[\theta]\,\text{Csc}[\theta]-4494\,\text{Csc}[\theta]^{2}\right)$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(18)]]**

Out[•]= $2736\,\sqrt{2}\,(-35+2\,\text{Cos}[\theta])\,\text{Csc}[\theta]^{3/2}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(19)]]**

Out[•]= $144\,(-105+2\,\text{Cos}[\theta])\,\text{Csc}[\theta]$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(20)]]**

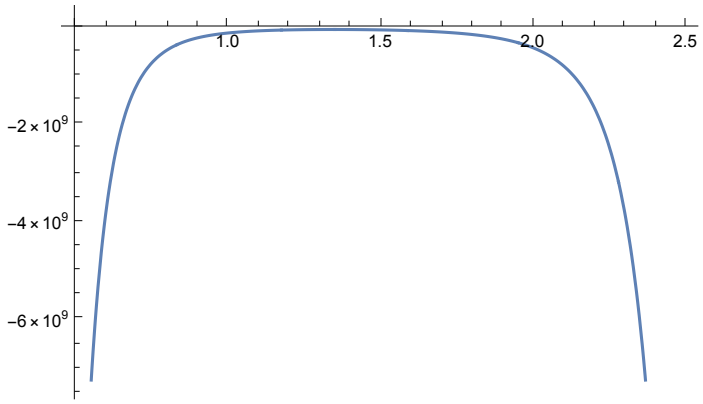
Out[•]= $-756\,\sqrt{2}\,\sqrt{\text{Csc}[\theta]}$

In[•]:= **FullSimplify[Coefficient[S[a, θ], a^(21)]]**

Out[•]= -36

```
(*For example, a plot of the coefficient of a^4.*)
Plot[-19 456 √2
  (6867 - 9024 Cos[θ] + 4060 Cos[2 θ] - 840 Cos[3 θ] + 62 Cos[4 θ]) Csc[θ]^(17/2), {θ, .5, 2.5}]
```

Out[ ]=



Code for the Creation of Figures 9 ad 10

In[50]:=
```
X1[θ_, ϕ_] := Sin[π - θ Degree] + .5 * (Abs[Sin[π - ϕ Degree] - Sin[π - θ Degree]]);

Y1[θ_, ϕ_] := -Cos[π - θ Degree] + .5 * (Abs[1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]]);

L1[θ_, ϕ_] := N[Abs[Sqrt[(1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]) ^ 2 +
       (Sin[π - ϕ Degree] - Sin[π - θ Degree]) ^ 2]]];

H1[θ_, ϕ_] := N[Abs[Sqrt[1 - (L1[θ, ϕ] / 2) ^ 2]]];

M1[θ_, ϕ_] := If[Sin[π - θ] == Sin[π - ϕ], 0,
   (1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]) / (Sin[π - ϕ Degree] - Sin[π - θ Degree])];

MHat1[θ_, ϕ_] := If[M1[θ, ϕ] == 0, 0, -1 / (M1[θ, ϕ])];

α1[θ_, ϕ_] := ArcTan[MHat1[θ, ϕ]];

Xa := 0;
Ya := 0;

Xa1 := 0;
Ya1 := 1;

Xb[θ_, ϕ_] := Sin[π - θ Degree];
Yb[θ_, ϕ_] := -Cos[π - θ Degree];

Xc[θ_, ϕ_] := Sin[π - ϕ Degree];
Yc[θ_, ϕ_] := 1 + Cos[π - ϕ Degree];

Xd[θ_, ϕ_] := X1[θ, ϕ] + H1[θ, ϕ] * Cos[α1[θ, ϕ]];
Yd[θ_, ϕ_] := Y1[θ, ϕ] + H1[θ, ϕ] * Sin[α1[θ, ϕ]];

Centroid[θ_, ϕ_] := RegionCentroid[Polygon[{{Xa, Ya}, {Xa1, Ya1},
     {Xc[θ, ϕ], Yc[θ, ϕ]}, {Xd[θ, ϕ], Yd[θ, ϕ]}, {Xb[θ, ϕ], Yb[θ, ϕ]}}]];

Tx[θ_, ϕ_] := Centroid[θ, ϕ][[1]];
Ty[θ_, ϕ_] := Centroid[θ, ϕ][[2]];

LineA[θ_, ϕ_, x_] := ((Yb[θ, ϕ] - Ya) / (Xb[θ, ϕ] - Xa)) * x;

LineB[θ_, ϕ_, x_] := ((Yc[θ, ϕ] - Ya1) / (Xc[θ, ϕ] - Xa1)) * x + Ya1;

LineC[θ_, ϕ_, x_] :=
  ((Yd[θ, ϕ] - Yb[θ, ϕ]) / (Xd[θ, ϕ] - Xb[θ, ϕ])) * (x - Xd[θ, ϕ]) + Yd[θ, ϕ];
```

```
LineD[θ_, ϕ_, x_] :=
  ((Yd[θ, ϕ] - Yc[θ, ϕ]) / (Xd[θ, ϕ] - Xc[θ, ϕ])) * (x - Xd[θ, ϕ]) + Yd[θ, ϕ];

CentX1[θ_, ϕ_] :=
  Sin[π - θ Degree] + .5 * (Abs[Sin[π - ϕ Degree] - Sin[π - θ Degree]]) - Tx[θ, ϕ];

CentY1[θ_, ϕ_] :=
  - Cos[π - θ Degree] + .5 * (Abs[1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]]) - Ty[θ, ϕ];

CentXa[θ_, ϕ_] := 0 - Tx[θ, ϕ];
CentYa[θ_, ϕ_] := 0 - Ty[θ, ϕ];

CentXa1[θ_, ϕ_] := 0 - Tx[θ, ϕ];
CentYa1[θ_, ϕ_] := 1 - Ty[θ, ϕ];

CentXb[θ_, ϕ_] := Sin[π - θ Degree] - Tx[θ, ϕ];
CentYb[θ_, ϕ_] := - Cos[π - θ Degree] - Ty[θ, ϕ];

CentXc[θ_, ϕ_] := Sin[π - ϕ Degree] - Tx[θ, ϕ];
CentYc[θ_, ϕ_] := 1 + Cos[π - ϕ Degree] - Ty[θ, ϕ];

CentXd[θ_, ϕ_] := X1[θ, ϕ] + H1[θ, ϕ] * Cos[α1[θ, ϕ]] - Tx[θ, ϕ];
CentYd[θ_, ϕ_] := Y1[θ, ϕ] + H1[θ, ϕ] * Sin[α1[θ, ϕ]] - Ty[θ, ϕ];

CentLineA[θ_, ϕ_, x_] := ((Yb[θ, ϕ] - Ya) / (Xb[θ, ϕ] - Xa)) * (x + Tx[θ, ϕ]) - Ty[θ, ϕ];

CentLineB[θ_, ϕ_, x_] :=
  ((Yc[θ, ϕ] - Ya1) / (Xc[θ, ϕ] - Xa1)) * (x + Tx[θ, ϕ]) + Ya1 - Ty[θ, ϕ];

CentLineC[θ_, ϕ_, x_] := ((Yd[θ, ϕ] - Yb[θ, ϕ]) / (Xd[θ, ϕ] - Xb[θ, ϕ])) *
    ((x + Tx[θ, ϕ]) - Xd[θ, ϕ]) + Yd[θ, ϕ] - Ty[θ, ϕ];

CentLineD[θ_, ϕ_, x_] := ((Yd[θ, ϕ] - Yc[θ, ϕ]) / (Xd[θ, ϕ] - Xc[θ, ϕ])) *
    ((x + Tx[θ, ϕ]) - Xd[θ, ϕ]) + Yd[θ, ϕ] - Ty[θ, ϕ];

Ic[m_, n_, θ_, ϕ_] := Integrate[x^m * y^n,
    {x, CentXa[θ, ϕ], CentXb[θ, ϕ]}, {y, CentLineA[θ, ϕ, x], CentLineB[θ, ϕ, x]}] +
   Integrate[x^m * y^n, {x, CentXb[θ, ϕ], CentXc[θ, ϕ]},
    {y, CentLineC[θ, ϕ, x], CentLineB[θ, ϕ, x]}] + Integrate[x^m * y^n,
    {x, CentXc[θ, ϕ], CentXd[θ, ϕ]}, {y, CentLineC[θ, ϕ, x], CentLineD[θ, ϕ, x]}];

Rho1[θ_, ϕ_] := (4 * ((Ic[2, 0, θ, ϕ] × Ic[0, 2, θ, ϕ]) - (Ic[1, 1, θ, ϕ])^2) /
      (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])) / Area[Pol[θ, ϕ]]^2;
```

```
c[θ_, ϕ_][m_, n_] := NIntegrate[(x + I * y) ^ m * (x - I * y) ^ n,
     {x, CentXa[θ, ϕ], CentXb[θ, ϕ]}, {y, CentLineA[θ, ϕ, x], CentLineB[θ, ϕ, x]}] +
    NIntegrate[(x + I * y) ^ m * (x - I * y) ^ n, {x, CentXb[θ, ϕ], CentXc[θ, ϕ]},
     {y, CentLineC[θ, ϕ, x], CentLineB[θ, ϕ, x]}] +
    NIntegrate[(x + I * y) ^ m * (x - I * y) ^ n, {x, CentXc[θ, ϕ], CentXd[θ, ϕ]},
     {y, CentLineC[θ, ϕ, x], CentLineD[θ, ϕ, x]}];

WbarR[n_, θ_, ϕ_] :=
  Join[Array[c[θ, ϕ], {n, n + 1}, 0], Array[c[θ, ϕ], {1, n + 1}, {0, 1}]];

Sigma[n_, θ_, ϕ_] := Join[Transpose[Array[c[θ, ϕ], {n, n}, {0, 0}]]];

WR[n_, θ_, ϕ_] := Array[c[θ, ϕ], {n + 1, n + 1}, 0];

IP[n_, θ_, ϕ_] :=
  (Det[Abs[WbarR[n, θ, ϕ]]]) ^ 2 / (Det[Sigma[n, θ, ϕ]] * Det[WR[n, θ, ϕ]]);

Rho[n_, θ_, ϕ_] := (c[θ, ϕ][1, 1] - Sum[IP[j, θ, ϕ], {j, 1, n}]) / Area[Pol[θ, ϕ]] ^ 2;

Pol[θ_, ϕ_] := Polygon[{{CentXa[θ, ϕ], CentYa[θ, ϕ]},
     {CentXa1[θ, ϕ], CentYa1[θ, ϕ]}, {CentXc[θ, ϕ], CentYc[θ, ϕ]},
     {CentXd[θ, ϕ], CentYd[θ, ϕ]}, {CentXb[θ, ϕ], CentYb[θ, ϕ]}}];
```

In[•]:=

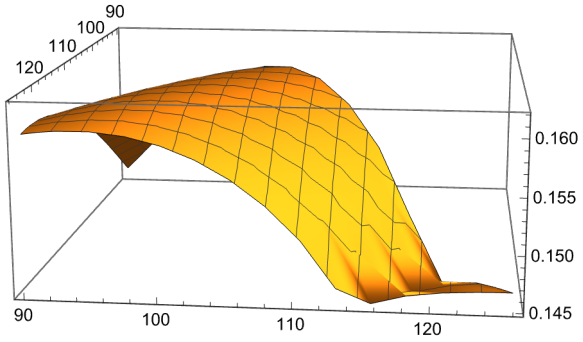
```
A[θ_, ϕ_] :=
 ((Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) / ((Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) (Ic[4, 0, θ, ϕ] + 2 Ic[2, 2, θ, ϕ] + Ic[0, 4, θ, ϕ]) -
     (Ic[3, 0, θ, ϕ] ^ 2 + 2 Ic[3, 0, θ, ϕ] * Ic[1, 2, θ, ϕ] + Ic[2, 1, θ, ϕ] ^ 2 +
       2 Ic[2, 1, θ, ϕ] * Ic[0, 3, θ, ϕ] + Ic[1, 2, θ, ϕ] ^ 2 + Ic[0, 3, θ, ϕ] ^ 2) - (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])
      (Ic[2, 0, θ, ϕ] ^ 2 - 2 Ic[2, 0, θ, ϕ] * Ic[0, 2, θ, ϕ] + 4 Ic[1, 1, θ, ϕ] ^ 2 + Ic[0, 2, θ, ϕ] ^ 2)))
```

In[101]:=
$$
\begin{aligned}
B[\theta\_, \phi\_] := {} & \mathrm{Ic}[3,0,\theta,\phi]^2 - 6\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi] + 9\,\mathrm{Ic}[1,2,\theta,\phi]^2 + \frac{\mathrm{Ic}[3,0,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \\
& \frac{2\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi]\,\mathrm{Ic}[2,0,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{\mathrm{Ic}[1,2,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \\
& \frac{2\,\mathrm{Ic}[3,0,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \frac{4\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi]\times\mathrm{Ic}[2,0,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \\
& \frac{2\,\mathrm{Ic}[1,2,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{\mathrm{Ic}[3,0,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \\
& \frac{2\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi]\,\mathrm{Ic}[0,2,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{\mathrm{Ic}[1,2,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \frac{2\,\mathrm{Ic}[3,0,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \\
& \frac{4\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi]\times\mathrm{Ic}[2,0,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \frac{6\,\mathrm{Ic}[1,2,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \frac{2\,\mathrm{Ic}[3,0,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} - \\
& \frac{4\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} - \frac{6\,\mathrm{Ic}[1,2,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + 9\,\mathrm{Ic}[2,1,\theta,\phi]^2 - \\
& 6\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi] + \mathrm{Ic}[0,3,\theta,\phi]^2 + \frac{\mathrm{Ic}[2,1,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \\
& \frac{2\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\,\mathrm{Ic}[2,0,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{\mathrm{Ic}[0,3,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \\
& \frac{4\,\mathrm{Ic}[3,0,\theta,\phi]^2\,\mathrm{Ic}[1,1,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{8\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi]\,\mathrm{Ic}[1,1,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \\
& \frac{4\,\mathrm{Ic}[1,2,\theta,\phi]^2\,\mathrm{Ic}[1,1,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \frac{2\,\mathrm{Ic}[2,1,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \\
& \frac{4\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\times\mathrm{Ic}[2,0,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \frac{2\,\mathrm{Ic}[0,3,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \\
& \frac{\mathrm{Ic}[2,1,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{2\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\,\mathrm{Ic}[0,2,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{\mathrm{Ic}[0,3,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} - \\
& \frac{6\,\mathrm{Ic}[2,1,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} - \frac{4\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\times\mathrm{Ic}[2,0,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \frac{2\,\mathrm{Ic}[0,3,\theta,\phi]^2\,\mathrm{Ic}[2,0,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} - \\
& \frac{8\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[1,1,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} - \frac{24\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[1,2,\theta,\phi]\times\mathrm{Ic}[1,1,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \\
& \frac{8\,\mathrm{Ic}[3,0,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\times\mathrm{Ic}[1,1,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} - \frac{8\,\mathrm{Ic}[1,2,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\times\mathrm{Ic}[1,1,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \\
& \frac{6\,\mathrm{Ic}[2,1,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \frac{4\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\times\mathrm{Ic}[0,2,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} - \frac{2\,\mathrm{Ic}[0,3,\theta,\phi]^2\,\mathrm{Ic}[0,2,\theta,\phi]}{\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi]} + \\
& \frac{4\,\mathrm{Ic}[2,1,\theta,\phi]^2\,\mathrm{Ic}[1,1,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{8\,\mathrm{Ic}[2,1,\theta,\phi]\times\mathrm{Ic}[0,3,\theta,\phi]\,\mathrm{Ic}[1,1,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2} + \frac{4\,\mathrm{Ic}[0,3,\theta,\phi]^2\,\mathrm{Ic}[1,1,\theta,\phi]^2}{(\mathrm{Ic}[2,0,\theta,\phi]+\mathrm{Ic}[0,2,\theta,\phi])^2}
\end{aligned}
$$

In[102]:=
$$
\mathrm{Rho2}[\theta\_, \phi\_] := \mathrm{Rho1}[\theta, \phi] - (\mathrm{A}[\theta, \phi] * \mathrm{B}[\theta, \phi])
$$

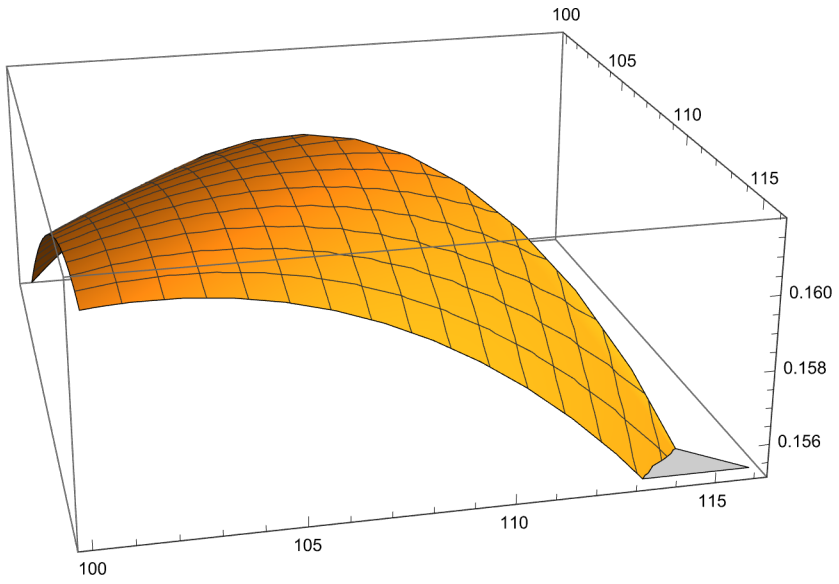
In[ ]:=
```
Plot3D[Rho1[θ, ϕ] / Area[Pol[θ, ϕ]]^2, {θ, 90, 126}, {ϕ, 90, θ}]
```

Out[ ]=




```
Plot3D[Rho[2, θ, ϕ], {θ, 100, 116}, {ϕ, 100, θ}]
```

Out[103]=



Code for the Calculation of ρ_{33} in Section 3

In[•]:=
```
X1[θ_, ϕ_] := Sin[π - θ Degree] + .5 * (Abs[Sin[π - ϕ Degree] - Sin[π - θ Degree]]);

Y1[θ_, ϕ_] := -Cos[π - θ Degree] + .5 * (Abs[1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]]);

L1[θ_, ϕ_] := N[Abs[Sqrt[(1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]) ^ 2 +
       (Sin[π - ϕ Degree] - Sin[π - θ Degree]) ^ 2]]];

H1[θ_, ϕ_] := N[Abs[Sqrt[1 - (L1[θ, ϕ] / 2) ^ 2]]];

M1[θ_, ϕ_] := If[Sin[π - θ] == Sin[π - ϕ], 0,
   (1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]) / (Sin[π - ϕ Degree] - Sin[π - θ Degree])];

MHat1[θ_, ϕ_] := If[M1[θ, ϕ] == 0, 0, -1 / (M1[θ, ϕ])];

α1[θ_, ϕ_] := ArcTan[MHat1[θ, ϕ]];

Xa := 0;
Ya := 0;

Xa1 := 0;
Ya1 := 1;

Xb[θ_, ϕ_] := Sin[π - θ Degree];
Yb[θ_, ϕ_] := -Cos[π - θ Degree];

Xc[θ_, ϕ_] := Sin[π - ϕ Degree];
Yc[θ_, ϕ_] := 1 + Cos[π - ϕ Degree];

Xd[θ_, ϕ_] := X1[θ, ϕ] + H1[θ, ϕ] * Cos[α1[θ, ϕ]];
Yd[θ_, ϕ_] := Y1[θ, ϕ] + H1[θ, ϕ] * Sin[α1[θ, ϕ]];

Centroid[θ_, ϕ_] := RegionCentroid[Polygon[{{Xa, Ya}, {Xa1, Ya1},
     {Xc[θ, ϕ], Yc[θ, ϕ]}, {Xd[θ, ϕ], Yd[θ, ϕ]}, {Xb[θ, ϕ], Yb[θ, ϕ]}}]];

Tx[θ_, ϕ_] := Centroid[θ, ϕ][[1]];
Ty[θ_, ϕ_] := Centroid[θ, ϕ][[2]];

LineA[θ_, ϕ_, x_] := ((Yb[θ, ϕ] - Ya) / (Xb[θ, ϕ] - Xa)) * x;

LineB[θ_, ϕ_, x_] := ((Yc[θ, ϕ] - Ya1) / (Xc[θ, ϕ] - Xa1)) * x + Ya1;

LineC[θ_, ϕ_, x_] :=
  ((Yd[θ, ϕ] - Yb[θ, ϕ]) / (Xd[θ, ϕ] - Xb[θ, ϕ])) * (x - Xd[θ, ϕ]) + Yd[θ, ϕ];
```

```
LineD[θ_, ϕ_, x_] :=
  ((Yd[θ, ϕ] - Yc[θ, ϕ]) / (Xd[θ, ϕ] - Xc[θ, ϕ])) * (x - Xd[θ, ϕ]) + Yd[θ, ϕ];

CentX1[θ_, ϕ_] :=
  Sin[π - θ Degree] + .5 * (Abs[Sin[π - ϕ Degree] - Sin[π - θ Degree]]) - Tx[θ, ϕ];

CentY1[θ_, ϕ_] :=
  -Cos[π - θ Degree] + .5 * (Abs[1 + Cos[π - ϕ Degree] + Cos[π - θ Degree]]) - Ty[θ, ϕ];

CentXa[θ_, ϕ_] := 0 - Tx[θ, ϕ];
CentYa[θ_, ϕ_] := 0 - Ty[θ, ϕ];

CentXa1[θ_, ϕ_] := 0 - Tx[θ, ϕ];
CentYa1[θ_, ϕ_] := 1 - Ty[θ, ϕ];

CentXb[θ_, ϕ_] := Sin[π - θ Degree] - Tx[θ, ϕ];
CentYb[θ_, ϕ_] := -Cos[π - θ Degree] - Ty[θ, ϕ];

CentXc[θ_, ϕ_] := Sin[π - ϕ Degree] - Tx[θ, ϕ];
CentYc[θ_, ϕ_] := 1 + Cos[π - ϕ Degree] - Ty[θ, ϕ];

CentXd[θ_, ϕ_] := X1[θ, ϕ] + H1[θ, ϕ] * Cos[α1[θ, ϕ]] - Tx[θ, ϕ];
CentYd[θ_, ϕ_] := Y1[θ, ϕ] + H1[θ, ϕ] * Sin[α1[θ, ϕ]] - Ty[θ, ϕ];

CentLineA[θ_, ϕ_, x_] := ((Yb[θ, ϕ] - Ya) / (Xb[θ, ϕ] - Xa)) * (x + Tx[θ, ϕ]) - Ty[θ, ϕ];

CentLineB[θ_, ϕ_, x_] :=
  ((Yc[θ, ϕ] - Ya1) / (Xc[θ, ϕ] - Xa1)) * (x + Tx[θ, ϕ]) + Ya1 - Ty[θ, ϕ];

CentLineC[θ_, ϕ_, x_] := ((Yd[θ, ϕ] - Yb[θ, ϕ]) / (Xd[θ, ϕ] - Xb[θ, ϕ])) *
    ((x + Tx[θ, ϕ]) - Xd[θ, ϕ]) + Yd[θ, ϕ] - Ty[θ, ϕ];

CentLineD[θ_, ϕ_, x_] := ((Yd[θ, ϕ] - Yc[θ, ϕ]) / (Xd[θ, ϕ] - Xc[θ, ϕ])) *
    ((x + Tx[θ, ϕ]) - Xd[θ, ϕ]) + Yd[θ, ϕ] - Ty[θ, ϕ];

Ic[m_, n_, θ_, ϕ_] := Integrate[x^m * y^n,
    {x, CentXa[θ, ϕ], CentXb[θ, ϕ]}, {y, CentLineA[θ, ϕ, x], CentLineB[θ, ϕ, x]}] +
   Integrate[x^m * y^n, {x, CentXb[θ, ϕ], CentXc[θ, ϕ]},
    {y, CentLineC[θ, ϕ, x], CentLineB[θ, ϕ, x]}] + Integrate[x^m * y^n,
    {x, CentXc[θ, ϕ], CentXd[θ, ϕ]}, {y, CentLineC[θ, ϕ, x], CentLineD[θ, ϕ, x]}];

Rho1[θ_, ϕ_] := (4 * ((Ic[2, 0, θ, ϕ] × Ic[0, 2, θ, ϕ]) - (Ic[1, 1, θ, ϕ])^2) /
       (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])) / Area[Pol[θ, ϕ]]^2;;
```

```
c[θ_, ϕ_][m_, n_] := NIntegrate[(x + I * y) ^ m * (x - I * y) ^ n,
     {x, CentXa[θ, ϕ], CentXb[θ, ϕ]}, {y, CentLineA[θ, ϕ, x], CentLineB[θ, ϕ, x]}] +
    NIntegrate[(x + I * y) ^ m * (x - I * y) ^ n, {x, CentXb[θ, ϕ], CentXc[θ, ϕ]},
     {y, CentLineC[θ, ϕ, x], CentLineB[θ, ϕ, x]}] +
    NIntegrate[(x + I * y) ^ m * (x - I * y) ^ n, {x, CentXc[θ, ϕ], CentXd[θ, ϕ]},
     {y, CentLineC[θ, ϕ, x], CentLineD[θ, ϕ, x]}];

WbarR[n_, θ_, ϕ_] :=
  Join[Array[c[θ, ϕ], {n, n + 1}, 0], Array[c[θ, ϕ], {1, n + 1}, {0, 1}]];

Sigma[n_, θ_, ϕ_] := Join[Transpose[Array[c[θ, ϕ], {n, n}, {0, 0}]]];

WR[n_, θ_, ϕ_] := Array[c[θ, ϕ], {n + 1, n + 1}, 0];

IP[n_, θ_, ϕ_] :=
  (Det[Abs[WbarR[n, θ, ϕ]]]) ^ 2 / (Det[Sigma[n, θ, ϕ]] * Det[WR[n, θ, ϕ]]);

Rho[n_, θ_, ϕ_] := (c[θ, ϕ][1, 1] - Sum[IP[j, θ, ϕ], {j, 1, n}]) / Area[Pol[θ, ϕ]] ^ 2;

Pol[θ_, ϕ_] := Polygon[{{CentXa[θ, ϕ], CentYa[θ, ϕ]},
     {CentXa1[θ, ϕ], CentYa1[θ, ϕ]}, {CentXc[θ, ϕ], CentYc[θ, ϕ]},
     {CentXd[θ, ϕ], CentYd[θ, ϕ]}, {CentXb[θ, ϕ], CentYb[θ, ϕ]}}];
```

Rho1 Regular Pentagon (area 1 centroid 0) came out to 0.161856
We want Rho_Infty Regular Pentagon roughly 0.14943

In[•]:=
```
m = 0;
n = 0;
i = 0;
j = 0;
For[m = 0, m ≤ 15, m++,
 For[n = 0, n ≤ 15, n++, For[i = 0, i ≤ 20, i++, For[j = 0, j ≤ 20, j++,
    If[i ≥ j, If[i == 10 && j == 10, c[107 + 0.1 * i, 107 + 0.1 * j][m, n] = c[108, 108][m, n],
      If[m > n, c[107 + 0.1 * i, 107 + 0.1 * j][m, n] =
        Conjugate[c[107 + 0.1 * i, 107 + 0.1 * j][n, m]],
       c[107 + 0.1 * i, 107 + 0.1 * j][m, n] = N[c[107 + 0.1 * i, 107 + 0.1 * j][m, n]]];
      Print[{"=θ" 107 + 0.1 * i, "=ϕ" 107 + 0.1 * j, "=m" m, "=n" n,
        "=c[θ,ϕ][m,n]" c[107 + 0.1 * i, 107 + 0.1 * j][m, n]}]], Print["nothing"]

    ]]]]]
```

In[•]:=
```
ReadList["Attempt1.txt"]
```

In[•]:=
```
m = 0;
n = 0;
j = 1;
While[j ≤ 256, While[n ≤ 15, c[108, 108][m, n] = Out[50][[j]][[3]];
  Print[{m, n, c[108, 108][m, n]}];
  j++;
  n++];
 n = 0;
 m++]
```

In[•]:=
```
ReadList["Attempt2.txt"]
```

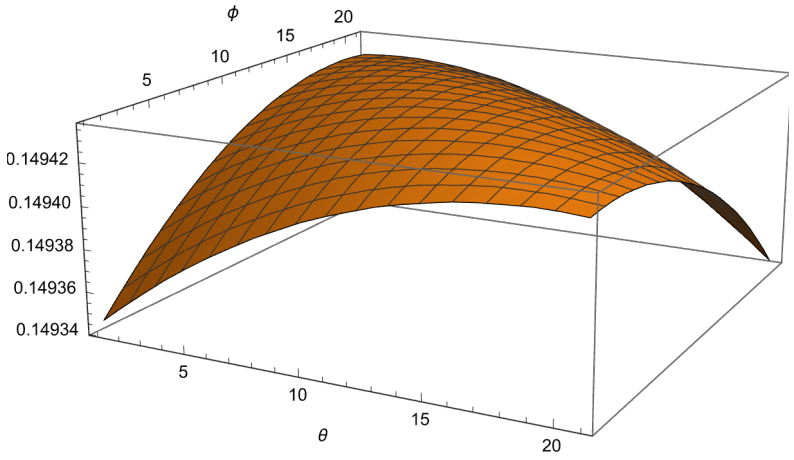
Out[•]=
```
{{0., 0., 0, 0, 1.71928}, nothing, nothing, nothing, nothing, nothing, nothing,
 nothing, ··· 112 625 ···, {2., 1.4, 15, 15, 0.0000902153 + 3.94538 × 10^-23 i},
 {2., 1.5, 15, 15, 0.0000909102 - 1.59901 × 10^-23 i},
 {2., 1.6, 15, 15, 0.0000916636 + 1.80412 × 10^-23 i},
 {2., 1.7, 15, 15, 0.0000924762 - 4.1542 × 10^-23 i},
 {2., 1.8, 15, 15, 0.0000933488 + 2.81796 × 10^-23 i},
 {2., 1.9, 15, 15, 0.0000942821 + 9.14892 × 10^-24 i},
 {2., 2., 15, 15, 0.0000952769 + 1.6848 × 10^-23 i}}
```

Full expression not available (original memory size: 12.7 MB)

```
In[•]:= m = 0;
        n = 0;
        i = 0;
        j = 0;
        k = 1;
        While[j ≤ 20, While[i ≤ 20,
          While[m ≤ 15, While[n ≤ 15, While[k ≤ 112 640, If[TrueQ[Out[55][[k]] == nothing], ,
              If[TrueQ[0.1 * i ≠ Out[55][[k]][[1]]], , If[TrueQ[0.1 * j ≠ Out[55][[k]][[2]]], ,
                If[TrueQ[m ≠ Out[55][[k]][[3]]], , If[TrueQ[n ≠ Out[55][[k]][[4]]], ,
                  c[107 + 0.1 * i, 107 + 0.1 * j][Out[55][[k]][[3]], Out[55][[k]][[4]] = Out[55][[k]][[5]];
                  Print[MatrixForm[{i, j, m, n, c[107 + 0.1 * i, 107 + 0.1 * j][
                       Out[55][[k]][[3]], Out[55][[k]][[4]]}]];
                  k = 112 641]]]]];
             k++];
            k = 1;
            n++];
           n = 0;
           m++];
          m = 0;
          i++];
         j++;
         i = j]

In[•]:= DumpSave["state.mx", "Global`"]

Out[•]= {Global`}

In[•]:= Get["state.mx"]

In[•]:= m = 0;
        n = 0;
        For[m = 0, m ≤ 15, m++, For[n = 0, n ≤ 15, n++,
          If[m > n, c[107.9 + .1, 107.9 + .1][m, n] = Conjugate[c[107.9 + .1, 107.9 + .1][n, m]],
           c[107.9 + .1, 107.9 + .1][m, n] = N[c[107.9 + .1, 107.9 + .1][m, n]]];
          Print[{m, n, c[107.9 + .1, 107.9 + .1][m, n]}]]]

In[•]:= data2 = Table[If[i > j, Abs[Rho[14, 107 + 0.1 i, 107 + 0.1 * j]],
            If[i == 10 && j == 10, Abs[Rho[14, 107.9 + 0.1, 107.9 + 0.1]],
             Abs[Rho[14, 107 + 0.1 j, 107 + 0.1 * i]]]], {i, 0, 20}, {j, 0, 20}];

In[•]:= data3 = Table[{x, y, 0.14943}, {x, 0, 20}, {y, 0, 20}];
```

In[•]:= `ListPlot3D[data2, Mesh → Automatic, AxesLabel → {"θ", "ϕ"}]`

Out[•]=



In[•]:= `MatrixForm[data2]`

Out[•]//MatrixForm=

0.149346	0.149354	0.149363	0.149371	0.149378	0.149384	0.14939	0.149396	0.149402
0.149354	0.149363	0.14937	0.149378	0.149385	0.149391	0.149397	0.149402	0.149407
0.149363	0.14937	0.149378	0.149385	0.149391	0.149397	0.149403	0.149408	0.149412
0.149371	0.149378	0.149385	0.149391	0.149398	0.149403	0.149408	0.149413	0.149417
0.149378	0.149385	0.149391	0.149398	0.149403	0.149409	0.149413	0.149417	0.149421
0.149384	0.149391	0.149397	0.149403	0.149409	0.149413	0.149417	0.149421	0.149424
0.14939	0.149397	0.149403	0.149408	0.149413	0.149417	0.149421	0.149425	0.149428
0.149396	0.149402	0.149408	0.149413	0.149417	0.149421	0.149425	0.149428	0.14943
0.149402	0.149407	0.149412	0.149417	0.149421	0.149424	0.149428	0.14943	0.149432
0.149407	0.149412	0.149416	0.149421	0.149424	0.149427	0.14943	0.149432	0.149434
0.149411	0.149415	0.14942	0.149424	0.149427	0.14943	0.149432	0.149434	0.149435
0.149414	0.149419	0.149423	0.149426	0.149429	0.149431	0.149433	0.149435	0.149436
0.149418	0.149422	0.149425	0.149428	0.149431	0.149433	0.149434	0.149435	0.149436
0.149421	0.149424	0.149427	0.14943	0.149432	0.149433	0.149435	0.149435	0.149435
0.149423	0.149426	0.149429	0.149431	0.149433	0.149434	0.149434	0.149435	0.149434
0.149425	0.149427	0.14943	0.149431	0.149433	0.149434	0.149434	0.149433	0.149433
0.149426	0.149428	0.14943	0.149432	0.149432	0.149433	0.149433	0.149432	0.149431
0.149427	0.149429	0.14943	0.149431	0.149432	0.149432	0.149431	0.14943	0.149428
0.149427	0.149429	0.14943	0.14943	0.14943	0.14943	0.149429	0.149427	0.149425
0.149427	0.149428	0.149429	0.149429	0.149428	0.149427	0.149426	0.149424	0.149422
0.149426	0.149427	0.149427	0.149427	0.149426	0.149425	0.149423	0.149421	0.149418

In[•]:= `ListPlot3D[{data2, data3}, Mesh → Automatic, AxesLabel → {"θ", "ϕ"}]`

It appears graphing Rho[4] should verify the regular pentagon maximizes Rho, up to an error of 5 degrees

It appears graphing Rho[9] should verify the regular pentagon maximizes Rho, up to an error of 1.5 degrees

It appears graphing Rho[14] should verify the regular pentagon maximizes Rho, up to an error of 1 degree

It appears graphing Rho[24] should verify the regular pentagon maximizes Rho, up to an error of 0.5

degrees

In[•]:=
```
m = 0;
n = 0;
For[m = 0, m ≤ 20, m++, For[n = 0, n ≤ 20, n++,
  Rho[1, 107 + 0.1 * m, 107 + 0.1 * n] = N[Rho[1, 107 + 0.1 * m, 107 + 0.1 * n]];
  Print[{m, n, Rho[1, 107 + 0.1 * m, 107 + 0.1 * n]}]]]
```

$\{0, 0, 0.161779 - 1.44601 \times 10^{-18}\, i\}$

In[•]:=
```
data = Table[If[i > j, Abs[Rho[1, 107 + 0.1 i, 107 + 0.1 * j]],
    Abs[Rho[1, 107 + 0.1 j, 107 + 0.1 * i]]], {i, 0, 20}, {j, 0, 20}];
```

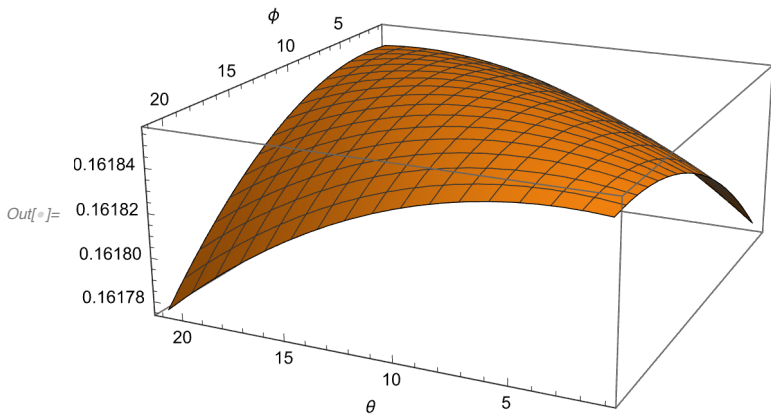
In[•]:=
```
MatrixForm[data]
```

Out[•]//MatrixForm=

0.161779	0.161787	0.161794	0.1618	0.161806	0.161812	0.161817	0.161822	0.161827
0.161787	0.161794	0.1618	0.161807	0.161812	0.161818	0.161823	0.161827	0.161832
0.161794	0.1618	0.161807	0.161813	0.161818	0.161823	0.161828	0.161832	0.161836
0.1618	0.161807	0.161813	0.161818	0.161823	0.161828	0.161833	0.161836	0.16184
0.161806	0.161812	0.161818	0.161823	0.161828	0.161833	0.161837	0.16184	0.161843
0.161812	0.161818	0.161823	0.161828	0.161833	0.161837	0.16184	0.161844	0.161846
0.161817	0.161823	0.161828	0.161833	0.161837	0.16184	0.161844	0.161847	0.161849
0.161822	0.161827	0.161832	0.161836	0.16184	0.161844	0.161847	0.161849	0.161851
0.161827	0.161832	0.161836	0.16184	0.161843	0.161846	0.161849	0.161851	0.161853
0.161831	0.161835	0.161839	0.161843	0.161846	0.161849	0.161851	0.161853	0.161854
0.161835	0.161839	0.161842	0.161846	0.161848	0.161851	0.161853	0.161854	0.161855
0.161838	0.161842	0.161845	0.161848	0.16185	0.161852	0.161854	0.161855	0.161856
0.161841	0.161844	0.161847	0.16185	0.161852	0.161853	0.161855	0.161856	0.161856
0.161843	0.161846	0.161849	0.161851	0.161853	0.161854	0.161855	0.161856	0.161856
0.161845	0.161848	0.16185	0.161852	0.161853	0.161854	0.161855	0.161855	0.161855
0.161847	0.161849	0.161851	0.161852	0.161854	0.161854	0.161854	0.161854	0.161853
0.161848	0.16185	0.161851	0.161853	0.161853	0.161854	0.161853	0.161853	0.161852
0.161848	0.16185	0.161851	0.161852	0.161853	0.161852	0.161852	0.161851	0.16185
0.161849	0.16185	0.161851	0.161851	0.161851	0.161851	0.16185	0.161849	0.161847
0.161849	0.16185	0.16185	0.16185	0.16185	0.161849	0.161848	0.161846	0.161844
0.161848	0.161849	0.161849	0.161849	0.161848	0.161847	0.161845	0.161843	0.161841

In[•]:=
```
ListPlot3D[data, Mesh → Automatic, AxesLabel → {"θ", "ϕ"}]
```

Out[•]=



```
In[•]:= Plot[{LineA[125, 98, x], LineB[125, 98, x], LineC[125, 98, x], LineD[125, 98, x]},
  {x, 0, 1.5}, PlotRange → Automatic, PlotLegends → "Expressions", AspectRatio → 1]
```

```
In[•]:= Plot[{CentLineA[155, 133, x], CentLineB[155, 133, x],
   CentLineC[155, 133, x], CentLineD[155, 133, x]},
  {x, -.75, .75}, PlotRange → Automatic, PlotLegends → "Expressions"]
```

```
In[•]:= Plot[{CentLineA[115, 107, x], CentLineB[115, 107, x], CentLineC[115, 107, x],
   CentLineD[115, 107, x]}, {x, CentXa1[115, 107], CentXa1[115, 107] + 1.5},
  PlotRange → {{-1.5, 1.5}, {-.95, .95}}, PlotLegends → "Expressions",
  Epilog → {InfiniteLine[{-Tx[115, 107], 0}, {0, 1}]}]
```

```
In[•]:= A[θ_, ϕ_] :=
 ((Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) / ((Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) (Ic[4, 0, θ, ϕ] + 2 Ic[2, 2, θ, ϕ] + Ic[0, 4, θ, ϕ]) -
     (Ic[3, 0, θ, ϕ] ^ 2 + 2 Ic[3, 0, θ, ϕ] * Ic[1, 2, θ, ϕ] + Ic[2, 1, θ, ϕ] ^ 2 +
       2 Ic[2, 1, θ, ϕ] * Ic[0, 3, θ, ϕ] + Ic[1, 2, θ, ϕ] ^ 2 + Ic[0, 3, θ, ϕ] ^ 2) - (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])
      (Ic[2, 0, θ, ϕ] ^ 2 - 2 Ic[2, 0, θ, ϕ] * Ic[0, 2, θ, ϕ] + 4 Ic[1, 1, θ, ϕ] ^ 2 + Ic[0, 2, θ, ϕ] ^ 2)))
```

In[•]:=
```
B[θ_, ϕ_] := Ic[3, 0, θ, ϕ]^2 - 6 Ic[3, 0, θ, ϕ] ×Ic[1, 2, θ, ϕ] + 9 Ic[1, 2, θ, ϕ]^2 + Ic[3, 0, θ, ϕ]^2 Ic[2, 0, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 +
 2 Ic[3, 0, θ, ϕ] ×Ic[1, 2, θ, ϕ] Ic[2, 0, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + Ic[1, 2, θ, ϕ]^2 Ic[2, 0, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 -
 2 Ic[3, 0, θ, ϕ]^2 Ic[2, 0, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 - 4 Ic[3, 0, θ, ϕ] ×Ic[1, 2, θ, ϕ] ×Ic[2, 0, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 -
 2 Ic[1, 2, θ, ϕ]^2 Ic[2, 0, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + Ic[3, 0, θ, ϕ]^2 Ic[0, 2, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 +
 2 Ic[3, 0, θ, ϕ] ×Ic[1, 2, θ, ϕ] Ic[0, 2, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + Ic[1, 2, θ, ϕ]^2 Ic[0, 2, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 - 2 Ic[3, 0, θ, ϕ]^2 Ic[2, 0, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) +
 4 Ic[3, 0, θ, ϕ] ×Ic[1, 2, θ, ϕ] ×Ic[2, 0, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) + 6 Ic[1, 2, θ, ϕ]^2 Ic[2, 0, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) + 2 Ic[3, 0, θ, ϕ]^2 Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) -
 4 Ic[3, 0, θ, ϕ] ×Ic[1, 2, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) - 6 Ic[1, 2, θ, ϕ]^2 Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) + 9 Ic[2, 1, θ, ϕ]^2 -
 6 Ic[2, 1, θ, ϕ] ×Ic[0, 3, θ, ϕ] + Ic[0, 3, θ, ϕ]^2 + Ic[2, 1, θ, ϕ]^2 Ic[2, 0, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 +
 2 Ic[2, 1, θ, ϕ] ×Ic[0, 3, θ, ϕ] Ic[2, 0, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + Ic[0, 3, θ, ϕ]^2 Ic[2, 0, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 +
 4 Ic[3, 0, θ, ϕ]^2 Ic[1, 1, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + 8 Ic[3, 0, θ, ϕ] ×Ic[1, 2, θ, ϕ] Ic[1, 1, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 +
 4 Ic[1, 2, θ, ϕ]^2 Ic[1, 1, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 - 2 Ic[2, 1, θ, ϕ]^2 Ic[2, 0, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 -
 4 Ic[2, 1, θ, ϕ] ×Ic[0, 3, θ, ϕ] ×Ic[2, 0, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 - 2 Ic[0, 3, θ, ϕ]^2 Ic[2, 0, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 +
 Ic[2, 1, θ, ϕ]^2 Ic[0, 2, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + 2 Ic[2, 1, θ, ϕ] ×Ic[0, 3, θ, ϕ] Ic[0, 2, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + Ic[0, 3, θ, ϕ]^2 Ic[0, 2, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 -
 6 Ic[2, 1, θ, ϕ]^2 Ic[2, 0, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) - 4 Ic[2, 1, θ, ϕ] ×Ic[0, 3, θ, ϕ] ×Ic[2, 0, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) + 2 Ic[0, 3, θ, ϕ]^2 Ic[2, 0, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) -
 8 Ic[3, 0, θ, ϕ] ×Ic[2, 1, θ, ϕ] ×Ic[1, 1, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) - 24 Ic[2, 1, θ, ϕ] ×Ic[1, 2, θ, ϕ] ×Ic[1, 1, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) +
 8 Ic[3, 0, θ, ϕ] ×Ic[0, 3, θ, ϕ] ×Ic[1, 1, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) - 8 Ic[1, 2, θ, ϕ] ×Ic[0, 3, θ, ϕ] ×Ic[1, 1, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) +
 6 Ic[2, 1, θ, ϕ]^2 Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) + 4 Ic[2, 1, θ, ϕ] ×Ic[0, 3, θ, ϕ] ×Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) - 2 Ic[0, 3, θ, ϕ]^2 Ic[0, 2, θ, ϕ] / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ]) +
 4 Ic[2, 1, θ, ϕ]^2 Ic[1, 1, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + 8 Ic[2, 1, θ, ϕ] ×Ic[0, 3, θ, ϕ] Ic[1, 1, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2 + 4 Ic[0, 3, θ, ϕ]^2 Ic[1, 1, θ, ϕ]^2 / (Ic[2, 0, θ, ϕ] + Ic[0, 2, θ, ϕ])^2
```

In[•]:=
```
Rho2[θ_, ϕ_] := Rho1[θ, ϕ] - (A[θ, ϕ] * B[θ, ϕ])
```

In[•]:=
```
m = 0;
n = 0;
For[m = 0, m ≤ 15, m++,
 For[n = 0, n ≤ 15, n++, If[m > n, c[108, 108][m, n] = Conjugate[c[108, 108][n, m]],
   c[108, 108][m, n] = N[c[108, 108][m, n]]];
  Print[{m, n, c[108, 108][m, n]}]]]
```

```
{0, 0, 1.72048}
```

$$\{0, 1, 2.77556 \times 10^{-16} - 2.33596 \times 10^{-16}\, i\}$$

$$\{0, 2, 1.38778 \times 10^{-17} - 4.97752 \times 10^{-17}\, i\}$$

$$\{0, 3, -2.22045 \times 10^{-16} - 6.88883 \times 10^{-17}\, i\}$$

$$\{0, 4, -4.16334 \times 10^{-17} - 2.64724 \times 10^{-16}\, i\}$$

$$\{0, 5, 0.036491 + 1.43154 \times 10^{-16}\, i\}$$

$$\{0, 6, 2.91434 \times 10^{-15} + 1.76648 \times 10^{-16}\, i\}$$

$$\{0, 7, -2.55351 \times 10^{-15} + 5.29838 \times 10^{-16}\, i\}$$

$$\{0, 8, -1.77358 \times 10^{-14} - 2.37905 \times 10^{-15}\, i\}$$

$$\{0, 9, 1.38778 \times 10^{-17} - 4.76119 \times 10^{-15}\, i\}$$

$$\{0, 10, 0.00517152 + 1.44833 \times 10^{-14}\, i\}$$

$$\{0, 11, 8.83738 \times 10^{-14} - 3.30078 \times 10^{-14}\, i\}$$

$$\{0, 12, -3.59157 \times 10^{-14} - 9.7358 \times 10^{-14}\, i\}$$

$$\{0, 13, 1.13909 \times 10^{-13} - 1.48501 \times 10^{-13}\, i\}$$

$$\{0, 14, 3.55105 \times 10^{-13} + 1.18993 \times 10^{-13}\, i\}$$

$$\{0, 15, 0.00111784 - 4.83127 \times 10^{-13}\, i\}$$

$$\{1, 0, 2.77556 \times 10^{-16} + 2.33596 \times 10^{-16}\, i\}$$

$$\{1, 1, 0.479101\}$$

$$\{1, 2, 2.22045 \times 10^{-16} - 2.25162 \times 10^{-16}\, i\}$$

$$\{1, 3, -1.249 \times 10^{-16} - 1.33413 \times 10^{-16}\, i\}$$

$$\{1, 4, 1.33921 \times 10^{-15} - 3.26731 \times 10^{-16}\, i\}$$

$$\{1, 5, -1.00614 \times 10^{-16} + 1.4999 \times 10^{-15}\, i\}$$

$$\{1, 6, 0.0211638 + 3.12145 \times 10^{-15}\, i\}$$

$$\{1, 7, 1.28127 \times 10^{-14} + 4.04544 \times 10^{-15}\, i\}$$

$$\{1, 8, 1.43358 \times 10^{-14} + 3.86739 \times 10^{-15}\, i\}$$

$$\{1, 9, -5.68053 \times 10^{-14} - 4.11046 \times 10^{-15}\, i\}$$

$$\{1, 10, -1.65319 \times 10^{-14} + 2.74377 \times 10^{-14}\, i\}$$

$$\{1, 11, 0.00330622 - 3.39451 \times 10^{-13}\, i\}$$

$$\{1, 12, -7.91339 \times 10^{-13} - 4.19556 \times 10^{-13}\, i\}$$

$$\{1, 13, -1.64568 \times 10^{-12} - 6.90124 \times 10^{-13}\, i\}$$

$$\{1, 14, -9.54965 \times 10^{-12} - 2.75021 \times 10^{-12}\, i\}$$

$$\{1, 15, -2.39422 \times 10^{-12} + 8.07868 \times 10^{-12}\, i\}$$

$$\{2, 0, 1.38778 \times 10^{-17} + 4.97752 \times 10^{-17}\, i\}$$

$$\{2, 1, 2.22045 \times 10^{-16} + 2.25162 \times 10^{-16}\, i\}$$

$$\{2, 2, 0.181073\}$$

$$\{2, 3, -5.06539 \times 10^{-16} - 7.25243 \times 10^{-16}\, i\}$$

$$\{2, 4, -1.72952 \times 10^{-15} - 3.22255 \times 10^{-16}\, i\}$$

$$\{2, 5, 3.54231 \times 10^{-15} - 1.40288 \times 10^{-15}\, i\}$$

$$\{2, 6, -1.59595 \times 10^{-16} - 1.60591 \times 10^{-15}\, i\}$$

$$\{2, 7, 0.0123982 - 6.26583 \times 10^{-15}\, i\}$$

$$\{2, 8, -1.70801 \times 10^{-14} - 1.80885 \times 10^{-14}\, i\}$$

$$\{2, 9, 8.77493 \times 10^{-14} + 7.86875 \times 10^{-14}\, i\}$$

$$\{2, 10, 3.12955 \times 10^{-13} + 5.19796 \times 10^{-15}\, i\}$$

$$\{2, 11, 3.93907 \times 10^{-13} + 4.29838 \times 10^{-15}\, i\}$$

$$\{2, 12, 0.00211565 - 9.44852 \times 10^{-13}\, i\}$$

$$\{2, 13, -1.97185 \times 10^{-12} + 1.46202 \times 10^{-12}\, i\}$$

$$\{2, 14, 1.21092 \times 10^{-12} + 1.69631 \times 10^{-12}\, i\}$$

$$\{2, 15, -2.63162 \times 10^{-12} + 2.24924 \times 10^{-11}\, i\}$$

$$\{3, 0, -2.22045 \times 10^{-16} + 6.88883 \times 10^{-17}\, i\}$$

$$\{3, 1, -1.249 \times 10^{-16} + 1.33413 \times 10^{-16}\, i\}$$

$$\{3, 2, -5.06539 \times 10^{-16} + 7.25243 \times 10^{-16}\, i\}$$

$$\{3, 3, 0.0784107\}$$

$$\{3, 4, -3.31332 \times 10^{-16} + 7.76299 \times 10^{-16}\, i\}$$

$$\{3, 5, -6.42195 \times 10^{-15} + 1.10393 \times 10^{-15}\, i\}$$

$$\{3, 6, 1.3458 \times 10^{-14} - 3.65823 \times 10^{-15}\, i\}$$

$$\{3, 7, 6.91634 \times 10^{-15} - 2.20658 \times 10^{-14}\, i\}$$

$$\{3, 8, 0.00733581 - 1.94849 \times 10^{-14}\, i\}$$

$$\{3, 9, -1.71706 \times 10^{-13} - 5.34337 \times 10^{-14}\, i\}$$

$$\{3, 10, -2.41026 \times 10^{-13} + 1.5501 \times 10^{-15}\, i\}$$

$$\{3, 11, 8.7242 \times 10^{-13} + 4.34862 \times 10^{-13}\, i\}$$

$$\{3, 12, 1.36394 \times 10^{-12} + 1.31248 \times 10^{-13}\, i\}$$

$$\{3, 13, 0.00135605 - 1.72732 \times 10^{-12}\, i\}$$

$$\{3, 14, 9.52714 \times 10^{-12} - 2.71518 \times 10^{-12}\, i\}$$

$$\{3, 15, 1.47838 \times 10^{-11} + 5.71361 \times 10^{-12}\, i\}$$

$$\{4, 0, -4.16334 \times 10^{-17} + 2.64724 \times 10^{-16}\, i\}$$

$$\{4, 1, 1.33921 \times 10^{-15} + 3.26731 \times 10^{-16}\, i\}$$

$$\{4, 2, -1.72952 \times 10^{-15} + 3.22255 \times 10^{-16}\, i\}$$

$$\{4, 3, -3.31332 \times 10^{-16} - 7.76299 \times 10^{-16}\, i\}$$

$$\{4, 4, 0.0368898\}$$

$$\{4, 5, -5.6101 \times 10^{-15} + 9.57777 \times 10^{-15}\, i\}$$

$$\{4, 6, -2.06857 \times 10^{-14} + 9.75183 \times 10^{-15}\, i\}$$

$$\{4, 7, 7.61891 \times 10^{-14} + 5.1962 \times 10^{-15}\, i\}$$

$$\{4, 8, 2.23693 \times 10^{-14} - 2.458 \times 10^{-14}\, i\}$$

$$\{4, 9, 0.00438328 + 1.75266 \times 10^{-14}\, i\}$$

$$\{4, 10, -6.42379 \times 10^{-13} + 3.82535 \times 10^{-13}\, i\}$$

$$\{4, 11, -5.83231 \times 10^{-13} - 6.68273 \times 10^{-13}\, i\}$$

$$\{4, 12, -4.98736 \times 10^{-12} - 1.00988 \times 10^{-12}\, i\}$$

$$\{4, 13, 5.84033 \times 10^{-12} + 1.76261 \times 10^{-12}\, i\}$$

$$\{4, 14, 0.000871096 + 1.90338 \times 10^{-11}\, i\}$$

$$\{4, 15, -6.90876 \times 10^{-11} + 3.54491 \times 10^{-11}\, i\}$$

$$\{5, 0, 0.036491 - 1.43154 \times 10^{-16}\, i\}$$

$$\{5, 1, -1.00614 \times 10^{-16} - 1.4999 \times 10^{-15}\, i\}$$

$$\{5, 2, 3.54231 \times 10^{-15} + 1.40288 \times 10^{-15}\, i\}$$

$$\{5, 3, -6.42195 \times 10^{-15} - 1.10393 \times 10^{-15}\, i\}$$

$$\{5, 4, -5.6101 \times 10^{-15} - 9.57777 \times 10^{-15}\, i\}$$

$$\{5, 5, 0.0184073\}$$

$$\{5, 6, 1.40773 \times 10^{-14} + 2.69365 \times 10^{-14}\, i\}$$

$$\{5, 7, -7.30475 \times 10^{-14} - 1.8793 \times 10^{-15}\, i\}$$

$$\{5, 8, -1.56698 \times 10^{-14} + 1.89019 \times 10^{-14}\, i\}$$

$$\{5, 9, 3.04074 \times 10^{-13} + 4.68821 \times 10^{-13}\, i\}$$

$$\{5, 10, 0.00264427 - 2.78508 \times 10^{-13}\, i\}$$

$$\{5, 11, 3.07347 \times 10^{-12} + 9.01558 \times 10^{-13}\, i\}$$

$$\{5, 12, 4.83841 \times 10^{-12} - 2.41748 \times 10^{-13}\, i\}$$

$$\{5, 13, 2.20726 \times 10^{-14} - 6.4027 \times 10^{-12}\, i\}$$

$$\{5, 14, 2.18392 \times 10^{-11} + 1.38026 \times 10^{-11}\, i\}$$

$$\{5, 15, 0.000561026 - 4.9374 \times 10^{-11}\, i\}$$

$$\{6, 0, 2.91434 \times 10^{-15} - 1.76648 \times 10^{-16}\, i\}$$

$$\{6, 1, 0.0211638 - 3.12145 \times 10^{-15}\, i\}$$

$$\{6, 2, -1.59595 \times 10^{-16} + 1.60591 \times 10^{-15}\, i\}$$

$$\{6, 3, 1.3458 \times 10^{-14} + 3.65823 \times 10^{-15}\, i\}$$

$$\{6, 4, -2.06857 \times 10^{-14} - 9.75183 \times 10^{-15}\, i\}$$

$$\{6, 5, 1.40773 \times 10^{-14} - 2.69365 \times 10^{-14}\, i\}$$

$$\{6, 6, 0.00961171\}$$

$$\{6, 7, 6.93751 \times 10^{-14} + 8.18003 \times 10^{-15}\, i\}$$

$$\{6, 8, -5.28363 \times 10^{-13} + 9.42718 \times 10^{-15}\, i\}$$

$$\{6, 9, -2.004 \times 10^{-13} + 5.0395 \times 10^{-13}\, i\}$$

$$\{6, 10, -1.83888 \times 10^{-13} + 7.77732 \times 10^{-13}\, i\}$$

$$\{6, 11, 0.00161004 + 2.54882 \times 10^{-12}\, i\}$$

$$\{6, 12, 1.63487 \times 10^{-11} + 3.20105 \times 10^{-12}\, i\}$$

$$\{6, 13, 2.66913 \times 10^{-11} + 7.4021 \times 10^{-12}\, i\}$$

$$\{6, 14, -1.13412 \times 10^{-10} + 1.34351 \times 10^{-11}\, i\}$$

$$\{6, 15, 1.79385 \times 10^{-10} + 7.7275 \times 10^{-11}\, i\}$$

$$\{7, 0, -2.55351 \times 10^{-15} - 5.29838 \times 10^{-16}\, i\}$$

$$\{7, 1, 1.28127 \times 10^{-14} - 4.04544 \times 10^{-15}\, i\}$$

$$\{7, 2, 0.0123982 + 6.26583 \times 10^{-15}\, i\}$$

$$\{7, 3, 6.91634 \times 10^{-15} + 2.20658 \times 10^{-14}\, i\}$$

$$\{7, 4, 7.61891 \times 10^{-14} - 5.1962 \times 10^{-15}\, i\}$$

$$\{7, 5, -7.30475 \times 10^{-14} + 1.8793 \times 10^{-15}\, i\}$$

$$\{7, 6, 6.93751 \times 10^{-14} - 8.18003 \times 10^{-15}\, i\}$$

$$\{7, 7, 0.00520693\}$$

$$\{7, 8, -3.85143 \times 10^{-13} - 1.09013 \times 10^{-13}\, i\}$$

$$\{7, 9, -2.30878 \times 10^{-12} - 1.5379 \times 10^{-13}\, i\}$$

$$\{7, 10, 3.44117 \times 10^{-12} - 9.94843 \times 10^{-13}\, i\}$$

$$\{7, 11, -2.74143 \times 10^{-12} + 2.1043 \times 10^{-12}\, i\}$$

$$\{7, 12, 0.000989078 - 2.96069 \times 10^{-12}\, i\}$$

$$\{7, 13, 7.1203 \times 10^{-11} + 2.35987 \times 10^{-11}\, i\}$$

$$\{7, 14, 2.78916 \times 10^{-10} - 2.22193 \times 10^{-11}\, i\}$$

$$\{7, 15, -1.04361 \times 10^{-9} + 1.19751 \times 10^{-10}\, i\}$$

$$\{8, 0, -1.77358 \times 10^{-14} + 2.37905 \times 10^{-15}\, i\}$$

$$\{8, 1, 1.43358 \times 10^{-14} - 3.86739 \times 10^{-15}\, i\}$$

$$\{8, 2, -1.70801 \times 10^{-14} + 1.80885 \times 10^{-14}\, i\}$$

$$\{8, 3, 0.00733581 + 1.94849 \times 10^{-14}\, i\}$$

$$\{8, 4, 2.23693 \times 10^{-14} + 2.458 \times 10^{-14}\, i\}$$

$$\{8, 5, -1.56698 \times 10^{-14} - 1.89019 \times 10^{-14}\, i\}$$

$$\{8, 6, -5.28363 \times 10^{-13} - 9.42718 \times 10^{-15}\, i\}$$

$$\{8, 7, -3.85143 \times 10^{-13} + 1.09013 \times 10^{-13}\, i\}$$

$$\{8, 8, 0.00290743\}$$

$$\{8, 9, 1.7862 \times 10^{-12} + 6.74844 \times 10^{-13}\, i\}$$

$$\{8, 10, -1.05924 \times 10^{-13} + 3.40416 \times 10^{-12}\, i\}$$

$$\{8, 11, 6.62967 \times 10^{-11} - 3.46389 \times 10^{-12}\, i\}$$

$$\{8, 12, -2.22261 \times 10^{-11} + 2.45462 \times 10^{-11}\, i\}$$

$$\{8, 13, 0.000612786 + 5.09005 \times 10^{-11}\, i\}$$

$$\{8, 14, 1.63099 \times 10^{-10} + 3.00638 \times 10^{-13}\, i\}$$

$$\{8, 15, 1.09355 \times 10^{-9} + 8.09325 \times 10^{-11}\, i\}$$

$$\{9, 0, 1.38778 \times 10^{-17} + 4.76119 \times 10^{-15}\, i\}$$

$$\{9, 1, -5.68053 \times 10^{-14} + 4.11046 \times 10^{-15}\, i\}$$

$$\{9, 2, 8.77493 \times 10^{-14} - 7.86875 \times 10^{-14}\, i\}$$

$$\{9, 3, -1.71706 \times 10^{-13} + 5.34337 \times 10^{-14}\, i\}$$

$$\{9, 4, 0.00438328 - 1.75266 \times 10^{-14}\, i\}$$

$$\{9, 5, 3.04074 \times 10^{-13} - 4.68821 \times 10^{-13}\, i\}$$

$$\{9, 6, -2.004 \times 10^{-13} - 5.0395 \times 10^{-13}\, i\}$$

$$\{9, 7, -2.30878 \times 10^{-12} + 1.5379 \times 10^{-13}\, i\}$$

$$\{9, 8, 1.7862 \times 10^{-12} - 6.74844 \times 10^{-13}\, i\}$$

$$\{9, 9, 0.00168762\}$$

$$\{9, 10, -2.26983 \times 10^{-12} + 1.45948 \times 10^{-11}\, i\}$$

$$\{9, 11, -6.11436 \times 10^{-11} - 4.3142 \times 10^{-12}\, i\}$$

$$\{9, 12, 2.60017 \times 10^{-10} + 1.81365 \times 10^{-11}\, i\}$$

$$\{9, 13, 3.00111 \times 10^{-11} + 1.92774 \times 10^{-11}\, i\}$$

$$\{9, 14, 0.000382718 - 9.40431 \times 10^{-11}\, i\}$$

$$\{9, 15, 4.98026 \times 10^{-11} + 3.91924 \times 10^{-10}\, i\}$$

$$\{10, 0, 0.00517152 - 1.44833 \times 10^{-14}\, i\}$$

$$\{10, 1, -1.65319 \times 10^{-14} - 2.74377 \times 10^{-14}\, i\}$$

$$\{10, 2, 3.12955 \times 10^{-13} - 5.19796 \times 10^{-15}\, i\}$$

$$\{10, 3, -2.41026 \times 10^{-13} - 1.5501 \times 10^{-15}\, i\}$$

$$\{10, 4, -6.42379 \times 10^{-13} - 3.82535 \times 10^{-13}\, i\}$$

$$\{10, 5, 0.00264427 + 2.78508 \times 10^{-13}\, i\}$$

$$\{10, 6, -1.83888 \times 10^{-13} - 7.77732 \times 10^{-13}\, i\}$$

$$\{10, 7, 3.44117 \times 10^{-12} + 9.94843 \times 10^{-13}\, i\}$$

$$\{10, 8, -1.05924 \times 10^{-13} - 3.40416 \times 10^{-12}\, i\}$$

$$\{10, 9, -2.26983 \times 10^{-12} - 1.45948 \times 10^{-11}\, i\}$$

$$\{10, 10, 0.00103964\}$$

$$\{10, 11, -1.17803 \times 10^{-10} - 2.39406 \times 10^{-11}\, i\}$$

$$\{10, 12, -1.28978 \times 10^{-10} - 1.04256 \times 10^{-10}\, i\}$$

$$\{10, 13, 1.66608 \times 10^{-9} - 1.14807 \times 10^{-10}\, i\}$$

$$\{10, 14, 1.15855 \times 10^{-9} + 8.54413 \times 10^{-11}\, i\}$$

$$\{10, 15, 0.000240849 + 1.35249 \times 10^{-9}\, i\}$$

$$\{11, 0, 8.83738 \times 10^{-14} + 3.30078 \times 10^{-14}\, i\}$$

$$\{11, 1, 0.00330622 + 3.39451 \times 10^{-13}\, i\}$$

$$\{11, 2, 3.93907 \times 10^{-13} - 4.29838 \times 10^{-15}\, i\}$$

$$\{11, 3, 8.7242 \times 10^{-13} - 4.34862 \times 10^{-13}\, i\}$$

$$\{11, 4, -5.83231 \times 10^{-13} + 6.68273 \times 10^{-13}\, i\}$$

$$\{11, 5, 3.07347 \times 10^{-12} - 9.01558 \times 10^{-13}\, i\}$$

$$\{11, 6, 0.00161004 - 2.54882 \times 10^{-12}\, i\}$$

$$\{11, 7, -2.74143 \times 10^{-12} - 2.1043 \times 10^{-12}\, i\}$$

$$\{11, 8, 6.62967 \times 10^{-11} + 3.46389 \times 10^{-12}\, i\}$$

$$\{11, 9, -6.11436 \times 10^{-11} + 4.3142 \times 10^{-12}\, i\}$$

$$\{11, 10, -1.17803 \times 10^{-10} + 2.39406 \times 10^{-11}\, i\}$$

$$\{11, 11, 0.00207543\}$$

$$\{11, 12, -7.04271 \times 10^{-10} - 1.68788 \times 10^{-11}\, i\}$$

$$\{11, 13, -4.17172 \times 10^{-9} - 3.06467 \times 10^{-10}\, i\}$$

$$\{11, 14, 4.75213 \times 10^{-9} + 4.46768 \times 10^{-10}\, i\}$$

$$\{11, 15, 4.59763 \times 10^{-9} + 6.94189 \times 10^{-9}\, i\}$$

$$\{12, 0, -3.59157 \times 10^{-14} + 9.7358 \times 10^{-14}\, i\}$$

$$\{12, 1, -7.91339 \times 10^{-13} + 4.19556 \times 10^{-13}\, i\}$$

$$\{12, 2, 0.00211565 + 9.44852 \times 10^{-13}\, i\}$$

$$\{12, 3, 1.36394 \times 10^{-12} - 1.31248 \times 10^{-13}\, i\}$$

$$\{12, 4, -4.98736 \times 10^{-12} + 1.00988 \times 10^{-12}\, i\}$$

$$\{12, 5, 4.83841 \times 10^{-12} + 2.41748 \times 10^{-13}\, i\}$$

$$\{12, 6, 1.63487 \times 10^{-11} - 3.20105 \times 10^{-12}\, i\}$$

$$\{12, 7, 0.000989078 + 2.96069 \times 10^{-12}\, i\}$$

$$\{12, 8, -2.22261 \times 10^{-11} - 2.45462 \times 10^{-11}\, i\}$$

$$\{12, 9, 2.60017 \times 10^{-10} - 1.81365 \times 10^{-11}\, i\}$$

$$\{12, 10, -1.28978 \times 10^{-10} + 1.04256 \times 10^{-10}\, i\}$$

$$\{12, 11, -7.04271 \times 10^{-10} + 1.68788 \times 10^{-11}\, i\}$$

$$\{12, 12, 0.0679533\}$$

$$\{12, 13, -1.49947 \times 10^{-9} + 1.31783 \times 10^{-9}\, i\}$$

$$\{12, 14, -1.1317 \times 10^{-8} + 4.25077 \times 10^{-11}\, i\}$$

$$\{12, 15, -1.59965 \times 10^{-9} + 6.52625 \times 10^{-9}\, i\}$$

$$\{13, 0, 1.13909 \times 10^{-13} + 1.48501 \times 10^{-13}\, i\}$$

$$\{13, 1, -1.64568 \times 10^{-12} + 6.90124 \times 10^{-13}\, i\}$$

$$\{13, 2, -1.97185 \times 10^{-12} - 1.46202 \times 10^{-12}\, i\}$$

$$\{13, 3, 0.00135605 + 1.72732 \times 10^{-12}\, i\}$$

$$\{13, 4, 5.84033 \times 10^{-12} - 1.76261 \times 10^{-12}\, i\}$$

$$\{13, 5, 2.20726 \times 10^{-14} + 6.4027 \times 10^{-12}\, i\}$$

$$\{13, 6, 2.66913 \times 10^{-11} - 7.4021 \times 10^{-12}\, i\}$$

$$\{13, 7, 7.1203 \times 10^{-11} - 2.35987 \times 10^{-11}\, i\}$$

$$\{13, 8, 0.000612786 - 5.09005 \times 10^{-11}\, i\}$$

$$\{13, 9, 3.00111 \times 10^{-11} - 1.92774 \times 10^{-11}\, i\}$$

$$\{13, 10, 1.66608 \times 10^{-9} + 1.14807 \times 10^{-10}\, i\}$$

$$\{13, 11, -4.17172 \times 10^{-9} + 3.06467 \times 10^{-10}\, i\}$$

$$\{13, 12, -1.49947 \times 10^{-9} - 1.31783 \times 10^{-9}\, i\}$$

$$\{13, 13, -3.30211\}$$

$$\{13, 14, -3.65107 \times 10^{-11} + 1.32539 \times 10^{-8}\, i\}$$

$$\{13, 15, -6.84819 \times 10^{-8} + 1.3249 \times 10^{-9}\, i\}$$

$$\{14, 0, 3.55105 \times 10^{-13} - 1.18993 \times 10^{-13}\, i\}$$

$$\{14, 1, -9.54965 \times 10^{-12} + 2.75021 \times 10^{-12}\, i\}$$

$$\{14, 2, 1.21092 \times 10^{-12} - 1.69631 \times 10^{-12}\, i\}$$

$$\{14, 3, 9.52714 \times 10^{-12} + 2.71518 \times 10^{-12}\, i\}$$

$$\{14, 4, 0.000871096 - 1.90338 \times 10^{-11}\, i\}$$

$$\{14, 5, 2.18392 \times 10^{-11} - 1.38026 \times 10^{-11}\, i\}$$

$$\{14, 6, -1.13412 \times 10^{-10} - 1.34351 \times 10^{-11}\, i\}$$

$$\{14, 7, 2.78916 \times 10^{-10} + 2.22193 \times 10^{-11}\, i\}$$

$$\{14, 8, 1.63099 \times 10^{-10} - 3.00638 \times 10^{-13}\, i\}$$

$$\{14, 9, 0.000382718 + 9.40431 \times 10^{-11}\, i\}$$

$$\{14, 10, 1.15855 \times 10^{-9} - 8.54413 \times 10^{-11}\, i\}$$

$$\{14, 11, 4.75213 \times 10^{-9} - 4.46768 \times 10^{-10}\, i\}$$

$$\{14, 12, -1.1317 \times 10^{-8} - 4.25077 \times 10^{-11}\, i\}$$

$$\{14, 13, -3.65107 \times 10^{-11} - 1.32539 \times 10^{-8}\, i\}$$

$$\{14, 14, 117.576\}$$

$$\{14, 15, -9.82919 \times 10^{-9} + 5.59845 \times 10^{-8}\, i\}$$

$$\{15, 0, 0.00111784 + 4.83127 \times 10^{-13}\, i\}$$

$$\{15, 1, -2.39422 \times 10^{-12} - 8.07868 \times 10^{-12}\, i\}$$

$$\{15, 2, -2.63162 \times 10^{-12} - 2.24924 \times 10^{-11}\, i\}$$

$$\{15, 3, 1.47838 \times 10^{-11} - 5.71361 \times 10^{-12}\, i\}$$

$$\{15, 4, -6.90876 \times 10^{-11} - 3.54491 \times 10^{-11}\, i\}$$

$$\{15, 5, 0.000561026 + 4.9374 \times 10^{-11}\, i\}$$

$$\{15, 6, 1.79385 \times 10^{-10} - 7.7275 \times 10^{-11}\, i\}$$

$$\{15, 7, -1.04361 \times 10^{-9} - 1.19751 \times 10^{-10}\, i\}$$

$$\{15, 8, 1.09355 \times 10^{-9} - 8.09325 \times 10^{-11}\, i\}$$

$$\{15, 9, 4.98026 \times 10^{-11} - 3.91924 \times 10^{-10}\, i\}$$

$$\{15, 10, 0.000240849 - 1.35249 \times 10^{-9}\, i\}$$

$$\{15, 11, 4.59763 \times 10^{-9} - 6.94189 \times 10^{-9}\, i\}$$

$$\{15, 12, -1.59965 \times 10^{-9} - 6.52625 \times 10^{-9}\, i\}$$

$$\{15, 13, -6.84819 \times 10^{-8} - 1.3249 \times 10^{-9}\, i\}$$

$$\{15, 14, -9.82919 \times 10^{-9} - 5.59845 \times 10^{-8}\, i\}$$

$$\{15, 15, -323.248\}$$

In[•]:=
```
MemoryInUse[]
```

Out[•]= 114 608 552

In[•]:=
```
MaxMemoryUsed[]
```

Out[•]= 162 171 896

In[•]:=
```
ClearSystemCache[]
```

In[•]:=
```
MemoryInUse[]
```

Out[•]= 691 004 376

In[•]:=
```
MaxMemoryUsed[]
```

Out[•]= 706 438 032

In[•]:=
```
SetSystemOptions["GraphOptions" → "CacheResults" → False]
```

Out[•]= GraphOptions → {CacheResults → False, EdgeCountThreshold → 10 000,
RenderingOrder → VertexFirst, VertexCountThreshold → 1000}

In[•]:=
```
MemoryInUse[]
```

Out[•]= 691 493 784