DISCONTINUOUS GALERKIN METHODS FOR FRACTIONAL DIFFUSION EQUATIONS

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Abstract. We consider the development and analysis of local discontinuous Galerkin methods for fractional diffusion problems, characterized by having fractional derivatives, parameterized by $\beta \in [1, 2]$. We show through analysis that one can construct a numerical flux which results in a scheme that exhibit optimal order of convergence $\mathcal{O}(h^{k+1})$ in the continuous range between pure advection ($\beta = 1$) and pure diffusion ($\beta = 2$). In these limits, known schemes are recovered. The analysis is confirmed through a few examples.

Key words. Fractional derivatives, discontinuous Galerkin methods, optimal convergence.

AMS subject classifications. 35R11, 65M60, 65M12

1. Introduction.

The basic ideas behind fractional calculus has a history that is similar and aligned with that of more classic calculus for three hundred years and attracted the interests of the mathematicians who contributed fundamentally to the development of classical calculus, including L'Hospital, Leibniz, Liouville, Riemann, Grünward, and Letnikov [4]. In spite of this, the development and analysis of fractional calculus and fractional differential equations are not as mature as that associated with classical calculus. However, during the last few decades this has begun to change as it has become clear that fractional calculus emerges as a natural description for a broad range of non-classical phenomena in the applied sciences and engineering. A striking example of this is as a model for anomalous transport processes and diffusion, leading to partial differential equations (PDEs) of fractional version [2, 16]. Applications of such models are found in wide range of applications such as porous flows, models of a variety of biological processes, and transport in fusion plasmas, to name a few.

With this emerging range of applications and models based on fractional calculus comes a need for the development of robust and accurate computational methods for solving these equations. A fundamental difference between problems in classic calculus and fractional calculus is the global nature of the latter formulations. Nevertheless, methods based on finite difference methods [17] and finite element formulations [10, 11] have been developed and successfully applied.

However, the global nature often leads computational techniques which are substantially more resource intense than those associated with more classic problems. Furthermore, the other hand, solutions of fractional diffusion problem are generally endowed with substantial smoothness. While these observations both suggest that a higher order accurate formulation may be attractive, there appears to be very limited work in that direction with notable exception being [15]. In this work a spectral method is used to discretize classical space derivative to solve time fractional PDEs and confirms the possible advantages of doing so.

In this work we develop discontinuous Galerkin (DG) methods for the fractional

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diffusion problem. This development is based on the extensive work on DG for problems founded in classic calculus [6, 7, 12, 13, 18]. In particular we consider the extension of the local discontinuous Galerkin (LDG) method [6], based on previous work in [3], to problems containing fractional spatial derivatives. We find that almost all the advantages or characteristics [13] of DG methods when used to solve classical PDEs carries over to fractional PDEs, i.e., the methods are naturally formulated for any order of accuracy in each element, flexibility in choosing element sizes in different places, allowing for adaptivity, and the mass matrix is local and easily invertible, leading an explicit formulation for time dependent problems. We shall also discuss, however, the choice of the numerical flux is essential to ensure the accuracy and stability of the scheme.

We shall in general discuss equations on the form

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(x,t), \tag{1.1}$$

subject to appropriate boundary conditions. Here $f(x,t)$ is a source term, $\beta \in [1,2]$, and $d \in R^+$, becoming a velocity in the advective limit ($\beta = 1$) and a pure diffusion coefficient in the pure diffusion limit of ($\beta = 2$). In the continues range of $\beta \in (1,2)$ we shall refer to it as the generalized diffusion coefficient. In this non-classical range, we shall by $\frac{\partial^\beta u}{\partial x^\beta}$ imply the Riemann-Liouville derivative of order β, defined as

$$\frac{\partial^\beta u}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_a^x (x-\xi)^{1-\beta}u(\xi,t)d\xi, \tag{1.2}$$

where a is the left boundary of $u(x,t)$ in (1.1). When $\beta = 1$ and 2, the definition (1.2) still makes sense and recovers exactly the first order and second order classical derivatives, respectively. Hence, within this framework we can unify the classical and fractional calculus and recover known formulations in special cases.

What remains of this paper is organized as follows. In Section 2, we review the definition and properties of fractional derivatives and introduce the appropriate functional setting. This sets the stage for 3 in which we introduce ways to specify appropriate initial and boundary conditions, propose the LDG schemes. Section 4 contains the detailed stability and error analyses of two different schemes. In Section 5 the analysis is confirmed through extensive numerical results and Section 6 contains a few concluding remarks and outlook for future work.

2. Preliminaries on fractional calculus. In the following we offer the formal definitions of fractional derivatives and associated functional setting, required to proceed with the analysis in the subsequent sections.

2.1. Fractional calculus. The formal definition of the fractional integral emerges as a natural generalization of multiple integration for which it is well-known that to integrate a function $v(x)$ n times amounts to

$$\int_a^x d\xi_n \int_a^{\xi_n} d\xi_{n-1} \cdots \int_a^{\xi_2} v(\xi_1)d\xi_1 = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1}v(\xi)d\xi, \quad x > a,$$

for which we introduce the notation

$$ {}_aD_x^{-n}v(x) = \frac{1}{\Gamma(n)} \int_a^x (x-\xi)^{n-1} v(\xi) d\xi, \quad x > a. \tag{2.1} $$

Clearly, (2.1) still makes sense if we replace n by α ($\in \mathbb{R}^+$), leading to the definition of the fractional integral,

$$ {}_aD_x^{-\alpha}v(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} v(\xi) d\xi, \quad x > a, \quad \alpha \in \mathbb{R}^+, \tag{2.2} $$

where $a \in \mathbb{R}$ and a can be $-\infty$.

With this insight, it would appear the related definition of the fractional derivative can be obtained by simply mixing fractional integration with classical derivatives. However, one quickly observes that there are several options for this definition. Indeed, there are three such definitions, not completely equivalent, of fractional derivatives and in this work we shall focus on the closely related Riemann-Liouville derivative, and the Caputo derivative, respectively. The third form, known as the Grünwald-Letnikov derivative is based on ideas of close to finite difference methods, is equivalent to the Riemann-Liouville derivative provided the functions they perform are sufficiently smooth.

The Riemann-Liouville derivative is recovered by first performing integration followed by classic differential, leading to the definition

$$ {}_aD_x^{\alpha}v(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\alpha-1} v(\xi) d\xi, \quad x > a, \quad \alpha \in [n-1, n). \tag{2.3} $$

Reversing the order of integration and differentiation results in what is known as the Caputo derivative on the form

$$ {}_a^CD_x^{\alpha}v(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n v(\xi)}{d\xi^n} d\xi, \quad x > a, \quad \alpha \in [n-1, n). \tag{2.4} $$

From an analysis point of view, the Riemann-Liouville derivative is natural since we can define the operator ${}_aD_x^{\alpha}$ ($\alpha \in R$) which 'connects' $-\infty$ to $+\infty$. When α are negative integers, zero, and positive integers, they correspond exactly to multiple integrations, the identity operator, and the classical derivatives [9].

More precisely, for $\alpha \in R^+$ and n a natural number, if we denote $\lim\limits_{\alpha \to n^+}$ as the right limit and $\lim\limits_{\alpha \to n^-}$ the left limit, respectively, we have

$$ \lim_{\alpha \to 0^+} {}_aD_x^{-\alpha}v(x) = \lim_{\alpha \to 0^+} {}_aD_x^{\alpha}v(x) = v(x), \tag{2.5} $$

$$ \lim_{\alpha \to (n-1)^+} {}_aD_x^{-\alpha}v(x) = {}_aD_x^{-(n-1)}v(x), \quad \lim_{\alpha \to n^-} {}_aD_x^{-\alpha}v(x) = {}_aD_x^{-n}v(x), \tag{2.6} $$

$$ \lim_{\alpha \to (n-1)^+} {}_aD_x^{\alpha}v(x) = \frac{d^{n-1}v(x)}{dx^{n-1}}, \quad \lim_{\alpha \to n^-} {}_aD_x^{\alpha}v(x) = \frac{d^n v(x)}{dx^n}, \tag{2.7} $$

$$(2.8)\quad \lim_{\alpha\to(n-1)^+} {}_a^C D_x^\alpha v(x) = \frac{d^{n-1}v(x)}{dx^{n-1}} - \left.\frac{d^{n-1}v(x)}{dx^{n-1}}\right|_{x=a}, \quad \lim_{\alpha\to n^-} {}_a^C D_x^\alpha v(x) = \frac{d^n v(x)}{dx^n}.$$

The goal in this work is the develop computational methods which mirror these attractive properties.

The main advantage of using the definition based on the Caputo derivative is the ease by which we can specify the proper initial and boundary conditions of the fractional differential equations. Following (2.4), we simply need to give the values of $\frac{d^k v(x)}{dx^k}$, $k = 0, 1, \cdots, n-1$ at the boundaries, similar to the situation for the classical differential equations. However, for the Riemann-Liouville derivative (2.3), we need to specify values of ${}_aD_x^{k+\alpha-n}v(x)$, $k = 0, 1, \cdots, n-1$ at the boundaries. While not a fundamental obstacle, it may appear as a practical problem when solving problems for which the simple and geometric interpretation of the classic derivatives are more natural.

In this paper, we will exploit the advantages of both of the two forms of fractional derivatives and focus on problems where (1.1) is equivalent under the two forms. Situations for which this true is understood through [14],

$$(2.9)\qquad {}_aD_x^\alpha v(x) = {}_a^C D_x^\alpha v(x) \;\text{ if }\; v^{(k)}(a) = 0, \quad k = 0, 1, \cdots, n-1,$$

where n is the smallest integer greater than or equal to α, and $v(x)$ is sufficiently smooth.

At times, (2.2) and (2.3) are referred to as the left Riemann-Liouville fractional integral and the left Riemann-Liouville fractional derivative, respectively. In some cases it is natural to consider the interval $[x, b]$ instead of $[a, x]$, leading to the right Riemann-Liouville fractional integral being defined as

$$(2.10)\qquad {}_xD_b^{-\alpha}v(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\xi - x)^{\alpha-1}v(\xi)d\xi, \quad x < b, \quad \alpha \in \mathbb{R}^+,$$

where $b \in \mathbb{R}$ and b can be $+\infty$. For the two operators (2.2) and (2.10), we have the following result

LEMMA 2.1 ([11]). *The left and the right Riemann-Liouville fractional integral operators are adjoint w.r.t. the $L^2(a, b)$ inner product, i.e.,*

$$(2.11)\qquad \left({}_aD_x^{-\alpha}u, v\right)_{L^2(a,b)} = \left(u, {}_xD_b^{-\alpha}v\right)_{L^2(a,b)} \quad \forall\alpha > 0, \; a < b.$$

2.2. Negative fractional norms and fractional integral spaces. Recall the Fourier transform $\hat{v}(\omega)$ of $v(x)$ being defined as

$$\mathcal{F}(v(x)) = \int_{-\infty}^{+\infty} e^{-ix\cdot\omega}v(x)dx = \hat{v}(\omega)$$

As in classic Fourier analysis, we define a higher norms for functions in $H^{-\alpha}(\mathbb{R})$ in terms of the Fourier transform.

DEFINITION 2.2. *Let $\alpha > 0$. Define the norm*

$$(2.12)\qquad \|v\|_{H^{-\alpha}(\mathbb{R})} := \| \, |\omega|^{-\alpha}\hat{v}\|_{L^2(\mathbb{R})},$$

and let $H^{-\alpha}(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ w.r.t. $\|\cdot\|_{H^{-\alpha}(\mathbb{R})}$.

In a similar fashion, we define the norm associated with the left Riemann-Liouville derivative

DEFINITION 2.3. *Let $\alpha > 0$. Define the norm*

$$\|v\|_{J_L^{-\alpha}(\mathbb{R})} := \|{}_{-\infty}D_x^{-\alpha} v\|_{L^2(\mathbb{R})}, \tag{2.13}$$

and let $J_L^{-\alpha}(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ w.r.t. $\|\cdot\|_{J_L^{-\alpha}(\mathbb{R})}$,

as well as the the norm associated with the left Riemann-Liouville derivative

DEFINITION 2.4. *Let $\alpha > 0$. Define the norm*

$$\|v\|_{J_R^{-\alpha}(\mathbb{R})} := \|{}_xD_\infty^{-\alpha} v\|_{L^2(\mathbb{R})}, \tag{2.14}$$

and let $J_L^{-\alpha}(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ w.r.t. $\|\cdot\|_{J_L^{-\alpha}(\mathbb{R})}$.

The three norms are closely related as stated in the following result

THEOREM 2.5. *The three spaces $H^{-\alpha}$, $J_L^{-\alpha}$, and $J_R^{-\alpha}$ are equal with equivalent norms.*

Proof. The Fourier transforms of the left and right Riemann-Liouville fractional integrals are, respectively,

$$\mathcal{F}({}_{-\infty}D_x^{-\alpha} v(x)) = (i\omega)^{-\alpha}\hat{v}(\omega); \quad \text{and} \quad \mathcal{F}({}_xD_\infty^{-\alpha} v(x)) = (-i\omega)^{-\alpha}\hat{v}(\omega).$$

Since

$$|(i\omega)^{-\alpha}| = |(-i\omega)^{-\alpha}| = |\omega|^{-\alpha}.$$

Applying Plancherel's theorem, we have

$$\|v\|_{H^{-\alpha}(\mathbb{R})} = \|\,|\omega|^{-\alpha}\hat{v}\|_{L^2(\mathbb{R})} = \|v\|_{J_L^{-\alpha}} = \|v\|_{J_R^{-\alpha}}.$$ ∎

Using the same idea in the proof of Lemma 2.4 of [11], we obtain

LEMMA 2.6.

$$({}_{-\infty}D_x^{-\alpha} v,\, {}_xD_\infty^{-\alpha} v) = \cos(\alpha\pi)\|D^{-\alpha}v\|^2_{L^2(\mathbb{R})} = \cos(\alpha\pi)\|v\|_{H^{-\alpha}(\mathbb{R})}. \tag{2.15}$$

Let us now restrict attention to the case in which $\mathrm{supp}(v) \subset \Omega = (a, b)$. Then ${}_{-\infty}D_x^{-\alpha} v = {}_aD_x^{-\alpha} v$, and ${}_xD_\infty^{-\alpha} v = {}_xD_b^{-\alpha} v$. Straightforward extension of the definitions given above yields

DEFINITION 2.7. *Define the spaces $H_0^{-\alpha}(\Omega)$, $J_{L,0}^{-\alpha}(\Omega)$, and $J_{R,0}^{-\alpha}(\Omega)$ as the closures of $C_0^\infty(\Omega)$ under their respective norms.*

The following theorem gives the relations among the fractional integral spaces with different α.

THEOREM 2.8. *If $-\alpha_2 < -\alpha_1 < 0$, then $J_{L,0}^{-\alpha_1}(\Omega)$ ($H_0^{-\alpha_1}(\Omega)$ or $J_{R,0}^{-\alpha_1}(\Omega)$) is embedded into $J_{L,0}^{-\alpha_2}(\Omega)$ ($H_0^{-\alpha_2}(\Omega)$ or $J_{R,0}^{-\alpha_2}(\Omega)$), and $L^2(\Omega)$ is embedded into both of them.*

Proof. Using the semigroup properties of fractional integral operators [4], we have

$$ {}_aD_x^{-\alpha_2}v = {}_aD_x^{-(\alpha_2-\alpha_1)}\,{}_aD_x^{-\alpha_1}v. $$

Further based on the Young's inequality [1] and the definition of the fractional integral, we have

$$
\begin{aligned}
\|v\|_{J_{L,0}^{-\alpha_2}(\Omega)} &= |{}_aD_x^{-\alpha_2}v|_{L^2(\Omega)} = |{}_aD_x^{-(\alpha_2-\alpha_1)}\,{}_aD_x^{-\alpha_1}v|_{L^2(\Omega)} \\
&= \tfrac{1}{\Gamma(\alpha_2-\alpha_1)}|x^{\alpha_2-\alpha_1-1} * {}_aD_x^{-\alpha_1}v|_{L^2(\Omega)} \\
&\le \tfrac{1}{\Gamma(\alpha_2-\alpha_1)}\|x^{\alpha_2-\alpha_1-1}\|_{L^1(\Omega)}\cdot|{}_aD_x^{-\alpha_1}v|_{L^2(\Omega)} \\
&= \tfrac{|a|^{\alpha_2-\alpha_1}+|b|^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2-\alpha_1+1)}\|v\|_{J_{L,0}^{-\alpha_1}(\Omega)}.
\end{aligned}
$$

Continuing in a similar fashion, we have

$$ \|v\|_{J_{L,0}^{-\alpha_1}(\Omega)} = \|{}_aD_x^{-\alpha_1}v\|_{L^2(\Omega)} \le \frac{|a|^{\alpha_1}+|b|^{\alpha_1}}{\Gamma(\alpha_1+1)}\|v\|_{L^2(\Omega)}, $$

from which the result follows. □.

3. The local discontinuous Galerkin schemes for the fractional diffusion equation. The fractional derivative in (1.2) can be treated in two ways: the Riemann-Liouville fractional derivative of order β, ${}_aD_x^{\beta}u(x)$ or, alternatively, one classical derivative acting on the Riemann-Liouville fractional derivative of order $\beta-1$. Assuming that $u(a,t)=0$ it follows from (2.9) that ${}_aD_x^{\beta-1}u(x,t) = {}_a^CD_x^{\beta-1}u(x,t)$, and we can easily identify and impose the boundary conditions of (1.1). This allows us to more clearly describe the model discussed here

$$ \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\, d\, {}_aD_x^{\beta-2}\frac{\partial}{\partial x}u(x,t) + f(x,t) \quad \text{in } \Omega_T = (a,b)\times(0,T), \tag{3.1} $$

with the initial condition

$$ u(x,0) = u_0(x) \quad \text{on } \Omega = (a,b), \tag{3.2} $$

and the Dirichlet boundary conditions

$$ u(a,t) = 0, \quad u(b,t) = g(t) \quad \text{on } (0,T), \tag{3.3} $$

where $\beta \in [1,2]$, $d>0$. In the special case of $\beta=1$, the equation degenerates to a pure convection problem only the boundary condition at $x=b$ is required.

It is worth noting that the decomposition in (3.1) is not unique, i.e., one could express the fractional derivative in several different ways such as

$$ \frac{\partial^2}{\partial x^2}\,{}_aD_x^{\beta-2}, \quad \frac{\partial}{\partial x}\,{}_aD_x^{\beta-2}\frac{\partial}{\partial x}, \quad {}_aD_x^{\beta-2}\frac{\partial^2}{\partial x^2}, $$

by combining the classic and Riemann-Liouville operators in different ways. The choice we propose here is not only related to the need to specify boundary conditions in a straightforward manner, but the order also impacts the properties of the numerical scheme.

3.1. The weak formulation. Following the standard approach for the development of local discontinuous Galerkin methods for problems with higher derivatives [6, 18, 13], let us introduce the auxiliary variables p and q, and rewrite (3.1)–(3.3) as

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - \sqrt{d}\frac{\partial q(x,t)}{\partial x} = f(x,t) && \text{in } \Omega_T,\\
&q - {}_aD_x^{\beta-2}p(x,t) = 0 && \text{in } \Omega_T,\\
&p - \sqrt{d}\frac{\partial u(x,t)}{\partial x} = 0 && \text{in } \Omega_T,\\
&u(x,0) = u_0(x) && \text{on } \Omega,\\
&u(a,t) = 0, \quad u(b,t) = g(t) && \text{on } (0,T).
\end{aligned}
\tag{3.4}
$$

Using standard notation, given the nodes $a = x_0 < x_1 < \cdots < x_{M-1} < x_M = b$, we define the mesh $\mathcal{T} = \{I_j = (x_{j-1}, x_j),\ j = 1, \ldots, M\}$ and set $h_j := |I_j| = x_j - x_{j-1}$; and $h := \max_{j=1}^{M} h_j$. Associated with the mesh $\mathcal{T}$, we define the broken Sobolev spaces

$$L^2(\Omega, \mathcal{T}) := \{v : \Omega \to \mathbb{R} \mid v|_{I_j} \in L^2(I_j),\ j = 1, \ldots, M\};$$

and

$$H^1(\Omega, \mathcal{T}) := \{v : \Omega \to \mathbb{R} \mid v|_{I_j} \in H^1(I_j),\ j = 1, \ldots, M\}.$$

For a function $v \in H^1(\Omega, \mathcal{T})$, we denote the one-sided limits at the nodes $\{x_j\}$ by

$$v^{\pm}(x_j) = v(x_j^{\pm}) := \lim_{x \to x_j^{\pm}} v(x). \tag{3.5}$$

We shall consider two slightly different schemes and will assume that the exact solution $\mathbf{w} = (u, p, q)$ of (3.4) belongs to

$$H^1(0,T;H^1(\Omega,\mathcal{T})) \times L^2(0,T;L^2(\Omega,\mathcal{T})) \times L^2(0,T;H^1(\Omega,\mathcal{T})) \tag{3.6}$$

and

$$H^1(0,T;H^1(\Omega,\mathcal{T})) \times L^2(0,T;H^1(\Omega,\mathcal{T})) \times L^2(0,T;H^1(\Omega,\mathcal{T})) \tag{3.7}$$

in two different schemes, respectively. This assumption is reasonable since both $H^1(\Omega)$ and $L^2(\Omega)$ are embedded in the fractional integral spaces.

Under these assumptions we require that $\mathbf{w}$ satisfies

$$\left(\frac{\partial u(x,t)}{\partial t}, v\right)_{I_j} + \sqrt{d}\left(q(x,t), \frac{\partial v}{\partial x}\right)_{I_j} - \sqrt{d}q(x,t)v\Big|_{x_{j-1}^+}^{x_j^-} = (f, v)_{I_j}, \tag{3.8}$$

$$(q, w)_{I_j} - \left(\frac{\partial^{\beta-2}p(x,t)}{\partial x^{\beta-2}}, w\right)_{I_j} = 0, \tag{3.9}$$

$$(p, z)_{I_j} + \sqrt{d}\left(u(x,t), \frac{\partial z}{\partial x}\right)_{I_j} - \sqrt{d}u(x,t)z\Big|_{x_{j-1}^+}^{x_j^-} = 0, \tag{3.10}$$

$$(u(\cdot,0), v)_{I_j} = (u_0(\cdot), v)_{I_j}, \tag{3.11}$$

for all test functions $w \in L^2(\Omega, \mathcal{T})$, and $v, z \in H^1(\Omega, \mathcal{T})$, and for $j = 1, \dots, M$; Here the time derivative is understood in the weak sense and $(u, v)_I = \int_I u(x)v(x)dx$ is the standard inner product over the element.

3.2. The numerical schemes. In the following we shall propose numerical schemes for (3.1)–(3.3) based on the equations (3.8)–(3.11). The global nature of the fractional derivative is reflected in (3.9) while (3.8) and (3.10) remain local as in a more traditional discontinuous Galerkin formulations.

We now restrict the trial and test functions v, w, and z to the finite dimensional subspaces $V \subset H^1(\Omega, \mathcal{T})$, and choose V to be the space of discontinuous, piecewise polynomial functions

$$V = \{v : \Omega \to \mathbb{R} |\, v|_{I_j} \in \mathcal{P}^k(I_j),\, j = 1, \dots, M\},$$

where $\mathcal{P}^k(I_j)$ denotes the set of all polynomials of degree less than or equal k ($\geqslant 1$) on I_j. Furthermore, we define U, P, and Q as the approximations of u, p, and q, respectively, in the space V. We then need to find $(U, P, Q) \in H^1(0, T; V) \times L^2(0, T; V) \times L^2(0, T; V)$ such that for all v, w, and $z \in V$, and for $j = 1, \dots, M$ the following holds:

$$\left(\frac{\partial U(x,t)}{\partial t}, v\right)_{I_j} + \sqrt{d}\left(Q(x,t), \frac{\partial v}{\partial x}\right)_{I_j} - \sqrt{d}\hat{Q}(x,t)v\Big|_{x_{j-1}^+}^{x_j^-} = (f, v)_{I_j}, \tag{3.12}$$

$$(Q, w)_{I_j} - \left(\frac{\partial^{\beta-2} P(x,t)}{\partial x^{\beta-2}}, w\right)_{I_j} = 0, \tag{3.13}$$

$$(P, z)_{I_j} + \sqrt{d}\left(U(x,t), \frac{\partial z}{\partial x}\right)_{I_j} - \sqrt{d}\hat{U}(x,t)z\Big|_{x_{j-1}^+}^{x_j^-} = 0, \tag{3.14}$$

$$(U(\cdot, 0), v)_{I_j} = (u_0(\cdot), v)_{I_j}. \tag{3.15}$$

To complete the formulation of the numerical schemes, we must define the numerical fluxes $\hat{Q}(x,t)$ and $\hat{U}(x,t)$. As with traditional local discontinuous Galerkin methods, this choice is the most delicate one as it determines not only locality but also consistency, stability, and order of convergence of the scheme. Seeking inspiration in the mixed formulation for the heat equation we use the 'alternating principle' [18] in choosing the numerical fluxes for (3.12) and (3.14), that is, we choose

$$\hat{Q}(x_j, t) = Q^+(x_j, t), \quad \hat{U}(x_j, t) = U^-(x_j, t); \tag{3.16}$$

or

$$\hat{Q}(x_j, t) = Q^-(x_j, t), \quad \hat{U}(x_j, t) = U^+(x_j, t); \tag{3.17}$$

at all interior boundaries. At the external boundaries we use

$$\hat{Q}(a, t) = Q^+(a, t) = Q^-(a, t), \quad \hat{Q}(b, t) = Q^-(b, t) = Q^+(b, t); \tag{3.18}$$

and

$$\hat{U}(a,t)=0, \quad \hat{U}(b,t)=g(t), \tag{3.19}$$

reflecting a Dirichlet boundary.

We shall shortly show, theoretically and numerically, that the scheme, (3.12)–(3.15) with fluxes (3.16)–(3.19), is stable for any $\beta \in [1,2]$; and it has the optimal convergent order $k+1$ for $\beta \in (1,2]$ and suboptimal convergent order k for $\beta = 1$.

To address this loss of optimality, we observe that when $\beta = 1$ the numerical dissipation term disappears. We therefore introduce dissipation to enhance the stability of the scheme at $\beta = 1$ by adding a penalty term in (3.13) to recover an alternative scheme

$$(Q,w)_{I_j} - \left(\frac{\partial^{\beta-2}P(x,t)}{\partial x^{\beta-2}}, w\right)_{I_j} + C(h,\beta)\cdot\left(\hat{P}(x,t)w\right)\Big|_{x_{j-1}^+}^{x_j^-} = 0, \tag{3.20}$$

where

$$\begin{aligned} \hat{P}(x_j^-,t) &= [P(x_j,t)] := P(x_j^+,t) - P(x_j^-,t); \\ \hat{P}(x_{j-1}^+,t) &= [P(x_{j-1},t)] := P(x_{j-1}^+,t) - P(x_{j-1}^-,t). \end{aligned} \tag{3.21}$$

Here $C(h,\beta)$ is a constant depending on β and h, the local cell size. As we shall confirm numerically, the order of convergence is optimal $k+1$ for any $\beta \in [1,2]$, but we are unable to establish this theoretically as the added penalty term does not improve control over the dominating error terms.

If the value of $C(h,\beta)$ is at least of order h, the order of convergence is $k+1$ for $\beta \in (1,2]$ (see Theorem 4.3). In the computational examples we take $C(h,\beta) = h^\beta$ in agreement with the scaling of the global operator

In the following, we refer to the scheme in (3.12)–(3.15) with fluxes (3.16) (or (3.17)) and (3.18)–(3.19) as Scheme I; and the penalized scheme (3.12), (3.20), (3.14), and (3.15) with fluxes (3.16) (or (3.17)), (3.21), and (3.18)–(3.19) as Scheme II.

4. Stability and error estimates. In this section we will develop the main theoretical analysis. Both schemes are expressed as: Find $(U,P,Q) \in H^1(0,T;V) \times L^2(0,T;V) \times L^2(0,T;V)$ such that for all $(v,w,z) \in H^1(0,T;V) \times L^2(0,T;V) \times L^2(0,T;V)$, the following holds

$$B_1(U,P,Q;\, v,w,z) = \mathcal{L}(v,w,z) \tag{4.1}$$

and

$$B_2(U,P,Q;\, v,w,z) = \mathcal{L}(v,w,z), \tag{4.2}$$

for Scheme I and II, respectively. Here $(U(\cdot,0), v(\cdot,0)) = (u_0(\cdot), v(\cdot,0))$ and the discrete bilinear form B_1 and B_2 are defined as

$$
\begin{aligned}
B_1(U,P,Q;\,v,w,z) \quad :=\; & \int_0^T \left(\frac{\partial U(\cdot,t)}{\partial t}, v(\cdot,t)\right) dt \\
& +\sqrt{d}\int_0^T \left(Q(x,t), \frac{\partial v(x,t)}{\partial x}\right) dt + \int_0^T \left(Q(\cdot,t), w(\cdot,t)\right) dt \\
& -\int_0^T \left(\frac{\partial^{\beta-2} P(x,t)}{\partial x^{\beta-2}}, w(x,t)\right) dt + \int_0^T (P(\cdot,t), z(\cdot,t)) dt \\
& +\sqrt{d}\int_0^T \left(U(x,t), \frac{\partial z(x,t)}{\partial x}\right) dt \\
& +\sqrt{d}\int_0^T \sum_{j=1}^{M-1} \hat{Q}(x_j,t)[v](x_j,t) dt \\
& +\sqrt{d}\int_0^T (Q^+(a,t)v^+(a,t) - Q^-(b,t)v^-(b,t)) dt \\
& +\sqrt{d}\int_0^T \sum_{j=1}^{M-1} \hat{U}(x_j,t)[z](x_j,t) dt,
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
B_2(U,P,Q;\,v,w,z) :=\; & B_1(U,P,Q;\,v,w,z) \\
& -C(h,\beta)\int_0^T \sum_{j=1}^{M-1} [P](x_j,t)[w](x_j,t) dt.
\end{aligned}
\tag{4.4}
$$

The discrete linear form $\mathcal{L}$ is given by

$$
\mathcal{L}(v,w,z) = \int_0^T (f(\cdot,t), v(\cdot,t)) dt + \sqrt{d}\int_0^T g(t) z(b,t) dt. \tag{4.5}
$$

Since Scheme I is consistent with (3.4), the exact solution (u,p,q) of (3.4) in the space (3.6) satisfies

$$
B_1(u,p,q;\,v,w,z) = \mathcal{L}(v,w,z) \tag{4.6}
$$

for all $(v,w,z) \in H^1(0,T;V) \times L^2(0,T;V) \times L^2(0,T;V)$. Likewise, Scheme II is consistent with (3.4), and the exact solution (u,p,q) of (3.4) in the space (3.7) satisfies

$$
B_2(u,p,q;\,v,w,z) = \mathcal{L}(v,w,z) \tag{4.7}
$$

for all $(v,w,z) \in H^1(0,T;V) \times L^2(0,T;V) \times L^2(0,T;V)$.

4.1. Numerical stability. Let $(\widetilde{U},\widetilde{P},\widetilde{Q}) \in H^1(0,T;V)\times L^2(0,T;V)\times L^2(0,T;V)$ be the approximate solution of (U,P,Q). We denote $e_U := \widetilde{U}-U$, $e_P := \widetilde{P}-P$, and $e_Q := \widetilde{Q}-Q$ as the errors. Stability of the two schemes is established in the following theorem.

THEOREM 4.1 (L^2 stability). *Schemes (4.1)-(4.2) are L^2 stable, and for all $t \in [0,T]$ their solutions satisfy*

$$\|e_U^2(\cdot,t)\|_{L^2(\Omega)} = \|e_U^2(\cdot,0)\|_{L^2(\Omega)} - 2\cos\left((\beta/2-1)\,\pi\right)\int_0^t \|e_P(\cdot,t)\|^2_{H^{-(1-\frac{\beta}{2})}(\Omega)}dt \tag{4.8}$$

and

$$\begin{aligned}\|e_U^2(\cdot,t)\|_{L^2(\Omega)} = \;& \|e_U^2(\cdot,0)\|_{L^2(\Omega)} - 2\cos\left((\beta/2-1)\,\pi\right)\int_0^t \|e_P(\cdot,t)\|^2_{H^{-(1-\frac{\beta}{2})}(\Omega)}dt \\ & -2\cdot C(h,\beta)\int_0^t \sum_{j=1}^{M-1}[e_P]^2(x_j,t)dt,\end{aligned} \tag{4.9}$$

respectively.

Note. If $\beta=1$ in (4.8), then $\|e_U^2(\cdot,t)\|_{L^2(\Omega)} = \|e_U^2(\cdot,0)\|_{L^2(\Omega)}$, that is numerical dissipation term disappears. In contrast to that, even if $\beta = 1$ in (4.9) the penalty terms maintains dissipation.

Proof of Theorem 4.1. We just prove the case $t=T$ (when $t=T$ the theorem holds then the theorem holds for any $t\in[0,T]$) and for (4.9). The corresponding result for (4.8) follows by taking $C(h,\beta)=0$.

From (4.2) we recover the perturbation equation

$$B_2(e_U,e_P,e_Q;\,v,w,z)=0, \tag{4.10}$$

for all $(v,w,z)\in H^1(0,T;V)\times L^2(0,T;V)\times L^2(0,T;V)$. Taking $v=e_U$, $w=-e_P$, $z=e_Q$, and using Lemma 2.1 and 2.6, we obtain,

$$\begin{aligned} 0 = \;& B_2(e_U,e_P,e_Q;\,e_U,-e_P,e_Q) \\ = \;& \frac{1}{2}\int_0^T \frac{\partial}{\partial t}\|e_U^2(\cdot,t)\|_{L^2(\Omega)}dt + \cos((\beta/2-1)\pi)\int_0^T \|e_P^2\|_{H^{-(1-\frac{\beta}{2})}(\Omega)}dt \\ & + C(h,\beta)\int_0^T \sum_{j=1}^{M-1}[e_P]^2(x_j,t)dt + \sqrt{d}\int_0^T\int_a^b \frac{\partial(e_U\cdot e_Q)}{\partial x}dxdt \\ & + \sqrt{d}\int_0^T \sum_{j=1}^{M-1}\hat{e}_Q(x_j,t)[e_U](x_j,t)dt + \sqrt{d}\int_0^T \sum_{j=1}^{M-1}\hat{e}_U(x_j,t)[e_Q](x_j,t)dt \\ & + \sqrt{d}\int_0^T \left(e_Q^+(a,t)e_U^+(a,t) - e_Q^-(b,t)e_U^-(b,t)\right)dt. \end{aligned} \tag{4.11}$$

In (4.11),

$$\int_0^T\int_a^b \frac{\partial(e_U\cdot e_Q)}{\partial x}dxdt = \int_0^T \Big(-e_Q^+(a,t)e_U^+(a,t)+e_Q^-(b,t)e_U^-(b,t)\Big)\,dt \tag{4.12}$$
$$-\int_0^T\sum_{j=1}^{M-1}[e_U\cdot e_Q](x_j,t),$$

and when $\hat{e}_Q=e_Q^+$ and $\hat{e}_U=e_U^-$ (or $\hat{e}_Q=e_Q^-$ and $\hat{e}_U=e_U^+$),

$$\int_0^T\sum_{j=1}^{M-1}\hat{e}_Q(x_j,t)[e_U](x_j,t)+\sum_{j=1}^{M-1}\hat{e}_U(x_j,t)[e_Q](x_j,t)dt \tag{4.13}$$
$$=\int_0^T\sum_{j=1}^{M-1}[e_U\cdot e_Q](x_j,t)dt.$$

Combining (4.11) and (4.12)–(4.13), the desired result is obtained. ∎

4.2. Error estimates. For the error analysis, we shall define the projection operators $\mathrm{P}^{\pm}$, S, and S′ from $H^1(\Omega,\mathcal{T})$ to V. For all the intervals $I_j=(x_{j-1},x_j)$, $j=1,2,\cdots,M$, $\mathrm{P}^{\pm}$ are defined to satisfy the $k+1$ conditions:

$$(\mathrm{P}^{\pm}p-p,w)_{I_j}=0 \quad \forall w\in\mathcal{P}^{k-1}(I_j),\ \text{if } k>0, \tag{4.14}$$
$$\mathrm{P}^-p(x_j)=p^-(x_j)\quad \mathrm{P}^+p(x_{j-1})=p^+(x_{j-1}).$$

S and S′ are the standard L^2-projections, which are defined, repectively, as

$$(\mathrm{S}u-u,v)_{I_j}=0\quad \forall v\in\mathcal{P}^k(I_j), \tag{4.15}$$

$$(\mathrm{S}'u-u,v)_{I_j}=0\quad \forall v\in\mathcal{P}^{k-1}(I_j),\ \text{if } k>0. \tag{4.16}$$

We are now ready to state our results and then prove them.

THEOREM 4.2 (Error estimate for Scheme I). *The error for the scheme (3.12)–(3.15) with fluxes (3.16)–(3.19) applied to the model (3.1)-(3.3) satisfies*

$$\sqrt{\int_a^b(u(x,t)-U(x,t))^2dx}\leqslant\begin{cases} c(\beta)h^{k+1} & \text{for } 1<\beta\leqslant 2,\\ ch^k & \text{for } \beta=1,\end{cases} \tag{4.17}$$

where $c(\beta)$ *and* c *depend on* $\frac{\partial^{k+1}U(x,t)}{\partial x^{k+1}}$, $\frac{\partial^{k+\beta-1}U(x,t)}{\partial x^{k+\beta-1}}$, $\frac{\partial^{k+\beta}U(x,t)}{\partial x^{k+\beta}}$, *and time* t.

Similarly we have the error estimate for Scheme II:

THEOREM 4.3 (Error estimate for Scheme II). *The error for the scheme (3.12), (3.20), (3.14), and (3.15) with fluxes (3.16)–(3.19) and (3.21) applied to the model (3.1)-(3.3) satisfies*

$$\sqrt{\int_a^b (u(x,t)-U(x,t))^2dx} \leqslant \begin{cases} \left(\sqrt{\frac{C(h,\beta)}{h}}c + c(\beta)\right) h^{k+1} & \text{for } 1<\beta\leqslant 2, \\ ch^k & \text{for } \beta = 1, \end{cases} \tag{4.18}$$

where $c(\beta)$ *and* c *depend on* $\frac{\partial^{k+1}U(x,t)}{\partial x^{k+1}}$, $\frac{\partial^{k+\beta-1}U(x,t)}{\partial x^{k+\beta-1}}$, $\frac{\partial^{k+\beta}U(x,t)}{\partial x^{k+\beta}}$, *and time* t.

We shall begin by proving Theorem 4.2 for Scheme I, and then give a sketch for the proof of Theorem 4.3 based on this.

Proof of Theorem 4.2. We denote

$$e_u = u(x,t)-U(x,t), \quad e_p = p(x,t)-P(x,t), \quad e_q = q(x,t)-Q(x,t).$$

From (4.1) and (4.6), we recover the error equation

$$B_1(e_u,e_p,e_q;\, v,w,z) = 0 \tag{4.19}$$

for all $(v,w,z)\in H^1(0,T;V)\times L^2(0,T;V)\times L^2(0,T;V)$. Take

$$v = \mathrm{P}^{\pm}u - U, \quad w = P - \mathrm{S}p, \quad z = \mathrm{P}^{\mp}q - Q$$

in (4.19). After rearranging terms, we obtain

$$B_1(v,-w,z;\, v,w,z) = B_1(v^e,-w^e,z^e;\, v,w,z), \tag{4.20}$$

where v^e, w^e, and z^e are given as

$$v^e = \mathrm{P}^{\pm}u - u, \quad w^e = p - \mathrm{S}p, \quad z^e = \mathrm{P}^{\mp}q - q.$$

Following the discussion in the proof of Theorem 4.1 the left hand side of (4.20) becomes

$$\begin{aligned} &B_1(v,-w,z;\, v,w,z) \\ &= \frac{1}{2}\int_0^T \frac{\partial}{\partial t}\|v(\cdot,t)\|^2_{L^2(\Omega)}dt + \cos((\beta/2-1)\pi)\int_0^T \|w(\cdot,t)\|^2_{H^{-(1-\frac{\beta}{2})}(\Omega)}dt. \end{aligned} \tag{4.21}$$

Using the notation in [18], the right hand side of (4.20) can be expressed as

$$B_1(v^e,-w^e,z^e;\, v,w,z) = \mathcal{I}+\mathcal{II}+\mathcal{III}+\mathcal{IV}+\mathcal{V}, \tag{4.22}$$

where

$$\mathcal{I} = \int_0^T \left(\frac{\partial v^e(\cdot,t)}{\partial t}, v(\cdot,t)\right)dt, \tag{4.23}$$

$$\mathcal{II} = \sqrt{d}\int_0^T \left(z^e(x,t), \frac{\partial v(x,t)}{\partial x}\right) + \sqrt{d}\int_0^T \left(v^e(x,t), \frac{\partial z(x,t)}{\partial x}\right) - \int_0^T (w^e(\cdot,t), z(\cdot,t)), \tag{4.24}$$

$$\mathcal{III} = \sqrt{d}\int_0^T \sum_{j=1}^{M-1} \hat{z}^e(x_j,t)[v](x_j,t)dt + \sqrt{d}\int_0^T \sum_{j=1}^{M-1} \hat{v}^e(x_j,t)[z](x_j,t)dt, \tag{4.25}$$

$$\mathcal{IV} = \sqrt{d}\int_0^T ((z^e)^+(a,t)v^+(a,t) - (z^e)^-(b,t)v^-(b,t))dt, \tag{4.26}$$

and

$$\mathcal{V} = \int_0^T (z^e(\cdot,t), w(\cdot,t))dt + \int_0^T \left(\frac{\partial^{\beta-2} w^e(x,t)}{\partial x^{\beta-2}}, w(x,t)\right) dt. \tag{4.27}$$

Using standard approximation theory [8], we obtain

$$\begin{aligned} \mathcal{I} &\leqslant \frac{1}{2}\int_0^T \int_a^b \left(\frac{\partial v^e(x,t)}{\partial t}\right)^2 dxdt + \int_0^T \int_a^b \left(\frac{v^2(x,t)}{2}\right) dxdt \\ &\leqslant ch^{2k+2} + \frac{1}{2}\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt, \end{aligned} \tag{4.28}$$

where c is constant.

All the terms in $\mathcal{II}$ are vanish due to Galerkin orthogonality, i.e., $p - \mathrm{S}p$ is orthogonal to all polynomials of degree up to k, and $\mathrm{P}^{\pm}u - u$ and $\mathrm{P}^{\pm}q - q$ to $k-1$. For the terms in $\mathcal{III}$, when taking $\hat{z}^e = (z^e)^-$ and $\hat{v}^e = (v^e)^+$, we use $z^e = \mathrm{P}^-q - q$ and $v^e = \mathrm{P}^+u - u$; and when giving $\hat{z}^e = (z^e)^+$ and $\hat{v}^e = (v^e)^-$, we choose $z^e = \mathrm{P}^+q - q$ and $v^e = \mathrm{P}^-u - u$. Hence, both terms in $\mathcal{III}$ are zero and so is $\mathcal{III}$. An application of the inequality $xy \leqslant \frac{1}{2}(x^2 + y^2)$, the standard approximation on point values of z^e, and the equivalence of norms in finite dimensional spaces implies

$$\begin{aligned} \mathcal{IV} &\leqslant \frac{\sqrt{d}}{2}\int_0^T \left(((z^e)^+(a,t))^2 + ((z^e)^-(b,t))^2 + (v^+(a,t))^2 + (v^-(b,t))^2\right) dt \\ &\leqslant ch^{2k+2} + c\int_0^T \|v(\cdot,t)\|^2_{L^\infty(\Omega)}dt \\ &\leqslant ch^{2k+2} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt. \end{aligned} \tag{4.29}$$

One of the terms in $\mathcal{IV}$ is exactly zero. When $z^e = \mathrm{P}^+q - q$, the first term is zero; and when $z^e = \mathrm{P}^-q - q$, the second term is zero. For $\mathcal{V}$, we recover two different kinds of estimates corresponding to $\beta = 1$ and $\beta \in (1, 2]$ respectively. Both of them use Lemma 2.1.

When $\beta = 1$, further applying the operators $\mathcal{S}$ and $\mathcal{S}'$ defined in (4.15) and (4.16) yields

$$
\begin{aligned}
\mathcal{V} &= \int_0^T (z^e(\cdot,t), w(\cdot,t) - \mathcal{S}'w(\cdot,t)) + (z^e(\cdot,t), \mathcal{S}'w(\cdot,t))dt \\
&+ \int_0^T \left(w^e(\cdot,t), \int_x^b w(\xi,t)d\xi - \mathcal{S}\int_x^b w(\xi,t)d\xi\right) + \left(w^e(\cdot,t), \mathcal{S}\int_x^b w(\xi,t)d\xi\right) dt \\
&= \int_0^T (z^e(\cdot,t), w(\cdot,t) - \mathcal{S}'w(\cdot,t)) + \left(w^e(\cdot,t), \int_x^b w(\xi,t)d\xi - \mathcal{S}\int_x^b w(\xi,t)d\xi\right) dt \\
&\leqslant \int_0^T \int_a^b \left(\frac{(z^e(x,t))^2}{2}\right) dxdt + \int_0^T \int_a^b \left(\frac{(w(x,t) - \mathcal{S}'w(x,t))^2}{2}\right) dxdt \\
&+ \frac{1}{2}\int_0^T \int_a^b \left(\frac{(w^e(x,t))^2}{2} + \left(\int_x^b w(\xi,t)dx - \mathcal{S}\int_x^b w(\xi,t)d\xi\right)^2\right) dxdt \\
&\leqslant ch^{2k+2} + ch^{2k} + ch^{2k+2} + ch^{2k+2} \\
&\leqslant ch^{2k}.
\end{aligned}
\tag{4.30}
$$

When $\beta \in (1,2]$, further using the inequality $xy \leqslant \frac{x^2}{2\epsilon} + \frac{\epsilon y^2}{2}$ and the norm-equivalence gives

$$
\begin{aligned}
\mathcal{V} &\leqslant \int_0^T \int_a^b \frac{(z^e(x,t))^2}{2\epsilon} dxdt + \frac{\epsilon}{2}\int_0^T \|w(\cdot,t)\|^2_{L^2(\Omega)} dt \\
&+ \int_0^T \int_a^b \frac{(w^e(x,t))^2}{2\epsilon} dxdt + \frac{\epsilon}{2}\int_0^T \|w(\cdot,t)\|^2_{H^{-(2-\beta)}(\Omega)} dt \\
&\leqslant (c/\epsilon)h^{2k+2} + c\epsilon \int_0^T \|w(\cdot,t)\|^2_{H^{-(1-\frac{\beta}{2})}(\Omega)} dt,
\end{aligned}
\tag{4.31}
$$

where ϵ is a small number, chosen such that the term in this equation can be controlled by the corresponding term in (4.21), namely, choosing a sufficiently small ϵ such that $c\epsilon < \cos((\beta/2 - 1)\pi)$. Combining all the above estimates we have, for $\beta \in (1,2]$,

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \frac{\partial}{\partial t}\|v(\cdot,t)\|^2_{L^2(\Omega)} + (\cos((\beta/2-1)\pi) - c\epsilon)\int_0^T \|w(\cdot,t)\|^2_{H^{-(1-\frac{\beta}{2})}(\Omega)} dt \\
&\leqslant (c/\epsilon)h^{2k+2} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)} dt,
\end{aligned}
\tag{4.32}
$$

which we express as

$$\begin{aligned}
&\frac{1}{2}\|v(\cdot,T)\|^2_{L^2(\Omega)} + (\cos((\beta/2-1)\pi) - c\epsilon)\int_0^T \|w(\cdot,t)\|^2_{H^{-(1-\frac{\beta}{2})}(\Omega)}dt \\
&\leqslant \frac{1}{2}\|v(\cdot,0)\|^2_{L^2(\Omega)} + (c/\epsilon)h^{2k+2} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt \\
&\leqslant (c/\epsilon)h^{2k+2} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt.
\end{aligned} \tag{4.33}$$

For the case of $\beta = 1$,

$$\frac{1}{2}\int_0^T \frac{\partial}{\partial t}\|v(\cdot,t)\|^2_{L^2(\Omega)} \leqslant ch^{2k} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt, \tag{4.34}$$

such that

$$\begin{aligned}
\frac{1}{2}\|v(\cdot,T)\|^2_{L^2(\Omega)} &\leqslant \frac{1}{2}\|v(\cdot,0)\|^2_{L^2(\Omega)} + ch^{2k} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt \\
&\leqslant ch^{2k} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt.
\end{aligned} \tag{4.35}$$

According to Grönwall's lemma [8] and the standard approximation on $v^e = \mathrm{P}u - u$ the desired estimate (4.17) is obtained. ◻

Note. Since the control term for w in (4.21) disappears when $\beta = 1$, an attempt to overcome this is in the first term of $\mathcal{V}$ to take $z^e = \mathrm{S}q - q$ instead of $z^e = \mathrm{P}^{\mp}q - q$, such that this term vanishes. However, in this case the first term of $\mathcal{III}$ no longer vanishes and we have

$$\begin{aligned}
\mathcal{III} &\leqslant \sqrt{d}\int_0^T \sum_{j=1}^{M-1}\left(\frac{(\hat{z}^e(x_j,t))^2}{2} + \frac{([v](x_j,t))^2}{2}\right)dt \\
&\leqslant \int_0^T \sum_{j=1}^{M-1}\left(ch^{2k+2} + c\|v(\cdot,t)\|^2_{L^\infty(I_j)}\right)dt \\
&\leqslant ch^{2k+1} + \int_0^T \frac{c}{h}\|v(\cdot,t)\|^2_{L^2(\Omega)}dt.
\end{aligned} \tag{4.36}$$

This results in the final estimate being

$$\frac{1}{2}\|v(\cdot,T)\|^2_{L^2(\Omega)} \leqslant h^{2k+1} + \int_0^T \frac{c}{h}\|v(\cdot,t)\|^2_{L^2(\Omega)}dt. \tag{4.37}$$

This estimate appears to be useless. From the proof of the estimates for $\beta \in (1,2]$ we observe that if β decreases then the constant c/ϵ in front of the convergent order increases. This is because $\cos((\beta/2-1)\pi)$ decreases when β decreases, i.e., we can expect that $c(\beta)$ in the Theorem 4.2 increases as β decreases towards one. The numerical results in the next section will confirm this.

Sketch of the proof of Theorem 4.3 After proving Theorem 4.2 we just give the estimates of the new terms. From (4.2) and (4.7), the error equation is

$$B_2(e_u, e_p, e_q;\, v, w, z) = 0.$$

or

$$B_1(e_u, e_p, e_q;\, v, w, z) - C(h,\beta)\int_0^T \sum_{j=1}^{M-1} [e_p](x_j,t)[w](x_j,t)dt = 0. \tag{4.38}$$

After rearranging terms, we obtain

$$B_2(v, -w, z;\, v, w, z) = B_2(v^e, -w^e, z^e;\, v, w, z).$$

or

$$B_1(v, -w, z;\, v, w, z) + \mathcal{VI}' = B_1(v^e, -w^e, z^e;\, v, w, z) + \mathcal{VI}, \tag{4.39}$$

where

$$\mathcal{VI}' = C(h,\beta)\int_0^T \sum_{j=1}^{M-1} [w]^2(x_j,t)dt \tag{4.40}$$

and

$$\mathcal{VI} = C(h,\beta)\int_0^T \sum_{j=1}^{M-1} [w^e](x_j,t)[w](x_j,t)dt. \tag{4.41}$$

We now need to estimate $\mathcal{VI}$.

$$\begin{aligned} \mathcal{VI} &\leqslant C(h,\beta)\int_0^T \frac{1}{2}\sum_{j=1}^{M-1} \left([w^e]^2(x_j,t) + [w]^2(x_j,t)\right) dt \\ &\leqslant C(h,\beta)ch^{2k+1} + \frac{1}{2}\mathcal{VI}'. \end{aligned} \tag{4.42}$$

We now have the final estimate, for $\beta \in (1,2]$,

$$\begin{aligned} &\frac{1}{2}\|v(\cdot,T)\|^2_{L^2(\Omega)} + (\cos((\beta/2-1)\pi) - c\epsilon)\int_0^T \|w(\cdot,t)\|^2_{H^{-(1-\frac{\beta}{2})}(\Omega)}dt \\ &+\frac{C(h,\beta)}{2}\int_0^T \sum_{j=1}^{M-1}[w]^2(x_j,t)dt \\ &\leqslant C(h,\beta)ch^{2k+1} + (c/\epsilon)h^{2k+2} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt \\ &\leqslant \left(\frac{C(h,\beta)}{h}c + (c/\epsilon)\right)h^{2k+2} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt. \end{aligned} \tag{4.43}$$

For the special case of $\beta = 1$,

$$
\begin{aligned}
&\frac{1}{2}\|v(\cdot,T)\|^2_{L^2(\Omega)} + \frac{C(h,\beta)}{2}\int_0^T \sum_{j=1}^{M-1} [w]^2(x_j,t)dt \\
&\leqslant C(h,\beta)ch^{2k+1} + ch^{2k} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt \qquad (4.44)\\
&\leqslant ch^{2k} + c\int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)}dt.
\end{aligned}
$$

Again using the Grönwall's lemma recovers the desired estimate (4.18). □.

The introduction of the terms $\mathcal{VI}'$ and $\mathcal{VI}$ does not appear to add any additional freedom to improve on the error estimates ($\mathcal{VI}'$ can't control $\|w\|^2$, so we can't improve the order of convergence when $\beta = 1$). On the contrary, we have to estimate the trace errors of w. It seems hard to choose a special projection to eliminate the traces errors, since at one point we have two traces of w. It is not clear whether the error estimate (4.18) is sharp. The numerical computations below show that the order of convergence is $k+1$ for any $\beta \in [1, 1+\varepsilon]$, where ε may be 0.01 or 0.001.

5. Numerical results. Let us now offer some numerical results to validate analysis. We mainly focus on the accuracy and stability of the spatial approximation although we shall also numerically study the CFL condition. We use a fourth order explicit Runge-Kutta method to solve the method-of-line fractional PDE, i.e., the classical ODE system. To ensure the overall error is dominated by space error, small time steps are used.

Example. On the computational domain $x \in \Omega = (0, 1)$, we consider the following equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\Gamma(6-\beta)}{\Gamma(6)}\frac{\partial^\beta u(x,t)}{\partial x^\beta} - e^{-t}(x^5 + x^{5-\beta}), \qquad (5.1)$$

$\Gamma(x)$ is the classic Gamma function. We consider the initial condition

$$u(x,0) = x^5, \qquad (5.2)$$

and the Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^{-t}, \qquad (5.3)$$

where $\beta \in [1, 2]$. The exact solution is given by $e^{-t}x^5$.

Tables c1-c4 demonstrate the errors and order of convergence of of Scheme I and confirm optimality for $\beta \in [1+\varepsilon, 2.0]$ with suboptimal convergence for $\beta = 1.0$ as predicted by Theorem 4.2. We also observe that the asymptotic order of convergence is reached at higher accuracy as $\beta \to 1$, confirming that the constant, $c(\beta)$ increases when β decreases. Tables c5-c8 display that when $\beta = 1.0$ and $\beta = 1.01$ Scheme II recovers the optimal order of convergence due to the penalty term.

Table c1(a). The error and order of convergence of Scheme I for first order polynomial approximation ($k = 1$) when β is not very close to 1. M denotes the number of elements.

β	$M = 2^5$	$M = 2^6$		$M = 2^7$		$M = 2^8$		$M = 2^9$	
	error	error	order	error	order	error	order	error	order
2.0	2.81e-04	7.82e-05	1.85	2.04e-05	1.94	5.20e-06	1.97	1.31e-06	1.99
1.8	3.37e-04	9.37e-05	1.85	2.44e-05	1.94	6.14e-06	1.99	1.52e-06	2.01
1.5	4.63e-04	1.27e-04	1.87	3.23e-05	1.98	7.93e-06	2.03	1.95e-06	2.02
1.2	9.82e-04	3.17e-04	1.63	8.32e-05	1.93	2.02e-05	2.05	5.09e-06	1.99

Table c1(b). The error and order of convergence of Scheme I for first order polynomial approximation ($k = 1$) when β is very close to 1. M denotes the number of elements.

β	$M = 2^6$	$M = 2^7$		$M = 2^8$		$M = 2^9$		$M = 2^{10}$	
	error	error	order	error	order	error	order	error	order
1.10	5.66e-04	1.90e-04	1.58	5.22e-05	1.86	1.32e-05	1.98	3.31e-06	2.00
1.09	5.98e-04	2.09e-04	1.51	5.96e-05	1.81	1.53e-05	1.96	3.86e-06	1.99
1.08	6.32e-04	2.31e-04	1.45	6.85e-05	1.76	1.78e-05	1.94	4.53e-06	1.98
1.07	6.66e-04	2.56e-04	1.38	7.94e-05	1.69	2.12e-05	1.90	5.40e-06	1.98
1.06	7.00e-04	2.82e-04	1.31	9.26e-05	1.61	2.57e-05	1.85	6.52e-06	1.98
1.05	7.35e-04	3.11e-04	1.24	1.09e-04	1.52	3.17e-05	1.78	8.11e-06	1.97
1.04	7.69e-04	3.43e-04	1.17	1.29e-04	1.41	4.01e-05	1.68	1.04e-05	1.95
1.03	8.03e-04	3.75e-04	1.10	1.52e-04	1.30	5.18e-05	1.55	1.45e-05	1.84
1.02	8.36e-04	4.10e-04	1.03	1.80e-04	1.19	6.89e-05	1.39	2.16e-05	1.67
1.01	8.69e-04	4.46e-04	0.96	2.12e-04	1.07	9.31e-05	1.19	3.44e-05	1.44
1.00	9.00e-04	4.83e-04	0.90	2.49e-04	0.95	1.27e-04	0.97	6.38e-05	0.99

Table c2(a). The error and order of convergence of Scheme I for second order polynomial approximation ($k = 2$) when β is not very close to 1. M denotes the number of elements.

β	$M = 2^3$	$M = 2^4$		$M = 2^5$		$M = 2^6$	
	error	error	order	error	order	error	order
2.0	4.92e-04	6.62e-05	2.89	8.61e-06	2.94	1.09e-06	2.98
1.8	5.19e-04	6.76e-05	2.94	8.50e-06	2.99	1.07e-06	2.99
1.5	5.53e-04	7.27e-05	2.93	9.08e-06	3.00	1.12e-06	3.02
1.2	6.04e-04	7.02e-05	3.11	8.73e-06	3.01	1.10e-06	2.99

Table c2(b). The error and order of convergence of Scheme I for second order polynomial approximation ($k = 2$) when β is very close to 1. M denotes the number of elements.

β	$M=2^3$	$M=2^4$		$M=2^5$		$M=2^6$	
	error	error	order	error	order	error	order
1.10	3.04e-04	8.28e-05	1.88	9.80e-06	3.08	9.74e-07	3.33
1.09	2.98e-04	8.20e-05	1.86	1.01e-05	3.01	8.97e-07	3.50
1.08	2.92e-04	8.10e-05	1.85	1.05e-05	2.95	7.99e-07	3.71
1.07	2.86e-04	7.99e-05	1.84	1.08e-05	2.88	6.80e-07	4.00
1.06	2.80e-04	7.87e-05	1.83	1.12e-05	2.82	5.51e-07	4.34
1.05	2.74e-04	7.75e-05	1.82	1.15e-05	2.75	4.36e-07	4.72
1.04	2.69e-04	7.62e-05	1.82	1.18e-05	2.69	3.63e-07	5.02
1.03	2.63e-04	7.48e-05	1.81	1.21e-05	2.63	3.69e-07	5.03
1.02	2.57e-04	7.34e-05	1.81	1.23e-05	2.58	4.28e-07	4.85
1.01	2.51e-04	7.19e-05	1.81	1.25e-05	2.53	5.08e-07	4.62
1.00	2.46e-04	7.04e-05	1.81	1.26e-05	2.48	5.79e-07	4.45

Table c3(a). The error and order of convergence of Scheme I for third order polynomial approximation ($k=3$) when β is not very close to 1. M denotes the number of elements.

β	$M=4$	$M=8$		$M=12$	
	error	error	order	error	order
2.0	1.49e-04	1.12e-05	3.74	2.51e-06	3.69
1.8	1.56e-04	1.13e-05	3.79	2.48e-06	3.73
1.5	2.00e-04	1.37e-05	3.87	2.74e-06	3.97
1.2	3.89e-04	2.98e-05	3.71	5.16e-06	4.33

Table c3(b). The error and order of convergence of Scheme I for third order polynomial approximation ($k=3$) when β is very close to 1. M denotes the number of elements.

β	$M=4$	$M=8$		$M=12$	
	error	error	order	error	order
1.10	5.45e-04	5.37e-05	3.34	1.04e-05	4.06
1.09	5.65e-04	5.78e-05	3.29	1.14e-05	4.00
1.08	5.85e-04	6.22e-05	3.23	1.26e-05	3.94
1.07	6.05e-04	6.72e-05	3.17	1.40e-05	3.89
1.06	6.26e-04	7.28e-05	3.11	1.56e-05	3.80
1.05	6.48e-04	7.89e-05	3.04	1.74e-05	3.72
1.04	6.71e-04	8.57e-05	2.97	1.96e-05	3.64
1.03	6.94e-04	9.32e-05	2.90	2.22e-05	3.54
1.02	7.18e-04	1.01e-04	2.82	2.51e-05	3.45
1.01	7.43e-04	1.11e-04	2.75	2.86e-05	3.34
1.00	7.70e-04	1.21e-04	2.67	3.27e-05	3.22

Table c4(a). The error and order of convergence of Scheme I for fourth order polynomial approximation ($k=4$) when β is not very close to 1. M denotes the number of elements.

β	$M=2$	$M=3$		$M=4$	
	error	error	order	error	order
2.0	6.55e-05	7.74e-06	5.27	1.60e-06	5.47
1.8	8.31e-05	9.83e-06	5.27	1.83e-06	5.85
1.5	1.24e-04	1.59e-05	5.07	2.32e-06	6.69
1.2	1.55e-04	3.08e-05	3.99	4.12e-06	6.99

Table c4(b). The error and order of convergence of Scheme I for fourth order polynomial approximation ($k = 4$) when β is very close to 1. M denotes the number of elements.

β	$M = 2$	$M = 3$		$M = 4$	
	error	error	order	error	order
1.10	1.56e-04	3.96e-05	3.38	7.53e-06	5.77
1.09	1.55e-04	4.06e-05	3.30	8.05e-06	5.63
1.08	1.53e-04	4.16e-05	3.22	8.60e-06	5.48
1.07	1.52e-04	4.27e-05	3.13	9.18e-06	5.34
1.06	1.50e-04	4.38e-05	3.03	9.80e-06	5.21
1.05	1.48e-04	4.50e-05	2.93	1.04e-05	5.08
1.04	1.45e-04	4.63e-05	2.82	1.11e-05	4.96
1.03	1.43e-04	4.77e-05	2.72	1.18e-05	4.84
1.02	1.42e-04	4.91e-05	2.62	1.26e-05	4.72
1.01	1.42e-04	5.06e-05	2.54	1.35e-05	4.59
1.00	1.43e-04	5.20e-05	2.50	1.45e-05	4.43

Table c5. The error and order of convergence of Scheme II for first order polynomial approximation ($k = 1$) with $C(h, \beta) = h^\beta$. M denotes the number of elements.

β	$M = 2^4$	$M = 2^5$		$M = 2^6$		$M = 2^7$		$M = 2^8$	
	error	error	order	error	order	error	order	error	order
1.01	1.70e-03	5.38e-04	1.66	1.43e-04	1.91	3.71e-05	1.95	9.45e-06	1.97
1.00	1.70e-03	5.38e-04	1.66	1.43e-04	1.91	3.71e-05	1.95	9.45e-06	1.97

Table c6. The error and order of convergence of Scheme II for second order polynomial approximation ($k = 2$) with $C(h, \beta) = h^\beta$. M denotes the number of elements.

β	$M = 2^3$	$M = 2^4$		$M = 2^5$		$M = 2^6$	
	error	error	order	error	order	error	order
1.01	1.50e-05	1.43e-06	3.41	1.10e-07	3.70	1.07e-08	3.36
1.00	1.52e-05	1.43e-06	3.39	1.10e-07	3.71	1.06e-08	3.36

Table c7. The error and order of convergence of Scheme II for third order polynomial approximation ($k = 3$) with $C(h, \beta) = h^\beta$. M denotes the number of elements.

β	$M = 8$	$M = 10$		$M = 12$		$M = 14$	
	error	error	order	error	order	error	order
1.01	3.25e-06	1.38e-06	3.84	6.79e-07	3.89	3.64e-07	4.05
1.00	3.25e-06	1.38e-06	3.84	6.79e-07	3.89	3.64e-07	4.05

Table c8. The error and order of convergence of Scheme II for fourth order polynomial approximation ($k = 4$) with $C(h, \beta) = h^\beta$. M denotes the number of elements.

β	$M = 2$	$M = 3$		$M = 4$	
	error	error	order	error	order
1.01	3.97e-04	8.60e-05	3.77	2.33e-05	4.54
1.00	4.03e-04	8.79e-05	3.75	2.42e-05	4.49

We finally consider the question of how the spectral radius scales with β and the spatial resolution. Based on the results in Table s1, we observe a scaling for solving the space discretized equation (1.1) using Scheme I as

$$\Delta t \sim \left(\frac{h}{k^2}\right)^{\beta},$$

where Δt is the time step size and k still is the order of the approximate polynomial. This is in agreement with expectations based on the experience for integer values of β [13].

Table s1. The time step sizes required with a fixed space step length $h = 0.01$ for different values of β for Scheme I with first and second order polynomial approximations. Here 'S' denotes stable and 'U' unstable.

Δt (1st order)	Δt(2nd order)	β =1.0	β =1.2	β =1.5	β =1.8	β =2.0
$2.0d-2$	$1.0d-2$	S	U	U	U	U
$1.0d-2$	$5.0d-3$	S	S	U	U	U
$1.0d-3$	$1.0d-3$	S	S	S	U	U
$5.0d-4$	$1.0d-4$	S	S	S	S	U
$1.0d-4$	$1.0d-5$	S	S	S	S	S

We also want to emphasize that, in agreement with our theoretical analyses, both fluxes (3.16) and (3.17) work well for the two schemes for any $\beta \in [1, 2]$. This may seem slightly counter intuitive when $\beta = 1$ since in this case one of the flux choices is similar to downwinding. This illustrates that analytically when $\beta = 1$ the single equation is equivalent to the system but in doing numerical computations a new mechanism is introduced when writing the pure diffusion equation as a system due to the introduction of the fluxes.

6. Concluding remarks. We have proposed a local discontinuous Galerkin method for the fractional diffusion equation, containing both limits of pure convection equation and pure diffusion. In the framework of fractional derivatives, the fractional diffusion equation 'bridges' the other two equations and we have shown that a single scheme can be used to accurately solve both types of problems.

We have introduced two schemes in this paper, the second one being a modified version of the first one, obtained by adding a penalty term to the global term. Both schemes allow a theoretical and numerical optimal order of convergent when $\beta \in (1, 2]$, but a suboptimal order of convergence can only be established in the extreme case $\beta = 1$. For the second scheme we can numerically recover the optimal order of convergence for any $\beta \in [1, 2]$, due to the added stability introduced by the penalty term. We have not, however, been able to confirm this in the analysis.

This paper marks the beginning of using DG methods to fractional derivatives and DG methods have significant potentials for such problems with fractional derivatives as they are flexible high order methods while we can maintain a large degree of locality in the formulation in contrast to most alternative methods. We hope to report on more general applications and extensions to multiple dimensions in future work.

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