Optimal error estimates to smooth solutions of the central discontinuous Galerkin methods for nonlinear scalar conservation laws

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Abstract

In this paper, we study the error estimates to sufficiently smooth solutions of the nonlinear scalar conservation laws for the semi-discrete central discontinuous Galerkin (DG) finite element methods on uniform Cartesian meshes. A general approach with an explicitly checkable condition is established for the proof of optimal L^2 error estimates of the semi-discrete CDG schemes, and this condition is checked to be valid in one and two dimensions for polynomials of degree up to $k = 8$. Numerical experiments are given to verify the theoretical results.

AMS subject classifications: 65M12, 65M15, 65M60

Key Words: central DG method; nonlinear conservation laws; optimal error estimates

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1 Introduction

In this paper, we study the central discontinuous Galerkin (DG) finite element method for solving scalar conservation laws [10]. The optimal error estimates of the central DG methods have been proved for linear conservation laws in [12]. In this paper, we present the optimal error estimates of central DG approximation based on tensor-product polynomials under suitable assumptions for the general nonlinear scalar conservation laws

$$\begin{cases} u_t + \sum_{i=1}^{d} (f_i(u))_{x_i} = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T] \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_d)$ and Ω is a bounded rectangular domain in $\mathbb{R}^d$. Here $u_0(\mathbf{x})$ is a given smooth function. We do not pay attention to boundary conditions in this paper; hence the exact solution is considered to be either periodic or compactly supported. We also assume the flux $f(u)$ is smooth in the variable u; for example, $f \in C^2$ is enough for our proof. The analysis in this paper is for the smooth solutions of (1.1). Discontinuous solutions with shocks are not considered here. We study the cases with $d = 1$ and 2, but the approach is applicable to any d.

The central scheme of Nessyahu and Tadmor [14] computes hyperbolic conservation laws on a staggered mesh and avoids the Riemann solver. In [3], Kurganov and Tadmor introduced a new type of central scheme without the large dissipation error related to the small time step size by using a variable control volume whose size depends on the time step size. To avoid the excessive numerical dissipation for small time steps, Liu [8] uses another coupling technique. The overlapping cell approach evolves two independent cell averages on overlapping cells, which opens up many new possibilities. The advantages of overlapping cells motivate the combination of the central scheme and the DG method, which results in the central DG methods [7, 9, 10]. The central DG method evolves two copies of approximating solutions defined on staggered meshes and avoids using numerical fluxes which can be complicated and costly [4]. Like some previous central schemes, the central DG method also avoids the excessive numerical dissipation for small time steps by a suitable choice of the numerical dissipation term. Besides, the central method carry many features of standard DG methods, such as compact stencil, easy parallel implementation, etc. It is generally understood that the central DG method allows for a larger CFL number compared to the regular DG method [10]. Later in [11], the central local discontinuous Galerkin method was introduced to solve diffusion equations, which is formulated based on the local discontinuous Galerkin scheme on overlapping cells. Recently, the central DG method has been used to solve systems of conservation laws in many applications [6, 5, 21, 17, 16].

In [12], suitable special projections for central DG methods were proposed to yield optimal error estimates for scalar linear conservation laws. The proper local projections were constructed according to the superconvergence property and the duality of overlapping cells, which also required uniform Cartesian meshes. Zhang and Shu firstly presented *a priori* error estimates for the fully discrete second order Runge-Kutta DG methods with smooth solutions for scalar nonlinear conservation laws [18] and symmetrizable systems [19]. The main techniques they used are Taylor expansion and energy estimates. Later these techniques are widely used in error estimates for DG-type methods of nonlinear equations, like the local DG methods for convection-diffusion and KdV equations [15], the ultra weak DG methods for equations with higher order derivatives [1], the third order Runge-Kutta DG methods for scalar conservation laws [20] and for symmetrizable systems [13].

In this paper, we combine the special projections in [12] and the techniques used in [18] to construct new projections to provide the optimal error estimates of the central DG methods on uniform Cartesian meshes for nonlinear scalar conservation laws with smooth solutions. In one dimension, we construct a proper local projection $\mathbb{P}_h^*$ similar to [12]. The existence and optimal approximation properties of this projection are proved by standard finite element techniques. Moreover, this projection has similar superconvergence property as the projections in [12]. By using this property we develop a general approach with an explicitly checkable condition, and this condition is checked to be valid in one dimension for polynomials of degree up to $k = 8$. The optimal convergence results is valid for uniform meshes and for polynomials of degree $k \geq 1$, while for $k = 0$ we need the convection flux to be linear to get the optimal results. For two-dimensional conservation laws, we follow the same arguments as in the one-dimensional case to construct a suitable projection $\mathbb{P}_h^*$ and to analyze its existence and approximation properties. This new projection utilizes Q^k, the space of tensor-product polynomials of degree at most k in each variable. Similarly, the optimal convergence result is valid for uniform meshes and for polynomials of degree $k \geq 2$ in the two-dimensional case, while for $k = 0,\ 1$ we need the convection flux to be linear to get the optimal results. The superconvergence result of $\mathbb{P}_h^*$ on uniform Cartesian meshes will help to yield optimal convergence results under some suitable assumptions. Similar approach with an explicitly checkable condition is established, and here we also check this condition for polynomials of degree up to $k = 8$. The approach is applicable to higher dimension d, but it will not be discussed in this paper.

The rest of the paper is organized as follows. In section 2, we recall the central DG method for one-dimensional conservation laws. Then we construct a special projection and study its existence, uniqueness and optimal approximation properties. With the help of this projection, we will prove the optimal error estimate for the semi-discrete central DG methods on uniform meshes for the nonlinear conservation laws in one dimension. In section 3, we extend the analysis to two-dimensions. Optimal error estimates are proved by following the same lines of the one dimensional case. We provide numerical examples to show our theoretical results in section 4. In section 5, we give a few concluding remarks and perspectives for future work. Finally, in the appendix we provide proofs for some of the more technical results of the error estimates.

2 The central DG method in one dimension

Here we consider the one-dimensional conservation law given by

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in [a,b] \times (0,T] \\ u(x,0) = u_0(x), & x \in [a,b] \end{cases} \tag{2.1}$$

with periodic boundary condition or compactly supported boundary condition.

2.1 Basic notations

For a given interval $I = [a,b]$, we divide it into N cells as follows:

$$a = x_0 < x_1 < \cdots < x_N = b. \tag{2.2}$$

We denote

$$x_{j+\frac{1}{2}} = \frac{x_j + x_{j+1}}{2},\ I_{j+\frac{1}{2}} = (x_j, x_{j+1}),\ h_{j+\frac{1}{2}} = x_{j+1} - x_j, \quad j = 0, \ldots N-1 \tag{2.3}$$

and similarly for the dual mesh

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),\ h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad j = 1, \ldots N-1. \tag{2.4}$$

We let $h = \max_j h_{j+\frac{1}{2}}$ and assume the mesh is regular. Define the approximation space as

$$\begin{aligned} V_h^k &= \{\varphi_h : (\varphi_h)|_{I_j} \in P^k(I_j), j = 1, ..., N\} \\ W_h^k &= \{\psi_h : (\psi_h)|_{I_{j+\frac{1}{2}}} \in P^k(I_{j+\frac{1}{2}}), j = 1, ..., N\} \end{aligned} \tag{2.5}$$

Here $P^k(I_j)$ denotes the set of all polynomials of degree at most k on I_j. For a function $\varphi_h \in V_h^k$, we use $(\varphi_h)^-_{j+\frac{1}{2}}$ or $(\varphi_h)^+_{j+\frac{1}{2}}$ to refer to the value of φ_h at $x_{j+\frac{1}{2}}$ from the left cell I_j and the right cell I_{j+1}, respectively. For $\psi_h \in W_h^k$, $(\psi_h)^-_j$ and $(\psi_h)^+_j$ have similar meanings. $[\varphi_h]$ or $[\psi_h]$ is used to denote $\varphi_h^+ - \varphi_h^-$ or $\psi_h^+ - \psi_h^-$, i.e. the jump of φ_h or ψ_h at cell interfaces. We denote by C a positive constant independent of h, which may depend on the solution of the problem and other parameters. Similar to [15, 18], to emphasize the nonlinearity of the flux $f(u)$, we use C_* to denote a non-negative constant depending on the maximum of $|f''|$. We remark $C_* = 0$ for linear fluxes $f(u) = cu$ with a constant c.

2.2 The central DG scheme

We propose the following semi-discrete central DG scheme for periodic boundary condition: find $u_h \in V_h^k$ and $v_h \in W_h^k$, such that for any $\varphi_h \in V_h^k$ and $\psi_h \in W_h^k$,

$$
\begin{aligned}
\int_{I_j} (u_h)_t \varphi_h dx =& \frac{1}{\tau_{max}} \int_{I_j} (v_h - u_h)\varphi_h dx + \int_{I_j} f(v_h)(\varphi_h)_x dx \\
& - (f(v_h)\varphi_h^-)_{j+\frac{1}{2}} + (f(v_h)\varphi_h^+)_{j-\frac{1}{2}}, \qquad (2.6a)\\
\int_{I_{j+\frac{1}{2}}} (v_h)_t \psi_h dx =& \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (u_h - v_h)\psi_h dx + \int_{I_{j+\frac{1}{2}}} f(u_h)(\psi_h)_x dx \\
& - (f(u_h)\psi_h^-)_{j+1} + (f(u_h)\psi_h^+)_{j}, \qquad (2.6b)
\end{aligned}
$$

where τ_{max} is an upper bound for the time step size due to the CFL restriction, that is, $\tau_{max} = c\ h$ with a given constant CFL number c dictated by stability. For the initial condition, we simply take $u_h(\cdot, 0) = \mathbb{P}_h u_0(\cdot),\ v_h(\cdot, 0) = \mathbb{Q}_h u_0(\cdot)$, where $\mathbb{P}_h$ and $\mathbb{Q}_h$ are the L^2 projections into V_h^k and W_h^k, respectively, and we have

$$
\begin{aligned}
\|u_0 - \mathbb{P}_h u_0\|_{L^2(I_j)} &\le C h^{k+1} \|u_0\|_{H^{k+1}(I_j)}, \\
\|u_0 - \mathbb{Q}_h u_0\|_{L^2(I_{j+\frac{1}{2}})} &\le C h^{k+1} \|u_0\|_{H^{k+1}(I_{j+\frac{1}{2}})}.
\end{aligned} \qquad (2.7)
$$

2.3 L^2 stability for the linear equation

In [10], the following stability result is proved for this scheme if $f(u)$ is linear. Without loss of generality, we take $f(u) = u$. Hence, we have

$$
\begin{cases}
u_t + u_x = 0, & (x,t) \in [a,b] \times (0,T] \\
u(x,0) = u_0(x), & x \in [a,b]
\end{cases} \qquad (2.8)
$$

with periodic boundary condition.

Theorem 2.1. *The numerical solutions u_h and v_h of the CDG scheme* (2.6) *for the equation* (2.8) *have the following L^2 stability property*

$$
\frac{1}{2}\frac{d}{dt}\int_a^b (u_h^2 + v_h^2) dx = -\frac{1}{\tau_{max}} \int_a^b (v_h - u_h)^2 dx \le 0. \qquad (2.9)
$$

2.4 Optimal L^2 error estimate

It is worth noting that the L^2 stability for CDG scheme for nonlinear problem is generally not available [10]. But under some assumptions we can still get the error estimate of the nonlinear case. In this subsection, we show *a priori* L^2 error estimate of the scheme (2.6) for the equation (2.1).

Here and below, we use $\|\cdot\|$ to denote the standard L^2 norm. For the proof, we recall the classical inverse and trace inequalities [2]. For any $w_h \in V_h^k$ or $w_h \in W_h^k$, there exists a positive constant C independent of w_h and h, such that

$$\|\partial_x w_h\| \le Ch^{-1}\|w_h\|, \ \|w_h\|_\Gamma \le Ch^{-\frac{1}{2}}\|w_h\|, \ \|w_h\|_\infty \le Ch^{-\frac{1}{2}}\|w_h\| \tag{2.10}$$

where Γ is the set of boundary points of all elements I_j or $I_{j+\frac{1}{2}}$.

First we introduce some notations. For the numerical solutions u_h and v_h of the CDG scheme (2.6) for equation (2.1), we define

$$\begin{aligned}\tilde{B}_j(u_h, v_h; \varphi_h; f, u) :=& \frac{1}{\tau_{max}}\int_{I_j}(v_h - u_h)\varphi_h dx + \int_{I_j} f'(u(x_j))v_h(\varphi_h)_x \\ &- f'(u(x_j))(v_h\varphi_h^-)_{j+\frac{1}{2}} + f'(u(x_j))(v_h\varphi_h^+)_{j-\frac{1}{2}},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\hat{B}_{j+\frac{1}{2}}(u_h, v_h; \psi_h; f, u) :=& \frac{1}{\tau_{max}}\int_{I_{j+\frac{1}{2}}}(u_h - v_h)\psi_h dx + \int_{I_{j+\frac{1}{2}}} f'(u(x_{j+\frac{1}{2}}))u_h(\psi_h)_x \\ &- f'(u(x_{j+\frac{1}{2}}))(u_h\psi_h^-)_{j+1} + f'(u(x_{j+\frac{1}{2}}))(u_h\psi_h^+)_j,\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}B_j(u_h, v_h; \varphi_h, \psi_h) :=& \int_{I_j}(u_h)_t\varphi_h dx + \int_{I_{j+\frac{1}{2}}}(v_h)_t\psi_h dx \\ &- \frac{1}{\tau_{max}}\int_{I_j}(v_h - u_h)\varphi_h dx - \frac{1}{\tau_{max}}\int_{I_{j+\frac{1}{2}}}(u_h - v_h)\psi_h dx,\end{aligned} \tag{2.13}$$

Obviously, we have

$$\begin{aligned}B_j(u_h, v_h; \varphi_h, \psi_h) =& \int_{I_j} f(v_h)(\varphi_h)_x dx + \int_{I_{j+\frac{1}{2}}} f(u_h)(\psi_h)_x dx - (f(v_h)\varphi_h^-)_{j+\frac{1}{2}} \\ &+ (f(v_h)\varphi_h^+)_{j-\frac{1}{2}} - (f(u_h)\psi_h^-)_{j+1} + (f(u_h)\psi_h^+)_j, \\ &\forall \varphi_h \in V_h^k, \psi_h \in W_h^k.\end{aligned} \tag{2.14}$$

It is also clear that the exact solution u of (2.1) satisfies

$$\begin{aligned}B_j(u,u;\varphi_h,\psi_h) = &\int_{I_j} f(u)(\varphi_h)_x dx + \int_{I_{j+\frac{1}{2}}} f(u)(\psi_h)_x dx - (f(u)\varphi_h^-)_{j+\frac{1}{2}} \\ &+ (f(u)\varphi_h^+)_{j-\frac{1}{2}} - (f(u)\psi_h^-)_{j+1} + (f(u)\psi_h^+)_j, \\ &\forall \varphi_h \in V_h^k, \psi_h \in W_h^k.\end{aligned} \tag{2.15}$$

Subtracting (2.14) from (2.15), we obtain the error equation

$$\begin{aligned}B_j(u-u_h,u-v_h;\varphi_h,\psi_h) = &\int_{I_j} (f(u)-f(v_h))(\varphi_h)_x dx + \int_{I_{j+\frac{1}{2}}} (f(u)-f(u_h))(\psi_h)_x dx \\ &- ((f(u)-f(v_h))\varphi_h^-)_{j+\frac{1}{2}} + ((f(u)-f(v_h))\varphi_h^+)_{j-\frac{1}{2}} \\ &- ((f(u)-f(u_h))\psi_h^-)_{j+1} + ((f(u)-f(u_h))\psi_h^+)_j \\ :=& H_j(f;u,u_h,v_h;\varphi_h,\psi_h) \\ &\forall \varphi_h \in V_h^k, \psi_h \in W_h^k.\end{aligned} \tag{2.16}$$

Summing over all j, the error equation becomes

$$\sum_j B_j(u-u_h,u-v_h;\varphi_h,\psi_h) = \sum_j H_j(f;u,u_h,v_h;\varphi_h,\psi_h), \quad \forall \varphi_h \in V_h^k, \psi_h \in W_h^k. \tag{2.17}$$

2.4.1 Projection operators

Similar to [12], we define $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ as the following projections onto V_h^k and W_h^k respectively on uniform meshes. That is, for a given function $w(x)$, we define $\mathbb{P}_h^* w \in V_h^k$, such that $\forall j$,

$$\int_{I_j} \mathbb{P}_h^* w dx = \int_{I_j} w dx \tag{2.18a}$$

$$\tilde{P}_h(\mathbb{P}_h^* w;\varphi_h;f,u)_j = \tilde{P}_h(w;\varphi_h;f,u)_j,\ \forall \varphi_h \in P^k(I_j) \tag{2.18b}$$

where $\tilde{P}_h(w;\varphi_h)_j$ is defined as follows

$$\begin{aligned}\tilde{P}_h(w;\varphi_h;f,u)_j =&\frac{1}{\tau_{max}}(\int_{x_{j-\frac{1}{2}}}^{x_j} w(x+\frac{h}{2})\varphi_h dx + \int_{x_j}^{x_{j+\frac{1}{2}}} w(x-\frac{h}{2})\varphi_h dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x)\varphi_h dx)\\ &+ \int_{x_{j-\frac{1}{2}}}^{x_j} f'(u(x_j))w(x+\frac{h}{2})(\varphi_h)_x dx\\ &+ \int_{x_j}^{x_{j+\frac{1}{2}}} f'(u(x_j))w(x-\frac{h}{2})(\varphi_h)_x dx\\ &- f'(u(x_j))w(x_j)(\varphi_h(x_{j+\frac{1}{2}}^-) - \varphi_h(x_{j-\frac{1}{2}}^+)).\end{aligned} \tag{2.19}$$

Similarly, we define $\mathbb{Q}_h^* w \in W_h^k$, such that $\forall j$,

$$\int_{I_{j+\frac{1}{2}}} \mathbb{Q}_h^* w dx = \int_{I_{j+\frac{1}{2}}} w dx \tag{2.20a}$$

$$\tilde{Q}_h(\mathbb{Q}_h^* w;\psi_h;f,u)_{j+\frac{1}{2}} = \tilde{Q}_h(w;\psi_h;f,u)_{j+\frac{1}{2}},\ \forall\psi_h \in P^k(I_{j+\frac{1}{2}}), \tag{2.20b}$$

where $\tilde{Q}_h(w;\psi_h)_{j+\frac{1}{2}}$ is defined as follows

$$\begin{aligned}\tilde{Q}_h(w;\varphi_h;f,u)_{j+\frac{1}{2}} =&\frac{1}{\tau_{max}}(\int_{x_j}^{x_{j+\frac{1}{2}}} w(x+\frac{h}{2})\psi_h dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w(x-\frac{h}{2})\psi_h dx - \int_{x_j}^{x_{j+1}} w(x)\psi_h dx)\\ &+ \int_{x_j}^{x_{j+\frac{1}{2}}} f'(u(x_{j+\frac{1}{2}}))w(x+\frac{h}{2})(\psi_h)_x dx\\ &+ \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} f'(u(x_{j+\frac{1}{2}}))w(x-\frac{h}{2})(\varphi_h)_x dx\\ &- f'(u(x_{j+\frac{1}{2}}))w(x_{j+\frac{1}{2}})(\varphi_h(x_{j+1}^-) - \varphi_h(x_j^+)).\end{aligned} \tag{2.21}$$

Next, we will discuss the properties of the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$. Without loss of generality we will only consider $\mathbb{P}_h^*$. The equation (2.18a) is required by conservation. Note that $\tilde{P}_h(w;\varphi_h;f,u)_j = 0$ for $\forall w$ when φ_h is a constant, so (2.18b) alone misses one condition which is provided by (2.18a). The following lemma gives the existence and uniqueness of the special projection $\mathbb{P}_h^*$.

Lemma 2.1. *The projection $\mathbb{P}_h^*$ defined by* (2.18) *exists and is unique for any smooth function $w(x)$, and the following inequality holds*

$$\|\mathbb{P}_h^* w\| \le C\|w\|_\infty, \tag{2.22}$$

for all k. The positive constant C depends on k, the bound of $f'(u)$, the constant c in the scheme (2.6) *and is independent of h and w.*

Proof. The proof of this lemma is given in Appendix A.1. □

Since $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ are k-th degree polynomial preserving local projections, standard approximation theory [2] implies, for smooth function w,

$$\begin{aligned} \|\mathbb{P}_h^* w - w\| + h\|\mathbb{P}_h^* w - w\|_\infty + h^{\frac{1}{2}}\|\mathbb{P}_h^* w - w\|_\Gamma \le Ch^{k+1}\|u\|_{H^{k+1}([a,b])}, \\ \|\mathbb{Q}_h^* w - w\| + h\|\mathbb{Q}_h^* w - w\|_\infty + h^{\frac{1}{2}}\|\mathbb{Q}_h^* w - w\|_\Gamma \le Ch^{k+1}\|u\|_{H^{k+1}([a,b])}, \end{aligned} \tag{2.23}$$

Besides the standard approximation results (2.23), the special projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ also have the following superconvergence result.

Proposition 2.1. *For $k = 0, 1..., 8$, assume that u is a $(k+1)$-th degree polynomial function in $P^{k+1}([a, b])$. For a uniform partition on the interval $[a, b]$, set $u_I = \mathbb{P}_h^* u \in V_h^k$ and $v_I = \mathbb{Q}_h^* u \in W_h^k$,. Then we have*

$$\begin{aligned} |\tilde{B}_j(u_I - u, v_I - u; \varphi_h; f, u)| \le Ch^{2k+3} + C\|\varphi_h\|^2_{L^2(I_j)},\ \forall \varphi_h \in P^k(I_j) \\ |\hat{B}_{j+\frac{1}{2}}(u_I - u, v_I - u; \psi_h; f, u)| \le Ch^{2k+3} + C\|\psi_h\|^2_{L^2(I_{j+\frac{1}{2}})},\ \forall \psi_h \in P^k(I_{j+\frac{1}{2}}). \end{aligned} \tag{2.24}$$

Proof. The proof of this proposition is given in Appendix A.2. □

2.4.2 *A priori* L^2 error estimates

Theorem 2.2. *For $k = 0, 1..., 8$, let $u(\cdot, t)$ be the exact solution of equation* (2.1)*, which is sufficiently smooth with bounded derivatives, and assume $f \in C^2$. The numerical solutions u_h and v_h of the CDG scheme* (2.6) *using uniform meshes satisfies the following L^2 error estimate*

$$\|u(\cdot, T) - u_h(\cdot, T)\|^2 + \|u(\cdot, T) - v_h(\cdot, T)\|^2 \le Ch^{2k+2}, \tag{2.25}$$

where k is the polynomial degree in the finite element spaces V_h^k and W_h^k, and the constant C depends on k, the final time T, $\|u\|_{H^{k+2}}$ and the bounds on the derivatives $|f^m|$, $m = 1, 2$, but is independent of the mesh size h. Here $\|u\|_{H^{k+2}}$ is the maximum $(k+2)$-th *order Sobolev norm of u over time in $[0,T]$. For $k=0$ we need $f(u)$ to be linear, i.e. $f(u)=cu$.*

Proof. Let $e_u = u - u_h,\ e_v = u - v_h$ be the error between the numerical and exact solutions. To deal with the nonlinearity of $f(u)$, we would like to first make the *a priori* assumption that, for small enough h, we have

$$\|u-u_h\| \le Ch^{\frac{3}{2}},\ \|u-v_h\| \le Ch^{\frac{3}{2}} \tag{2.26}$$

and then by the interpolation property,

$$\begin{aligned} &\|e_u\|_\infty \le Ch \quad \text{and} \quad \|\mathbb{P}_h^* u - u_h\|_\infty \le Ch \\ &\|e_v\|_\infty \le Ch \quad \text{and} \quad \|\mathbb{Q}_h^* u - u_h\|_\infty \le Ch \end{aligned} \tag{2.27}$$

This assumption is not necessary for linear f. We will verify this assumption for $k \ge 1$ later.

By taking

$$\varphi_h = \mathbb{P}_h^* u - u_h,\ \psi_h = \mathbb{Q}_h^* u - v_h, \varphi^e = \mathbb{P}_h^* u - u,\ \psi^e = \mathbb{Q}_h^* u - u, \tag{2.28}$$

we obtain the energy equality

$$\sum_j B_j(\varphi_h - \varphi^e, \psi_h - \psi^e; \varphi_h, \psi_h) = \sum_j H_j(f; u, u_h, v_h; \varphi_h, \psi_h). \tag{2.29}$$

From the definition of B_j, we can obtain

$$\begin{aligned} \sum_j B_j(\varphi_h, \psi_h; \varphi_h, \psi_h) =& \sum_j B_j(\varphi^e, \psi^e; \varphi_h, \psi_h) + \sum_j H_j(f; u, u_h, v_h; \varphi_h, \psi_h) \\ =& \sum_j \int_{I_j} (\psi^e)_t \varphi_h dx + \sum_j \int_{I_{j+\frac{1}{2}}} (\varphi^e)_t \psi_h dx \\ &- \sum_j \frac{1}{\tau_{max}} \int_{I_j} (\psi^e - \varphi^e)\varphi_h dx - \sum_j \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (\varphi^e - \psi^e)\psi_h dx \\ &+ \sum_j \int_{I_j} (f(u) - f(v_h))(\varphi_h)_x dx + \sum_j ((f(u) - f(v_h))[\varphi_h])_{j+\frac{1}{2}} \\ &+ \sum_j \int_{I_{j+\frac{1}{2}}} (f(u) - f(u_h))(\psi_h)_x dx + \sum_j ((f(u) - f(u_h))[\psi_h])_j. \end{aligned} \tag{2.30}$$

For the left-hand side of (2.30), we follow the L^2 stability proof in Theorem 2.1 for linear case to conclude

$$\sum_j B_j(\varphi_h,\psi_h;\varphi_h,\psi_h) = \frac{1}{2}\frac{d}{dt}\int_a^b(\varphi_h^2+\psi_h^2)dx + \frac{1}{\tau_{max}}\int_a^b(\varphi_h-\psi_h)^2dx. \tag{2.31}$$

Similar to [18] and [15], to deal with the nonlinear part of (2.30) we would like to use the following Taylor expansions:

$$\begin{aligned} f(u)-f(u_h) =& f'(u)\varphi_h - f'(u)\varphi^e - \frac{1}{2}f''_u(\varphi_h-\varphi^e)^2, \\ f(u)-f(v_h) =& f'(u)\psi_h - f'(u)\psi^e - \frac{1}{2}f''_v(\psi_h-\psi^e)^2, \end{aligned} \tag{2.32}$$

where f''_u and f''_v are the mean values. These imply the following representation,

$$\begin{aligned} &\sum_j B_j(\varphi^e,\psi^e;\varphi_h,\psi_h) + \sum_j H_j(f;u,u_h,v_h;\varphi_h,\psi_h) \\ =& L + N_1 + N_2 + N_3 + N_4 \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} L =& \sum_j\int_{I_j}(\psi^e)_t\varphi_h dx + \sum_j\int_{I_{j+\frac{1}{2}}}(\varphi^e)_t\psi_h dx \\ N_1 =& -\sum_j\frac{1}{\tau_{max}}\int_{I_j}(\psi^e-\varphi^e)\varphi_h dx - \sum_j\int_{I_j}f'(u)\psi^e(\varphi_h)_x dx - \sum_j(f'(u)\psi^e[\varphi_h])_{j+\frac{1}{2}} \\ N_2 =& -\sum_j\frac{1}{\tau_{max}}\int_{I_{j+\frac{1}{2}}}(\varphi^e-\psi^e)\psi_h dx - \sum_j\int_{I_{j+\frac{1}{2}}}f'(u)\varphi^e(\psi_h)_x dx - \sum_j(f'(u)\varphi^e[\psi_h])_j \\ N_3 =& \sum_j\int_{I_j}f'(u)\psi_h(\varphi_h)_x dx + \sum_j(f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} \\ &+ \sum_j\int_{I_{j+\frac{1}{2}}}f'(u)\varphi_h(\psi_h)_x dx + \sum_j(f'(u)\varphi_h[\psi_h])_j \\ N_4 =& -\frac{1}{2}\Big(\sum_j\int_{I_j}f''_v(\psi_h-\psi^e)^2(\varphi_h)_x dx + \sum_j\int_{I_{j+\frac{1}{2}}}f''_u(\varphi_h-\varphi^e)^2(\psi_h)_x dx \\ &+ \sum_j(f''_v(\psi_h-\psi^e)^2[\varphi_h])_{j+\frac{1}{2}} + \sum_j(f''_u(\varphi_h-\varphi^e)^2[\psi_h])_j\Big) \end{aligned}$$

By Young's inequality and (2.23), we have

$$L \le C(\|\varphi_h\|^2+\|\psi_h\|^2) + Ch^{2k+2}\|u\|^2_{H^{k+1}([a,b])}. \tag{2.34}$$

Next we estimate the nonlinear part. First for the N_1 term, we can rewrite it in the form

$$
\begin{aligned}
N_1 = & -\sum_j \frac{1}{\tau_{max}} \int_{I_j} (\psi^e - \varphi^e)\varphi_h dx - \sum_j \int_{I_j} f'(u(x_j))\psi^e(\varphi_h)_x dx \\
& - \sum_j (f'(u(x_j))\psi^e[\varphi_h])_{j+\frac{1}{2}} + \sum_j \int_{I_j} (f'(u(x_j)) - f'(u))\psi^e(\varphi_h)_x dx \\
& - \sum_j (f'(u(x_j)) - f'(u))\psi^e[\varphi_h])_{j+\frac{1}{2}} \\
= & -\sum_j \tilde{B}_j(\varphi^e, \psi^e; \varphi_h) + \sum_j \int_{I_j} (f'(u(x_j)) - f'(u))\psi^e(\varphi_h)_x dx \\
& - \sum_j (f'(u(x_j)) - f'(u))\psi^e[\varphi_h])_{j+\frac{1}{2}}
\end{aligned}
$$

By the inequality in (2.10), (2.23) and $\|f'(u(x_j)) - f'(u)\|_{L^\infty(I_j)} = O(h)$, we have

$$
N_1 \leq -\sum_j \tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) + C_*\|\varphi_h\|^2 + C_* h^{2k+2}\|u\|^2_{H^{k+1}([a,b])} \tag{2.35}
$$

For $\tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u)$, let $\widehat{u_I}$ be the Taylor polynomial of order $k+1$ of u near x_j i.e. $\widehat{u_I}^j = \sum_{i=0}^{k+1} \frac{1}{i!} u^{(i)}(x_j)(x - x_j)^i, \quad x \in (x_{j-1}, x_{j+1})$. Let r_u denote the residual term i.e. $r_u^j = u - \widehat{u_I}^j$. Recalling the Bramble-Hilbert lemma [2], we have

$$
\|r_u^j\|_{L^\infty(I_j)} \leq C h^{k+\frac{3}{2}} |u|_{H^{k+2}(I_j)}. \tag{2.36}
$$

Then we rewrite φ^e and ψ^e

$$
\begin{aligned}
\varphi^e &= \mathbb{P}_h^* u - u = \mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{P}_h^* r_u^j - r_u^j, \\
\psi^e &= \mathbb{Q}_h^* u - u = \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{Q}_h^* r_u^j - r_u^j.
\end{aligned} \tag{2.37}
$$

Hence, using Proposition 2.1, we have

$$
\begin{aligned}
\tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) = & \tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) \\
= & \tilde{B}_j(\mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h; f, u) \\
= & \tilde{B}_j(\mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j, \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j; \varphi_h; f, u) \\
& + \tilde{B}_j(\mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h; f, u)
\end{aligned}
$$

$$=\tilde{B}_j(\mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h; f, u) + Ch^{2k+3} + C\|\varphi_h\|^2_{L^2(I_j)}. \quad (2.38)$$

Therefore, by using Young's inequality, (2.23), the inequality in (2.10) and (2.36), we have

$$-\sum_j \tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) \le Ch^{2k+2}|u|_{H^{k+2}([a,b])} + C\|\varphi_h\|^2. \quad (2.39)$$

Hence, for N_1 we have

$$N_1 \le (C + C_*)\|\varphi_h\|^2 + (C + C_*)h^{2k+2}\|u\|^2_{H^{k+2}([a,b])} \quad (2.40)$$

Similarly, for N_2 we have

$$N_2 \le (C + C_*)\|\psi_h\|^2 + (C + C_*)h^{2k+2}\|u\|^2_{H^{k+2}([a,b])} \quad (2.41)$$

The N_3 term can be rewritten as the following form

$$\begin{aligned}
N_3 =& \sum_j \Big(\int_{x_j}^{x_{j+\frac{1}{2}}} f'(u)(\psi_h\varphi_h)_x dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} f'(u)(\psi_h\varphi_h)_x dx \Big) \\
& + \sum_j (f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} + \sum_j (f'(u)\varphi_h[\psi_h])_j \\
=& \sum_j \Big((f'(u)\psi_h\varphi_h^-)_{j+\frac{1}{2}} - (f'(u)\varphi_h\psi_h^+)_j + (f'(u)\varphi_h\psi_h^-)_{j+1} \\
& - (f'(u)\psi_h\varphi_h^+)_{j+\frac{1}{2}} + (f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} + (f'(u)\varphi_h[\psi_h])_j \Big) \\
& - \sum_j \int_{x_j}^{x_{j+1}} (f'(u))_x \psi_h\varphi_h dx \\
=& -\sum_j \int_{x_j}^{x_{j+1}} (f'(u))_x \psi_h\varphi_h dx \\
\le& C\|\psi_h\|\|\varphi_h\| \le C(\|\psi_h\|^2 + \|\varphi_h\|^2).
\end{aligned} \quad (2.42)$$

N_4 is the high order term in Taylor expansion, it is easy to show that

$$\begin{aligned}
N_4 \le& C_* h^{-1}(\|e_v\|_\infty \|e_v\|\|\varphi_h\| + \|e_u\|_\infty\|e_u\|\|\psi_h\|) \\
\le& C_* h^{-1}\Big(\|e_v\|_\infty(\|\varphi_h\|\|\psi_h\| + \|\varphi_h\|\|\psi^e\|) + \|e_v\|_\infty(\|\psi_h\|\|\varphi_h\| + \|\psi_h\|\|\varphi^e\|) \Big) \\
\le& C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty)(\|\varphi_h\|^2 + \|\psi_h\|^2) \\
& + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty)h^{2k+2}\|u\|^2_{H^{k+1}([a,b])}
\end{aligned} \quad (2.43)$$

Hence, combining (2.34), (2.40), (2.41), (2.42), (2.43), (2.31), we obtain from (2.30)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b(\varphi_h^2+\psi_h^2)dx \leq & (C+C_*(h^{-1}\|e_v\|_\infty+h^{-1}\|e_u\|_\infty))(\|\varphi_h\|^2+\|\psi_h\|^2) \\ & +(C+C_*(h^{-1}\|e_v\|_\infty+h^{-1}\|e_u\|_\infty))h^{2k+2}\|u\|^2_{H^{k+2}([a,b])}\end{aligned} \tag{2.44}$$

When $k \geq 1$, by using *a priori* assumption (2.26) we have

$$\frac{1}{2}\frac{d}{dt}\int_a^b(\varphi_h^2+\psi_h^2)dx \leq (C+C_*)(\|\varphi_h\|^2+\|\psi_h\|^2)+(C+C_*)h^{2k+2}\|u\|^2_{H^{k+2}([a,b])} \tag{2.45}$$

Finally, by Gronwall's inequality and a fact that $\|\varphi_h(\cdot,0)\| \leq Ch^{k+1}$, $\|\psi_h(\cdot,0)\| \leq Ch^{k+1}$ we can get

$$\int_a^b(\varphi_h^2+\psi_h^2)dx \leq Ch^{2k+2}. \tag{2.46}$$

This, together with the approximation result (2.23), implies the desired error estimate.

For the case of $k=0$, we assume that the convection term is linear, namely $f(u)=cu$. This is to avoid the need of the *a priori* assumption (2.26) which is no longer justifiable since our L^2 error estimate is only of order $O(h)$ in this case. The proof is similar to that for $k \geq 1$ case given above, and the only difference is $C_* = 0$ in this case. By similar lines of proof, we have

$$\frac{1}{2}\frac{d}{dt}\int_a^b(\varphi_h^2+\psi_h^2)dx \leq C(\|\varphi_h\|^2+\|\psi_h\|^2)+Ch^2. \tag{2.47}$$

An application of Gronwalls inequality give us that

$$\int_a^b(\varphi_h^2+\psi_h^2)dx \leq Ch^2. \tag{2.48}$$

This, together with the approximation result (2.23), implies the desired error estimate.

Finally, let us justify the *a priori* assumption (2.26) for $k \geq 1$. Similar to [18] and [1], we can verify this by a proof by contradiction. By (2.25), we can consider h small enough so that $Ch^{k+1} < \frac{1}{2}h^{\frac{3}{2}}$, where C is the constant in (2.25) determined by the final time T. Define $t^* = sup\{t : \|u(\cdot,t)-u_h(\cdot,t)\|+\|u(\cdot,t)-v_h(\cdot,t)\| \leq h^{\frac{3}{2}}\}$, then we have $\|u(\cdot,t^*)-u_h(\cdot,t^*)\|+\|u(\cdot,t^*)-v_h(\cdot,t^*)\| = h^{\frac{3}{2}}$ by continuity if t^* is finite. Clearly, (2.25) holds for $t \leq t^*$, in particular, $\|u(\cdot,t^*)-u_h(\cdot,t^*)\|+\|u(\cdot,t^*)-v_h(\cdot,t^*)\| \leq Ch^{k+1} < \frac{1}{2}h^{\frac{3}{2}}$. This is a contradiction if $t^* < T$. Hence, $t^* \geq T$ and our *a priori* assumption is justified. □

3 The central DG method in multi-dimensions

In this section, we consider the semi-discrete central DG method for multidimensional nonlinear conservation laws. Without loss of generality, we will show our central DG scheme and prove the optimal *a priori* error estimates in two dimensions ($d = 2$); all the arguments we present in our analysis depend on the tensor product structure of the mesh and finite element space and can be easily extended to the more general cases $d > 2$. Now we consider the following two-dimensional problem,

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \quad (x, y, t) \in \Omega \times (0, T] \\ u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega \end{cases} \tag{3.1}$$

with periodic boundary condition or compactly supported boundary condition.

3.1 Basic notations

Let $\{K_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\}$ be a partition of Ω into uniform square cells, depicted by the solid lines in Fig. 3.1, and tagged by their cell centroid at (x_i, y_j). Define $h = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$. Let $X_h^k := \{v \in L^2(\Omega) : v|_{K_{i,j}} \in Q^k(K_{i,j}), \quad \forall (i,j)\}$, where $Q^k(K_{i,j})$ is the tensor-product polynomials of degrees at most k in each variable defined on $K_{i,j}$ and no continuity is assumed across cell boundaries. Let $K_{i+\frac{1}{2},j+\frac{1}{2}}$ be the dual mesh which consists of a $\frac{h}{2}$ shift of the $K_{i,j}$, depicted by the dashed lines in Fig. 3.1. Let $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ be the cell centroid of the cell $K_{i+\frac{1}{2},j+\frac{1}{2}}$ and let $Y_h^k := \{v \in L^2(\Omega) : v|_{K_{i,j}} \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}), \quad \forall (i,j)\}$ denotes the space of tensor-product polynomials of degrees at most k in each variable defined on $K_{i+\frac{1}{2},j+\frac{1}{2}}$ and no continuity is assumed across the cell boundary. For a function $\varphi_h \in X_h^k$, we use $(\varphi_h)^+_{i+\frac{1}{2},y}$ and $(\varphi_h)^-_{i+\frac{1}{2},y}$ to denote the values of φ_h at $(x_{i+\frac{1}{2}}, y)$ from the right cell $K_{i+1,j}$ and the left cell $K_{i,j}$, respectively, when $y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ on all vertical edges. And for $\psi_h \in Y_h^k$, we use $(\psi_h)^+_{i,y}$ and $(\psi_h)^-_{i,y}$ to denote the values of ψ_h at (x_i, y) from the right cell $K_{i+\frac{1}{2},j+\frac{1}{2}}$ and the left cell $K_{i-\frac{1}{2},j+\frac{1}{2}}$, respectively, when $y \in [y_j, y_{j+1}]$ on all vertical edges. The notation $[\varphi_h]_{i+\frac{1}{2},y}$ or $[\psi_h]_{i+1,y}$ denote $(\varphi_h)^+_{i+\frac{1}{2},y} - (\varphi_h)^-_{i+\frac{1}{2},y}$ or $(\psi_h)^+_{i,y} - (\psi_h)^-_{i,y}$,i.e. the jump of φ_h at

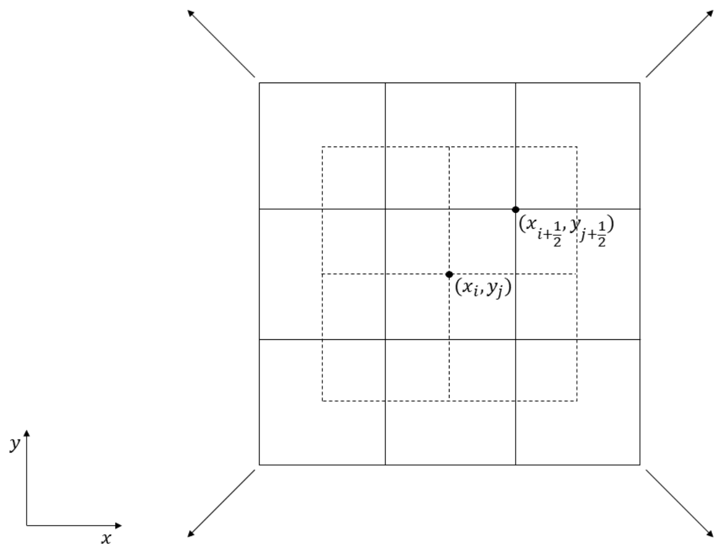

Fig. 3.1. 2D overlapping cells formed by collapsing the staggered dual cells on two adjacent time levels to one time level.

$(x_{i+\frac{1}{2}}, y)$ when $y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ or the jump of ψ_h at (x_i, y) when $y \in [y_j, y_{j+1}]$. Similarly, we can define $(\varphi_h)^+_{x,j+\frac{1}{2}}$, $(\varphi_h)^-_{x,j+\frac{1}{2}}$, $(\psi_h)^+_{x,j}$, $(\psi_h)^-_{x,j}$, $[\varphi_h]_{x,j+\frac{1}{2}}$ and $[\psi_h]_{x,j}$.

3.2 The central DG scheme

We propose the following semi-discrete CDG scheme for periodic boundary condition: find $u_h \in X_h^k$ and $v_h \in Y_h^k$, such that for any $\varphi_h \in X_h^k$ and $\psi_h \in Y_h^k$,

$$
\begin{aligned}
\int_{K_{i,j}} (u_h)_t \varphi_h dxdy =& \frac{1}{\tau_{max}} \int_{K_{i,j}} (v_h - u_h)\varphi_h dxdy \\
&+ \int_{K_{i,j}} (f(v_h)(\varphi_h)_x + g(v_h)(\varphi_h)_y) dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f(v_h)\varphi_h^-)_{i+\frac{1}{2},y} - (f(v_h)\varphi_h^+)_{i-\frac{1}{2},y}) dy \\
&- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g(v_h)\varphi_h^-)_{x,j+\frac{1}{2}} - (g(v_h)\varphi_h^+)_{x,j+\frac{1}{2}}) dx, \qquad (3.2a)
\end{aligned}
$$

$$
\begin{aligned}
\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (v_h)_t \psi_h dxdy =& \frac{1}{\tau_{max}} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (u_h - v_h)\psi_h dxdy \\
&+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u_h)(\psi_h)_x + g(u_h)(\psi_h)_y) dxdy
\end{aligned}
$$

$$-\int_{y_j}^{y_{j+1}} ((f(u_h)\psi_h^-)_{i+1,y} - (f(u_h)\psi_h^+)_{i,y})dy$$
$$-\int_{x_i}^{x_{i+1}} ((g(u_h)\psi_h^-)_{x,j+1} - (g(u_h)\psi_h^+)_{x,j})dx, \tag{3.2b}$$

where τ_{max} is a max step size, determined by $\tau_{max} =$ (CFL factor)$\times h/$(maximum characteristic speed), in which the CFL constant should be less than 1/2. Similarly, for the initial condition we simply take $u_h(\cdot,\cdot,0) = \mathbb{P}_h u_0(\cdot,\cdot),\ v_h(\cdot,\cdot,0) = \mathbb{Q}_h u_0(\cdot,\cdot)$, where $\mathbb{P}_h$ and $\mathbb{Q}_h$ are the L^2 projections into V_h^k and W_h^k, respectively, and we have

$$\begin{aligned} \|u_0 - \mathbb{P}_h u_0\|_{L^2(K_{i,j})} &\le Ch^{k+1}\|u_0\|_{H^{k+1}(K_{i,j})}, \\ \|u_0 - \mathbb{Q}_h u_0\|_{L^2(K_{i+\frac{1}{2},j+\frac{1}{2}})} &\le Ch^{k+1}\|u_0\|_{H^{k+1}(K_{i+\frac{1}{2},j+\frac{1}{2}})}. \end{aligned} \tag{3.3}$$

3.3 L^2 Stability for linear equation

The L^2-stability is proved for the CDG scheme (3.2) in [10] if $f(u)$ and $g(u)$ are linear. Without loss of generality, we take $f(u) = g(u) = u$. Hence, we have

$$\begin{cases} u_t + u_x + u_y = 0, & (x,y,t) \in \Omega \times (0,T] \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega \end{cases} \tag{3.4}$$

with periodic boundary condition.

Theorem 3.1. *The numerical solutions u_h and v_h of the semi-discrete CDG scheme* (3.2) *for the equation* (3.4) *have the following L^2 stability property*

$$\|u_h(\cdot,\cdot,T)\|_{L^2(\Omega)}^2 + \|v_h(\cdot,\cdot,T)\|_{L^2(\Omega)}^2 \le \|u_h(\cdot,\cdot,0)\|_{L^2(\Omega)}^2 + \|v_h(\cdot,\cdot,0)\|_{L^2(\Omega)}^2. \tag{3.5}$$

3.4 Optimal L^2 error estimate

In this subsection, we show the *a priori* L^2 error estimate of the scheme (3.2) for the equation (3.1).

Here and below, we again use $\|\cdot\|$ to denote the standard L^2 norm. Similar to the one-dimensional case, we recall the classical inverse and trace inequalities [2]. For any $w_h \in X_h^k$ or $w_h \in Y_h^k$, there exists a positive constant C independent of w_h and h, such that

$$\|\partial_x w_h\| \le Ch^{-1}\|w_h\|,\ \|w_h\|_\Gamma \le Ch^{-\frac{1}{2}}\|w_h\|,\ \|w_h\|_\infty \le Ch^{-1}\|w_h\| \tag{3.6}$$

where Γ is the set of boundaries of all elements $K_{i,j}$ or $K_{i+\frac{1}{2},j+\frac{1}{2}}$.

Similar to the one-dimensional case, we first introduce some notations. Assume u_h and v_h are the numerical solutions of CDG scheme (3.2) for equation (3.1), we define

$$\begin{aligned}
\tilde{B}_{i,j}(u_h, v_h; \varphi_h; f, g, u) :=& \frac{1}{\tau_{max}} \int_{K_{i,j}} (v_h - u_h)\varphi_h dxdy \\
&+ \int_{K_{i,j}} (f'(u(x_i, y_j))(\varphi_h)_x + g'(u(x_i, y_j))(\varphi_h)_y) v_h dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f'(u(x_i, y_j))((v_h\varphi_h^-)_{i+\frac{1}{2},y} - (v_h\varphi_h^+)_{i-\frac{1}{2},y})dy \\
&- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g'(u(x_i, y_j))((v_h\varphi_h^-)_{x,j+\frac{1}{2}} - (v_h\varphi_h^+)_{x,j+\frac{1}{2}})dx,
\end{aligned} \tag{3.7a}$$

$$\begin{aligned}
\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_h, v_h; \psi_h; f, g, u) :=& \frac{1}{\tau_{max}} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (u_h - v_h)\psi_h dxdy \\
&+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))(\psi_h)_x \\
&+ g'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))(\psi_h)_y) u_h dxdy \\
&- \int_{y_j}^{y_{j+1}} f'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))((u_h\psi_h^-)_{i+1,y} - (u_h\psi_h^+)_{i,y})dy \\
&- \int_{x_i}^{x_{i+1}} g'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))((u_h\psi_h^-)_{x,j+1} - (u_h\psi_h^+)_{x,j})dx,
\end{aligned} \tag{3.7b}$$

and

$$\begin{aligned}
B_{i,j}(u_h, v_h; \varphi_h, \psi_h) =& \int_{K_{i,j}} (u_h)_t\varphi_h dxdy + \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (v_h)_t\psi_h dxdy \\
&- \frac{1}{\tau_{max}} \int_{K_{i,j}} (v_h - u_h)\varphi_h dxdy - \frac{1}{\tau_{max}} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (u_h - v_h)\psi_h dx,
\end{aligned} \tag{3.8}$$

Obviously, we have

$$
\begin{aligned}
B_{i,j}(u_h, v_h; \varphi_h, \psi_h) = &\int_{K_{i,j}} (f(v_h)(\varphi_h)_x + g(v_h)(\varphi_h)_y) dxdy \\
&+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u_h)(\psi_h)_x + g(u_h)(\psi_h)_y) dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f(v_h)\varphi_h^-)_{i+\frac{1}{2},y} - (f(v_h)\varphi_h^+)_{i-\frac{1}{2},y}) dy \\
&- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g(v_h)\varphi_h^-)_{x,j+\frac{1}{2}} - (g(v_h)\varphi_h^+)_{x,j+\frac{1}{2}}) dx \\
&- \int_{y_j}^{y_{j+1}} ((f(u_h)\psi_h^-)_{i+1,y} - (f(u_h)\psi_h^+)_{i,y}) dy \\
&- \int_{x_i}^{x_{i+1}} ((g(u_h)\psi_h^-)_{x,j+1} - (g(u_h)\psi_h^+)_{x,j}) dx, \\
&\forall \varphi_h \in Q^k(K_{i,j}), \quad \forall \psi_h \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.9}
$$

Let u be the exact solution of equation (3.1), clearly we have

$$
\begin{aligned}
B_{i,j}(u, u; \varphi_h, \psi_h) = &\int_{K_{i,j}} (f(u)(\varphi_h)_x + g(u)(\varphi_h)_y) dxdy \\
&+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u)(\psi_h)_x + g(u)(\psi_h)_y) dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f(u)\varphi_h^-)_{i+\frac{1}{2},y} - (f(u)\varphi_h^+)_{i-\frac{1}{2},y}) dy \\
&- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g(u)\varphi_h^-)_{x,j+\frac{1}{2}} - (g(u)\varphi_h^+)_{x,j+\frac{1}{2}}) dx \\
&- \int_{y_j}^{y_{j+1}} ((f(u)\psi_h^-)_{i+1,y} - (f(u)\psi_h^+)_{i,y}) dy \\
&- \int_{x_i}^{x_{i+1}} ((g(u)\psi_h^-)_{x,j+1} - (g(u)\psi_h^+)_{x,j}) dx, \\
&\forall \varphi_h \in Q^k(K_{i,j}), \quad \forall \psi_h \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.10}
$$

Subtracting (3.9) from (3.10), we get the error equation for two-dimensional case,

$$
\begin{aligned}
&B_{i,j}(u-u_h, u-v_h; \varphi_h, \psi_h) = \\
&\int_{K_{i,j}} (f(u)-f(v_h))(\varphi_h)_x + (g(u)-g(v_h))(\varphi_h)_y dxdy \\
&+\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u)-f(u_h))(\psi_h)_x + (g(u)-g(u_h))(\psi_h)_y dxdy \\
&-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} [((f(u)-f(v_h))\varphi_h^-)_{i+\frac{1}{2},y} - ((f(u)-f(v_h))\varphi_h^+)_{i-\frac{1}{2},y}]dy \\
&-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [((g(u)-g(v_h))\varphi_h^-)_{x,j+\frac{1}{2}} - ((g(u)-g(v_h))\varphi_h^+)_{x,j+\frac{1}{2}}]dx \\
&-\int_{y_j}^{y_{j+1}} [((f(u)-f(u_h))\psi_h^-)_{i+1,y} - ((f(u)-f(u_h))\psi_h^+)_{i,y}]dy \\
&-\int_{x_i}^{x_{i+1}} [((g(u)-g(u_h))\psi_h^-)_{x,j+1} - ((g(u)-g(u_h))\psi_h^+)_{x,j}]dx \\
&:=H_{i,j}(f; u, u_h, v_h; \varphi_h, \psi_h), \\
&\forall \varphi_h \in Q^k(K_{i,j}), \quad \forall \psi_h \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.11}
$$

Summing over all i and j, the error equation becomes

$$
\begin{aligned}
&\sum_{i,j} B_{i,j}(u-u_h, u-v_h; \varphi_h, \psi_h) = \sum_{i,j} H_{i,j}(f; u, u_h, v_h; \varphi_h, \psi_h), \\
&\forall \varphi_h \in Q^k(K_{i,j}), \quad \forall \psi_h \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.12}
$$

3.4.1 Projection operators

To prove the error estimates for two-dimensional problems in uniform Cartesian meshes, we need two suitable projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ similar to the one-dimensional case. By applying the shifting technique in the two-dimensional case, for x and y variables respectively, we define $\mathbb{P}_h^*$ from $w \in L^\infty(K_{i,j})$ into $\mathbb{P}_h^* w \in Q^k(K_{i,j})$ over $K_{i,j}$ satisfying the following two equations,

$$\int_{K_{i,j}} \mathbb{P}_h^* w dxdy = \int_{K_{i,j}} w dxdy \tag{3.13a}$$

$$\tilde{P}_h(\mathbb{P}_h^* w; \varphi_h; f, g, u)_{i,j} = \tilde{P}_h(w; \varphi_h; f, g, u)_{i,j}, \ \forall \varphi_h \in Q^k(K_{i,j}) \tag{3.13b}$$

where $\tilde{P}_h(w; \varphi_h; f, g, u)_{i,j}$ is defined as follows,

$$\tilde{P}_h(w; \varphi_h; f, g, u)_{i,j} = \frac{1}{\tau_{max}} \Bigg(\int_{y_{j-\frac{1}{2}}}^{y_j} \int_{x_{i-\frac{1}{2}}}^{x_i} w(x+\frac{h}{2}, y+\frac{h}{2}) \varphi_h dxdy$$

$$
\begin{aligned}
&+\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_i}^{x_{i+\frac{1}{2}}} w(x-\frac{h}{2},y+\frac{h}{2})\varphi_h dxdy\\
&+\int_{y_j}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_i} w(x+\frac{h}{2},y-\frac{h}{2})\varphi_h dxdy\\
&+\int_{y_j}^{y_{j+\frac{1}{2}}}\int_{x_i}^{x_{i+\frac{1}{2}}} w(x-\frac{h}{2},y-\frac{h}{2})\varphi_h dxdy\\
&-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,y)\varphi_h dxdy\Bigg)\\
&+\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} w(x+\frac{h}{2},y+\frac{h}{2})(f'(u(x_i,y_j))\partial_x\varphi_h+g'(u(x_i,y_j))\partial_y\varphi_h)dxdy\\
&+\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_i}^{x_{i+\frac{1}{2}}} w(x-\frac{h}{2},y+\frac{h}{2})(f'(u(x_i,y_j))\partial_x\varphi_h+g'(u(x_i,y_j))\partial_y\varphi_h)dxdy\\
&+\int_{y_j}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_i} w(x+\frac{h}{2},y-\frac{h}{2})(f'(u(x_i,y_j))\partial_x\varphi_h+g'(u(x_i,y_j))\partial_y\varphi_h)dxdy\\
&+\int_{y_j}^{y_{j+\frac{1}{2}}}\int_{x_i}^{x_{i+\frac{1}{2}}} w(x-\frac{h}{2},y-\frac{h}{2})(f'(u(x_i,y_j))\partial_x\varphi_h+g'(u(x_i,y_j))\partial_y\varphi_h)dxdy\\
&-\int_{y_{j-\frac{1}{2}}}^{y_j} f'(u(x_i,y_j))w(x_i,y+\frac{h}{2})\Big(\varphi_h(x^-_{i+\frac{1}{2}},y)-\varphi_h(x^+_{i-\frac{1}{2}},y)\Big)dy\\
&-\int_{y_j}^{y_{j+\frac{1}{2}}} f'(u(x_i,y_j))w(x_i,y-\frac{h}{2})\Big(\varphi_h(x^-_{i+\frac{1}{2}},y)-\varphi_h(x^+_{i-\frac{1}{2}},y)\Big)dy\\
&-\int_{x_{i-\frac{1}{2}}}^{x_i} g'(u(x_i,y_j))w(x+\frac{h}{2},y_j)\Big(\varphi_h(x,y^-_{j+\frac{1}{2}})-\varphi_h(x,y^+_{j-\frac{1}{2}})\Big)dx\\
&-\int_{x_i}^{x_{i+\frac{1}{2}}} g'(u(x_i,y_j))w(x-\frac{h}{2},y_j)\Big(\varphi_h(x,y^-_{j+\frac{1}{2}})-\varphi_h(x,y^+_{j-\frac{1}{2}})\Big)dx
\end{aligned}
\tag{3.14}
$$

Similarly, we can define the projection $\mathbb{Q}_h^*$ from $w \in L^\infty(K_{i+\frac{1}{2},j+\frac{1}{2}})$ into $\mathbb{Q}_h^* w \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}})$ over $K_{i+\frac{1}{2},j+\frac{1}{2}}$. Next we will discuss the properties of these two special projections. Without loss of generality we will only consider $\mathbb{P}_h^*$. The equation (3.13a) is required by conservation. Note that $\tilde{P}_h(w;\varphi_h)_{i,j} = 0$ for $\forall w$ when φ_h is a constant, so (3.13b) alone misses one condition which is provided by (3.13a), just like the one-dimensional case. Existence and optimal approximate property of the projection $\mathbb{P}_h^*$ are established in the following lemma.

Lemma 3.1. *The projection* $\mathbb{P}_h^*$ *defined by* (3.13) *exists and is unique for any smooth* *function $w(x)$, and the following inequality holds*

$$\|\mathbb{P}_h^* w - w\| + h\|\mathbb{P}_h^* w - w\|_\infty + h^{\frac{1}{2}}\|\mathbb{P}_h^* w - w\|_\Gamma \leq Ch^{k+1}\|w\|_{H^{k+1}(\Omega)}, \tag{3.15}$$

for all k. The positive constant C depends on k, the bound of $f'(u)$, $g'(u)$, the constant c and is independent of h and w.

Proof. The proof of this lemma is given in Appendix A.3. □

Similarly, for $\mathbb{Q}_h^*$ we have

$$\|\mathbb{Q}_h^* w - w\| + h\|\mathbb{Q}_h^* w - w\|_\infty + h^{\frac{1}{2}}\|\mathbb{Q}_h^* w - w\|_\Gamma \leq Ch^{k+1}\|w\|_{H^{k+1}(\Omega)}, \tag{3.16}$$

if w is a smooth function.

Again, the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ satisfy the following superconvergence result.

Lemma 3.2. *For $m = 0, 1..., 8$, assume that $u = x^{k+1}$ or y^{k+1}, let $u_I = \mathbb{P}_h^* u$ and $v_I = \mathbb{Q}_h^* u$ then*

$$|\tilde{B}_{i,j}(u_I - u, v_I - u; \varphi_h; f, g, u)| \leq Ch^{2k+4} + C\|\varphi_h\|^2_{L^2(K_{i,j})} \tag{3.17}$$

$$|\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_I - u, v_I - u; \psi_h; f, g, u)| \leq Ch^{2k+4} + C\|\psi_h\|^2_{L^2(K_{i+\frac{1}{2},j+\frac{1}{2}})} \tag{3.18}$$

Proof. The proof of this lemma is given in Appendix A.4. □

3.5 *A priori* L^2 error estimates

Now let us give the *a priori* error estimate for the two-dimensional case.

Theorem 3.2. *For $k = 0, 1..., 8$, let $u(\cdot,\cdot,t)$ be the exact solution of equation (3.1), which is sufficiently smooth with bounded derivatives, and assume $f \in C^2$. The numerical solutions u_h and v_h of the CDG scheme (3.2) using uniform meshes satisfies the following L^2 error estimate*

$$\|u(\cdot,\cdot,T) - u_h(\cdot,\cdot,T)\|^2 + \|u(\cdot,\cdot,T) - v_h(\cdot,\cdot,T)\|^2 \leq Ch^{2k+2}, \tag{3.19}$$

where k is the polynomial degree in the finite element spaces X_h^k and Y_h^k, and the constant C depends on k, the final time T, $\|u\|_{H^{k+2}}$ and the bounds on the derivatives $|f^{(m)}|$, $|g^{(m)}|$, $m = 1, 2$, but is independent of the mesh size h. Here $\|u\|_{H^{k+2}}$ is the maximum $(k+2)$-th order Sobolev norm of u over time in $[0, T]$. For $k = 0$ and 1 we need $f(u)$ and $g(u)$ to be linear, i.e. $f(u) = c_1 u$ and $g(u) = c_2 u$ with constants c_1 and c_2.

Proof. Let $e_u = u - u_h$, $e_v = u - v_h$ be the error between the numerical and exact solutions. Similar to the one-dimensional case, to deal with the nonlinearity of $f(u)$ and $g(u)$, we would like first make *a priori* assumption that, for small enough h, we have

$$\|u - u_h\| \leq Ch^2, \;\; \|u - v_h\| \leq Ch^2 \tag{3.20}$$

and then by the interpolation property,

$$\begin{aligned} &\|e_u\|_\infty \leq Ch \quad \text{and} \quad \|\mathbb{P}_h^* u - u_h\|_\infty \leq Ch \\ &\|e_v\|_\infty \leq Ch \quad \text{and} \quad \|\mathbb{Q}_h^* u - u_h\|_\infty \leq Ch \end{aligned} \tag{3.21}$$

This assumption is not necessary for linear f and g. We will verify this assumption for $k \geq 2$ later.

By taking

$$\varphi_h = \mathbb{P}_h^* u - u_h, \;\; \psi_h = \mathbb{Q}_h^* u - v_h, \varphi^e = \mathbb{P}_h^* u - u, \;\; \psi^e = \mathbb{Q}_h^* u - u, \tag{3.22}$$

we obtain the energy equality

$$\sum_{i,j} B_{i,j}(\varphi_h - \varphi^e, \psi_h - \psi^e; \varphi_h, \psi_h) = \sum_{i,j} H_{i,j}(f; u, u_h, v_h; \varphi_h, \psi_h). \tag{3.23}$$

From the definition of $B_{i,j}$, we can obtain

$$\begin{aligned}
&\sum_{i,j} B_{i,j}(\varphi_h, \psi_h; \varphi_h, \psi_h) \\
=&\sum_{i,j} B_{i,j}(\varphi^e, \psi^e; \varphi_h, \psi_h) + \sum_{i,j} H_{i,j}(f; u, u_h, v_h; \varphi_h, \psi_h) \\
=&\sum_{i,j} \int_{K_{i,j}} (\psi^e)_t \varphi_h dxdy + \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (\varphi^e)_t \psi_h dxdy \\
&- \sum_{i,j} \frac{1}{\tau_{max}} \int_{K_{i,j}} (\psi^e - \varphi^e)\varphi_h dxdy - \sum_{i,j} \frac{1}{\tau_{max}} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (\varphi^e - \psi^e)\psi_h dxdy \\
&+ \sum_{i,j} \int_{K_{i,j}} (f(u) - f(v_h))(\varphi_h)_x + (g(u) - g(v_h))(\varphi_h)_y dxdy \\
&+ \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u) - f(u_h))(\psi_h)_x + (g(u) - g(u_h))(\psi_h)_y dxdy \\
&+ \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f(u) - f(v_h))[\varphi_h])_{i+\frac{1}{2},y} dy + \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g(u) - g(v_h))[\varphi_h])_{x,j+\frac{1}{2}} dx \\
&+ \sum_{i,j} \int_{y_j}^{y_{j+1}} ((f(u) - f(u_h))[\psi_h])_{i,y} dy + \sum_{i,j} \int_{x_i}^{x_{i+1}} ((g(u) - g(u_h))[\psi_h])_{x,j} dx.
\end{aligned} \tag{3.24}$$

For the left-hand side of (3.24), we follow the L^2 stability proof in Theorem 3.1 for linear case to conclude

$$\sum_{i,j} B_{i,j}(\varphi_h, \psi_h; \varphi_h, \psi_h) = \frac{1}{2}\frac{d}{dt}\int_\Omega (\varphi_h^2 + \psi_h^2)dx + \frac{1}{\tau_{max}}\int_\Omega (\varphi_h - \psi_h)^2 dx. \tag{3.25}$$

Similar to the proof in [18] and [15], to deal with the nonlinear part of (3.24) we would like to use the following Taylor expansions:

$$\begin{aligned}
f(u) - f(u_h) =& f'(u)\varphi_h - f'(u)\varphi^e - \frac{1}{2}f''_u(\varphi_h - \varphi^e)^2, \\
f(u) - f(v_h) =& f'(u)\psi_h - f'(u)\psi^e - \frac{1}{2}f''_v(\psi_h - \psi^e)^2, \\
g(u) - g(u_h) =& g'(u)\varphi_h - g'(u)\varphi^e - \frac{1}{2}g''_u(\varphi_h - \varphi^e)^2, \\
g(u) - g(v_h) =& g'(u)\psi_h - g'(u)\psi^e - \frac{1}{2}g''_v(\psi_h - \psi^e)^2,
\end{aligned} \tag{3.26}$$

where $f''_u,\ f''_v$ and $g''_u,\ g''_v$ are the mean values. These imply the following representation,

$$\begin{aligned}
&\sum_{i,j} B_{i,j}(\varphi^e, \psi^e; \varphi_h, \psi_h) + \sum_{i,j} H_{i,j}(f; u, u_h, v_h; \varphi_h, \psi_h) \\
=&\mathcal{L} + \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4
\end{aligned} \tag{3.27}$$

where

$$
\begin{aligned}
\mathcal{L} =& \sum_{i,j} \int_{K_{i,j}} (\psi^e)_t \varphi_h dxdy + \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (\varphi^e)_t \psi_h dxdy \\
\mathcal{N}_1 =& -\sum_{i,j} \frac{1}{\tau_{max}} \int_{K_{i,j}} (\psi^e - \varphi^e)\varphi_h dxdy \\
& - \sum_{i,j} \int_{K_{i,j}} (f'(u)\psi^e(\varphi_h)_x + g'(u)\psi^e(\varphi_h)_y) dxdy \\
& - \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (f'(u)\psi^e[\varphi_h])_{i+\frac{1}{2},y} dy - \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (g'(u)\psi^e[\varphi_h])_{x,j+\frac{1}{2}} dx \\
\mathcal{N}_2 =& -\sum_{i,j} \frac{1}{\tau_{max}} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (\varphi^e - \psi^e)\psi_h dxdy \\
& - \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f'(u)\varphi^e(\psi_h)_x + g'(u)\varphi^e(\psi_h)_y) dxdy \\
& - \sum_{i,j} \int_{y_j}^{y_{j+1}} (f'(u)\varphi^e[\psi_h])_{i,y} dy - \sum_{i,j} \int_{x_i}^{x_{i+1}} (g'(u)\varphi^e[\psi_h])_{x,j} dx \\
\mathcal{N}_3 =& \sum_{i,j} \int_{K_{i,j}} (f'(u)\psi_h(\varphi_h)_x + g'(u)\psi_h(\varphi_h)_y) dxdy \\
& + \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f'(u)\varphi_h(\psi_h)_x + g'(u)\varphi_h(\psi_h)_y) dxdy \\
& + \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (f'(u)\psi_h[\varphi_h])_{i+\frac{1}{2},y} dy + \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (g'(u)\psi_h[\varphi_h])_{x,j+\frac{1}{2}} dx \\
& + \sum_{i,j} \int_{y_j}^{y_{j+1}} (f'(u)\varphi_h[\psi_h])_{i,y} dy + \sum_{i,j} \int_{x_i}^{x_{i+1}} (g'(u)\varphi_h[\psi_h])_{x,j} dx \\
\mathcal{N}_4 =& -\frac{1}{2}\Big(\sum_{i,j} \int_{K_{i,j}} (f''_v(\psi_h - \psi^e)^2(\varphi_h)_x + g''_v(\psi_h - \psi^e)^2(\varphi_h)_y) dxdy \\
& + \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f''_u(\varphi_h - \varphi^e)^2(\psi_h)_x + g''_u(\varphi_h - \varphi^e)^2(\psi_h)_y) dxdy \\
& + \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (f''_v(\psi_h - \psi^e)^2[\varphi_h])_{i+\frac{1}{2},y} dy + \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (g''_v(\psi_h - \psi^e)^2[\varphi_h])_{x,j+\frac{1}{2}} dx \\
& + \sum_{i,j} \int_{y_j}^{y_{j+1}} (f''_u(\varphi_h - \varphi^e)^2[\psi_h])_{i,y} dy + \sum_{i,j} \int_{x_i}^{x_{i+1}} (g''_u(\varphi_h - \varphi^e)^2[\psi_h])_{x,j} dx \Big)
\end{aligned}
$$

By Young's inequality and (3.15), (3.16) we have

$$
\mathcal{L} \leq C(\|\varphi_h\|^2 + \|\psi_h\|^2) + Ch^{2k+2}\|u\|^2_{H^{k+1}(\Omega)}. \tag{3.28}
$$

Next we estimate the nonlinear part. First for the $\mathcal{N}_1$ term, we can rewrite it as

$$
\begin{aligned}
\mathcal{N}_1 = & -\sum_{i,j} \frac{1}{\tau_{max}} \int_{K_{i,j}} (\psi^e - \varphi^e)\varphi_h dxdy \\
& - \sum_{i,j} \int_{K_{i,j}} (f'(u(x_i, y_j))\psi^e(\varphi_h)_x + g'(u(x_i, y_j))\psi^e(\varphi_h)_y) dxdy \\
& - \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (f'(u(x_i, y_j))\psi^e[\varphi_h])_{i+\frac{1}{2},y} dy - \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (g'(u(x_i, y_j))\psi^e[\varphi_h])_{x,j+\frac{1}{2}} dx \\
& + \sum_{i,j} \int_{K_{i,j}} ((f'(u(x_i, y_j)) - f'(u))\psi^e(\varphi_h)_x + (g'(u(x_i, y_j)) - g'(u))\psi^e(\varphi_h)_y) dxdy \\
& + \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f'(u(x_i, y_j)) - f'(u))\psi^e[\varphi_h])_{i+\frac{1}{2},y} dy \\
& + \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g'(u(x_i, y_j)) - g'(u))\psi^e[\varphi_h])_{x,j+\frac{1}{2}} dx \\
= & -\sum_{i,j} \tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h) \\
& + \sum_{i,j} \int_{K_{i,j}} ((f'(u(x_i, y_j)) - f'(u))\psi^e(\varphi_h)_x + (g'(u(x_i, y_j)) - g'(u))\psi^e(\varphi_h)_y) dxdy \\
& + \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f'(u(x_i, y_j)) - f'(u))\psi^e[\varphi_h])_{i+\frac{1}{2},y} dy \\
& + \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g'(u(x_i, y_j)) - g'(u))\psi^e[\varphi_h])_{x,j+\frac{1}{2}} dx
\end{aligned}
$$

By using the inequality in (3.6), (3.15), (3.16) and $\|f'(u(x_i, y_j)) - f'(u)\|_{L^\infty(K_{i,j})} = O(h), \|g'(u(x_i, y_j)) - g'(u)\|_{L^\infty(K_{i,j})} = O(h)$, we have

$$
N_1 \leq -\sum_j \tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h) + C_*\|\varphi_h\|^2 + C_* h^{2k+2}\|u\|^2_{H^{k+1}([a,b])} \tag{3.29}
$$

For $\tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h)$, we know that for an arbitrary element $K_{i,j}$, we can obtain the following results from Lemma 3.2, for $\forall u \in P^{k+1}([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}])$, $\forall \varphi_h \in Q^k(K_{i,j})$

$$
|\tilde{B}_{i,j}(\mathbb{P}_h^* u - u, \mathbb{Q}_h^* u - u; \varphi_h; f, g, u)| \leq Ch^{2k+4} + C\|\varphi_h\|^2_{L^2(K_{i,j})}, \tag{3.30}
$$

On each element $K_{i,j}$ we consider the following Taylor expansion of u around (x_i, y_j),

$$
u = Tu + Ru, \tag{3.31}
$$

where

$$Tu = \sum_{l=0}^{k+1} \sum_{m=0}^{l} \frac{1}{m!(l-m)!} \frac{\partial^l u(x_i, y_j)}{\partial x^{l-m} \partial y^m} (x-x_i)^{l-m} (y-y_j)^m, \tag{3.32}$$

$$Ru = \sum_{m=0}^{k+2} \frac{(k+2)(x-x_i)^{k+2-m}(y-y_j)^m}{m!(k+2-m)!} \int_0^1 (1-s)^{k+1} \frac{\partial^{k+2} u(x_i^{(s)}, y_j^{(s)})}{} \partial x^{k+2-m} \partial y^m ds. \tag{3.33}$$

with $x_i^{(s)} = x_i + s(x - x_i),\ y_j^{(s)} = y_j + s(y - y_j)$. It is obvious that $Tu \in P^k([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}])$. Note that the operator $\mathbb{P}_h^*$ is a linear operator and $\mathbb{P}_h^* u = \mathbb{P}_h^* Tu + \mathbb{P}_h^* Ru$, we obtain from (3.30) that

$$\begin{aligned} \tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h; f, g, u) =& \tilde{B}_{i,j}(\mathbb{P}_h^* Tu - Tu + \mathbb{P}_h^* Ru - Ru, \\ & \mathbb{Q}_h^* Tu - Tu + \mathbb{Q}_h^* Ru - Ru; \varphi_h; f, g, u) \\ =& \tilde{B}_{i,j}(\mathbb{P}_h^* Tu - Tu, \mathbb{Q}_h^* Tu - Tu; \varphi_h; f, g, u) \\ & + \tilde{B}_{i,j}(\mathbb{P}_h^* Ru - Ru, \mathbb{Q}_h^* Ru - Ru; \varphi_h; f, g, u) \\ =& \tilde{B}_{i,j}(\mathbb{P}_h^* Ru - Ru, \mathbb{Q}_h^* Ru - Ru; \varphi_h; f, g, u) \\ & + Ch^{2k+4} + C\|\varphi_h\|^2_{L^2(K_{i,j})}. \end{aligned} \tag{3.34}$$

Recalling the Bramble-Hilbert lemma [2], we have

$$\|Ru\|_{L^\infty(K_{i,j})} \le Ch^{k+1} |u|_{H^{k+2}(K_{i,j})}. \tag{3.35}$$

Therefore, by using Young's inequality, (3.15), (3.16), (3.6) and (3.35), we have

$$-\sum_{i,j} \tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h; f, g, u) \le Ch^{2k+2} \|u\|_{H^{k+2}(\Omega)} + C\|\varphi_h\|^2. \tag{3.36}$$

Hence, for N_1 we have

$$\mathcal{N}_1 \le (C + C_*)\|\varphi_h\|^2 + (C + C_*) h^{2k+2} \|u\|^2_{H^{k+2}(\Omega)} \tag{3.37}$$

Similarly, for N_2 we have

$$\mathcal{N}_2 \le (C + C_*)\|\psi_h\|^2 + (C + C_*) h^{2k+2} \|u\|^2_{H^{k+2}(\Omega)} \tag{3.38}$$

Similar to the one-dimensional case, the $\mathcal{N}_3$ term can be rewritten as

$$
\begin{aligned}
\mathcal{N}_3 =& \sum_{i,j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_i} f'(u)(\psi_h\varphi_h)_x dxdy + \sum_{i,j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_i}^{x_{i+\frac{1}{2}}} f'(u)(\psi_h\varphi_h)_x dxdy \\
&+ \sum_{i,j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j} g'(u)(\psi_h\varphi_h)_y dydx + \sum_{i,j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}} g'(u)(\psi_h\varphi_h)_y dydx \\
&+ \sum_{i,j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}(f'(u)\psi_h[\varphi_h])_{i+\frac{1}{2},y}dy + \sum_{i,j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}(g'(u)\psi_h[\varphi_h])_{x,j+\frac{1}{2}}dx \\
&+ \sum_{i,j}\int_{y_j}^{y_{j+1}}(f'(u)\varphi_h[\psi_h])_{i,y}dy + \sum_{i,j}\int_{x_i}^{x_{i+1}}(g'(u)\varphi_h[\psi_h])_{x,j}dx \\
=& \sum_{i,j}\Big(\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}((f'(u)\psi_h\varphi_h^-)_{i+\frac{1}{2},y} - (f'(u)\psi_h\varphi_h^+)_{i-\frac{1}{2},y} \\
&+ (f'(u)\varphi_h\psi_h^-)_{i,y} - (f'(u)\varphi_h\psi_h^+)_{i,y} + (f'(u)\psi_h[\varphi_h])_{i+\frac{1}{2},y})dy \\
&+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}((g'(u)\psi_h\varphi_h^-)_{x,j+\frac{1}{2}} - (g'(u)\psi_h\varphi_h^+)_{x,j-\frac{1}{2}} \\
&+ (g'(u)\varphi_h\psi_h^-)_{x,j} - (g'(u)\varphi_h\psi_h^+)_{x,j} + (g'(u)\psi_h[\varphi_h])_{x,j+\frac{1}{2}})dx \\
&+ \int_{y_j}^{y_{j+1}}(f'(u)\varphi_h[\psi_h])_{i,y}dy + \int_{x_i}^{x_{i+1}}(g'(u)\varphi_h[\psi_h])_{x,j}dx \\
&- \int_{K_{i,j}}((f'(u))_x + (g'(u))_y)\varphi_h\psi_h dxdy\Big) \\
=& -\sum_{i,j}\int_{K_{i,j}}((f'(u))_x + (g'(u))_y)\varphi_h\psi_h dxdy \\
\leq& C_*\|\varphi_h\|\|\psi_h\| \leq C_*(\|\varphi_h\|^2 + \|\psi_h\|^2)
\end{aligned}
\tag{3.39}
$$

$\mathcal{N}_4$ is the high order term in Taylor expansion, its easy to show that

$$
\begin{aligned}
\mathcal{N}_4 \leq& C_* h^{-1}(\|e_v\|_\infty\|e_v\|\|\varphi_h\| + \|e_u\|_\infty\|e_u\|\|\psi_h\|) \\
\leq& C_* h^{-1}\Big(\|e_v\|_\infty(\|\varphi_h\|\|\psi_h\| + \|\varphi_h\|\|\psi^e\|) + \|e_v\|_\infty(\|\psi_h\|\|\varphi_h\| + \|\psi_h\|\|\varphi^e\|)\Big) \\
\leq& C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty)(\|\varphi_h\|^2 + \|\psi_h\|^2) \\
&+ C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty)h^{2k+2}\|u\|^2_{H^{k+1}(\Omega)}
\end{aligned}
\tag{3.40}
$$

Then by combining (3.28), (3.37), (3.38), (3.39), (3.40), (3.25), we obtain from (3.24)

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega(\varphi_h^2 + \psi_h^2)dxdy \leq& (C + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty))(\|\varphi_h\|^2 + \|\psi_h\|^2) \\
&+ (C + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty))h^{2k+2}\|u\|^2_{H^{k+2}(\Omega)}
\end{aligned}
\tag{3.41}
$$

When $k \geq 2$, by using *a priori* assumption (3.21) we have

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega(\varphi_h^2 + \psi_h^2)dxdy \leq (C + C_*)(\|\varphi_h\|^2 + \|\psi_h\|^2) + (C + C_*)h^{2k+2}\|u\|^2_{H^{k+2}(\Omega)} \tag{3.42}
$$

Finally, by Gronwalls inequality and a fact that $\|\varphi_h(\cdot,\cdot,0)\| \le Ch^{k+1},\ \|\psi_h(\cdot,\cdot,0)\| \le Ch^{k+1}$ we can get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\varphi_h^2+\psi_h^2)dxdy \le Ch^{2k+2} \tag{3.43}$$

This, together with the approximation result (3.15), (3.16) implies the desired error estimate.

For the case of $k=0$ or 1, we assume that $f(u)$ and $g(u)$ are linear fluxes, namely $f(u)=c_1u,\ g(u)=c_2u$ with constants $c_1,\ c_2$. This is to avoid the need of the *a priori* assumption (3.20) which is no longer justifiable in this case. By similar lines of proof and noting that $C_*=0$ in this case, we can obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\varphi_h^2+\psi_h^2)dxdy \le C(\|\varphi_h\|^2+\|\psi_h\|^2)+Ch^{2k+2},\ k=0,\ 1 \tag{3.44}$$

By using the Gronwalls inequality we have

$$\int_\Omega(\varphi_h^2+\psi_h^2)dxdy \le Ch^{2k+2},\ k=0,\ 1 \tag{3.45}$$

This, together with the approximation result (3.15), (3.16), implies the desired error estimate for $k=0,\ 1$ with linear fluxes.

Just like the one-dimensional case, let us justify the *a priori* assumption (3.20) with $k\ge 2$. Similar to [18] and [1], we can verify this by a proof by contradiction. By (3.19), we can consider h small enough so that $Ch^{k+1}<\frac{1}{2}h^2$, where C is the constant in (3.19) determined by the final time T. Define $t^*=sup\{t:\|u(\cdot,\cdot,t)-u_h(\cdot,\cdot,t)\|+\|u(\cdot,\cdot,t)-v_h(\cdot,\cdot,t)\|\le h^2\}$, then we have $\|u(\cdot,\cdot,t^*)-u_h(\cdot,\cdot,t^*)\|+\|u(\cdot,\cdot,t^*)-v_h(\cdot,\cdot,t^*)\|=h^2$ by continuity if t^* is finite. Clearly, (3.19) holds for $t\le t^*$, in particular, $\|u(\cdot,\cdot,t^*)-u_h(\cdot,\cdot,t^*)\|+\|u(\cdot,\cdot,t^*)-v_h(\cdot,\cdot,t^*)\|\le Ch^{k+1}<\frac{1}{2}h^2$. This is a contradiction if $t^*<T$. Hence, $t^*\ge T$ and our *a priori* assumption is justified. □

4 Numerical examples

In this section, we present numerical examples to verify our theoretical findings. Uniform meshes are used in all examples. The schemes are integrated in time with the third order SSP Runge-Kutta method. We would like to compute on elements of degree $k = 0, 1, 2, 3$. We set the CFL number to be 0.05. For $k = 0, 1, 2$ we let $\Delta t = CFL \cdot h$ and $\Delta t = CFL \cdot h^{\frac{4}{3}}$ for $k = 3$ where h is the characteristic length of the mesh, so that time error will be dominated by the spatial error.

Example 4.1. *We solve the one-dimensional Burgers equation given by*

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, \ \ x \in [-\pi, \pi] \\ u(x, 0) = \sin(x), \ \ x \in [-\pi, \pi] \\ u(-\pi, t) = u(\pi, t). \end{cases} \tag{4.1}$$

The exact solution is obtained by Newton iteration. In this example, we use $\tau_{max} = \frac{h}{2k+1}$, $h = \frac{2\pi}{N}$ to test the numerical schemes. The errors and numerical order of accuracy at $T = 0.5$ with $0 \leq k \leq 3$ are listed in Tables 4.1.

Table 4.1 shows that the order of convergence of the error achieves the expected $(k + 1)$-th order of accuracy.

Example 4.2. *We solve the two-dimensional Burgers equation given by*

$$\begin{cases} u_t + (\frac{u^2}{2})_x + (\frac{u^2}{2})_y = 0, \ \ (x, y) \in [-\pi, \pi]^2 \\ u(x, y, 0) = \sin(x + y), \ \ (x, y) \in [-\pi, \pi]^2, \end{cases} \tag{4.2}$$

with periodic boundary condition. The exact solution follows from the solution of one-dimensional Burgers equation with $\xi = x + y$. In this example, we use $\tau_{max} = \frac{h}{2k+1}$, $h = \frac{2\pi}{N}$ to test the numerical schemes. The central DG scheme is evolved up to $T = 0.2$ when the solution is still smooth. The errors and numerical order of accuracy with $0 \leq k \leq 3$ are listed in Tables 4.2.

Table 4.2 shows that the order of convergence of the error achieves the expected $(k + 1)$-th order of accuracy.

5 Concluding remarks

In this paper, *a priori* optimal L^2 error estimates to central DG methods on uniform meshes applied to nonlinear conservation laws with smooth solutions are proved with

k	N	L^1 error	order	L^2 error	order	L^∞ error	order
0	10	6.73E-001	-	3.65E-001	-	5.60E-001	-
	20	3.34E-001	1.01	1.83E-001	0.99	3.04E-001	0.88
	40	1.66E-001	1.00	9.19E-002	1.00	1.56E-001	0.97
	80	8.31E-002	1.00	4.60E-002	1.00	7.90E-002	0.98
	160	4.15E-002	1.00	2.30E-002	1.00	3.97E-002	0.99
1	10	6.90E-002	-	4.40E-002	-	8.69E-002	-
	20	1.86E-002	1.89	1.25E-002	1.81	2.58E-002	1.75
	40	4.73E-003	1.98	3.21E-003	1.97	7.34E-003	1.81
	80	1.19E-003	1.99	8.11E-004	1.98	1.95E-003	1.92
	160	2.98E-004	2.00	2.04E-004	1.99	4.94E-004	1.98
2	10	9.68E-003	-	8.58E-003	-	2.53E-002	-
	20	8.97E-004	3.43	9.29E-004	3.21	4.24E-003	2.58
	40	1.13E-004	2.99	1.14E-004	3.02	6.03E-004	2.82
	80	1.42E-005	2.99	1.44E-005	2.98	7.87E-005	2.94
	160	1.78E-006	3.00	1.81E-006	2.99	9.99E-006	2.98
3	10	6.06E-04	-	6.47E-04	-	3.26E-03	-
	20	6.17E-05	3.30	6.91E-05	3.23	2.73E-04	3.58
	40	4.54E-06	3.77	5.54E-06	3.64	3.21E-05	3.09
	80	2.86E-07	3.99	3.49E-07	3.99	2.06E-06	3.96
	160	1.79E-08	4.00	2.19E-08	4.00	1.30E-07	3.99

Table 4.1. Errors and numerical orders of accuracy for Example 4.1 on a uniform mesh of N cells. Here $\tau_{max} = \frac{h}{2k+1}$ and final time $T = 0.5$.

polynomial degrees of $k \leq 8$. The main techniques used in this paper are special projections and Taylor expansions. Our analysis is carried out both in one dimension and in two-dimensions for uniform Cartesian meshes and tensor-product polynomial spaces. We also give some numerical examples to verify the results of our theoretical analysis. The error estimates for nonlinear conservation laws in this paper were obtained using stability for the linear case and the smoothness of the exact solution. It is not clear whether stability holds for the scalar nonlinear conservation laws with general non-smooth solutions. Such a stability proof for the central DG schemes and the extension of this work to non-uniform meshes are interesting and challenging, and constitutes our going work.

A Appendix: Collection of technical proofs

In this appendix, we collect the proofs of some technical lemmas and propositions.

k	N× N	L^1 error	order	L^2 error	order	L^∞ error	order
0	10×10	5.57E+00	-	1.22E+00	-	8.16E-01	-
	20×20	2.76E+00	1.01	6.17E-01	0.98	4.87E-01	0.74
	40×40	1.37E+00	1.01	3.09E-01	1.00	2.57E-01	0.92
	80×80	6.81E-01	1.01	1.54E-01	1.00	1.30E-01	0.98
	160×160	3.40E-01	1.00	7.72E-02	1.00	6.54E-02	0.99
1	10×10	9.12E-01	-	2.34E-01	-	2.60E-01	-
	20×20	2.37E-01	1.94	6.25E-02	1.90	8.19E-02	1.67
	40×40	5.99E-02	1.99	1.60E-02	1.97	2.19E-02	1.90
	80×80	1.50E-02	2.00	4.02E-03	1.99	5.71E-03	1.94
	160×160	3.75E-03	2.00	1.01E-03	2.00	1.45E-03	1.98
2	10×10	1.49E-01	-	5.03E-02	-	1.22E-01	-
	20×20	1.91E-02	2.97	6.44E-03	2.97	2.14E-02	2.52
	40×40	2.38E-03	3.00	8.33E-04	2.95	3.00E-03	2.83
	80×80	3.00E-04	2.99	1.05E-04	2.98	3.87E-04	2.96
	160×160	3.77E-05	2.99	1.33E-05	2.99	4.87E-05	2.99
3	10×10	2.06E-02	-	7.45E-03	-	2.20E-02	-
	20×20	2.04E-03	3.33	8.72E-04	3.09	3.30E-03	2.74
	40×40	1.48E-04	3.79	6.09E-05	3.84	2.50E-04	3.72
	80×80	9.70E-06	3.93	4.02E-06	3.92	1.78E-05	3.81
	160×160	6.19E-07	3.97	2.62E-07	3.94	1.17E-06	3.92

Table 4.2. Errors and numerical orders of accuracy for Example 4.2 on a uniform mesh of $N \times N$ cells. Here $\tau_{max} = \frac{h}{2k+1}$ and final time $T = 0.2$.

A.1 Proof of Lemma 2.1

Proof. We only consider $\mathbb{P}_h^*$, while the proof for $\mathbb{Q}_h^*$ follows similar lines. For $\forall j$, we let $\xi = \frac{2(x-x_j)}{h}$ on I_j, for a smooth function $\omega(x)$ and a k-th order polynomial $\varphi_h(x)$ on I_j, and define

$$\begin{aligned} \tilde{\omega}(\xi) &= \omega(\frac{h}{2}\xi + x_j) = \omega(x), \\ \phi_h(\xi) &= \varphi_h(\frac{h}{2}\xi + x_j) = \varphi_h(x). \end{aligned} \tag{A.1}$$

Note that the procedure to find the $\mathbb{P}_h^*\tilde{\omega} \in \mathbb{P}^k([-1,1])$ is to solve for a linear system, so the existence and uniqueness are equivalent. Thus, we only need to prove the uniqueness of the projection $\mathbb{P}_h^*$. We set $\omega_I(\xi) = \mathbb{P}_h^*\tilde{\omega}(\xi) = \mathbb{P}_h^*\omega(x)$ with $\tilde{\omega}(\xi) = \omega(x) = 0$, and would like to prove $\omega_I(\xi) = 0$. Then by the definition of the projection $\mathbb{P}_h^*$, we have:

$$\tilde{P}_h(\omega_I; \phi_h; f, u)_j = \frac{h}{2\tau_{max}}(\int_{-1}^{0} \omega_I(\xi+1)\phi_h(\xi)d\xi + \int_{0}^{1} \omega_I(\xi-1)\phi_h(\xi)d\xi$$

$$
\begin{aligned}
&- \int_{-1}^{1} \omega_I(\xi)\phi_h(\xi)d\xi + \int_{-1}^{0} f'(u(x_j))\omega_I(\xi+1)(\phi_h(\xi))_\xi d\xi \\
&+ \int_{0}^{1} f'(u(x_j))\omega_I(\xi-1)(\phi_h(\xi))_\xi d\xi \\
&- f'(u(x_j))\omega_I(0)(\phi_h(1) - \phi_h(-1)),
\end{aligned} \tag{A.2a}
$$

$$
\frac{h}{2}\int_{-1}^{1} \omega_I(\xi)d\xi = 0. \tag{A.2b}
$$

Let $\phi_h(\xi) = \omega_I(\xi) \in \mathbb{P}^k([-1,1])$, we get

$$
\begin{aligned}
\tilde{P}_h(\omega_I;\omega_I;f,u)_j =& \frac{h}{2\tau_{max}}\left(\int_{-1}^{0} \omega_I(\xi+1)\omega_I(\xi)d\xi + \int_{0}^{1} \omega_I(\xi-1)\omega_I(\xi)d\xi - \int_{-1}^{1} \omega_I(\xi)^2 d\xi\right) \\
&+ \int_{-1}^{0} f'(u(x_j))\omega_I(\xi+1)(\omega_I(\xi))_\xi d\xi + \int_{0}^{1} f'(u(x_j))\omega_I(\xi-1)(\omega_I(\xi))_\xi d\xi \\
&- f'(u(x_j))\omega_I(0)\left(\omega_I(1) - \omega_I(-1)\right) = 0.
\end{aligned} \tag{A.3}
$$

We rewrite $\tilde{P}_h(\omega_I;\omega_I;f,u)_j$ by a change of variable $\xi \to \xi + 1$ for the integrations over $[-1,0]$ to get

$$
\begin{aligned}
\tilde{P}_h(\omega_I;\omega_I;f,u)_j =& \frac{h}{2\tau_{max}}\left(2\int_{0}^{1} \omega_I(\xi-1)\omega_I(\xi)d\xi - \int_{0}^{1} \omega_I(\xi-1)^2 d\xi - \int_{0}^{1} \omega_I(\xi)^2 d\xi\right) \\
&+ \int_{0}^{1} f'(u(x_j))\omega_I(\xi)(\omega_I(\xi-1))_\xi d\xi + \int_{0}^{1} f'(u(x_j))\omega_I(\xi-1)(\omega_I(\xi))_\xi d\xi \\
&- f'(u(x_j))\omega_I(0)(\omega_I(1) - \omega_I(-1)) \\
=& - \frac{h}{2\tau_{max}}\int_{0}^{1} (\omega_I(\xi) - \omega_I(\xi-1))^2 d\xi = 0.
\end{aligned} \tag{A.4}
$$

Thus,

$$
\omega_I(\xi) = \omega_I(\xi-1), \quad \forall \xi \in (0,1) \tag{A.5}
$$

$$
\omega_I(\xi+1) = \omega_I(\xi), \quad \forall \xi \in (-1,0) \tag{A.6}
$$

which indicates that $\omega_I(\xi)$ is a constant on $[-1,1]$. Hence, by (A.2b), we have

$$
\frac{h}{2}\int_{-1}^{1} \omega_I(\xi)d\xi = h\omega_I(\xi) = 0 \tag{A.7}
$$

which implies $\omega_I(\xi) \equiv 0$ on $[-1,1]$.

We have now finished the proof of uniqueness. Next we move to prove the boundedness. Let $\omega_I(x) = \mathbb{P}_h^*\omega(x) = \sum_{i=0}^{k} a_i x^i$ and set the test functions $\varphi_h = x, x^2, ..., x^k$. Then we have

$$\tilde{P}_h(\omega_I; x^l; f, u)_j = \sum_{i-0}^{k} \alpha_{il} a_i, \ 1 \le l \le k \tag{A.8}$$

$$\int_{-1}^{1} w_I(x)dx = \sum_{i=0}^{k} \frac{1^{i+1} - (-1)^{i+1}}{i+1} a_i = \sum_{i=0}^{k} \alpha_{i0} a_i \tag{A.9}$$

By calculation, for $1 \le l \le k$ we have

$$\begin{aligned}
\alpha_{il} =& \frac{h}{2\tau_{max}} [\frac{i!l!\left((-1)^i + (-1)^l\right)}{(i+l+1)!} + \frac{(-1)^{i+l}+1}{i+l+1}] \\
&+ f'(u(x_j)) \frac{li!(l-1)!\left((-1)^i + (-1)^{l+1}\right)}{(i+l)!} \\
=& \frac{h}{2\tau_{max}} \mu_{il} + f'(u(x_j))\eta_{il}
\end{aligned} \tag{A.10}$$

where

$$\begin{aligned}
\mu_{il} =& \frac{i!l!\left((-1)^i + (-1)^l\right)}{(i+l+1)!} + \frac{(-1)^{i+l}+1}{i+l+1} \\
\eta_{il} =& \frac{li!(l-1)!\left((-1)^i + (-1)^{l+1}\right)}{(i+l)!}
\end{aligned} \tag{A.11}$$

We denote $\beta = (a_0, ..., a_k)^T,\ A(i,l) = \alpha_{il},\ 0 \le i \le k,\ 0 \le l \le k$ and $b_0 = \int_{-1}^{1} w(x)dx,\ b_l = \tilde{P}_h(w; x^l; f, u),\ 1 \le l \le k,\ B = (b_0, ..., b_k)^T$. We will solve the following linear system to get the coefficients β,

$$A^T \beta = B \tag{A.12}$$

We can rewrite A as the following form,

$$A = \frac{h}{2\tau_{max}} \mathcal{M} + f'(u(x_j))\mathcal{H} + \mathcal{C} \tag{A.13}$$

where

$$\mathcal{M}(i,l) = \begin{cases} \mu_{il},\ 0 \le i \le k,\ 1 \le l \le k \\ 0,\ 0 \le i \le k,\ l = 0 \end{cases} \tag{A.14}$$

$$\mathcal{H}(i,l) = \begin{cases} \eta_{il},\ 0 \le i \le k,\ 1 \le l \le k \\ 0,\ 0 \le i \le k,\ l = 0 \end{cases} \tag{A.15}$$

$$\mathcal{C}(i,l) = \begin{cases} 0,\ 0 \le i \le k,\ 1 \le l \le k \\ \alpha_{i0},\ 0 \le i \le k,\ l = 0 \end{cases} \tag{A.16}$$

From the formulation of the scheme (2.6) we have $\tau_{max} = c\ h$, here c is a constant dictated by stability. Then we have

$$A^T = \frac{1}{2c}\mathcal{M}^T + f'(u(x_j))\mathcal{H}^T + \mathcal{C}^T \tag{A.17}$$

From (A.11) we know that

$$\mu_{il} = \begin{cases} \dfrac{2((i+l)! + i!l!)}{(i+l+1)!}, \ if\ i\ and\ l\ are\ even \\ \dfrac{2((i+l)! - i!l!)}{(i+l+1)!}, \ if\ i\ and\ l\ are\ odd \\ 0,\ if\ (i+l)\ is\ odd \end{cases} \tag{A.18}$$

$$\eta_{il} = \begin{cases} \dfrac{-2li!(l-1)!}{(i+l)!}, \ if\ i\ is\ odd\ and\ l\ is\ even \\ \dfrac{2li!(l-1)!}{(i+l)!}, \ if\ i\ is\ even\ and\ l\ is\ odd \\ 0,\ if\ (i+l)\ is\ even \end{cases} \tag{A.19}$$

and from (A.9) we have

$$\alpha_{i0} = \begin{cases} \dfrac{2}{i+1}, \ if\ i\ is\ even \\ 0,\ if\ i\ is\ odd \end{cases} \tag{A.20}$$

Hence, we can estimate the infinity norm of A^T,

$$\begin{aligned} \|A^T\|_\infty =& \|\frac{1}{2c}\mathcal{M}^T + f'(u(x_j))\mathcal{H}^T + \mathcal{C}^T\|_\infty \\ =& \max\{\sum_{i=0}^{k}|\alpha_{i0}|, \max_{1\le l\le k}\sum_{i=0}^{k}(\frac{1}{2c}|\mu_{il}| + |f'(u(x_j))\eta_{il}|)\} \end{aligned} \tag{A.21}$$

Since $\mu_{il} > 0$ for $(i+l)$ is even and $f'(u(x_j))$ is bounded, then we have

$$\|A^T\|_\infty \le \mathcal{E} \tag{A.22}$$

where $\mathcal{E}$ is a constant which depends on polynomial degree k, the bound of $f'(u(x_j))$ and constant c. Since the first row of the matrix A^T are constants α_{i0} which only depends on degree k and the other elements of A^T either only contain $\frac{1}{2c}$ or only $f'(u(x_j))$, the by the definition of determinant we have

$$\det(A^T) = \sum_{i=0}^{k}\mathcal{D}_i(k)(\frac{1}{2c})^i(f'(u(x_j)))^{k-i} \tag{A.23}$$

where $\mathcal{D}_i(k)$ is a constant which only depends on degree k. Notice that if $f'(u(x_j)) = 0$ in (A.23), then $\det(A^T) = \mathcal{D}_k(k)(\frac{1}{2c})^k$. From the previous proof of the existence and uniqueness of the projection, we know that A^T is always invertible which means $\det(A^T) \neq 0$ holds for any value of $f'(u(x_j))$. Hence, here we have $\mathcal{D}_k(k) \neq 0$. Then by taking sufficiently small constant c we can make

$$|\det(A^T)| \geq \frac{|\mathcal{D}_k(k)|}{2}(\frac{1}{2c})^k > 0 \tag{A.24}$$

holds for all $f'(u(x_j))$. Next let $\sigma_i(A^T)$ denotes the $i-$th singular value of A^T which are in descending order from 0 to k, $\sigma_{\max}(A^T)$ and $\sigma_{\min}(A^T)$ represent the largest and smallest singular value of matrix A^T. Then we have

$$\begin{aligned}
\|A^{-T}\|_2 =& \frac{1}{\sigma_{\min}(A^T)} \\
\leq& \frac{1}{\sigma_{\min}(A^T)} \cdot (\prod_{i=0}^{k-1} \frac{\sigma_{\max}(A^T)}{\sigma_i(A^T)}) \\
=& \frac{(\sigma_{\max}(A^T))^k}{\prod_{i=0}^{k} \sigma_i(A^T)} \\
=& \frac{\|A^T\|_2^k}{|\det(A^T)|} \\
\leq& \frac{2(2c)^k}{\mathcal{D}_k(k)}\|A^T\|_2^k
\end{aligned} \tag{A.25}$$

By the equivalence of norms

$$\|A^T\|_2 \leq \sqrt{k+1}\|A^T\|_\infty \tag{A.26}$$

$$\|A^{-T}\|_\infty \leq \sqrt{k+1}\|A^{-T}\|_2 \tag{A.27}$$

we have

$$\|A^{-T}\|_\infty \leq \frac{2(2c)^k(k+1)^{\frac{k+1}{2}}}{\mathcal{D}_k(k)}\mathcal{E}^k \tag{A.28}$$

It is obvious that $\|B\|_\infty \leq \tilde{C}\|w\|_\infty$ due to the boundedness of $f'(u(x_j))$. Here $\tilde{C}$ is a constant which depends on degree k and the bound of $f'(u(x_j))$. Hence, for the coefficients β we have

$$\|\beta\|_\infty \leq \|A^{-T}\|_\infty\|B\|_\infty \leq \frac{2(2c)^k(k+1)^{\frac{k+1}{2}}}{\mathcal{D}_k(k)}\mathcal{E}^k\tilde{C}\|w\|_\infty \tag{A.29}$$

which immediately results in the boundedness of $\mathbb{P}_h^* w$. □

A.2 Proof of Proposition 2.1

Let $u_I = \mathbb{P}_h^* u \in V_h^k$, $v_I = \mathbb{Q}_h^* u \in W_h^k$, $a_j = f'(u(x_j))$, $a_{j+\frac{1}{2}} = f'(u(x_{j+\frac{1}{2}}))$, by the definition of $\tilde{B}_j$ and $\hat{B}_{j+\frac{1}{2}}$, we have

$$
\begin{aligned}
&\tilde{B}_j(u_I, v_I; \varphi_h; f, u) - \tilde{B}_j(u, u; \varphi_h; f, u)\\
=&\frac{1}{\tau_{max}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (v_I - u_I)\varphi_h dx + a_j[\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (v_I - u)(\varphi_h)_x\\
&- (v_I(x_{j+\frac{1}{2}}) - u(x_{j+\frac{1}{2}}))\varphi_h(x_{j+\frac{1}{2}}^-) + (v_I(x_{j-\frac{1}{2}}) - u(x_{j-\frac{1}{2}}))\varphi_h(x_{j-\frac{1}{2}}^+)]\\
=&\tilde{P}_h(u_I - u; \varphi_h; f, u)_j + \frac{1}{\tau_{max}}[\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - u - u_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))\varphi_h dx\\
&+ \int_{x_{j-\frac{1}{2}}}^{x_j} (v_I - u - u_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))\varphi_h dx]\\
&+ a_j[\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - u - u_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))(\varphi_h)_x dx\\
&+ \int_{x_{j-\frac{1}{2}}}^{x_j} (v_I - u - u_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))(\varphi_h)_x dx\\
&- (v_I(x_{j+\frac{1}{2}}) - u(x_{j+\frac{1}{2}}) - u_I(x_j) + u(x_j))\varphi_h(x_{j+\frac{1}{2}}^-)\\
&+ (v_I(x_{j-\frac{1}{2}}) - u(x_{j-\frac{1}{2}}) - u_I(x_j) + u(x_j))\varphi_h(x_{j-\frac{1}{2}}^+)]
\end{aligned} \tag{A.30}
$$

and

$$
\begin{aligned}
&\hat{B}_{j+\frac{1}{2}}(u_I, v_I; \psi_h; f, u) - \hat{B}_{j+\frac{1}{2}}(u, u; \psi_h; f, u)\\
=&\frac{1}{\tau_{max}} \int_{x_j}^{x_{j+1}} (u_I - v_I)\psi_h dx + a_{j+\frac{1}{2}}[\int_{x_j}^{x_{j+1}} (u_I - u)(\psi_h)_x\\
&- (u_I(x_{j+1}) - u(x_{j+1}))\psi_h(x_{j+1}^-) + (u_I(x_j) - u(x_j))\psi_h(x_j^+)]\\
=&\tilde{Q}_h(v_I - u; \psi_h; f, u)_{j+\frac{1}{2}} + \frac{1}{\tau_{max}}[\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} (u_I - u - v_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))\psi_h dx\\
&+ \int_{x_j}^{x_{j+\frac{1}{2}}} (u_I - u - v_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))\psi_h dx]\\
&+ a_{j+\frac{1}{2}}[\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} (u_I - u - v_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))(\psi_h)_x dx\\
&+ \int_{x_j}^{x_{j+\frac{1}{2}}} (u_I - u - v_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))(\psi_h)_x dx\\
&- (u_I(x_{j+1}) - u(x_{j+1}) - v_I(x_{j+\frac{1}{2}}) + u(x_{j+\frac{1}{2}}))\psi_h(x_{j+1}^-)\\
&+ (u_I(x_j) - u(x_j) - v_I(x_{j+\frac{1}{2}}) + u(x_{j+\frac{1}{2}}))\psi_h(x_j^+)].
\end{aligned} \tag{A.31}
$$

For $u = x^{k+1}$, to get the desired result we need to estimate $\|v_I - x^{k+1} - u_I(x+\frac{h}{2}) + (x+\frac{h}{2})^{k+1}\|_{L^2(x_{j-\frac{1}{2}},x_j)}$, $\|v_I - x^{k+1} - u_I(x-\frac{h}{2}) + (x-\frac{h}{2})^{k+1}\|_{L^2(x_j,x_{j+\frac{1}{2}})}$ and $\|u_I - x^{k+1} - v_I(x+\frac{h}{2}) + (x+\frac{h}{2})^{k+1}\|_{L^2(x_j,x_{j+\frac{1}{2}})}$, $\|u_I - x^{k+1} - v_I(x-\frac{h}{2}) + (x-\frac{h}{2})^{k+1}\|_{L^2(x_{j+\frac{1}{2}},x_{j+1})}$. We will only show that $\|v_I - x^{k+1} - u_I(x-\frac{h}{2}) + (x-\frac{h}{2})^{k+1}\|_{L^2(x_j,x_{j+\frac{1}{2}})} \le Ch^{2k+5}$ with $k = 0, 1..., 8$, as the other cases are similar.

For $k = 0, 1..., 8$, by using the definition of the projection and the property that $\|a_j - a_{j+\frac{1}{2}}\|_{L^\infty(I_j)} = \|a_j - a_{j-\frac{1}{2}}\|_{L^\infty(I_j)} = O(h)$ we have the following results. For $u = x^{k+1}$, by the definition,

$$\begin{aligned}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^2 dx\\ \tilde{P}_h(u_I; x^l; f, u)_j &= \tilde{P}_h(x^{k+1}; x^l; f, u)_j,\ \ l = 1, ..., k\\ \int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x dx\\ \tilde{Q}_h(v_I; x^l; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^{k+1}; x^l; f, u)_{j+\frac{1}{2}},\ \ l = 1, ..., k\end{aligned} \tag{A.32}$$

then we have

$$\begin{aligned}u_I &= \sum_{l=0}^{k} \alpha_l x^l,\ \ \forall x \in I_j.\\ v_I &= \sum_{l=0}^{k} \beta_l x^l,\ \ \forall x \in I_{j+\frac{1}{2}}.\end{aligned} \tag{A.33}$$

Hence,

$$\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^2 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^2)^2 dx = O(h^{2k+5}) \tag{A.34}$$

We leave the detailed calculations and formulas in a separate file, attached here, since they are too lengthy. Hence, for $k = 0, 1.,,, 8$ we can prove that

$$\|v_I - x^{k+1} - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^{k+1}\|^2_{L^2(x_j,x_{j+\frac{1}{2}})} \le Ch^{2k+5} \tag{A.35}$$

Then by using Holder's inequality and Young's inequality, we obtain from (A.30)

$$|\tilde{B}_j(u_I, v_I; \varphi_h; f, u) - \tilde{B}_j(u, u; \varphi_h; f, u)| \le Ch^{2k+3} + C\|\varphi_h\|^2_{L^2(I_j)} \tag{A.36}$$

Similarly, for $\hat{B}_{j+\frac{1}{2}}$ we have

$$|\hat{B}_{j+\frac{1}{2}}(u_I, v_I; \psi_h; f, u) - \hat{B}_{j+\frac{1}{2}}(u, u; \psi_h; f, u)| \le Ch^{2k+3} + C\|\psi_h\|^2_{L^2(I_{j+\frac{1}{2}})} \tag{A.37}$$

A.3 Proof of Lemma 3.1

Proof. Let u_I denote $\mathbb{P}_h^* u$. Assume that $u \equiv 0$. Take $\varphi_h = u_I$ in (3.14), we get

$$
\begin{aligned}
0 = \tilde{P}_h(u_I, u_I)_{i,j} =& \frac{1}{\tau_{max}}\Big(\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} 2u_I(x+\frac{h}{2}, y+\frac{h}{2})u_I(x,y) \\
&+ 2u_I(x+\frac{h}{2}, y)u_I(x, y+\frac{h}{2})\,dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} u_I(x,y)^2 + u_I(x, y+\frac{h}{2})^2 \\
&+ u_I(x+\frac{h}{2}, y)^2 + u_I(x+\frac{h}{2}, y+\frac{h}{2})^2\,dxdy\Big) \\
=& -\frac{1}{\tau_{max}}\Big(\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} (u_I(x+\frac{h}{2}, y+\frac{h}{2}) - u_I(x,y))^2\,dxdy \\
&+ \int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} (u_I(x+\frac{h}{2}, y) - u_I(x, y+\frac{h}{2}))^2\,dxdy\Big)
\end{aligned} \tag{A.38}
$$

where we have again used change of variable to shift all the integration regions to the same subcell $(x_{i-\frac{1}{2}}, x_i) \times (y_{j-\frac{1}{2}}, y_j)$ to simplify the formulation. Then

$$
u_I(x,y) = u_I(x+\frac{h}{2}, y+\frac{h}{2}), \quad u_I(x+\frac{h}{2}, y) = u_I(x, y+\frac{h}{2}), \forall (x,y) \in (x_{i-\frac{1}{2}}, x_i) \times (y_{j-\frac{1}{2}}, y_j)
$$

Thus $u_I(x,y) \equiv c_0$ on $K_{i,j}$, c_0 is a constant. By (3.13) we immediately get $u_I \equiv 0$, and we have finished the proof of uniqueness, hence also existence. We note that this projection is a local projection, hence we can make a change of variables to the reference element $[-1,1] \times [-1,1]$ by taking $\xi = \frac{2(x-x_i)}{h}$ and $\eta = \frac{2(y-y_j)}{h}$. Taking a similar derivation as in the proof of (A.1), we obtain

$$
\|u_I\|_{L^\infty(K_{i,j})} \le C(k)\|u\|_{L^\infty(K_{i,j})} \tag{A.39}
$$

Again standard approximation theory [2] implies the optimal approximating estimates.

□

A.4 Proof of Lemma 3.2

Proof. Let $u_I = \mathbb{P}_h^* u \in X_h^k$, $v_I = \mathbb{Q}_h^* u \in Y_h^k$, and $a_{i,j} = f'(u(x_i, y_j))$, $b_{i,j} = g'(u(x_i, y_j))$, $a_{i+\frac{1}{2},j+\frac{1}{2}} = f'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))$, $b_{i+\frac{1}{2},j+\frac{1}{2}} = g'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))$, then by the definition of $\tilde{B}_{i,j}$

and $\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}$, we have

$$
\begin{aligned}
&\tilde{B}_{i,j}(u_I,v_I;\varphi_h;f,g,u)-\tilde{B}_{i,j}(u,u;\varphi_h;f,g,u)\\
=&\frac{1}{\tau_{max}}\int_{K_{i,j}}(v_I-u_I)\varphi_h dxdy+a_{i,j}\Bigg[\int_{K_{i,j}}(v_I-u)(\varphi_h)_x dxdy\\
&-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\Big((v_I-u)(x_{i+\frac{1}{2}},y)\varphi_h(x^-_{i+\frac{1}{2}},y)-(v_I-u)(x_{i-\frac{1}{2}},y)\varphi_h(x^+_{i-\frac{1}{2}},y)\Big)\Bigg]\\
&+b_{i,j}\Bigg[\int_{K_{i,j}}(v_I-u)(\varphi_h)_y dxdy-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\Big((v_I-u)(x,y_{j+\frac{1}{2}})\varphi_h(x,y^-_{j+\frac{1}{2}})\\
&-(v_I-u)(x,y_{j-\frac{1}{2}})\varphi_h(x,y^+_{j-\frac{1}{2}})\Big)\Bigg]\\
=&\tilde{P}_h(u_I-u;\varphi_h;f,g,u)_{i,j}\\
&+\frac{1}{\tau_{max}}\Bigg[\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)\varphi_h dxdy\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)\varphi_h dxdy\\
&+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)\varphi_h dxdy\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)\varphi_h dxdy\Bigg]\\
&+a_{i,j}\Bigg[\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_x dxdy\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_x dxdy\\
&+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_x dxdy\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_x dxdy\\
&-\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x_{i+\frac{1}{2}},y)-u(x_{i+\frac{1}{2}},y)-u_I(x_i,y-\frac{h}{2})+u(x_i,y-\frac{h}{2})\Big)\varphi_h(x^-_{i+\frac{1}{2}},y)dy\\
&-\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x_{i+\frac{1}{2}},y)-u(x_{i+\frac{1}{2}},y)-u_I(x_i,y+\frac{h}{2})+u(x_i,y+\frac{h}{2})\Big)\varphi_h(x^-_{i+\frac{1}{2}},y)dy\\
&+\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x_{i-\frac{1}{2}},y)-u(x_{i-\frac{1}{2}},y)-u_I(x_i,y-\frac{h}{2})+u(x_i,y-\frac{h}{2})\Big)\varphi_h(x^+_{i-\frac{1}{2}},y)dy
\end{aligned}
$$

$$+\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x_{i-\frac{1}{2}},y)-u(x_{i-\frac{1}{2}},y)-u_I(x_j,y+\frac{h}{2})+u(x_j,y+\frac{h}{2})\Big)\varphi_h(x^+_{i-\frac{1}{2}},y)dy\Bigg]$$

$$+b_{i,j}\Bigg[\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_y dxdy$$

$$+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_y dxdy$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_y dxdy$$

$$+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_y dxdy$$

$$-\int_{x_i}^{x_{i+\frac{1}{2}}}\Big(v_I(x,y_{j+\frac{1}{2}})-u(x,y_{j+\frac{1}{2}})-u_I(x-\frac{h}{2},y_j)+u(x-\frac{h}{2},y_j)\Big)\varphi_h(x,y^-_{j+\frac{1}{2}})dx$$

$$-\int_{x_{i-\frac{1}{2}}}^{x_i}\Big(v_I(x,y_{j+\frac{1}{2}})-u(x,y_{j+\frac{1}{2}})-u_I(x+\frac{h}{2},y_j)+u(x+\frac{h}{2},y_j)\Big)\varphi_h(x,y^-_{j+\frac{1}{2}})dx$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\Big(v_I(x,y_{j-\frac{1}{2}})-u(x,y_{j-\frac{1}{2}})-u_I(x-\frac{h}{2},y_j)+u(x-\frac{h}{2},y_j)\Big)\varphi_h(x,y^+_{j-\frac{1}{2}})dx$$

$$+\int_{x_{i-\frac{1}{2}}}^{x_i}\Big(v_I(x,y_{j-\frac{1}{2}})-u(x,y_{j-\frac{1}{2}})-u_I(x+\frac{h}{2},y_j)+u(x+\frac{h}{2},y_j)\Big)\varphi_h(x,y^+_{j-\frac{1}{2}})dx\Bigg]. \tag{A.40}$$

$$\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_I,v_I;\psi_h;f,u)-\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u,u;\psi_h;f,u)$$

$$=\frac{1}{\tau_{max}}\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(u_I-v_I)\psi_h dxdy+a_{i+\frac{1}{2},j+\frac{1}{2}}\Bigg[\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(v_I-u)(\psi_h)_x dxdy$$

$$-\int_{y_j}^{y_{j+1}}\Big((u_I-u)(x_{i+\frac{1}{2}},y)\psi_h(x^-_{i+\frac{1}{2}},y)-(u_I-u)(x_{i-\frac{1}{2}},y)\psi_h(x^+_{i-\frac{1}{2}},y)\Big)\Bigg]$$

$$+b_{i,j}\Bigg[\int_{K_{i,j}}(u_I-u)(\psi_h)_y dxdy-\int_{x_i}^{x_{i+1}}\Big((u_I-u)(x,y_{j+\frac{1}{2}})\psi_h(x,y^-_{j+\frac{1}{2}})$$

$$-(u_I-u)(x,y_{j-\frac{1}{2}})\psi_h(x,y^+_{j-\frac{1}{2}})\Big)\Bigg]$$

$$=\tilde{Q}_h(v_I-u;\psi_h;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+\frac{1}{\tau_{max}}\Bigg[\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{y_{j+\frac{1}{2}}}^{y_{j+1}}\Big(u_I(x,y)-u(x,y)-v_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)\psi_h dxdy$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j+\frac{1}{2}}}^{y_{j+1}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)\psi_h dxdy$$

$$+\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x,y)-u(x,y)-v_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)\psi_h dxdy$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)\psi_h dxdy\Bigg]$$

$$+a_{i+\frac{1}{2},j+\frac{1}{2}}\Bigg[\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{y_{j+\frac{1}{2}}}^{y_{j+1}}\Big(u_I(x,y)-u(x,y)-v_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)(\psi_h)_x dxdy$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j+\frac{1}{2}}}^{y_{j+1}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)(\psi_h)_x dxdy$$

$$+\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x,y)-u(x,y)-v_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)(\psi_h)_x dxdy$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)(\psi_h)_x dxdy$$

$$-\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x_{i+1},y)-u(x_{i+1},y)-v_I(x_{i+\frac{1}{2}},y-\frac{h}{2})+u(x_{i+\frac{1}{2}},y-\frac{h}{2})\Big)\psi_h(x_{i+1}^-,y)dy$$

$$-\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(u_I(x_{i+1},y)-u(x_{i+1},y)-v_I(x_{i+\frac{1}{2}},y+\frac{h}{2})+u(x_{i+\frac{1}{2}},y+\frac{h}{2})\Big)\psi_h(x_{i+1}^-,y)dy$$

$$+\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x_i,y)-u(x_i,y)-v_I(x_{i+\frac{1}{2}},y-\frac{h}{2})+u(x_{i+\frac{1}{2}},y-\frac{h}{2})\Big)\psi_h(x_i^+,y)dy$$

$$+\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(u_I(x_i,y)-u(x_i,y)-v_I(x_{i+\frac{1}{2}},y+\frac{h}{2})+u(x_{i+\frac{1}{2}},y+\frac{h}{2})\Big)\psi_h(x_i^+,y)dy\Bigg]$$

$$+b_{i+\frac{1}{2},j+\frac{1}{2}}\Bigg[\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{y_{j+\frac{1}{2}}}^{y_{j+1}}\Big(u_I(x,y)-u(x,y)-v_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)(\psi_h)_y dxdy$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j+\frac{1}{2}}}^{y_{j+1}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)(\psi_h)_y dxdy$$

$$+\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x,y)-u(x,y)-v_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)(\psi_h)_y dxdy$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)(\psi_h)_y dxdy$$

$$-\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\Big(u_I(x,y_{j+1})-u(x,y_{j+1})-v_I(x-\frac{h}{2},y_{j+\frac{1}{2}})+u(x-\frac{h}{2},y_{j+\frac{1}{2}})\Big)\psi_h(x,y_{j+1}^-)dx$$

$$-\int_{x_i}^{x_{i+\frac{1}{2}}}\Big(u_I(x,y_{j+1})-u(x,y_{j+1})-v_I(x+\frac{h}{2},y_{j+\frac{1}{2}})+u(x+\frac{h}{2},y_{j+\frac{1}{2}})\Big)\psi_h(x,y_{j+1}^-)dx$$

$$+\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\Big(u_I(x,y_j)-u(x,y_j)-v_I(x-\frac{h}{2},y_{j+\frac{1}{2}})+u(x-\frac{h}{2},y_{j+\frac{1}{2}})\Big)\psi_h(x,y_j^+)dx$$

$$+\int_{x_i}^{x_{i+\frac{1}{2}}}\Big(u_I(x,y_j)-u(x,y_j)-v_I(x+\frac{h}{2},y_{j+\frac{1}{2}})+u(x+\frac{h}{2},y_{j+\frac{1}{2}})\Big)\psi_h(x,y_j^+)dx\Bigg]. \tag{A.41}$$

For $u(x,y)=x^{k+1}$ *or* y^{k+1}, we only need to estimate $\|v_I(x,y)-x^{k+1}-u_I(x-\frac{h}{2},y-\frac{h}{2})+(x-\frac{h}{2})^{k+1}\|_{L^2((x_i,x_{i+\frac{1}{2}})\times(y_j,y_{j+\frac{1}{2}}))}$ and $\|v_I(x,y)-y^{k+1}-u_I(x-\frac{h}{2},y-\frac{h}{2})+(y-\frac{h}{2})^{k+1}\|_{L^2((x_i,x_{i+\frac{1}{2}})\times(y_j,y_{j+\frac{1}{2}}))}$ as the other cases are similar.

For $k=0,1...,8$, by using the definition of the projection and the property that $\|a_{i,j}-a_{i+\frac{1}{2},j+\frac{1}{2}}\|_{L^\infty(K_{i,j})}=O(h)$, $\|b_{i,j}-b_{i+\frac{1}{2},j+\frac{1}{2}}\|_{L^\infty(K_{i,j})}=O(h)$ we have the following results:

1) $u=x^{k+1}$, by the definition of the projection,

$$\begin{aligned}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}u_I dxdy=&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}x^{k+1}dxdy\\ \tilde{P}_h(u_I;x^my^n;f,g,u)_{i,j}=&\tilde{P}_h(x^{k+1};x^my^n;f,g,u)_{i,j},\ m,n=0,...,k\\ \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}v_I dxdy=&\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}x^{k+1}dxdy\\ \tilde{Q}_h(v_I;x^my^n;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}}=&\tilde{Q}_h(x^{k+1};x^my^n;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}},\ m,n=0,...,k\end{aligned}$$

then we have

$$u_I=\sum_{m=0}^{k}\sum_{n=0}^{k}\alpha_{m,n}x^my^n,\ \forall(x,y)\in K_{i,j} \tag{A.42}$$

$$v_I=\sum_{m=0}^{k}\sum_{n=0}^{k}\beta_{m,n}x^my^n,\ \forall(x,y)\in K_{i+\frac{1}{2},j+\frac{1}{2}} \tag{A.43}$$

Hence, by some calculation we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^{k+1}-u_I(x-\frac{h}{2},y-\frac{h}{2})+(x-\frac{h}{2})^{k+1})^2dxdy=O(h^{2k+6}) \tag{A.44}$$

2) $u=y^{k+1}$, by the definition of the projection,

$$\begin{aligned}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}u_I dxdy=&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}y^{k+1}dxdy\\ \tilde{P}_h(u_I;x^my^n;f,g,u)_{i,j}=&\tilde{P}_h(x^{k+1};x^my^n;f,g,u)_{i,j},\ m,n=0,...,k\end{aligned}$$

$$\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} v_I dxdy = \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} y^{k+1} dxdy$$
$$\tilde{Q}_h(v_I; x^m y^n; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} = \tilde{Q}_h(y^{k+1}; x^m y^n; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}},\ m, n = 0, ..., k$$

then we have

$$u_I = \sum_{m=0}^{k}\sum_{n=0}^{k} \alpha_{m,n} x^m y^n,\ \forall (x, y) \in K_{i,j} \tag{A.45}$$

$$v_I = \sum_{m=0}^{k}\sum_{n=0}^{k} \beta_{m,n} x^m y^n,\ \forall (x, y) \in K_{i+\frac{1}{2},j+\frac{1}{2}} \tag{A.46}$$

Hence, by some calculation we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - y^{k+1} - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (y - \frac{h}{2})^{k+1})^2 dxdy = O(h^{2k+6}) \tag{A.47}$$

Again, we leave the detailed calculations and formulas in a separate file, attached here, since they are too lengthy. Hence, for $k = 0, 1., , , 8$ we have proved that

$$\|v_I(x, y) - x^{k+1} - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^{k+1}\|^2_{L^2((x_i, x_{i+\frac{1}{2}})\times(y_j, y_{j+\frac{1}{2}}))} \leq Ch^{2k+6} \tag{A.48}$$

$$\|v_I(x, y) - y^{k+1} - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (y - \frac{h}{2})^{k+1}\|^2_{L^2((x_i, x_{i+\frac{1}{2}})\times(y_j, y_{j+\frac{1}{2}}))} \leq Ch^{2k+6} \tag{A.49}$$

Then by using Holder's inequality and Youngs inequality, we obtain from (A.40)

$$|\tilde{B}_{i,j}(u_I, v_I; \varphi_h; f, g, u) - \tilde{B}_{i,j}(u, u; \varphi_h; f, g, u)| \leq Ch^{2k+4} + C\|\varphi_h\|^2_{L^2(K_{i,j})} \tag{A.50}$$

Similarly, for $\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}$ we have

$$|\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_I, v_I; \psi_h; f, g, u) - \hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u, u; \psi_h; f, g, u)| \leq Ch^{2k+4} + C\|\psi_h\|^2_{L^2(K_{i+\frac{1}{2},j+\frac{1}{2}})} \tag{A.51}$$

□

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In this attachment we give detailed proofs and formulas for Proposition 2.1 and Lemma 3.2.

1 Proof of Proposition 2.1

Here we give the detail proof of $\|v_I - x^{k+1} - u_I(x-\frac{h}{2}) + (x-\frac{h}{2})^{k+1}\|_{L^2(x_j, x_{j+\frac{1}{2}})} \leq Ch^{2k+5}$ with $k = 0, 1, ..., 8$.

1. $k = 0,\ u = x$, by the definition,

$$\begin{aligned} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x dx \\ \int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x dx \end{aligned} \tag{1.1}$$

then we have

$$\begin{aligned} u_I =& x_j,\ \forall x \in I_j, \\ v_I =& x_{j+\frac{1}{2}},\ \forall x \in I_{j+\frac{1}{2}}. \end{aligned} \tag{1.2}$$

Hence, for $x \in (x_j, x_{j+\frac{1}{2}})$,

$$v_I - x^1 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^1 = x_{j+\frac{1}{2}} - x_j - \frac{h}{2} = 0 \tag{1.3}$$

2. $k = 1,\ u = x^2$, by the definition,

$$\begin{aligned} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^2 dx \\ \tilde{P}_h(u_I; x; f, u)_j =& \tilde{P}_h(x^2; x; f, u)_j \\ \int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x dx \\ \tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^2; x; f, u)_{j+\frac{1}{2}} \end{aligned} \tag{1.4}$$

then we have

$$\begin{aligned} u_I =& (2x_j - \frac{2}{3}a_j\tau_{max})x + \frac{1}{12}h^2 + \frac{2}{3}a_j x_j \tau_{max} - x_j^2,\ \forall x \in I_j, \\ v_I =& (2x_{j+\frac{1}{2}} - \frac{2}{3}a_{j+\frac{1}{2}}\tau_{max})x + \frac{1}{12}h^2 + \frac{2}{3}a_{j+\frac{1}{2}} x_{j+\frac{1}{2}} \tau_{max} - x_{j+\frac{1}{2}}^2,\ \forall x \in I_{j+\frac{1}{2}}. \end{aligned} \tag{1.5}$$

Hence,

$$
\begin{aligned}
&\int_{x_j}^{x_{j+\frac{1}{2}}}(v_I - x^2 - u_I(x-\frac{h}{2}) + (x-\frac{h}{2})^2)^2 \\
=&\frac{1}{54}h^3\tau_{max}^2(a_j - a_{j+\frac{1}{2}})^2 \\
=&O(h^7)
\end{aligned}
\tag{1.6}
$$

3. $k = 2,\ u = x^3$, by the definition

$$
\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx =& \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^3 dx \\
\tilde{P}_h(u_I; x; f, u)_j =& \tilde{P}_h(x^3; x; f, u)_j \\
\tilde{P}_h(u_I; x^2; f, u)_j =& \tilde{P}_h(x^3; x^2; f, u)_j \\
\int_{x_j}^{x_{j+1}} v_I dx =& \int_{x_j}^{x_{j+1}} x^3 dx \\
\tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^3; x; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^3; x^2; f, u)_{j+\frac{1}{2}}
\end{aligned}
\tag{1.7}
$$

then we have

$$
\begin{aligned}
u_I =& \alpha_2 x^2 + \alpha_1 x + \alpha_0,\ \forall x \in I_j. \\
v_I =& \beta_2 x^2 + \beta_1 x + \beta_0,\ \forall x \in I_{j+\frac{1}{2}}.
\end{aligned}
\tag{1.8}
$$

where

$$
\begin{aligned}
\alpha_2 =& (3(80a_j^2\tau_{max}^2 x_j - 2a_j h^2\tau_{max} + 15h^2 x_j))/(5(16a_j^2\tau_{max}^2 + 3h^2)) \\
\alpha_1 =& (3(16a_j^2 h^2\tau_{max}^2 - 640a_j^2\tau_{max}^2 x_j^2 + 32a_j h^2\tau_{max} x_j + 5h^4 - 120h^2 x_j^2)) \\
& /(40(16a_j^2\tau_{max}^2 + 3h^2)) \\
\alpha_0 =& -(48a_j^2 h^2\tau_{max}^2 x_j - 640a_j^2\tau_{max}^2 x_j^3 - 4a_j h^4\tau_{max} \\
& + 48a_j h^2\tau_{max} x_j^2 + 15h^4 x_j - 120h^2 x_j^3) \\
& /(40(16a_j^2\tau_{max}^2 + 3h^2)) \\
\beta_2 =& (3(80a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}} - 2a_{j+\frac{1}{2}} h^2\tau_{max} + 15h^2 x_{j+\frac{1}{2}}))/(5(16a_{j+\frac{1}{2}}^2\tau_{max}^2 + 3h^2)) \\
\beta_1 =& (3(16a_{j+\frac{1}{2}}^2 h^2\tau_{max}^2 - 640a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^2 + 32a_{j+\frac{1}{2}} h^2\tau_{max} x_{j+\frac{1}{2}} + 5h^4 - 120h^2 x_{j+\frac{1}{2}}^2)) \\
& /(40(16a_{j+\frac{1}{2}}^2\tau_{max}^2 + 3h^2))
\end{aligned}
$$

$$\beta_0 = -(48a_{j+\frac{1}{2}}^2 h^2 \tau_{max}^2 x_{j+\frac{1}{2}} - 640a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}}^3 - 4a_{j+\frac{1}{2}} h^4 \tau_{max}$$
$$+ 48a_{j+\frac{1}{2}} h^2 \tau_{max} x_{j+\frac{1}{2}}^2 + 15h^4 x_{j+\frac{1}{2}} - 120h^2 x_{j+\frac{1}{2}}^3)$$
$$/(40(16a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 3h^2))$$

Similarly, we have

$$\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^3 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^3)^2$$

$$=(h^9 \tau_{max}^2 (a_j - a_{j+\frac{1}{2}})^2 (24h^2 \tau_{max}^2 (5a_j^2 + 2a_j a_{j+\frac{1}{2}} + 5a_{j+\frac{1}{2}}^2) + 512 a_j^2 a_{j+\frac{1}{2}}^2 \tau_{max}^4$$
$$- 45h^3 \tau_{max}(a_j + a_{j+\frac{1}{2}}) + 240 a_j a_{j+\frac{1}{2}} h \tau_{max}^3 (a_j + a_{j+\frac{1}{2}}) + 18h^4))$$
$$/(500(16a_j^2 \tau_{max}^2 + 3h^2)^2 (16a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 3h^2)^2)$$
$$=O(h^9)$$

4. $k = 3$, $u = x^4$, by the definition

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^4 dx \\
\tilde{P}_h(u_I; x; f, u)_j &= \tilde{P}_h(x^4; x; f, u)_j \\
\tilde{P}_h(u_I; x^2; f, u)_j &= \tilde{P}_h(x^4; x^2; f, u)_j \\
\tilde{P}_h(u_I; x^3; f, u)_j &= \tilde{P}_h(x^4; x^3; f, u)_j \\
\int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^4 dx \\
\tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^4; x; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^4; x^3; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^3; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^4; x^3; f, u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.9}$$

then we have

$$\begin{aligned}
u_I &= \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, \ \forall x \in I_j. \\
v_I &= \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0, \ \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.10}$$

where

$$\alpha_3 = (4(176a_j^3 \tau_{max}^3 + 1008a_j^2 \tau_{max}^2 x_j + 19a_j h^2 \tau_{max} + 105h^2 x_j))/(21(48a_j^2 \tau_{max}^2 + 5h^2))$$

$$
\begin{aligned}
\alpha_2 =& -(704a_j^3\tau_{max}^3x_j - 68a_j^2h^2\tau_{max}^2 + 2016a_j^2\tau_{max}^2x_j^2 + 76a_jh^2\tau_{max}x_j - 7h^4 + 210h^2x_j^2)\\
&/(7(48a_j^2\tau_{max}^2 + 5h^2))\\
\alpha_1 =& -(2(44a_j^3h^2\tau_{max}^3 - 1760a_j^3\tau_{max}^3x_j^2 + 340a_j^2h^2\tau_{max}^2x_j - 3360a_j^2\tau_{max}^2x_j^3 + 5a_jh^4\tau_{max}\\
&- 190a_jh^2\tau_{max}x_j^2 + 35h^4x_j - 350h^2x_j^3))\\
&/(35(48a_j^2\tau_{max}^2 + 5h^2))\\
\alpha_0 =& -(-4224a_j^3h^2\tau_{max}^3x_j + 56320a_j^3\tau_{max}^3x_j^3 + 352a_j^2h^4\tau_{max}^2 - 16320a_j^2h^2\tau_{max}^2x_j^2\\
&+ 80640a_j^2\tau_{max}^2x_j^4 - 480a_jh^4\tau_{max}x_j + 6080a_jh^2\tau_{max}x_j^3\\
&+ 35h^6 - 1680h^4x_j^2 + 8400h^2x_j^4)\\
&/(1680(48a_j^2\tau_{max}^2 + 5h^2))\\
\beta_3 =&(4(176a_{j+\frac{1}{2}}^3\tau_{max}^3 + 1008a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 19a_{j+\frac{1}{2}}h^2\tau_{max} + 105h^2x_{j+\frac{1}{2}}))\\
&/(21(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))\\
\beta_2 =& -(704a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} - 68a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 2016a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2\\
&+ 76a_{j+\frac{1}{2}}h^2\tau_{max}x_{j+\frac{1}{2}} - 7h^4 + 210h^2x_{j+\frac{1}{2}}^2)\\
&/(7(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))\\
\beta_1 =& -(2(44a_{j+\frac{1}{2}}^3h^2\tau_{max}^3 - 1760a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2 + 340a_{j+\frac{1}{2}}^2h^2\tau_{max}^2x_{j+\frac{1}{2}}\\
&- 3360a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 + 5a_{j+\frac{1}{2}}h^4\tau_{max}\\
&- 190a_{j+\frac{1}{2}}h^2\tau_{max}x_{j+\frac{1}{2}}^2 + 35h^4x_{j+\frac{1}{2}} - 350h^2x_{j+\frac{1}{2}}^3))\\
&/(35(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))\\
\beta_0 =& -(-4224a_{j+\frac{1}{2}}^3h^2\tau_{max}^3x_{j+\frac{1}{2}} + 56320a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^3 + 352a_{j+\frac{1}{2}}^2h^4\tau_{max}^2\\
&- 16320a_{j+\frac{1}{2}}^2h^2\tau_{max}^2x_{j+\frac{1}{2}}^2 + 80640a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^4 - 480a_{j+\frac{1}{2}}h^4\tau_{max}x_{j+\frac{1}{2}}\\
&+ 6080a_{j+\frac{1}{2}}h^2\tau_{max}x_{j+\frac{1}{2}}^3 + 35h^6 - 1680h^4x_{j+\frac{1}{2}}^2 + 8400h^2x_{j+\frac{1}{2}}^4)\\
&/(1680(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int_{x_j}^{x_{j+\frac{1}{2}}}(v_I - x^4 - u_I(x-\frac{h}{2}) + (x-\frac{h}{2})^4)^2 \\
=&(h^7\tau_{max}^2(a_j - a_{j+\frac{1}{2}})^2(660160512a_j^4a_{j+\frac{1}{2}}^4\tau_{max}^8 + 80h^6\tau_{max}^2(18762a_j^2 - 101a_ja_{j+\frac{1}{2}} \\
&+ 18762a_{j+\frac{1}{2}}^2) - 1400h^5\tau_{max}^3(a_j + a_{j+\frac{1}{2}})(77a_j^2 - a_ja_{j+\frac{1}{2}} + 77a_{j+\frac{1}{2}}^2) \\
&- 1034880a_j^2a_{j+\frac{1}{2}}^2h^3\tau_{max}^5(a_j + a_{j+\frac{1}{2}}) + 743424a_j^2a_{j+\frac{1}{2}}^2h^2\tau_{max}^6(185a_j^2 - a_ja_{j+\frac{1}{2}} \\
&+ 185a_{j+\frac{1}{2}}^2) + 320h^4\tau_{max}^4(22385a_j^4 - 242a_j^3a_{j+\frac{1}{2}} + 89799a_j^2a_{j+\frac{1}{2}}^2 - 242a_ja_{j+\frac{1}{2}}^3 \\
&+ 22385a_{j+\frac{1}{2}}^4) - 11375h^7\tau_{max}(a_j + a_{j+\frac{1}{2}}) + 78625h^8)) \\
&/(617400(48a_j^2\tau_{max}^2 + 5h^2)^2(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2)^2) \\
=&O(h^{11})
\end{aligned}
$$

5. $k = 4,\ u = x^5$, by the definition

$$
\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx =& \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^5 dx \\
\tilde{P}_h(u_I; x; f, u)_j =& \tilde{P}_h(x^5; x; f, u)_j \\
\tilde{P}_h(u_I; x^2; f, u)_j =& \tilde{P}_h(x^5; x^2; f, u)_j \\
\tilde{P}_h(u_I; x^3; f, u)_j =& \tilde{P}_h(x^5; x^3; f, u)_j \\
\tilde{P}_h(u_I; x^4; f, u)_j =& \tilde{P}_h(x^5; x^4; f, u)_j \\
\int_{x_j}^{x_{j+1}} v_I dx =& \int_{x_j}^{x_{j+1}} x^5 dx \\
\tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^5; x; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^5; x^2; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^3; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^5; x^3; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^4; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^5; x^4; f, u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.11}
$$

then we have

$$
\begin{aligned}
u_I =& \alpha_4x^4 + \alpha_3x^3 + \alpha_2x^2 + \alpha_1x + \alpha_0,\ \forall x \in I_j. \\
v_I =& \beta_4x^4 + \beta_3x^3 + \beta_2x^2 + \beta_1x + \beta_0,\ \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.12}
$$

where

$$
\alpha_4 = (464640a_j^4\tau_{max}^4x_j + 80a_j^2h^2\tau_{max}^2(178a_j\tau_{max} + 795x_j) + 75h^4(20a_j\tau_{max} + 21x_j))
$$

$$/(92928a_j^4\tau_{max}^4 + 12720a_j^2h^2\tau_{max}^2 + 315h^4)$$

$$\begin{aligned}\alpha_3 =&(5(-2230272a_j^4\tau_{max}^4x_j^2 - 24h^4(-262a_j^2\tau_{max}^2 + 600a_j\tau_{max}x_j + 315x_j^2)\\ &+ 128a_j^2h^2\tau_{max}^2(286a_j^2\tau_{max}^2 - 1068a_j\tau_{max}x_j - 2385x_j^2) + 225h^6))\\ &/(36(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4))\end{aligned}$$

$$\begin{aligned}\alpha_2 =&-(5(-5203968a_j^4\tau_{max}^4x_j^3 + 896a_j^2h^2\tau_{max}^2x_j(286a_j^2\tau_{max}^2 - 534a_j\tau_{max}x_j - 795x_j^2)\\ &- 8h^4(-1856a_j^3\tau_{max}^3 - 5502a_j^2\tau_{max}^2x_j + 6300a_j\tau_{max}x_j^2 + 2205x_j^3)\\ &+ 63h^6(26a_j\tau_{max} + 25x_j)))\\ &/(84(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4))\end{aligned}$$

$$\begin{aligned}\alpha_1 =&-(5(20815872a_j^4\tau_{max}^4x_j^4 - 56h^6(-86a_j^2\tau_{max}^2 + 468a_j\tau_{max}x_j + 225x_j^2)\\ &- 3584a_j^2h^2\tau_{max}^2x_j^2(572a_j^2\tau_{max}^2 - 712a_j\tau_{max}x_j - 795x_j^2)\\ &+ 32h^4(440a_j^4\tau_{max}^4 - 7424a_j^3\tau_{max}^3x_j - 11004a_j^2\tau_{max}^2x_j^2\\ &+ 8400a_j\tau_{max}x_j^3 + 2205x_j^4) + 231h^8))\\ &/(672(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4))\end{aligned}$$

$$\begin{aligned}\alpha_0 =&(62447616a_j^4\tau_{max}^4x_j^5 - 17920a_j^2h^2\tau_{max}^2x_j^3(572a_j^2\tau_{max}^2 - 534a_j\tau_{max}x_j - 477x_j^2)\\ &- 8h^6(-3608a_j^3\tau_{max}^3 - 9030a_j^2\tau_{max}^2x_j + 24570a_j\tau_{max}x_j^2 + 7875x_j^3)\\ &+ 480h^4x_j(440a_j^4\tau_{max}^4 - 3712a_j^3\tau_{max}^3x_j - 3668a_j^2\tau_{max}^2x_j^2\\ &+ 2100a_j\tau_{max}x_j^3 + 441x_j^4) + 315h^8(12a_j\tau_{max} + 11x_j))\\ &/(2016(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4))\end{aligned}$$

$$\begin{aligned}\beta_4 =&(464640a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}} + 80a_{j+\frac{1}{2}}^2h^2\tau_{max}^2(178a_{j+\frac{1}{2}}\tau_{max} + 795x_{j+\frac{1}{2}})\\ &+ 75h^4(20a_{j+\frac{1}{2}}\tau_{max} + 21x_{j+\frac{1}{2}}))\\ &/(92928a_{j+\frac{1}{2}}^4\tau_{max}^4 + 12720a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 315h^4)\end{aligned}$$

$$\begin{aligned}\beta_3 =&(5(-2230272a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^2 - 24h^4(-262a_{j+\frac{1}{2}}^2\tau_{max}^2 + 600a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 315x_{j+\frac{1}{2}}^2)\\ &+ 128a_{j+\frac{1}{2}}^2h^2\tau_{max}^2(286a_{j+\frac{1}{2}}^2\tau_{max}^2 - 1068a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 2385x_{j+\frac{1}{2}}^2) + 225h^6))\\ &/(36(30976a_{j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4))\end{aligned}$$

$$\beta_2 =- (5(-5203968a^4_{j+\frac{1}{2}}\tau^4_{max}x^3_{j+\frac{1}{2}} + 896a^2_{j+\frac{1}{2}}h^2\tau^2_{max}x_{j+\frac{1}{2}}(286a^2_{j+\frac{1}{2}}\tau^2_{max}$$

$$- 534a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 795x^2_{j+\frac{1}{2}}) - 8h^4(-1856a^3_{j+\frac{1}{2}}\tau^3_{max} - 5502a^2_{j+\frac{1}{2}}\tau^2_{max}x_{j+\frac{1}{2}}$$

$$+ 6300a_{j+\frac{1}{2}}\tau_{max}x^2_{j+\frac{1}{2}} + 2205x^3_{j+\frac{1}{2}}) + 63h^6(26a_{j+\frac{1}{2}}\tau_{max} + 25x_{j+\frac{1}{2}})))$$

$$/(84(30976a^4_{j+\frac{1}{2}}\tau^4_{max} + 4240a^2_{j+\frac{1}{2}}h^2\tau^2_{max} + 105h^4))$$

$$\beta_1 =- (5(20815872a^4_{j+\frac{1}{2}}\tau^4_{max}x^4_{j+\frac{1}{2}} - 56h^6(-86a^2_{j+\frac{1}{2}}\tau^2_{max} + 468a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 225x^2_{j+\frac{1}{2}})$$

$$- 3584a^2_{j+\frac{1}{2}}h^2\tau^2_{max}x^2_{j+\frac{1}{2}}(572a^2_{j+\frac{1}{2}}\tau^2_{max} - 712a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 795x^2_{j+\frac{1}{2}})$$

$$+ 32h^4(440a^4_{j+\frac{1}{2}}\tau^4_{max} - 7424a^3_{j+\frac{1}{2}}\tau^3_{max}x_{j+\frac{1}{2}} - 11004a^2_{j+\frac{1}{2}}\tau^2_{max}x^2_{j+\frac{1}{2}}$$

$$+ 8400a_{j+\frac{1}{2}}\tau_{max}x^3_{j+\frac{1}{2}} + 2205x^4_{j+\frac{1}{2}}) + 231h^8))$$

$$/(672(30976a^4_{j+\frac{1}{2}}\tau^4_{max} + 4240a^2_{j+\frac{1}{2}}h^2\tau^2_{max} + 105h^4))$$

$$\beta_0 =(62447616a^4_{j+\frac{1}{2}}\tau^4_{max}x^5_{j+\frac{1}{2}} - 17920a^2_{j+\frac{1}{2}}h^2\tau^2_{max}x^3_{j+\frac{1}{2}}(572a^2_{j+\frac{1}{2}}\tau^2_{max}$$

$$- 534a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 477x^2_{j+\frac{1}{2}}) - 8h^6(-3608a^3_{j+\frac{1}{2}}\tau^3_{max} - 9030a^2_{j+\frac{1}{2}}\tau^2_{max}x_{j+\frac{1}{2}}$$

$$+ 24570a_{j+\frac{1}{2}}\tau_{max}x^2_{j+\frac{1}{2}} + 7875x^3_{j+\frac{1}{2}})$$

$$+ 480h^4x_{j+\frac{1}{2}}(440a^4_{j+\frac{1}{2}}\tau^4_{max} - 3712a^3_{j+\frac{1}{2}}\tau^3_{max}x_{j+\frac{1}{2}} - 3668a^2_{j+\frac{1}{2}}\tau^2_{max}x^2_{j+\frac{1}{2}}$$

$$+ 2100a_{j+\frac{1}{2}}\tau_{max}x^3_{j+\frac{1}{2}} + 441x^4_{j+\frac{1}{2}}) + 315h^8(12a_{j+\frac{1}{2}}\tau_{max} + 11x_{j+\frac{1}{2}}))$$

$$/(2016(30976a^4_{j+\frac{1}{2}}\tau^4_{max} + 4240a^2_{j+\frac{1}{2}}h^2\tau^2_{max} + 105h^4))$$

Hence, we have

$$\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^5 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^5)^2$$

$$=(h^{13}\tau^2_{max}(a_j - a_{j+\frac{1}{2}})^2(74297776099295232a^6_j a^6_{j+\frac{1}{2}}\tau^{12}_{max}$$

$$- 17160990584995840a^5_j a^5_{j+\frac{1}{2}}h\tau^{11}_{max}(a_j + a_{j+\frac{1}{2}})$$

$$+ 189000h^{10}\tau^2_{max}(1679899a^2_j - 1244578a_j a_{j+\frac{1}{2}} + 1679899a^2_{j+\frac{1}{2}})$$

$$+ 7938000h^9\tau^3_{max}(a_j + a_{j+\frac{1}{2}})(110956a^2_j - 146113a_j a_{j+\frac{1}{2}} + 110956a^2_{j+\frac{1}{2}})$$

$$+ 317194240a^4_j a^4_{j+\frac{1}{2}}h^2\tau^{10}_{max}(51995327a^2_j - 6974930a_j a_{j+\frac{1}{2}} + 51995327a^2_{j+\frac{1}{2}})$$

$$- 7208960a^3_j a^3_{j+\frac{1}{2}}h^3\tau^9_{max}(a_j + a_{j+\frac{1}{2}})(411149046a^2_j - 110513705a_j a_{j+\frac{1}{2}}$$

$$+ 411149046a^2_{j+\frac{1}{2}}) + 806400h^8\tau^4_{max}(4609881a^4_j - 5666392a^3_j a_{j+\frac{1}{2}}$$

$$
\begin{aligned}
&+ 21455275a_j^2a_{j+\frac{1}{2}}^2 - 5666392a_ja_{j+\frac{1}{2}}^3 + 4609881a_{j+\frac{1}{2}}^4)\\
&+ 403200h^7\tau_{max}^5(a_j + a_{j+\frac{1}{2}})(11867779a_j^4 - 59803436a_j^3a_{j+\frac{1}{2}}\\
&+ 53324708a_j^2a_{j+\frac{1}{2}}^2 - 59803436a_ja_{j+\frac{1}{2}}^3 + 11867779a_{j+\frac{1}{2}}^4)\\
&- 47308800a_ja_{j+\frac{1}{2}}h^5\tau_{max}^7(a_j + a_{j+\frac{1}{2}})(2637558a_j^4 - 2676409a_j^3a_{j+\frac{1}{2}}\\
&+ 11391631a_j^2a_{j+\frac{1}{2}}^2 - 2676409a_ja_{j+\frac{1}{2}}^3 + 2637558a_{j+\frac{1}{2}}^4)\\
&+ 1966080a_j^2a_{j+\frac{1}{2}}^2h^4\tau_{max}^8(532478408a_j^4 - 225653868a_j^3a_{j+\frac{1}{2}}\\
&+ 1900203439a_j^2a_{j+\frac{1}{2}}^2 - 225653868a_ja_{j+\frac{1}{2}}^3 + 532478408a_{j+\frac{1}{2}}^4)\\
&+ 307200h^6\tau_{max}^6(45865050a_j^6 - 72215220a_j^5a_{j+\frac{1}{2}} + 800405235a_j^4a_{j+\frac{1}{2}}^2\\
&- 293286178a_j^3a_{j+\frac{1}{2}}^3 + 800405235a_j^2a_{j+\frac{1}{2}}^4 - 72215220a_ja_{j+\frac{1}{2}}^5 + 45865050a_{j+\frac{1}{2}}^6)\\
&+ 40059613125h^{11}\tau_{max}(a_j + a_{j+\frac{1}{2}}) + 8661350250h^{12}))\\
&/(254016(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4)^2(30976a_{j+\frac{1}{2}}^4\tau_{max}^4\\
&+ 4240a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)^2)\\
=&O(h^{13})
\end{aligned}
$$

6. $k=5,\ u=x^6$, by the definition

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^6 dx \\
\tilde{P}_h(u_I;x;f,u)_j &= \tilde{P}_h(x^6;x;f,u)_j \\
\tilde{P}_h(u_I;x^2;f,u)_j &= \tilde{P}_h(x^6;x^2;f,u)_j \\
\tilde{P}_h(u_I;x^3;f,u)_j &= \tilde{P}_h(x^6;x^3;f,u)_j \\
\tilde{P}_h(u_I;x^4;f,u)_j &= \tilde{P}_h(x^6;x^4;f,u)_j \\
\tilde{P}_h(u_I;x^5;f,u)_j &= \tilde{P}_h(x^6;x^5;f,u)_j \\
\int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^6 dx \\
\tilde{Q}_h(v_I;x;f,u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6;x;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^2;f,u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6;x^2;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^3;f,u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6;x^3;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^4;f,u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6;x^4;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^5;f,u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6;x^5;f,u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.13}$$

then we have

$$\begin{aligned}
u_I &= \alpha_5 x^5 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0,\ \forall x \in I_j. \\
v_I &= \beta_5 x^5 + \beta_4 x^4 + \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0,\ \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.14}$$

where

$$\begin{aligned}
\alpha_5 =&(630(33x_j + 145a_j\tau_{max})h^4 + 288a_j^2\tau_{max}^2(70455x_j + 481a_j\tau_{max})h^2 \\
&+ 5632a_j^4\tau_{max}^4(35193x_j - 1331a_j\tau_{max})) \\
&/(11(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4)) \\
\alpha_4 =&(5(3465h^6 - 12(10395x_j^2 + 91350a_j\tau_{max}x_j - 191168a_j^2\tau_{max}^2)h^4 \\
&+ 64a_j^2\tau_{max}^2(-1902285x_j^2 - 25974a_j\tau_{max}x_j \\
&+ 348082a_j^2\tau_{max}^2)h^2 + 33792a_j^4x_j\tau_{max}^4(2662a_j\tau_{max} - 35193x_j))) \\
&/(132(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4))
\end{aligned}$$

$$
\begin{aligned}
\alpha_3 =&(5(-9(1155x_j + 5531a_j\tau_{max})h^6 + 12(10395x_j^3 + 137025a_j\tau_{max}x_j^2 \\
&- 573504a_j^2\tau_{max}^2x_j - 21856a_j^3\tau_{max}^3)h^4 + 64a_j^2\tau_{max}^2(1902285x_j^3 \\
&+ 38961a_j\tau_{max}x_j^2 - 1044246a_j^2\tau_{max}^2x_j + 34606a_j^3\tau_{max}^3)h^2 \\
&- 101376a_j^4x_j^2\tau_{max}^4(1331a_j\tau_{max} - 11731x_j))) \\
&/(99(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4)) \\
\alpha_2 =&-(5(2871h^8 - 48(3465x_j^2 + 33186a_j\tau_{max}x_j - 23006a_j^2\tau_{max}^2)h^6 \\
&+ 32(31185x_j^4 + 548100a_j\tau_{max}x_j^3 - 3441024a_j^2\tau_{max}^2x_j^2 - 262272a_j^3\tau_{max}^3x_j \\
&+ 329432a_j^4\tau_{max}^4)h^4 + 512a_j^2x_j\tau_{max}^2(1902285x_j^3 + 51948a_j\tau_{max}x_j^2 \\
&- 2088492a_j^2\tau_{max}^2x_j + 138424a_j^3\tau_{max}^3)h^2 \\
&- 270336a_j^4x_j^3\tau_{max}^4(5324a_j\tau_{max} - 35193x_j))) \\
&/(1056(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4)) \\
\alpha_1 =&(3465(29x_j + 149a_j\tau_{max})h^8 - 336(5775x_j^3 + 82965a_j\tau_{max}x_j^2 \\
&- 115030a_j^2\tau_{max}^2x_j - 12178a_j^3\tau_{max}^3)h^6 + 160(43659x_j^5 + 959175a_j\tau_{max}x_j^4 \\
&- 8029056a_j^2\tau_{max}^2x_j^3 - 917952a_j^3\tau_{max}^3x_j^2 + 2306024a_j^4\tau_{max}^4x_j - 53240a_j^5\tau_{max}^5)h^4 \\
&+ 17920a_j^2x_j^2\tau_{max}^2(380457x_j^3 + 12987a_j\tau_{max}x_j^2 - 696164a_j^2\tau_{max}^2x_j \\
&+ 69212a_j^3\tau_{max}^3)h^2 - 1892352a_j^4x_j^4\tau_{max}^4(6655a_j\tau_{max} - 35193x_j)) \\
&/(3696(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4)) \\
\alpha_0 =&(17325h^{10} - 84(14355x_j^2 + 147510a_j\tau_{max}x_j - 37792a_j^2\tau_{max}^2)h^8 + 32(363825x_j^4 \\
&+ 6969060a_j\tau_{max}x_j^3 - 14493780a_j^2\tau_{max}^2x_j^2 - 3068856a_j^3\tau_{max}^3x_j + 873136a_j^4\tau_{max}^4)h^6 \\
&- 1920x_j(14553x_j^5 + 383670a_j\tau_{max}x_j^4 - 4014528a_j^2\tau_{max}^2x_j^3 - 611968a_j^3\tau_{max}^3x_j^2 \\
&+ 2306024a_j^4\tau_{max}^4x_j - 106480a_j^5\tau_{max}^5)h^4 - 14336a_j^2x_j^3\tau_{max}^2(1902285x_j^3 \\
&+ 77922a_j\tau_{max}x_j^2 - 5221230a_j^2\tau_{max}^2x_j + 692120a_j^3\tau_{max}^3)h^2 \\
&+ 22708224a_j^4x_j^5\tau_{max}^4(2662a_j\tau_{max} - 11731x_j)) \\
&/(88704(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4))
\end{aligned}
$$

$$
\begin{aligned}
\beta_5 =&(630(33x_{j+\frac{1}{2}} + 145a_{j+\frac{1}{2}}\tau_{max})h^4 + 288a_{j+\frac{1}{2}}^2\tau_{max}^2(70455x_{j+\frac{1}{2}} + 481a_{j+\frac{1}{2}}\tau_{max})h^2 \\
&+ 5632a_{j+\frac{1}{2}}^4\tau_{max}^4(35193x_{j+\frac{1}{2}} - 1331a_{j+\frac{1}{2}}\tau_{max})) \\
&/(11(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4)) \\
\beta_4 =&(5(3465h^6 - 12(10395x_{j+\frac{1}{2}}^2 + 91350a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 191168a_{j+\frac{1}{2}}^2\tau_{max}^2)h^4 \\
&+ 64a_{j+\frac{1}{2}}^2\tau_{max}^2(-1902285x_{j+\frac{1}{2}}^2 - 25974a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} \\
&+ 348082a_{j+\frac{1}{2}}^2\tau_{max}^2)h^2 + 33792a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}\tau_{max}^4(2662a_{j+\frac{1}{2}}\tau_{max} - 35193x_{j+\frac{1}{2}}))) \\
&/(132(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4)) \\
\beta_3 =&(5(-9(1155x_{j+\frac{1}{2}} + 5531a_{j+\frac{1}{2}}\tau_{max})h^6 + 12(10395x_{j+\frac{1}{2}}^3 + 137025a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 \\
&- 573504a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} - 21856a_{j+\frac{1}{2}}^3\tau_{max}^3)h^4 + 64a_{j+\frac{1}{2}}^2\tau_{max}^2(1902285x_{j+\frac{1}{2}}^3 \\
&+ 38961a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 1044246a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 34606a_{j+\frac{1}{2}}^3\tau_{max}^3)h^2 \\
&- 101376a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^2\tau_{max}^4(1331a_{j+\frac{1}{2}}\tau_{max} - 11731x_{j+\frac{1}{2}}))) \\
&/(99(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4)) \\
\beta_2 =&-(5(2871h^8 - 48(3465x_{j+\frac{1}{2}}^2 + 33186a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 23006a_{j+\frac{1}{2}}^2\tau_{max}^2)h^6 \\
&+ 32(31185x_{j+\frac{1}{2}}^4 + 548100a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 - 3441024a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
&- 262272a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} + 329432a_{j+\frac{1}{2}}^4\tau_{max}^4)h^4 + 512a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2(1902285x_{j+\frac{1}{2}}^3 \\
&+ 51948a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 2088492a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 138424a_{j+\frac{1}{2}}^3\tau_{max}^3)h^2 \\
&- 270336a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^3\tau_{max}^4(5324a_{j+\frac{1}{2}}\tau_{max} - 35193x_{j+\frac{1}{2}}))) \\
&/(1056(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4)) \\
\beta_1 =&(3465(29x_{j+\frac{1}{2}} + 149a_{j+\frac{1}{2}}\tau_{max})h^8 - 336(5775x_{j+\frac{1}{2}}^3 + 82965a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 \\
&- 115030a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} - 12178a_{j+\frac{1}{2}}^3\tau_{max}^3)h^6 + 160(43659x_{j+\frac{1}{2}}^5 \\
&+ 959175a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^4 - 8029056a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 - 917952a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2 \\
&+ 2306024a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}} - 53240a_{j+\frac{1}{2}}^5\tau_{max}^5)h^4 \\
&+ 17920a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^2\tau_{max}^2(380457x_{j+\frac{1}{2}}^3 + 12987a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 696164a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} \\
&+ 69212a_{j+\frac{1}{2}}^3\tau_{max}^3)h^2 - 1892352a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^4\tau_{max}^4(6655a_{j+\frac{1}{2}}\tau_{max} - 35193x_{j+\frac{1}{2}}))
\end{aligned}
$$

$$/(3696(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2 h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4))$$

$$\begin{aligned}
\beta_0 =&(17325h^{10} - 84(14355x_{j+\frac{1}{2}}^2 + 147510a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 37792a_{j+\frac{1}{2}}^2\tau_{max}^2)h^8 \\
&+ 32(363825x_{j+\frac{1}{2}}^4 + 6969060a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 - 14493780a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
&- 3068856a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} + 873136a_{j+\frac{1}{2}}^4\tau_{max}^4)h^6 \\
&- 1920x_{j+\frac{1}{2}}(14553x_{j+\frac{1}{2}}^5 + 383670a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^4 - 4014528a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 \\
&- 611968a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2 + 2306024a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}} - 106480a_{j+\frac{1}{2}}^5\tau_{max}^5)h^4 \\
&- 14336a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^3\tau_{max}^2(1902285x_{j+\frac{1}{2}}^3 + 77922a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 \\
&- 5221230a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 692120a_{j+\frac{1}{2}}^3\tau_{max}^3)h^2 \\
&+ 22708224a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^5\tau_{max}^4(2662a_{j+\frac{1}{2}}\tau_{max} - 11731x_{j+\frac{1}{2}})) \\
&/(88704(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\int_{x_j}^{x_{j+\frac{1}{2}}}(v_I - x^6 - u_I(x-\frac{h}{2}) + (x-\frac{h}{2})^6)^2 \\
=&(h^{11}\tau_{max}^2(a_j - a_{j+\frac{1}{2}})^2(19501200013171323158958545305 6a_j^8a_{j+\frac{1}{2}}^8\tau_{max}^{16} \\
&- 20189195085233133391655 7312a_j^6a_{j+\frac{1}{2}}^6h^3\tau_{max}^{13}(a_j + a_{j+\frac{1}{2}}) \\
&+ 1800338400h^{14}\tau_{max}^2(13438284110a_j^2 - 51320885931a_ja_{j+\frac{1}{2}} \\
&+ 13438284110a_{j+\frac{1}{2}}^2) + 100416141472907132928a_j^6a_{j+\frac{1}{2}}^6h^2\tau_{max}^{14}(397624535a_j^2 \\
&- 18450573a_ja_{j+\frac{1}{2}} + 397624535a_{j+\frac{1}{2}}^2) - 2200413600h^{13}\tau_{max}^3(10662687321a_j^3 \\
&+ 292187561882a_j^2a_{j+\frac{1}{2}} + 292187561882a_ja_{j+\frac{1}{2}}^2 + 10662687321a_{j+\frac{1}{2}}^3) \\
&+ 228614400h^{12}\tau_{max}^4(2094049467654a_j^4 - 8864817048804a_j^3a_{j+\frac{1}{2}} \\
&+ 295334595970901a_j^2a_{j+\frac{1}{2}}^2 - 8864817048804a_ja_{j+\frac{1}{2}}^3 + 2094049467654a_{j+\frac{1}{2}}^4) \\
&- 23253221376a_j^4a_{j+\frac{1}{2}}^4h^5\tau_{max}^{11}(2316625977228695a_j^3 \\
&+ 3103819172036596a_j^2a_{j+\frac{1}{2}} + 3103819172036596a_ja_{j+\frac{1}{2}}^2 \\
&+ 2316625977228695a_{j+\frac{1}{2}}^3) + 1056964608a_j^4a_{j+\frac{1}{2}}^4h^4\tau_{max}^{12}(197232718998897488 5a_j^4 \\
&- 357721171163321350a_j^3a_{j+\frac{1}{2}} + 8331528893607301417a_j^2a_{j+\frac{1}{2}}^2
\end{aligned}$$

$$
\begin{aligned}
&- 357721171163321350 a_j a_{j+\frac{1}{2}}^3 + 1972327189988974885 a_{j+\frac{1}{2}}^4) \\
&- 11735539200 h^{11} \tau_{max}^5 (54815097606 a_j^5 + 1107739544932 a_j^4 a_{j+\frac{1}{2}} + 4065675623387 a_j^3 a_{j+\frac{1}{2}}^2 \\
&+ 4065675623387 a_j^2 a_{j+\frac{1}{2}}^3 + 1107739544932 a_j a_{j+\frac{1}{2}}^4 + 54815097606 a_{j+\frac{1}{2}}^5) \\
&- 19074908160 a_j^2 a_{j+\frac{1}{2}}^2 h^7 \tau_{max}^9 (179294959298488 a_j^5 + 308410865284400 a_j^4 a_{j+\frac{1}{2}} \\
&+ 770619223416953 a_j^3 a_{j+\frac{1}{2}}^2 + 770619223416953 a_j^2 a_{j+\frac{1}{2}}^3 + 308410865284400 a_j a_{j+\frac{1}{2}}^4 \\
&+ 179294959298488 a_{j+\frac{1}{2}}^5) + 487710720 h^{10} \tau_{max}^6 (5441989209404 a_j^6 \\
&- 26664212415965 a_j^5 a_{j+\frac{1}{2}} + 2805981567667107 a_j^4 a_{j+\frac{1}{2}}^2 - 492603505952389 a_j^3 a_{j+\frac{1}{2}}^3 \\
&+ 2805981567667107 a_j^2 a_{j+\frac{1}{2}}^4 - 26664212415965 a_j a_{j+\frac{1}{2}}^5 + 5441989209404 a_{j+\frac{1}{2}}^6) \\
&+ 1189085184 a_j^2 a_{j+\frac{1}{2}}^2 h^6 \tau_{max}^{10} (3522080594245675 a_j^6 - 16385982633633415 a_j^5 a_{j+\frac{1}{2}} \\
&+ 466662656914525505 a_j^4 a_{j+\frac{1}{2}}^2 - 655805535133354431 a_j^3 a_{j+\frac{1}{2}}^3 + 466662656914525505 a_j^2 a_{j+\frac{1}{2}}^4 \\
&- 16385982633633415 a_j a_{j+\frac{1}{2}}^5 + 3522080594245675 a_{j+\frac{1}{2}}^6) \\
&- 1788272640 h^9 \tau_{max}^7 (1947374673365 a_j^7 + 36401168398920 a_j^6 a_{j+\frac{1}{2}} \\
&+ 422084292605924 a_j^5 a_{j+\frac{1}{2}}^2 + 595382564504327 a_j^4 a_{j+\frac{1}{2}}^3 + 595382564504327 a_j^3 a_{j+\frac{1}{2}}^4 \\
&+ 422084292605924 a_j^2 a_{j+\frac{1}{2}}^5 + 36401168398920 a_j a_{j+\frac{1}{2}}^6 + 1947374673365 a_{j+\frac{1}{2}}^7) \\
&+ 185794560 h^8 \tau_{max}^8 (11547805227035 a_j^8 - 106377383972860 a_j^7 a_{j+\frac{1}{2}} \\
&+ 39707200376096864 a_j^6 a_{j+\frac{1}{2}}^2 - 22325282290601846 a_j^5 a_{j+\frac{1}{2}}^3 \\
&+ 259867644987323578 a_j^4 a_{j+\frac{1}{2}}^4 - 22325282290601846 a_j^3 a_{j+\frac{1}{2}}^5 \\
&+ 39707200376096864 a_j^2 a_{j+\frac{1}{2}}^6 - 106377383972860 a_j a_{j+\frac{1}{2}}^7 + 11547805227035 a_{j+\frac{1}{2}}^8) \\
&+ 604417549765837500 h^{15} \tau_{max} (a_j + a_{j+\frac{1}{2}}) + 71270165866433625 h^{16})) \\
&/(10819049472 (3003136 a_j^4 \tau_{max}^4 + 307440 a_j^2 h^2 \tau_{max}^2 + 315 h^4)^2 (3003136 a_{j+\frac{1}{2}}^4 \tau_{max}^4 \\
&+ 307440 a_{j+\frac{1}{2}}^2 h^2 \tau_{max}^2 + 315 h^4)^2) \\
=& O(h^{15})
\end{aligned}
$$

7. $k=6,\ u=x^7$, by the definition

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx = &\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^7 dx \\
\tilde{P}_h(u_I;x;f,u)_j =&\tilde{P}_h(x^7;x;f,u)_j \\
\tilde{P}_h(u_I;x^2;f,u)_j =&\tilde{P}_h(x^7;x^2;f,u)_j \\
\tilde{P}_h(u_I;x^3;f,u)_j =&\tilde{P}_h(x^7;x^3;f,u)_j \\
\tilde{P}_h(u_I;x^4;f,u)_j =&\tilde{P}_h(x^7;x^4;f,u)_j \\
\tilde{P}_h(u_I;x^5;f,u)_j =&\tilde{P}_h(x^7;x^5;f,u)_j \\
\tilde{P}_h(u_I;x^6;f,u)_j =&\tilde{P}_h(x^7;x^6;f,u)_j \\
\int_{x_j}^{x_{j+1}} v_I dx = &\int_{x_j}^{x_{j+1}} x^7 dx \\
\tilde{Q}_h(v_I;x;f,u)_{j+\frac{1}{2}} =&\tilde{Q}_h(x^7;x;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^2;f,u)_{j+\frac{1}{2}} =&\tilde{Q}_h(x^7;x^2;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^3;f,u)_{j+\frac{1}{2}} =&\tilde{Q}_h(x^7;x^3;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^4;f,u)_{j+\frac{1}{2}} =&\tilde{Q}_h(x^7;x^4;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^5;f,u)_{j+\frac{1}{2}} =&\tilde{Q}_h(x^7;x^5;f,u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I;x^6;f,u)_{j+\frac{1}{2}} =&\tilde{Q}_h(x^7;x^6;f,u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.15}$$

then we have

$$\begin{aligned}
u_I =&\alpha_6 x^6+\alpha_5 x^5+\alpha_4 x^4+\alpha_3 x^3+\alpha_2 x^2+\alpha_1 x+\alpha_0,\ \forall x\in I_j. \\
v_I =&\beta_6 x^6+\beta_5 x^5+\beta_4 x^4+\beta_3 x^3+\beta_2 x^2+\beta_1 x+\beta_0,\ \forall x\in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.16}$$

where

$$\begin{aligned}
\alpha_6 =&(7(94332080128a_j^6\tau_{max}^6x_j-1280a_j^4h^2\tau_{max}^4(4415546a_j\tau_{max}-42028623x_j) \\
&-672a_j^2h^4\tau_{max}^2(789340a_j\tau_{max}-6671691x_j)+63h^6(67718a_j\tau_{max}+2145x_j))) \\
&/(13(7256313856a_j^6\tau_{max}^6+4138202880a_j^4h^2\tau_{max}^4+344875104a_j^2h^4\tau_{max}^2+10395h^6)) \\
\alpha_5 =&(21(-754656641024a_j^6\tau_{max}^6x_j^2-504h^6(-1271288a_j^2\tau_{max}^2+135436a_j\tau_{max}x_j+2145x_j^2) \\
&+256a_j^2h^4\tau_{max}^2(28301533a_j^2\tau_{max}^2+33152280a_j\tau_{max}x_j-140105511x_j^2)
\end{aligned}$$

$$
\begin{aligned}
&+ 2048a_j^4h^2\tau_{max}^4(4448202a_j^2\tau_{max}^2 + 44155460a_j\tau_{max}x_j - 210143115x_j^2) + 28665h^8))\\
&/(104(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))\\
\alpha_4 =& -(35(-8301223051264a_j^6\tau_{max}^6x_j^3 + 67584a_j^4h^2\tau_{max}^4x_j(4448202a_j^2\tau_{max}^2\\
&+ 22077730a_j\tau_{max}x_j - 70047705x_j^2) - 24h^6(103797556a_j^3\tau_{max}^3 - 881002584a_j^2\tau_{max}^2x_j\\
&+ 46928574a_j\tau_{max}x_j^2 + 495495x_j^3) - 256a_j^2h^4\tau_{max}^2(108151636a_j^3\tau_{max}^3\\
&- 933950589a_j^2\tau_{max}^2x_j - 547012620a_j\tau_{max}x_j^2 + 1541160621x_j^3)\\
&+ 2079h^8(15216a_j\tau_{max} + 455x_j)))\\
&/(1144(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))\\
\alpha_3 =& -(35(16602446102528a_j^6\tau_{max}^6x_j^4 - 132h^8(-6457912a_j^2\tau_{max}^2 + 1917216a_j\tau_{max}x_j\\
&+ 28665x_j^2) - 45056a_j^4h^2\tau_{max}^4x_j^2(26689212a_j^2\tau_{max}^2 + 88310920a_j\tau_{max}x_j - 210143115x_j^2)\\
&+ 16h^6(569473600a_j^4\tau_{max}^4 + 1245570672a_j^3\tau_{max}^3x_j - 5286015504a_j^2\tau_{max}^2x_j^2\\
&+ 187714296a_j\tau_{max}x_j^3 + 1486485x_j^4) + 512a_j^2h^4\tau_{max}^2(13427128a_j^4\tau_{max}^4\\
&+ 432606544a_j^3\tau_{max}^3x_j - 1867901178a_j^2\tau_{max}^2x_j^2 - 729350160a_j\tau_{max}x_j^3\\
&+ 1541160621x_j^4) + 63063h^{10}))\\
&/(2288(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))\\
\alpha_2 =& (7(49807338307584a_j^6\tau_{max}^6x_j^5 - 675840a_j^4h^2\tau_{max}^4x_j^3(8896404a_j^2\tau_{max}^2 + 22077730a_j\tau_{max}x_j\\
&- 42028623x_j^2) - 180h^8(7154768a_j^3\tau_{max}^3 - 71037032a_j^2\tau_{max}^2x_j\\
&+ 10544688a_j\tau_{max}x_j^2 + 105105x_j^3) + 1536a_j^2h^4\tau_{max}^2x_j(67135640a_j^4\tau_{max}^4\\
&+ 1081516360a_j^3\tau_{max}^3x_j - 3113168630a_j^2\tau_{max}^2x_j^2 - 911687700a_j\tau_{max}x_j^3\\
&+ 1541160621x_j^4) + 16h^6(-999477488a_j^5\tau_{max}^5 + 8542104000a_j^4\tau_{max}^4x_j\\
&+ 9341780040a_j^3\tau_{max}^3x_j^2 - 26430077520a_j^2\tau_{max}^2x_j^3 + 703928610a_j\tau_{max}x_j^4 + 4459455x_j^5)\\
&+ 99h^{10}(334172a_j\tau_{max} + 9555x_j)))\\
&/(2288(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))\\
\alpha_1 =& (7(-796917412921344a_j^6\tau_{max}^6x_j^6 - 1584h^{10}(-2907158a_j^2\tau_{max}^2 + 2005032a_j\tau_{max}x_j
\end{aligned}
$$

$$+ 28665x_j^2) + 32440320a_j^4h^2\tau_{max}^4x_j^4(4448202a_j^2\tau_{max}^2 + 8831092a_j\tau_{max}x_j$$
$$- 14009541x_j^2) + 864h^8(54537656a_j^4\tau_{max}^4 + 143095360a_j^3\tau_{max}^3x_j - 710370320a_j^2\tau_{max}^2x_j^2$$
$$+ 70297920a_j\tau_{max}x_j^3 + 525525x_j^4) - 24576a_j^2h^4\tau_{max}^2x_j^2(201406920a_j^4\tau_{max}^4$$
$$+ 2163032720a_j^3\tau_{max}^3x_j - 4669752945a_j^2\tau_{max}^2x_j^2 - 1094025240a_j\tau_{max}x_j^3 + 1541160621x_j^4)$$
$$- 256h^6(-55859408a_j^6\tau_{max}^6 - 5996864928a_j^5\tau_{max}^5x_j + 25626312000a_j^4\tau_{max}^4x_j^2$$
$$+ 18683560080a_j^3\tau_{max}^3x_j^3 - 39645116280a_j^2\tau_{max}^2x_j^4 + 844714332a_j\tau_{max}x_j^5$$
$$+ 4459455x_j^6) + 637065h^{12}))$$
$$/(109824(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))$$
$$\alpha_0 =(796917412921344a_j^6\tau_{max}^6x_j^7 - 15138816a_j^4h^2\tau_{max}^4x_j^5(13344606a_j^2\tau_{max}^2$$
$$+ 22077730a_j\tau_{max}x_j - 30020445x_j^2) + 1008h^{10}(1438204a_j^3\tau_{max}^3 - 31978738a_j^2\tau_{max}^2x_j$$
$$+ 11027676a_j\tau_{max}x_j^2 + 105105x_j^3) + 172032a_j^2h^4\tau_{max}^2x_j^3(67135640a_j^4\tau_{max}^4$$
$$+ 540758180a_j^3\tau_{max}^3x_j - 933950589a_j^2\tau_{max}^2x_j^2 - 182337540a_j\tau_{max}x_j^3 + 220165803x_j^4)$$
$$- 32h^8(-967818016a_j^5\tau_{max}^5 + 10307616984a_j^4\tau_{max}^4x_j + 13522511520a_j^3\tau_{max}^3x_j^2$$
$$- 44753330160a_j^2\tau_{max}^2x_j^3 + 3321576720a_j\tau_{max}x_j^4 + 19864845x_j^5)$$
$$+ 1792h^6x_j(-55859408a_j^6\tau_{max}^6 - 2998432464a_j^5\tau_{max}^5x_j + 8542104000a_j^4\tau_{max}^4x_j^2$$
$$+ 4670890020a_j^3\tau_{max}^3x_j^3 - 7929023256a_j^2\tau_{max}^2x_j^4 + 140785722a_j\tau_{max}x_j^5$$
$$+ 637065x_j^6) - 693h^{12}(232088a_j\tau_{max} + 6435x_j))$$
$$/(109824(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))$$
$$\beta_6 =(7(94332080128a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}} - 1280a_{j+\frac{1}{2}}^4h^2\tau_{max}^4(4415546a_{j+\frac{1}{2}}\tau_{max} - 42028623x_{j+\frac{1}{2}})$$
$$- 672a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(789340a_{j+\frac{1}{2}}\tau_{max} - 6671691x_{j+\frac{1}{2}}) + 63h^6(67718a_{j+\frac{1}{2}}\tau_{max} + 2145x_{j+\frac{1}{2}})))$$
$$/(13(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))$$
$$\beta_5 =(21(-754656641024a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}}^2 - 504h^6(-1271288a_{j+\frac{1}{2}}^2\tau_{max}^2$$
$$+ 135436a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 2145x_{j+\frac{1}{2}}^2)$$
$$+ 256a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(28301533a_{j+\frac{1}{2}}^2\tau_{max}^2 + 33152280a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 140105511x_{j+\frac{1}{2}}^2)$$

$$+ 2048a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4(4448202a_{j+\frac{1}{2}}^2\tau_{max}^2 + 44155460a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}$$

$$- 210143115x_{j+\frac{1}{2}}^2) + 28665h^8))$$

$$/(104(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_4 = -(35(-8301223051264a_{j+\frac{1}{2}}^6\tau_{max}^6 x_{j+\frac{1}{2}}^3 + 67584a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 x_{j+\frac{1}{2}}(4448202a_{j+\frac{1}{2}}^2\tau_{max}^2$$

$$+ 22077730a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 70047705x_{j+\frac{1}{2}}^2) - 24h^6(103797556a_{j+\frac{1}{2}}^3\tau_{max}^3$$

$$- 881002584a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}$$

$$+ 46928574a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 495495x_{j+\frac{1}{2}}^3) - 256a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2(108151636a_{j+\frac{1}{2}}^3\tau_{max}^3$$

$$- 933950589a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}} - 547012620a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 1541160621x_{j+\frac{1}{2}}^3)$$

$$+ 2079h^8(15216a_{j+\frac{1}{2}}\tau_{max} + 455x_{j+\frac{1}{2}})))$$

$$/(1144(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_3 = -(35(16602446102528a_{j+\frac{1}{2}}^6\tau_{max}^6 x_{j+\frac{1}{2}}^4 - 132h^8(-6457912a_{j+\frac{1}{2}}^2\tau_{max}^2 + 1917216a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}$$

$$+ 28665x_{j+\frac{1}{2}}^2) - 45056a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 x_{j+\frac{1}{2}}^2(26689212a_{j+\frac{1}{2}}^2\tau_{max}^2$$

$$+ 88310920a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 210143115x_{j+\frac{1}{2}}^2)$$

$$+ 16h^6(569473600a_{j+\frac{1}{2}}^4\tau_{max}^4 + 1245570672a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}} - 5286015504a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^2$$

$$+ 187714296a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 + 1486485x_{j+\frac{1}{2}}^4) + 512a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2(13427128a_{j+\frac{1}{2}}^4\tau_{max}^4$$

$$+ 432606544a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}} - 1867901178a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^2 - 729350160a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3$$

$$+ 1541160621x_{j+\frac{1}{2}}^4) + 63063h^{10}))$$

$$/(2288(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_2 = (7(49807338307584a_{j+\frac{1}{2}}^6\tau_{max}^6 x_{j+\frac{1}{2}}^5 - 675840a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 x_{j+\frac{1}{2}}^3(8896404a_{j+\frac{1}{2}}^2\tau_{max}^2$$

$$+ 22077730a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}$$

$$- 42028623x_{j+\frac{1}{2}}^2) - 180h^8(7154768a_{j+\frac{1}{2}}^3\tau_{max}^3 - 71037032a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}$$

$$+ 10544688a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 105105x_{j+\frac{1}{2}}^3) + 1536a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 x_{j+\frac{1}{2}}(67135640a_{j+\frac{1}{2}}^4\tau_{max}^4$$

$$+ 1081516360a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}} - 3113168630a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^2 - 911687700a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3$$

$$+ 1541160621x_{j+\frac{1}{2}}^4) + 16h^6(-999477488a_{j+\frac{1}{2}}^5\tau_{max}^5 + 8542104000a_{j+\frac{1}{2}}^4\tau_{max}^4 x_{j+\frac{1}{2}}$$

$$+ 9341780040a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}}^2 - 26430077520a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^3$$

$$+ 703928610a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^4 + 4459455x_{j+\frac{1}{2}}^5)$$

$$+ 99h^{10}(334172a_{j+\frac{1}{2}}\tau_{max} + 9555x_{j+\frac{1}{2}})))$$

$$/(2288(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4$$

$$+ 344875104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_1 =(7(-796917412921344a_{j+\frac{1}{2}}^6\tau_{max}^6 x_{j+\frac{1}{2}}^6 - 1584h^{10}(-2907158a_{j+\frac{1}{2}}^2\tau_{max}^2 + 2005032a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}$$

$$+ 28665x_{j+\frac{1}{2}}^2) + 32440320a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 x_{j+\frac{1}{2}}^4(4448202a_{j+\frac{1}{2}}^2\tau_{max}^2 + 8831092a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}$$

$$- 14009541x_{j+\frac{1}{2}}^2) + 864h^8(54537656a_{j+\frac{1}{2}}^4\tau_{max}^4$$

$$+ 143095360a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}} - 710370320a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^2$$

$$+ 70297920a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^3 + 525525x_{j+\frac{1}{2}}^4) - 24576a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 x_{j+\frac{1}{2}}^2(201406920a_{j+\frac{1}{2}}^4\tau_{max}^4$$

$$+ 2163032720a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}} - 4669752945a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^2$$

$$- 1094025240a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^3 + 1541160621x_{j+\frac{1}{2}}^4)$$

$$- 256h^6(-55859408a_{j+\frac{1}{2}}^6\tau_{max}^6 - 5996864928a_{j+\frac{1}{2}}^5\tau_{max}^5 x_{j+\frac{1}{2}} + 25626312000a_{j+\frac{1}{2}}^4\tau_{max}^4 x_{j+\frac{1}{2}}^2$$

$$+ 18683560080a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}}^3 - 39645116280a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^4 + 844714332a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^5$$

$$+ 4459455x_{j+\frac{1}{2}}^6) + 637065h^{12}))$$

$$/(109824(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4$$

$$+ 344875104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_0 =(796917412921344a_{j+\frac{1}{2}}^6\tau_{max}^6 x_{j+\frac{1}{2}}^7 - 15138816a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 x_{j+\frac{1}{2}}^5(13344606a_{j+\frac{1}{2}}^2\tau_{max}^2$$

$$+ 22077730a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}} - 30020445x_{j+\frac{1}{2}}^2)$$

$$+ 1008h^{10}(1438204a_{j+\frac{1}{2}}^3\tau_{max}^3 - 31978738a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}$$

$$+ 11027676a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^2 + 105105x_{j+\frac{1}{2}}^3) + 172032a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 x_{j+\frac{1}{2}}^3(67135640a_{j+\frac{1}{2}}^4\tau_{max}^4$$

$$+ 540758180a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}} - 933950589a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^2$$

$$- 182337540a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^3 + 220165803x_{j+\frac{1}{2}}^4)$$

$$- 32h^8(-967818016a_{j+\frac{1}{2}}^5\tau_{max}^5 + 10307616984a_{j+\frac{1}{2}}^4\tau_{max}^4 x_{j+\frac{1}{2}} + 13522511520a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}}^2$$

$$
\begin{aligned}
&- 44753330160a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^3 + 3321576720a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^4 + 19864845x_{j+\frac{1}{2}}^5) \\
&+ 1792h^6 x_{j+\frac{1}{2}}(-55859408a_{j+\frac{1}{2}}^6\tau_{max}^6 - 2998432464a_{j+\frac{1}{2}}^5\tau_{max}^5 x_{j+\frac{1}{2}} + 8542104000a_{j+\frac{1}{2}}^4\tau_{max}^4 x_{j+\frac{1}{2}}^2 \\
&+ 4670890020a_{j+\frac{1}{2}}^3\tau_{max}^3 x_{j+\frac{1}{2}}^3 - 7929023256a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}^4 + 140785722a_{j+\frac{1}{2}}\tau_{max} x_{j+\frac{1}{2}}^5 \\
&+ 637065x_{j+\frac{1}{2}}^6) - 693h^{12}(232088a_{j+\frac{1}{2}}\tau_{max} + 6435x_{j+\frac{1}{2}})) \\
&/(109824(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 \\
&+ 344875104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6))
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^7 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^7)^2 \\
=&((a_j - a_{j+\frac{1}{2}})^2 h^{17}\tau_{max}^2(4374030469942367957781925h^{20} \\
&+ 8859873555495770497489800(a_j + a_{j+\frac{1}{2}})\tau_{max} h^{19} \\
&+ 1485279180(1699073037709623145a_j^2 - 1615895394979739357 4a_{j+\frac{1}{2}}a_j \\
&+ 1699073037709623145a_{j+\frac{1}{2}}^2)\tau_{max}^2 h^{18} - 2973528918360(1414536046558650a_j^3 \\
&+ 10168933573634476 1a_{j+\frac{1}{2}}a_j^2 + 10168933573634476 1a_{j+\frac{1}{2}}^2 a_j \\
&+ 1414536046558650a_{j+\frac{1}{2}}^3)\tau_{max}^3 h^{17} + 3168595584(20961537958383611375a_j^4 \\
&- 32242548744010154460 0a_{j+\frac{1}{2}}a_j^3 + 151880015288861843693378a_{j+\frac{1}{2}}^2 a_j^2 \\
&- 32242548744010154460 0a_{j+\frac{1}{2}}^3 a_j + 20961537958383611375a_{j+\frac{1}{2}}^4)\tau_{max}^4 h^{16} \\
&- 257448391200(4202929509789698 49a_j^5 + 1707257092732227470 5a_{j+\frac{1}{2}}a_j^4 \\
&+ 5867688044883488522 56a_{j+\frac{1}{2}}^2 a_j^3 + 5867688044883488522 56a_{j+\frac{1}{2}}^3 a_j^2 \\
&+ 1707257092732227470 5a_{j+\frac{1}{2}}^4 a_j + 4202929509789698 49a_{j+\frac{1}{2}}^5)\tau_{max}^5 h^{15} \\
&+ 3840721920(1565954360007405135 90a_j^6 - 3699346840438621615628a_{j+\frac{1}{2}}a_j^5 \\
&+ 310992455849545045853146 1a_{j+\frac{1}{2}}^2 a_j^4 - 3855737499325029678375014a_{j+\frac{1}{2}}^3 a_j^3 \\
&+ 310992455849545045853146 1a_{j+\frac{1}{2}}^4 a_j^2 - 3699346840438621615628a_{j+\frac{1}{2}}^5 a_j \\
&+ 1565954360007405135 90a_{j+\frac{1}{2}}^6)\tau_{max}^6 h^{14} - 549223234560(1593068096417356625a_j^7 \\
&- 1132561118253725926 7a_{j+\frac{1}{2}}a_j^6 + 7442033876711042429922a_{j+\frac{1}{2}}^2 a_j^5
\end{aligned}
$$

$$- 16581821408728408340361 2a^3_{j+\frac{1}{2}}a^4_j - 165818214087284083403612a^4_{j+\frac{1}{2}}a^3_j$$

$$+ 744203387671104242992 2a^5_{j+\frac{1}{2}}a^2_j - 113256111825372592 67a^6_{j+\frac{1}{2}}a_j$$

$$+ 1593068096417356625a^7_{j+\frac{1}{2}})\tau^7_{max}h^{13} + 585252864(366796085696145646530 0a^8_j$$

$$- 1267862006464011966029 00a_{j+\frac{1}{2}}a^7_j + 1679601412916286171135979 54a^2_{j+\frac{1}{2}}a^6_j$$

$$- 5524196517709204696690038 40a^3_{j+\frac{1}{2}}a^5_j + 1145773697291063997025997 9445a^4_{j+\frac{1}{2}}a^4_j$$

$$- 5524196517709204696690038 40a^5_{j+\frac{1}{2}}a^3_j + 1679601412916286171135979 54a^6_{j+\frac{1}{2}}a^2_j$$

$$- 1267862006464011966029 00a^7_{j+\frac{1}{2}}a_j + 366796085696145646530 0a^8_{j+\frac{1}{2}})\tau^8_{max}h^{12}$$

$$- 19020718080(1168836801955071939 25a^9_j - 12913068248935915017815a_{j+\frac{1}{2}}a^8_j$$

$$+ 1986166975590076367754380a^2_{j+\frac{1}{2}}a^7_j - 10723798433951653276905773 6a^3_{j+\frac{1}{2}}a^6_j$$

$$+ 676505105051936160968017607a^4_{j+\frac{1}{2}}a^5_j + 676505105051936160968017607a^5_{j+\frac{1}{2}}a^4_j$$

$$- 10723798433951653276905773 6a^6_{j+\frac{1}{2}}a^3_j + 1986166975590076367754380a^7_{j+\frac{1}{2}}a^2_j$$

$$- 12913068248935915017815a^8_{j+\frac{1}{2}}a_j + 1168836801955071939 25a^9_{j+\frac{1}{2}})\tau^9_{max}h^{11}$$

$$+ 3901685760(65385477659831406148 5a^{10}_j - 263475376934901477227 42a_{j+\frac{1}{2}}a^9_j$$

$$+ 7700900476506496242232516 3a^2_{j+\frac{1}{2}}a^8_j - 5274771991351045102863607 72a^3_{j+\frac{1}{2}}a^7_j$$

$$+ 37103073127910502944858616329a^4_{j+\frac{1}{2}}a^6_j - 71356452354496691768307042 6a^5_{j+\frac{1}{2}}a^5_j$$

$$+ 37103073127910502944858616329a^6_{j+\frac{1}{2}}a^4_j - 5274771991351045102863607 72a^7_{j+\frac{1}{2}}a^3_j$$

$$+ 7700900476506496242232516 3a^8_{j+\frac{1}{2}}a^2_j - 263475376934901477227 42a^9_{j+\frac{1}{2}}a_j$$

$$+ 65385477659831406148 5a^{10}_{j+\frac{1}{2}})\tau^{10}_{max}h^{10} + 101443829760a_ja_{j+\frac{1}{2}}(455972135396566303436 9a^9_j$$

$$- 1154079108387189138422257a_{j+\frac{1}{2}}a^8_j + 129370543069772118288464708a^2_{j+\frac{1}{2}}a^7_j$$

$$- 27670842410343721232033068 78a^3_{j+\frac{1}{2}}a^6_j - 21420665061035095834297631 49a^4_{j+\frac{1}{2}}a^5_j$$

$$- 21420665061035095834297631 49a^5_{j+\frac{1}{2}}a^4_j - 27670842410343721232033068 78a^6_{j+\frac{1}{2}}a^3_j$$

$$+ 129370543069772118288464708a^7_{j+\frac{1}{2}}a^2_j - 1154079108387189138422257a^8_{j+\frac{1}{2}}a_j$$

$$+ 455972135396566303436 9a^9_{j+\frac{1}{2}})\tau^{11}_{max}h^9$$

$$+ 16647192576a^2_ja^2_{j+\frac{1}{2}}(184612839713117411661925 42a^8_j$$

$$- 1830110024921776710278962400a_{j+\frac{1}{2}}a_j^7 + 56697160731182582899942819925a_{j+\frac{1}{2}}^2a_j^6$$

$$+ 23726284012411370865033375200a_{j+\frac{1}{2}}^3a_j^5 + 190469561341043092690908870388a_{j+\frac{1}{2}}^4a_j^4$$

$$+ 23726284012411370865033375200a_{j+\frac{1}{2}}^5a_j^3 + 56697160731182582899942819925a_{j+\frac{1}{2}}^6a_j^2$$

$$- 1830110024921776710278962400a_{j+\frac{1}{2}}^7a_j + 184612839713117411661925420a_{j+\frac{1}{2}}^8)\tau_{max}^{12}h^8$$

$$+ 1082067517440a_j^3a_{j+\frac{1}{2}}^3(18321852318992025713799428a_j^7$$

$$- 1321003108505324002163493396a_{j+\frac{1}{2}}a_j^6$$

$$- 498594225191753361165098406a_{j+\frac{1}{2}}^2a_j^5 - 4512798124101130032009521037a_{j+\frac{1}{2}}^3a_j^4$$

$$- 4512798124101130032009521037a_{j+\frac{1}{2}}^4a_j^3 - 498594225191753361165098406a_{j+\frac{1}{2}}^5a_j^2$$

$$- 1321003108505324002163493396a_{j+\frac{1}{2}}^6a_j + 18321852318992025713799428a_{j+\frac{1}{2}}^7)\tau_{max}^{13}h^7$$

$$+ 31708938240a_j^4a_{j+\frac{1}{2}}^4(48802141800690392120765929181a_j^6$$

$$+ 11048478168216937331149943798a_{j+\frac{1}{2}}a_j^5 + 651595638176670808303246681584a_{j+\frac{1}{2}}^2a_j^4$$

$$+ 43557999647137173043199301698a_{j+\frac{1}{2}}^3a_j^3 + 651595638176670808303246681584a_{j+\frac{1}{2}}^4a_j^2$$

$$+ 11048478168216937331149943798a_{j+\frac{1}{2}}^5a_j + 48802141800690392120765929181a_{j+\frac{1}{2}}^6)\tau_{max}^{14}h^6$$

$$+ 1097319516733440a_j^5a_{j+\frac{1}{2}}^5(3198843901481228518023451a_j^5$$

$$- 22843767273592440900178964a_{j+\frac{1}{2}}a_j^4 - 13165800443519527815052054a_{j+\frac{1}{2}}^2a_j^3$$

$$- 13165800443519527815052054a_{j+\frac{1}{2}}^3a_j^2 - 22843767273592440900178964a_{j+\frac{1}{2}}^4a_j$$

$$+ 3198843901481228518023451a_{j+\frac{1}{2}}^5)\tau_{max}^{15}h^5$$

$$+ 67645734912a_j^6a_{j+\frac{1}{2}}^6(492258158772799267571707352258a_j^4$$

$$+ 115400788701858795739154476200a_{j+\frac{1}{2}}a_j^3 + 1968592966632694779138376243595a_{j+\frac{1}{2}}^2a_j^2$$

$$+ 115400788701858795739154476200a_{j+\frac{1}{2}}^3a_j + 492258158772799267571707352258a_{j+\frac{1}{2}}^4)\tau_{max}^{16}h^4$$

$$+ 5852370755911680a_j^7a_{j+\frac{1}{2}}^7(10862152607395926487179716a_j^3$$

$$- 13228624002089287227191317a_{j+\frac{1}{2}}a_j^2 - 13228624002089287227191317a_{j+\frac{1}{2}}^2a_j$$

$$+ 10862152607395926487179716a_{j+\frac{1}{2}}^3)\ \tau_{max}^{17}h^3$$

$$+ 798923638576250880a_j^8a_{j+\frac{1}{2}}^8(258161172004769893077187a_j^2$$

$$
\begin{aligned}
&+ 500572501284991495982 70 a_{j+\frac{1}{2}} a_j + 2581611720047698930771 87 a_{j+\frac{1}{2}}^2) \tau_{max}^{18} h^2 \\
&+ 30453904949648581259039734418445472628736 0 a_j^9 a_{j+\frac{1}{2}}^9 (a_j + a_{j+\frac{1}{2}}) \tau_{max}^{19} h \\
&+ 27677441037502616784149029191661732967219 2 a_j^{10} a_{j+\frac{1}{2}}^{10} \tau_{max}^{20})) \\
&/(6124884480(10395h^6 + 344875104 a_j^2 \tau_{max}^2 h^4 + 4138202880 a_j^4 \tau_{max}^4 h^2 \\
&+ 7256313856 a_j^6 \tau_{max}^6)^2 (10395h^6 + 344875104 a_{j+\frac{1}{2}}^2 \tau_{max}^2 h^4 + 4138202880 a_{j+\frac{1}{2}}^4 \tau_{max}^4 h^2 \\
&+ 7256313856 a_{j+\frac{1}{2}}^6 \tau_{max}^6)^2) \\
=& O(h^{17})
\end{aligned}
$$

8. $k = 7, u = x^8$, by the definition

$$
\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx =& \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^8 dx \\
\tilde{P}_h(u_I; x^m; f, u)_j =& \tilde{P}_h(x^8; x^m; f, u)_j, \;\; m = 1, ..., 7 \\
\int_{x_j}^{x_{j+1}} v_I dx =& \int_{x_j}^{x_{j+1}} x^8 dx \\
\tilde{Q}_h(v_I; x^m; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^8; x^m; f, u)_{j+\frac{1}{2}}, \;\; m = 1, ..., 7
\end{aligned}
\tag{1.17}
$$

then we have

$$
\begin{aligned}
u_I =& \sum_{m=0}^{7} \alpha_m x^m, \;\; \forall x \in I_j. \\
v_I =& \sum_{m=0}^{7} \beta_m x^m, \;\; \forall x \in I_{j+\frac{1}{2}}.
\end{aligned}
\tag{1.18}
$$

where

$$
\begin{aligned}
\alpha_7 =& (8(4096 a_j^6 \tau_{max}^6 (2354497739 a_j \tau_{max} + 239935274463 x_j) \\
& - 6912 a_j^4 h^2 \tau_{max}^4 (879791707 a_j \tau_{max} - 21360056375 x_j) \\
& - 54432 a_j^2 h^4 \tau_{max}^2 (13002767 a_j \tau_{max} - 89468841 x_j) \\
& + 51975 h^6 (14369 a_j \tau_{max} + 39 x_j))) \\
& /(3(327591628066816 a_j^6 \tau_{max}^6 + 49213569888000 a_j^4 h^2 \tau_{max}^4 \\
& + 1623322651104 a_j^2 h^4 \tau_{max}^2 + 675675 h^6)) \\
\alpha_6 =& (28(-20480 a_j^6 \tau_{max}^6 x_j (4708995478 a_j \tau_{max} + 239935274463 x_j)
\end{aligned}
$$

$$- 945h^6(-407585262a_j^2\tau_{max}^2 + 7902950a_j\tau_{max}x_j + 10725x_j^2)$$

$$+ 288a_j^2h^4\tau_{max}^2(36101338556a_j^2\tau_{max}^2 + 24575229630a_j\tau_{max}x_j - 84548054745x_j^2)$$

$$+ 1280a_j^4h^2\tau_{max}^4(50856691162a_j^2\tau_{max}^2 + 47508752178a_j\tau_{max}x_j$$

$$- 576721522125x_j^2) + 259875h^8))$$

$$/(15(327591628066816a_j^6\tau_{max}^6 + 49213569888000a_j^4h^2\tau_{max}^4$$

$$+ 1623322651104a_j^2h^4\tau_{max}^2 + 675675h^6))$$

$$\alpha_5 = -(56(-266240a_j^6\tau_{max}^6x_j^2(2354497739a_j\tau_{max} + 79978424821x_j)$$

$$- 9h^6(95613443942a_j^3\tau_{max}^3 - 556353882630a_j^2\tau_{max}^2x_j + 5393763375a_j\tau_{max}x_j^2$$

$$+ 4879875x_j^3) - 288a_j^2h^4\tau_{max}^2(27266771676a_j^3\tau_{max}^3 - 469317401228a_j^2\tau_{max}^2x_j$$

$$- 159738992595a_j\tau_{max}x_j^2 + 366374903895x_j^3) + 1280a_j^4h^2\tau_{max}^4(5911894398a_j^3\tau_{max}^3$$

$$+ 661136985106a_j^2\tau_{max}^2x_j + 308806889157a_j\tau_{max}x_j^2 - 2499126595875x_j^3)$$

$$+ 61425h^8(21053a_j\tau_{max} + 55x_j)))$$

$$/(65(327591628066816a_j^6\tau_{max}^6 + 49213569888000a_j^4h^2\tau_{max}^4$$

$$+ 1623322651104a_j^2h^4\tau_{max}^2 + 675675h^6))$$

$$\alpha_4 = -((7(2641275h^{10} + 2129920a_j^6x_j^3\tau_{max}^6(239935274463x_j + 9417990956a_j\tau_{max})$$

$$- 5616h^8(28875x_j^2 + 22105650a_jx_j\tau_{max} - 383499197a_j^2\tau_{max}^2)$$

$$- 30720a_j^4h^2x_j\tau_{max}^4(-2499126595875x_j^3 + 411742518876a_jx_j^2\tau_{max}$$

$$+ 1322273970212a_j^2x_j\tau_{max}^2 + 23647577592a_j^3\tau_{max}^3)$$

$$+ 72h^6(14639625x_j^4 + 21575053500a_jx_j^3\tau_{max} - 3338123295780a_j^2x_j^2\tau_{max}^2$$

$$+ 1147361327304a_j^3x_j\tau_{max}^3 + 687203235848a_j^4\tau_{max}^4) + 256a_j^2h^4\tau_{max}^2(9892122405165x_j^4$$

$$- 5750603733420a_jx_j^3\tau_{max} - 25343139666312a_j^2x_j^2\tau_{max}^2 + 2944811341008a_j^3x_j\tau_{max}^3$$

$$+ 1087949803400a_j^4\tau_{max}^4)))$$

$$/(156(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2$$

$$+ 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6)))$$

$$
\begin{aligned}
\alpha_3 =&(7(96525h^{10}(301x_j + 118347a_j\tau_{max}) + 4685824a_j^6x_j^4\tau_{max}^6(239935274463x_j \\
&+ 11772488695a_j\tau_{max}) - 4752h^8(125125x_j^3 + 143686725a_jx_j^2\tau_{max} \\
&- 4985489561a_j^2x_j\tau_{max}^2 + 985840581a_j^3\tau_{max}^3) - 112640a_j^4h^2x_j^2\tau_{max}^4(-1499475957525x_j^3 \\
&+ 308806889157a_jx_j^2\tau_{max} + 1322273970212a_j^2x_j\tau_{max}^2 + 35471366388a_j^3\tau_{max}^3) \\
&+ 72h^6(32207175x_j^5 + 59331397125a_jx_j^4\tau_{max} - 12239785417860a_j^2x_j^3\tau_{max}^2 \\
&+ 6310487300172a_j^3x_j^2\tau_{max}^3 + 7559235594328a_j^4x_j\tau_{max}^4 - 625703362520a_j^5\tau_{max}^5) \\
&+ 256a_j^2h^4\tau_{max}^2(21762669291363x_j^5 - 15814160266905a_jx_j^4\tau_{max} - 92924845443144a_j^2x_j^3\tau_{max}^2 \\
&+ 16196462375544a_j^3x_j^2\tau_{max}^3 + 11967447837400a_j^4x_j\tau_{max}^4 + 89226796360a_j^5\tau_{max}^5))) \\
&/(429(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2 \\
&+ 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6)) \\
\alpha_2 =&-((7(-2413125h^{12} + 28114944a_j^6x_j^5\tau_{max}^6(79978424821x_j + 4708995478a_j\tau_{max}) \\
&+ 38610h^{10}(4515x_j^2 + 3550410a_jx_j\tau_{max} - 25259534a_j^2\tau_{max}^2) \\
&- 135168a_j^4h^2x_j^3\tau_{max}^4(-2499126595875x_j^3 + 617613778314a_jx_j^2\tau_{max} \\
&+ 3305684925530a_j^2x_j\tau_{max}^2 + 118237887960a_j^3\tau_{max}^3) - 4752h^8(375375x_j^4 \\
&+ 574746900a_jx_j^3\tau_{max} - 29912937366a_j^2x_j^2\tau_{max}^2 + 11830086972a_j^3x_j\tau_{max}^3 \\
&+ 3794229608a_j^4\tau_{max}^4) + 1536a_j^2h^4x_j\tau_{max}^2(7254223097121x_j^5 - 6325664106762a_jx_j^4\tau_{max} \\
&- 46462422721572a_j^2x_j^3\tau_{max}^2 + 10797641583696a_j^3x_j^2\tau_{max}^3 + 11967447837400a_j^4x_j\tau_{max}^4 \\
&+ 178453592720a_j^5\tau_{max}^5) + 16h^6(289864575x_j^6 + 640779088950a_jx_j^5\tau_{max} \\
&- 165237103141110a_j^2x_j^4\tau_{max}^2 + 113588771403096a_j^3x_j^3\tau_{max}^3 + 204099361046856a_j^4x_j^2\tau_{max}^4 \\
&- 33787981576080a_j^5x_j\tau_{max}^5 - 5178530366512a_j^6\tau_{max}^6))) \\
&/(1716(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2 \\
&+ 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6))) \\
\alpha_1 =&(-675675h^{12}(375x_j + 149849a_j\tau_{max}) + 140574720a_j^6x_j^6\tau_{max}^6(239935274463x_j \\
&+ 16481484173a_j\tau_{max}) + 4158h^{10}(1467375x_j^3 + 1730824875a_jx_j^2\tau_{max}
\end{aligned}
$$

$$
\begin{aligned}
&- 24628045650a_j^2x_j\tau_{max}^2 + 5177901778a_j^3\tau_{max}^3) - 14192640a_j^4h^2x_j^4\tau_{max}^4(-357018085125x_j^3\\
&+ 102935629719a_jx_j^2\tau_{max} + 661136985106a_j^2x_j\tau_{max}^2 + 29559471990a_j^3\tau_{max}^3)\\
&- 720h^8(52026975x_j^5 + 99574900425a_jx_j^4\tau_{max} - 6909888531546a_j^2x_j^3\tau_{max}^2\\
&+ 4099125135798a_j^3x_j^2\tau_{max}^3 + 2629401118344a_j^4x_j\tau_{max}^4 - 303683537672a_j^5\tau_{max}^5)\\
&+ 10752a_j^2h^4x_j^2\tau_{max}^2(15544763779545x_j^5 - 15814160266905a_jx_j^4\tau_{max}\\
&- 139387268164716a_j^2x_j^3\tau_{max}^2 + 40491155938860a_j^3x_j^2\tau_{max}^3 + 59837239187000a_j^4x_j\tau_{max}^4\\
&+ 1338401945400a_j^5\tau_{max}^5) + 560h^6(124227675x_j^7 + 320389544475a_jx_j^6\tau_{max}\\
&- 99142261884666a_j^2x_j^5\tau_{max}^2 + 85191578552322a_j^3x_j^4\tau_{max}^3 + 204099361046856a_j^4x_j^3\tau_{max}^4\\
&- 50681972364120a_j^5x_j^2\tau_{max}^5 - 15535591099536a_j^6x_j\tau_{max}^6 - 74240090992a_j^7\tau_{max}^7))\\
&/(12870(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2\\
&+ 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6))
\end{aligned}
$$

$$
\begin{aligned}
\alpha_0 = &- ((211486275h^14 + 2249195520a_j^6x_j^7\tau_{max}^6(239935274463x_j + 18835981912a_j\tau_{max})\\
&- 1729728h^{12}(9375x_j^2 + 7492450a_jx_j\tau_{max} - 22675009a_j^2\tau_{max}^2)\\
&- 10813440a_j^4h^2x_j^5\tau_{max}^4(-7497379787625x_j^3 + 2470455113256a_jx_j^2\tau_{max}\\
&+ 18511835582968a_j^2x_j\tau_{max}^2 + 993198258864a_j^3\tau_{max}^3) + 6336h^{10}(30814875x_j^4\\
&+ 48463096500a_jx_j^3\tau_{max} - 1034377917300a_j^2x_j^2\tau_{max}^2 + 434943749352a_j^3x_j\tau_{max}^3\\
&+ 87945430928a_j^4\tau_{max}^4) + 172032a_j^2h^4x_j^3\tau_{max}^2(15544763779545x_j^5\\
&- 18073326019320a_jx_j^4\tau_{max} - 185849690886288a_j^2x_j^3\tau_{max}^2 + 64785849502176a_j^3x_j^2\tau_{max}^3\\
&+ 119674478374000a_j^4x_j\tau_{max}^4 + 3569071854400a_j^5\tau_{max}^5) - 5120h^8(156080925x_j^6\\
&+ 358469641530a_jx_j^5\tau_{max} - 31094498391957a_j^2x_j^4\tau_{max}^2 + 24594750814788a_j^3x_j^3\tau_{max}^3\\
&+ 23664610065096a_j^4x_j^2\tau_{max}^4 - 5466303678096a_j^5x_j\tau_{max}^5 - 321570260696a_j^6\tau_{max}^6)\\
&+ 1792h^6x_j(621138375x_j^7 + 1830797397000a_jx_j^6\tau_{max} - 660948412564440a_j^2x_j^5\tau_{max}^2\\
&+ 681532628418576a_j^3x_j^4\tau_{max}^3 + 2040993610468560a_j^4x_j^3\tau_{max}^4 - 675759631521600a_j^5x_j^2\tau_{max}^5\\
&- 310711821990720a_j^6x_j\tau_{max}^6 - 2969603639680a_j^7\tau_{max}^7))
\end{aligned}
$$

$$/(1647360(675675h^6 + 1623322651104a_j^2 h^4\tau_{max}^2$$

$$+ 49213569888000a_j^4 h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6)))$$

$$\beta_7 =(8(4096a_{j+\frac{1}{2}}^6\tau_{max}^6(2354497739a_{j+\frac{1}{2}}\tau_{max} + 239935274463x_{j+\frac{1}{2}})$$

$$- 6912a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4(879791707a_{j+\frac{1}{2}}\tau_{max} - 21360056375x_{j+\frac{1}{2}})$$

$$- 54432a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2(13002767a_{j+\frac{1}{2}}\tau_{max} - 89468841x_{j+\frac{1}{2}})$$

$$+ 51975h^6(14369a_{j+\frac{1}{2}}\tau_{max} + 39x_{j+\frac{1}{2}})))$$

$$/(3(327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4$$

$$+ 1623322651104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 675675h^6))$$

$$\beta_6 =(28(-20480a_{j+\frac{1}{2}}^6\tau_{max}^6 x_{j+\frac{1}{2}}(4708995478a_{j+\frac{1}{2}}\tau_{max} + 239935274463x_{j+\frac{1}{2}})$$

$$- 945h^6(-407585262a_{j+\frac{1}{2}}^2\tau_{max}^2 + 7902950a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 10725x_{j+\frac{1}{2}}^2)$$

$$+ 288a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2(36101338556a_{j+\frac{1}{2}}^2\tau_{max}^2 + 24575229630a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 84548054745x_{j+\frac{1}{2}}^2)$$

$$+ 1280a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4(50856691162a_{j+\frac{1}{2}}^2\tau_{max}^2 + 47508752178a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}$$

$$- 576721522125x_{j+\frac{1}{2}}^2) + 259875h^8))$$

$$/(15(327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4$$

$$+ 1623322651104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 675675h^6))$$

$$\beta_5 =-(56(-266240a_{j+\frac{1}{2}}^6\tau_{max}^6 x_{j+\frac{1}{2}}^2(2354497739a_{j+\frac{1}{2}}\tau_{max} + 79978424821x_{j+\frac{1}{2}})$$

$$- 9h^6(95613443942a_{j+\frac{1}{2}}^3\tau_{max}^3 - 556353882630a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}} + 5393763375a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2$$

$$+ 4879875x_{j+\frac{1}{2}}^3) - 288a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2(27266771676a_{j+\frac{1}{2}}^3\tau_{max}^3 - 469317401228a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}}$$

$$- 159738992595a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 366374903895x_{j+\frac{1}{2}}^3) + 1280a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4(5911894398a_{j+\frac{1}{2}}^3\tau_{max}^3$$

$$+ 661136985106a_{j+\frac{1}{2}}^2\tau_{max}^2 x_{j+\frac{1}{2}} + 308806889157a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 2499126595875x_{j+\frac{1}{2}}^3)$$

$$+ 61425h^8(21053a_{j+\frac{1}{2}}\tau_{max} + 55x_{j+\frac{1}{2}})))$$

$$/(65(327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4$$

$$+ 1623322651104a_{j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 675675h^6))$$

$$\beta_4 =-((7(2641275h^{10} + 2129920a_{j+\frac{1}{2}}^6 x_{j+\frac{1}{2}}^3\tau_{max}^6(239935274463x_{j+\frac{1}{2}} + 9417990956a_{j+\frac{1}{2}}\tau_{max})$$

$$
\begin{aligned}
&- 5616h^8(28875x_{j+\frac{1}{2}}^2 + 22105650a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\tau_{max} - 383499197a_{j+\frac{1}{2}}^2\tau_{max}^2) \\
&- 30720a_{j+\frac{1}{2}}^4h^2x_{j+\frac{1}{2}}\tau_{max}^4(-2499126595875x_{j+\frac{1}{2}}^3 + 411742518876a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2\tau_{max} \\
&+ 1322273970212a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2 + 23647577592a_{j+\frac{1}{2}}^3\tau_{max}^3) \\
&+ 72h^6(14639625x_{j+\frac{1}{2}}^4 + 21575053500a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^3\tau_{max} - 3338123295780a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^2\tau_{max}^2 \\
&+ 1147361327304a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}\tau_{max}^3 + 687203235848a_{j+\frac{1}{2}}^4\tau_{max}^4) \\
&+ 256a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(9892122405165x_{j+\frac{1}{2}}^4 - 5750603733420a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^3\tau_{max} \\
&- 25343139666312a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^2\tau_{max}^2 + 2944811341008a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}\tau_{max}^3 \\
&+ 1087949803400a_{j+\frac{1}{2}}^4\tau_{max}^4))) \\
&/(156(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
&+ 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6)))
\end{aligned}
$$

$$
\begin{aligned}
\beta_3 =&(7(96525h^{10}(301x_{j+\frac{1}{2}} + 118347a_{j+\frac{1}{2}}\tau_{max}) + 4685824a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}^4\tau_{max}^6(239935274463x_{j+\frac{1}{2}} \\
&+ 11772488695a_{j+\frac{1}{2}}\tau_{max}) - 4752h^8(125125x_{j+\frac{1}{2}}^3 + 143686725a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2\tau_{max} \\
&- 4985489561a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2 + 985840581a_{j+\frac{1}{2}}^3\tau_{max}^3) \\
&- 112640a_{j+\frac{1}{2}}^4h^2x_{j+\frac{1}{2}}^2\tau_{max}^4(-1499475957525x_{j+\frac{1}{2}}^3 \\
&+ 308806889157a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2\tau_{max} + 1322273970212a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2 + 35471366388a_{j+\frac{1}{2}}^3\tau_{max}^3) \\
&+ 72h^6(32207175x_{j+\frac{1}{2}}^5 + 59331397125a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4\tau_{max} - 12239785417860a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^3\tau_{max}^2 \\
&+ 6310487300172a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^2\tau_{max}^3 + 7559235594328a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}\tau_{max}^4 - 625703362520a_{j+\frac{1}{2}}^5\tau_{max}^5) \\
&+ 256a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(21762669291363x_{j+\frac{1}{2}}^5 - 15814160266905a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4\tau_{max} \\
&- 92924845443144a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^3\tau_{max}^2 \\
&+ 16196462375544a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^2\tau_{max}^3 + 11967447837400a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}\tau_{max}^4 + 89226796360a_{j+\frac{1}{2}}^5\tau_{max}^5))) \\
&/(429(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
&+ 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6))
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 =& - ((7(-2413125h^{12} + 28114944a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}^5\tau_{max}^6(79978424821x_{j+\frac{1}{2}} + 4708995478a_{j+\frac{1}{2}}\tau_{max}) \\
&+ 38610h^{10}(4515x_{j+\frac{1}{2}}^2 + 3550410a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\tau_{max} - 25259534a_{j+\frac{1}{2}}^2\tau_{max}^2)
\end{aligned}
$$

$$
\begin{aligned}
&- 135168a_{j+\frac{1}{2}}^4 h^2 x_{j+\frac{1}{2}}^3 \tau_{max}^4(-2499126595875x_{j+\frac{1}{2}}^3 + 617613778314a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^2 \tau_{max} \\
&+ 3305684925530a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}} \tau_{max}^2 + 118237887960a_{j+\frac{1}{2}}^3 \tau_{max}^3) - 4752h^8(375375x_{j+\frac{1}{2}}^4 \\
&+ 574746900a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^3 \tau_{max} - 29912937366a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^2 \tau_{max}^2 + 11830086972a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}} \tau_{max}^3 \\
&+ 3794229608a_{j+\frac{1}{2}}^4 \tau_{max}^4) + 1536a_{j+\frac{1}{2}}^2 h^4 x_{j+\frac{1}{2}} \tau_{max}^2(7254223097121x_{j+\frac{1}{2}}^5 \\
&- 6325664106762a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^4 \tau_{max} \\
&- 46462422721572a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^3 \tau_{max}^2 + 10797641583696a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^2 \tau_{max}^3 \\
&+ 11967447837400a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}} \tau_{max}^4 \\
&+ 178453592720a_{j+\frac{1}{2}}^5 \tau_{max}^5) + 16h^6(289864575x_{j+\frac{1}{2}}^6 + 640779088950a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^5 \tau_{max} \\
&- 165237103141110a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^4 \tau_{max}^2 + 113588771403096a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^3 \tau_{max}^3 \\
&+ 204099361046856a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}}^2 \tau_{max}^4 \\
&- 33787981576080a_{j+\frac{1}{2}}^5 x_{j+\frac{1}{2}} \tau_{max}^5 - 5178530366512a_{j+\frac{1}{2}}^6 \tau_{max}^6))) \\
&/(1716(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 \\
&+ 49213569888000a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6 \tau_{max}^6)))
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 =&(-675675h^{12}(375x_{j+\frac{1}{2}} + 149849a_{j+\frac{1}{2}} \tau_{max}) + 140574720a_{j+\frac{1}{2}}^6 x_{j+\frac{1}{2}}^6 \tau_{max}^6(239935274463x_{j+\frac{1}{2}} \\
&+ 16481484173a_{j+\frac{1}{2}} \tau_{max}) + 4158h^{10}(1467375x_{j+\frac{1}{2}}^3 + 1730824875a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^2 \tau_{max} \\
&- 24628045650a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}} \tau_{max}^2 + 5177901778a_{j+\frac{1}{2}}^3 \tau_{max}^3) \\
&- 14192640a_{j+\frac{1}{2}}^4 h^2 x_{j+\frac{1}{2}}^4 \tau_{max}^4(-357018085125x_{j+\frac{1}{2}}^3 \\
&+ 102935629719a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^2 \tau_{max} + 661136985106a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}} \tau_{max}^2 + 29559471990a_{j+\frac{1}{2}}^3 \tau_{max}^3) \\
&- 720h^8(52026975x_{j+\frac{1}{2}}^5 + 99574900425a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^4 \tau_{max} - 6909888531546a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^3 \tau_{max}^2 \\
&+ 4099125135798a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^2 \tau_{max}^3 + 2629401118344a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}} \tau_{max}^4 - 303683537672a_{j+\frac{1}{2}}^5 \tau_{max}^5) \\
&+ 10752a_{j+\frac{1}{2}}^2 h^4 x_{j+\frac{1}{2}}^2 \tau_{max}^2(15544763779545x_{j+\frac{1}{2}}^5 - 15814160266905a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^4 \tau_{max} \\
&- 139387268164716a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^3 \tau_{max}^2 + 40491155938860a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^2 \tau_{max}^3 \\
&+ 59837239187000a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}} \tau_{max}^4 \\
&+ 1338401945400a_{j+\frac{1}{2}}^5 \tau_{max}^5) + 560h^6(124227675x_{j+\frac{1}{2}}^7 + 320389544475a_{j+\frac{1}{2}} x_{j+\frac{1}{2}}^6 \tau_{max}
\end{aligned}
$$

$$
\begin{aligned}
&- 99142261884666a_{j+\frac{1}{2}}^{2}x_{j+\frac{1}{2}}^{5}\tau_{max}^{2} + 85191578552322a_{j+\frac{1}{2}}^{3}x_{j+\frac{1}{2}}^{4}\tau_{max}^{3}\\
&+ 204099361046856a_{j+\frac{1}{2}}^{4}x_{j+\frac{1}{2}}^{3}\tau_{max}^{4}\\
&- 50681972364120a_{j+\frac{1}{2}}^{5}x_{j+\frac{1}{2}}^{2}\tau_{max}^{5} - 15535591099536a_{j+\frac{1}{2}}^{6}x_{j+\frac{1}{2}}\tau_{max}^{6} - 74240090992a_{j+\frac{1}{2}}^{7}\tau_{max}^{7}))\\
&/(12870(675675h^{6} + 1623322651104a_{j+\frac{1}{2}}^{2}h^{4}\tau_{max}^{2}\\
&+ 49213569888000a_{j+\frac{1}{2}}^{4}h^{2}\tau_{max}^{4} + 327591628066816a_{j+\frac{1}{2}}^{6}\tau_{max}^{6}))
\end{aligned}
$$

$$
\begin{aligned}
\beta_0 =&- ((211486275h^{1}4 + 2249195520a_{j+\frac{1}{2}}^{6}x_{j+\frac{1}{2}}^{7}\tau_{max}^{6}(239935274463x_{j+\frac{1}{2}} + 18835981912a_{j+\frac{1}{2}}\tau_{max})\\
&- 1729728h^{12}(9375x_{j+\frac{1}{2}}^{2} + 7492450a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\tau_{max} - 22675009a_{j+\frac{1}{2}}^{2}\tau_{max}^{2})\\
&- 10813440a_{j+\frac{1}{2}}^{4}h^{2}x_{j+\frac{1}{2}}^{5}\tau_{max}^{4}(-7497379787625x_{j+\frac{1}{2}}^{3} + 2470455113256a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^{2}\tau_{max}\\
&+ 18511835582968a_{j+\frac{1}{2}}^{2}x_{j+\frac{1}{2}}\tau_{max}^{2} + 993198258864a_{j+\frac{1}{2}}^{3}\tau_{max}^{3}) + 6336h^{10}(30814875x_{j+\frac{1}{2}}^{4}\\
&+ 48463096500a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^{3}\tau_{max} - 1034377917300a_{j+\frac{1}{2}}^{2}x_{j+\frac{1}{2}}^{2}\tau_{max}^{2} + 434943749352a_{j+\frac{1}{2}}^{3}x_{j+\frac{1}{2}}\tau_{max}^{3}\\
&+ 87945430928a_{j+\frac{1}{2}}^{4}\tau_{max}^{4}) + 172032a_{j+\frac{1}{2}}^{2}h^{4}x_{j+\frac{1}{2}}^{3}\tau_{max}^{2}(15544763779545x_{j+\frac{1}{2}}^{5}\\
&- 18073326019320a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^{4}\tau_{max} - 185849690886288a_{j+\frac{1}{2}}^{2}x_{j+\frac{1}{2}}^{3}\tau_{max}^{2}\\
&+ 64785849502176a_{j+\frac{1}{2}}^{3}x_{j+\frac{1}{2}}^{2}\tau_{max}^{3}\\
&+ 119674478374000a_{j+\frac{1}{2}}^{4}x_{j+\frac{1}{2}}\tau_{max}^{4} + 3569071854400a_{j+\frac{1}{2}}^{5}\tau_{max}^{5})\\
&- 5120h^{8}(156080925x_{j+\frac{1}{2}}^{6}\\
&+ 358469641530a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^{5}\tau_{max} - 31094498391957a_{j+\frac{1}{2}}^{2}x_{j+\frac{1}{2}}^{4}\tau_{max}^{2}\\
&+ 24594750814788a_{j+\frac{1}{2}}^{3}x_{j+\frac{1}{2}}^{3}\tau_{max}^{3}\\
&+ 23664610065096a_{j+\frac{1}{2}}^{4}x_{j+\frac{1}{2}}^{2}\tau_{max}^{4} - 5466303678096a_{j+\frac{1}{2}}^{5}x_{j+\frac{1}{2}}\tau_{max}^{5} - 321570260696a_{j+\frac{1}{2}}^{6}\tau_{max}^{6})\\
&+ 1792h^{6}x_{j+\frac{1}{2}}(621138375x_{j+\frac{1}{2}}^{7} + 1830797397000a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^{6}\tau_{max}\\
&- 660948412564440a_{j+\frac{1}{2}}^{2}x_{j+\frac{1}{2}}^{5}\tau_{max}^{2}\\
&+ 681532628418576a_{j+\frac{1}{2}}^{3}x_{j+\frac{1}{2}}^{4}\tau_{max}^{3} + 2040993610468560a_{j+\frac{1}{2}}^{4}x_{j+\frac{1}{2}}^{3}\tau_{max}^{4}\\
&- 675759631521600a_{j+\frac{1}{2}}^{5}x_{j+\frac{1}{2}}^{2}\tau_{max}^{5}\\
&- 310711821990720a_{j+\frac{1}{2}}^{6}x_{j+\frac{1}{2}}\tau_{max}^{6} - 2969603639680a_{j+\frac{1}{2}}^{7}\tau_{max}^{7}))\\
&/(1647360(675675h^{6} + 1623322651104a_{j+\frac{1}{2}}^{2}h^{4}\tau_{max}^{2}
\end{aligned}
$$

$$+ 49213569888000a_{j+\frac{1}{2}}^4 h^2\tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6)))$$

Hence, we have

$$\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^8 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^8)^2$$

$$=((a_j - a_{j+\frac{1}{2}})^2 h^{15}\tau_{max}^2(64076116972171871270411246601562 50h^{24}$$

$$+ 1496959745748072653734491363152343 75(a_j + a_{j+\frac{1}{2}})\tau h^{23}$$

$$+ 141932907121005000000(1021984055960323244 06a_j^2$$

$$+ 102198405596032324406a_{j+\frac{1}{2}}^2)\tau_{max}^2 h^{22} - 5125354979369625000(3424750007914959768 13a_j^3$$

$$+ 7085426149722570197 3887a_{j+\frac{1}{2}}a_j^2 + 70854261497225701973887a_{j+\frac{1}{2}}^2 a_j$$

$$+ 342475000791495976813a_{j+\frac{1}{2}}^3)\tau_{max}^3 h^{21} + 7646217218640000(116529188781377048742572 61a_j^4$$

$$- 35682589886546522278068 4354a_{j+\frac{1}{2}}a_j^3 + 4836984475588643523349889892863a_{j+\frac{1}{2}}^2 a_j^2$$

$$- 356825898865465222780684354a_{j+\frac{1}{2}}^3 a_j + 11652918878137704874257261a_{j+\frac{1}{2}}^4)\tau_{max}^4 h^{20}$$

$$- 84108389405040000(915648732630103373510011a_j^5 - 940788992670330964903312 7a_{j+\frac{1}{2}}a_j^4$$

$$+ 50390082001645985867302150063a_{j+\frac{1}{2}}^2 a_j^3 + 50390082001645985867302150063a_{j+\frac{1}{2}}^3 a_j^2$$

$$- 9407889926703309649033127a_{j+\frac{1}{2}}^4 a_j + 915648732630103373510011a_{j+\frac{1}{2}}^5)\tau_{max}^5 h^{19}$$

$$+ 11950131916800(16488418522200533380531 9210300a_j^6$$

$$- 66303427066165015934218 08756675a_{j+\frac{1}{2}}a_j^5$$

$$+ 1880762399511109233889594020533446 98a_{j+\frac{1}{2}}^2 a_j^4$$

$$- 2085751500818496796485771477245214 36a_{j+\frac{1}{2}}^3 a_j^3$$

$$+ 1880762399511109233889594020533446 98a_{j+\frac{1}{2}}^4 a_j^2$$

$$- 66303427066165015934218 08756675a_{j+\frac{1}{2}}^5 a_j$$

$$+ 164884185222005333805319210300a_{j+\frac{1}{2}}^6)\tau_{max}^6 h^{18}$$

$$- 116513786188800(9030355544280902800247270875a_j^7$$

$$- 373970812972841082020185538 1800a_{j+\frac{1}{2}}a_j^6$$

$$+ 16396719622720606444403866151826 08a_{j+\frac{1}{2}}^2 a_j^5$$

$$- 46810843593735883988285511555310999 a_{j+\frac{1}{2}}^3 a_j^4$$

$$- 46810843593735883988285511555310999 a_{j+\frac{1}{2}}^4 a_j^3$$

$$+ 16396719622720606444403866151826 08 a_{j+\frac{1}{2}}^5 a_j^2$$

$$- 3739708129728410820201855381800 a_{j+\frac{1}{2}}^6 a_j$$

$$+ 903035554428090280024727087 5 a_{j+\frac{1}{2}}^7) \tau_{max}^7 h^{17}$$

$$+ 1042920603648 (1769322984435893253982620487625 0 a_j^8$$

$$- 85101692632460906138022351923000 0 a_{j+\frac{1}{2}} a_j^7$$

$$+ 4712534454501658515695052210939758305 0 a_{j+\frac{1}{2}}^2 a_j^6$$

$$- 9752961928277414134671935082379313292 5 a_{j+\frac{1}{2}}^3 a_j^5$$

$$+ 52558575055726706146827465274022762601 22 a_{j+\frac{1}{2}}^4 a_j^4$$

$$- 9752961928277414134671935082379313292 5 a_{j+\frac{1}{2}}^5 a_j^3$$

$$+ 4712534454501658515695052210939758305 0 a_{j+\frac{1}{2}}^6 a_j^2$$

$$- 85101692632460906138022351923000 0 a_{j+\frac{1}{2}}^7 a_j$$

$$+ 1769322984435893253982620487625 0 a_{j+\frac{1}{2}}^8) \tau_{max}^8 h^{16}$$

$$- 49712548773888 (1010407720237332326172862056 25 a_j^9$$

$$- 14010521258252280584316069314375 0 a_{j+\frac{1}{2}} a_j^8$$

$$+ 51582530719180029823261911267645250 a_{j+\frac{1}{2}}^2 a_j^7$$

$$- 441636496570132835224680495743578429 25 a_{j+\frac{1}{2}}^3 a_j^6$$

$$+ 2447086399967470767042438160520728442 57 a_{j+\frac{1}{2}}^4 a_j^5$$

$$+ 2447086399967470767042438160520728442 57 a_{j+\frac{1}{2}}^5 a_j^4$$

$$- 441636496570132835224680495743578429 25 a_{j+\frac{1}{2}}^6 a_j^3$$

$$+ 51582530719180029823261911267645250 a_{j+\frac{1}{2}}^7 a_j^2$$

$$- 14010521258252280584316069314375 0 a_{j+\frac{1}{2}}^8 a_j$$

$$+ 1010407720237332326172862056 25 a_{j+\frac{1}{2}}^9) \tau_{max}^9 h^{15}$$

$$+ 1854081073152(335843312915646043010818586700000a_j^{10}$$

$$- 19061801949153125992729075756937 50a_{j+\frac{1}{2}}a_j^9$$

$$+ 2457641393125471768600595379003016598 00a_{j+\frac{1}{2}}^2a_j^8$$

$$- 70766394279754532914465185963189551 2600a_{j+\frac{1}{2}}^3a_j^7$$

$$+ 134163063816345518949074583247554347787928a_{j+\frac{1}{2}}^4a_j^6$$

$$+ 263786013929245432326130453052567724354 07a_{j+\frac{1}{2}}^5a_j^5$$

$$+ 134163063816345518949074583247554347787928a_{j+\frac{1}{2}}^6a_j^4$$

$$- 70766394279754532914465185963189551 2600a_{j+\frac{1}{2}}^7a_j^3$$

$$+ 2457641393125471768600595379003016598 00a_{j+\frac{1}{2}}^8a_j^2$$

$$- 19061801949153125992729075756937 50a_{j+\frac{1}{2}}^9a_j$$

$$+ 335843312915646043010818586700000a_{j+\frac{1}{2}}^{10})\tau_{max}^{10}h^{14}$$

$$- 927040536576(4806491069858999270093254178125a_j^{11}$$

$$- 32905119135827149176686917250697500a_{j+\frac{1}{2}}a_j^{10}$$

$$+ 1194725551322270110832843511753630520 0a_{j+\frac{1}{2}}^2a_j^9$$

$$- 2951637273449536863450817484274942845 3225a_{j+\frac{1}{2}}^3a_j^8$$

$$+ 716471911435411288350986492547285708111 55a_{j+\frac{1}{2}}^4a_j^7$$

$$- 515027236544736694707493486699984121084966a_{j+\frac{1}{2}}^5a_j^6$$

$$- 515027236544736694707493486699984121084966a_{j+\frac{1}{2}}^6a_j^5$$

$$+ 716471911435411288350986492547285708111 55a_{j+\frac{1}{2}}^7a_j^4$$

$$- 2951637273449536863450817484274942845 3225a_{j+\frac{1}{2}}^8a_j^3$$

$$+ 1194725551322270110832843511753630520 0a_{j+\frac{1}{2}}^9a_j^2$$

$$- 32905119135827149176686917250697500a_{j+\frac{1}{2}}^{10}a_j$$

$$+ 4806491069858999270093254178125a_{j+\frac{1}{2}}^{11})\tau_{max}^{11}h^{13}$$

$$+ 117719433216(1541604544580000163965512122 8125a_j^{12}$$

$$- 11248959872916054394746123802068750 a_{j+\frac{1}{2}} a_j^{11}$$

$$+ 1291629292868388271317120016409027487277 5 a_{j+\frac{1}{2}}^2 a_j^{10}$$

$$- 4892988345819519147167133830738518226400 0 a_{j+\frac{1}{2}}^3 a_j^9$$

$$+ 2909007446697316403571950192573791017510984 9 a_{j+\frac{1}{2}}^4 a_j^8$$

$$+ 8725725200094031043177646404131246876282890 a_{j+\frac{1}{2}}^5 a_j^7$$

$$+ 8700591092947165746393767886435344277370563 1 a_{j+\frac{1}{2}}^6 a_j^6$$

$$+ 8725725200094031043177646404131246876282890 a_{j+\frac{1}{2}}^7 a_j^5$$

$$+ 2909007446697316403571950192573791017510984 9 a_{j+\frac{1}{2}}^8 a_j^4$$

$$- 4892988345819519147167133830738518226400 0 a_{j+\frac{1}{2}}^9 a_j^3$$

$$+ 1291629292868388271317120016409027487277 5 a_{j+\frac{1}{2}}^{10} a_j^2$$

$$- 11248959872916054394746123802068750 a_{j+\frac{1}{2}}^{11} a_j$$

$$+ 1541604544580000163965512122812 5 a_{j+\frac{1}{2}}^{12}) \tau_{max}^{12} h^{12}$$

$$- 706316599296 a_j^2 a_{j+\frac{1}{2}}^2 (4265942923575175849094670782573247875 a_j^9$$

$$- 15386261374490700354593001927791158697972 5 a_{j+\frac{1}{2}} a_j^8$$

$$- 3474807299101645926175246054336032343331902 a_{j+\frac{1}{2}}^2 a_j^7$$

$$- 1837235194635440438056432529288108697262943 8 a_{j+\frac{1}{2}}^3 a_j^6$$

$$- 2322522097230268884310949308306206435660818 3 a_{j+\frac{1}{2}}^4 a_j^5$$

$$- 2322522097230268884310949308306206435660818 3 a_{j+\frac{1}{2}}^5 a_j^4$$

$$- 1837235194635440438056432529288108697262943 8 a_{j+\frac{1}{2}}^6 a_j^3$$

$$- 3474807299101645926175246054336032343331902 a_{j+\frac{1}{2}}^7 a_j^2$$

$$- 15386261374490700354593001927791158697972 5 a_{j+\frac{1}{2}}^8 a_j$$

$$+ 4265942923575175849094670782573247875 a_{j+\frac{1}{2}}^9) \tau_{max}^{13} h^{11}$$

$$+ 2054739197952 a_j^2 a_{j+\frac{1}{2}}^2 (42438637502740137369170564184456626 25 a_j^{10}$$

$$- 154770398349271645497495693523684339462 5 a_{j+\frac{1}{2}} a_j^9$$

$$+ 7727279661360167998007807567476980062903913a_{j+\frac{1}{2}}^{2} a_j^{8}$$

$$+ 2721712884336354977154964974396418615333980a_{j+\frac{1}{2}}^{3} a_j^{7}$$

$$+ 66282203069516127842005808949034716669693715a_{j+\frac{1}{2}}^{4} a_j^{6}$$

$$+ 10699144073071358121723076542916022358448199a_{j+\frac{1}{2}}^{5} a_j^{5}$$

$$+ 66282203069516127842005808949034716669693715a_{j+\frac{1}{2}}^{6} a_j^{4}$$

$$+ 2721712884336354977154964974396418615333980a_{j+\frac{1}{2}}^{7} a_j^{3}$$

$$+ 7727279661360167998007807567476980062903913a_{j+\frac{1}{2}}^{8} a_j^{2}$$

$$- 1547703983492716454974956935236843394625a_{j+\frac{1}{2}}^{9} a_j$$

$$+ 4243863750274013736917056418445662625a_{j+\frac{1}{2}}^{10})\tau_{max}^{14} h^{10}$$

$$+ 15068087451648a_j^{4} a_{j+\frac{1}{2}}^{4}(1226440005433813283100595708201779028363970a_j^{7}$$

$$+ 4637165397258385176176955537957113705171438a_{j+\frac{1}{2}} a_j^{6}$$

$$+ 12794089326376582464931337782336108624614973a_{j+\frac{1}{2}}^{2} a_j^{5}$$

$$+ 22775067494022410799181373159209801943311225a_{j+\frac{1}{2}}^{3} a_j^{4}$$

$$+ 22775067494022410799181373159209801943311225a_{j+\frac{1}{2}}^{4} a_j^{3}$$

$$+ 12794089326376582464931337782336108624614973a_{j+\frac{1}{2}}^{5} a_j^{2}$$

$$+ 4637165397258385176176955537957113705171438a_{j+\frac{1}{2}}^{6} a_j$$

$$+ 1226440005433813283100595708201779028363970a_{j+\frac{1}{2}}^{7})\tau_{max}^{15} h^{9}$$

$$+ 1826434842624a_j^{4} a_{j+\frac{1}{2}}^{4}(5735376305254471497397007185633322552332115a_j^{8}$$

$$+ 1709501352109512008062597478578959919914800a_{j+\frac{1}{2}} a_j^{7}$$

$$+ 355614542756479161277327105199371564782709331a_{j+\frac{1}{2}}^{2} a_j^{6}$$

$$+ 70479522127743247319870522725960510761860785a_{j+\frac{1}{2}}^{3} a_j^{5}$$

$$+ 1002829664307157485423082511164031759353903459a_{j+\frac{1}{2}}^{4} a_j^{4}$$

$$+ 70479522127743247319870522725960510761860785a_{j+\frac{1}{2}}^{5} a_j^{3}$$

$$+ 355614542756479161277327105199371564782709331a_{j+\frac{1}{2}}^{6} a_j^{2}$$

$$+ 170950135210951200806259747857895991991480 0a_{j+\frac{1}{2}}^{7}a_j$$

$$+ 5735376305254471497397007185633322552332115a_{j+\frac{1}{2}}^{8})\tau_{max}^{16}h^8$$

$$+ 3652869685248a_j^6a_{j+\frac{1}{2}}^{6}(2033845656576368611194235637475310843001638 75a_j^5$$

$$+ 4723009039093474130556308629583955525137509 87a_{j+\frac{1}{2}}a_j^4$$

$$+ 8454390596984357082793239001725835719714028 15a_{j+\frac{1}{2}}^{2}a_j^3$$

$$+ 8454390596984357082793239001725835719714028 15a_{j+\frac{1}{2}}^{3}a_j^2$$

$$+ 4723009039093474130556308629583955525137509 87a_{j+\frac{1}{2}}^{4}a_j$$

$$+ 2033845656576368611194235637475310843001638 75a_{j+\frac{1}{2}}^{5})\tau_{max}^{17}h^7$$

$$+ 463856467968a_j^6a_{j+\frac{1}{2}}^{6}(1369249982086609549862770039779934986011183875a_j^6$$

$$+ 2781397990077232182020029214629025859759634 95a_{j+\frac{1}{2}}a_j^5$$

$$+ 2048598339354440126566869311058136785775111 0947a_{j+\frac{1}{2}}^{2}a_j^4$$

$$+ 1906917249724283981846234551210738981775181 093a_{j+\frac{1}{2}}^{3}a_j^3$$

$$+ 2048598339354440126566869311058136785775111 0947a_{j+\frac{1}{2}}^{4}a_j^2$$

$$+ 2781397990077232182020029214629025859759634 95a_{j+\frac{1}{2}}^{5}a_j$$

$$+ 1369249982086609549862770039779934986011183875a_{j+\frac{1}{2}}^{6})\tau_{max}^{18}h^6$$

$$+ 5566277615616a_j^8a_{j+\frac{1}{2}}^{8}(1664951346635098223961807292973869362140155765a_j^3$$

$$+ 2526101350040567759312558807452293135318345361a_{j+\frac{1}{2}}a_j^2$$

$$+ 2526101350040567759312558807452293135318345361a_{j+\frac{1}{2}}^{2}a_j$$

$$+ 1664951346635098223961807292973869362140155765a_{j+\frac{1}{2}}^{3})\tau_{max}^{19}h^5$$

$$+ 11132555231232a_j^8a_{j+\frac{1}{2}}^{8}(1244577120142977931757886187009203449137132565a_j^4$$

$$+ 1457031354732656033672639260047506549650091 50a_{j+\frac{1}{2}}a_j^3$$

$$+ 5277782995676146054115129935872458903443701703a_{j+\frac{1}{2}}^{2}a_j^2$$

$$+ 1457031354732656033672639260047506549650091 50a_{j+\frac{1}{2}}^{3}a_j$$

$$+ 1244577120142977931757886187009203449137132565a_{j+\frac{1}{2}}^{4})\tau_{max}^{20}h^4$$

$$
\begin{aligned}
&+ 3686511569102240210086141738855256645938106005395288883200 0a_j^{10}a_{j+\frac{1}{2}}^{10}(a_j + a_{j+\frac{1}{2}})\tau_{max}^{21}h^3 \\
&+ 495319724397344709239777495037562060 80a_j^{10}a_{j+\frac{1}{2}}^{10}(2587661596660313943875a_j^2 \\
&+ 1306926345645831572 33a_{j+\frac{1}{2}}a_j + 2587661596660313943875a_{j+\frac{1}{2}}^2)\tau_{max}^{22}h^2 \\
&+ 4265903554930177686569851382970618482224629899296947935641 60a_j^{12}a_{j+\frac{1}{2}}^{12}\tau_{max}^{24})) \\
&/(127209139200(675675h^6 + 1623322651104a_j^2\tau_{max}^2h^4 + 49213569888000a_j^4\tau_{max}^4h^2 \\
&+ 327591628066816a_j^6\tau^6)^2(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2\tau_{max}^2h^4 \\
&+ 49213569888000a_{j+\frac{1}{2}}^4\tau_{max}^4h^2 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6)^2) \\
=&O(h^{19})
\end{aligned}
$$

9. $k = 8, u = x^9$, by the definition

$$
\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx =& \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^9 dx \\
\tilde{P}_h(u_I; x^m; f, u)_j =& \tilde{P}_h(x^9; x^m; f, u)_j, \ m = 1, ..., 8 \\
\int_{x_j}^{x_{j+1}} v_I dx =& \int_{x_j}^{x_{j+1}} x^8 dx \\
\tilde{Q}_h(v_I; x^m; f, u)_{j+\frac{1}{2}} =& \tilde{Q}_h(x^9; x^m; f, u)_{j+\frac{1}{2}}, \ m = 1, ..., 8
\end{aligned}
\tag{1.19}
$$

then we have

$$
\begin{aligned}
u_I =& \sum_{m=0}^{8} \alpha_m x^m, \ \forall x \in I_j. \\
v_I =& \sum_{m=0}^{8} \beta_m x^m, \ \forall x \in I_{j+\frac{1}{2}}.
\end{aligned}
\tag{1.20}
$$

By similar calculation we have

$$
\begin{aligned}
&\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^9 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^9)^2 \\
=&(3(a_j - a_{j+\frac{1}{2}})^2h^{21}\tau_{max}^2(1100247162677562372226747959055445824218750h^{28} \\
&+ 362716434886413039714947456336468263183593 75(a_j + a_{j+\frac{1}{2}})\tau_{max}h^{27} \\
&+ 4759258195128937500 0(6033585373784707170719220520 91a_j^2 \\
&- 1877690942672617678425015736837 0a_{j+\frac{1}{2}}a_j \\
&+ 6033585373784707170719220520 91a_{j+\frac{1}{2}}^2)\tau_{max}^2h^{26}
\end{aligned}
$$

$$- 17799625649782226250000(392265732763604422564840797a_j^3$$

$$+ 881527885756504264593056283964a_{j+\frac{1}{2}}a_j^2$$

$$+ 881527885756504264593056283964a_{j+\frac{1}{2}}^2a_j$$

$$+ 392265732763604422564840797a_{j+\frac{1}{2}}^3)\tau_{max}^3h^{25}$$

$$+ 15620129460936000000(24768446340696859107906815323295 85a_j^4$$

$$- 1012882469025845166697212823193072 40a_{j+\frac{1}{2}}a_j^3$$

$$+ 1315806201584544410351027611800885447831 74a_{j+\frac{1}{2}}^2a_j^2$$

$$- 1012882469025845166697212823193072 40a_{j+\frac{1}{2}}^3a_j$$

$$+ 24768446340696859107906815323295 85a_{j+\frac{1}{2}}^4)\tau_{max}^4h^{24}$$

$$- 863012152716714000000(775084225556119253134715403330 5a_j^5$$

$$- 24002984726861235502615709120783935a_{j+\frac{1}{2}}a_j^4$$

$$+ 3496819670060157374076563457341675 6678a_{j+\frac{1}{2}}^2a_j^3$$

$$+ 3496819670060157374076563457341675 6678a_{j+\frac{1}{2}}^3a_j^2$$

$$- 24002984726861235502615709120783935a_{j+\frac{1}{2}}^4a_j$$

$$+ 775084225556119253134715403330 5a_{j+\frac{1}{2}}^5)\tau_{max}^5h^{23}$$

$$+ 119010510178560000(1381088270529089528757979201207871490a_j^6$$

$$- 7645786068379630752224672410545183 3700a_{j+\frac{1}{2}}a_j^5$$

$$+ 232527795648485692055919616044493067786121087a_{j+\frac{1}{2}}^2a_j^4$$

$$- 137642650758131781459680385519949473952301362a_{j+\frac{1}{2}}^3a_j^3$$

$$+ 232527795648485692055919616044493067786121087a_{j+\frac{1}{2}}^4a_j^2$$

$$- 7645786068379630752224672410545183 3700a_{j+\frac{1}{2}}^5a_j$$

$$+ 1381088270529089528757979201207871490a_{j+\frac{1}{2}}^6)\tau_{max}^6h^{22}$$

$$- 202317867303552000 0(907242316623421820384032867222132 5a_j^7$$

$$- 1090166767535681435363470110534277424 55a_{j+\frac{1}{2}}a_j^6$$

$$+ 14147134108499900142539343321753197882029\,4a_{j+\frac{1}{2}}^2 a_j^5$$

$$- 8936187066684299019975093623491015152726699\,84a_{j+\frac{1}{2}}^3 a_j^4$$

$$- 893618706668429901997509362349101515272669984a_{j+\frac{1}{2}}^4 a_j^3$$

$$+ 141471341084999001425393433217531978820294a_{j+\frac{1}{2}}^5 a_j^2$$

$$- 10901667675356814353634701105342774245\,5a_{j+\frac{1}{2}}^6 a_j$$

$$+ 9072423166234218203840328672221325a_{j+\frac{1}{2}}^7)\tau_{max}^7 h^{21}$$

$$+ 177544817049600(1299813736540544765814417418951248101750\,0a_j^8$$

$$- 113515680261686998936734184712466826093750\,0a_{j+\frac{1}{2}} a_j^7$$

$$+ 66147727801571655366927955372883169976086045788\,70a_{j+\frac{1}{2}}^2 a_j^6$$

$$- 972288855871408344486981540655877972153396629136\,0a_{j+\frac{1}{2}}^3 a_j^5$$

$$+ 2311879244110624884078502406341183990639503622335887a_{j+\frac{1}{2}}^4 a_j^4$$

$$- 9722888558714083444869815406558779721533966291360a_{j+\frac{1}{2}}^5 a_j^3$$

$$+ 6614772780157165536692795537288316997608604578870a_{j+\frac{1}{2}}^6 a_j^2$$

$$- 1135156802616869989367341847124668260937500a_{j+\frac{1}{2}}^7 a_j$$

$$+ 12998137365405447658144174189512481017500a_{j+\frac{1}{2}}^8)\tau_{max}^8 h^{20}$$

$$- 215805725123788800(888457065097635122437710169588178125a_j^9$$

$$- 139584489748797144049151304944357848893\,75a_{j+\frac{1}{2}} a_j^8$$

$$+ 349047083393646541749246528149387526803591\,50a_{j+\frac{1}{2}}^2 a_j^7$$

$$- 6211804114705462697323402159174107384990473593\,50a_{j+\frac{1}{2}}^3 a_j^6$$

$$- 6137736630581073285363696286829426419551582444623a_{j+\frac{1}{2}}^4 a_j^5$$

$$- 6137736630581073285363696286829426419551582444623a_{j+\frac{1}{2}}^5 a_j^4$$

$$- 621180411470546269732340215917410738499047359350a_{j+\frac{1}{2}}^6 a_j^3$$

$$+ 34904708339364654174924652814938752680359150a_{j+\frac{1}{2}}^7 a_j^2$$

$$- 13958448974879714404915130494435784889375a_{j+\frac{1}{2}}^8 a_j$$

$$+ 888457065097635122437710169588178125a_{j+\frac{1}{2}}^{9})\tau_{max}^{9}h^{19}$$

$$+ 6763612078080(163452863680517390647354939191764402504062 5a_{j}^{10}$$

$$- 2825524842635056936599350987799562105759687 50a_{j+\frac{1}{2}}a_{j}^{9}$$

$$+ 2432160368250335153225589561159951054248108071082375a_{j+\frac{1}{2}}^{2}a_{j}^{8}$$

$$- 8310677190402830036475253936167190697765323270636500a_{j+\frac{1}{2}}^{3}a_{j}^{7}$$

$$+ 7356127678092772811353630086301910924963391405438894565a_{j+\frac{1}{2}}^{4}a_{j}^{6}$$

$$+ 7689971029108327923821124643338240856991632311163316 94a_{j+\frac{1}{2}}^{5}a_{j}^{5}$$

$$+ 7356127678092772811353630086301910924963391405438894565a_{j+\frac{1}{2}}^{6}a_{j}^{4}$$

$$- 8310677190402830036475253936167190697765323270636500a_{j+\frac{1}{2}}^{7}a_{j}^{3}$$

$$+ 2432160368250335153225589561159951054248108071082375a_{j+\frac{1}{2}}^{8}a_{j}^{2}$$

$$- 282552484263505693659935098779956210575968750a_{j+\frac{1}{2}}^{9}a_{j}$$

$$+ 1634528636805173906473549391917644025040625a_{j+\frac{1}{2}}^{10})\tau_{max}^{10}h^{18}$$

$$- 229962810654720(28781236274846811816242643887363892 50000a_{j}^{11}$$

$$- 1481202486925716958029636281810440272114062 5a_{j+\frac{1}{2}}a_{j}^{10}$$

$$+ 319879411192695470451622148491886177201553056875a_{j+\frac{1}{2}}^{2}a_{j}^{9}$$

$$- 8019635559103183879993033159378623968758372354886250a_{j+\frac{1}{2}}^{3}a_{j}^{8}$$

$$- 7576428435205941580623299943584075584986104421329216 50a_{j+\frac{1}{2}}^{4}a_{j}^{7}$$

$$- 10520410399471187623331385600533678783577237411681804 91a_{j+\frac{1}{2}}^{5}a_{j}^{6}$$

$$- 10520410399471187623331385600533678783577237411681804 91a_{j+\frac{1}{2}}^{6}a_{j}^{5}$$

$$- 7576428435205941580623299943584075584986104421329216 50a_{j+\frac{1}{2}}^{7}a_{j}^{4}$$

$$- 8019635559103183879993033159378623968758372354886250a_{j+\frac{1}{2}}^{8}a_{j}^{3}$$

$$+ 319879411192695470451622148491886177201553056875a_{j+\frac{1}{2}}^{9}a_{j}^{2}$$

$$- 1481202486925716958029636281810440272114062 5a_{j+\frac{1}{2}}^{10}a_{j}$$

$$+ 28781236274846811816242643887363892 50000a_{j+\frac{1}{2}}^{11})\tau_{max}^{11}h^{17}$$

$$+ 28858078199808(2942684922260313786679699660779164782812 50a_j^{12}$$

$$- 2361615064818495460702339761062741771125000000a_{j+\frac{1}{2}}a_j^{11}$$

$$+ 2713350071991957938204503838551143251438027831688750a_{j+\frac{1}{2}}^{2}a_j^{10}$$

$$- 23530211804947575547179775426357352139634642830010000a_{j+\frac{1}{2}}^{3}a_j^{9}$$

$$+ 63725509242974404237389260236210954684251780492952630125a_{j+\frac{1}{2}}^{4}a_j^{8}$$

$$+ 8744721099627909795672887753416351889485809029059518800a_{j+\frac{1}{2}}^{5}a_j^{7}$$

$$+ 209273760833645673576341575313464830902246817989905451396a_{j+\frac{1}{2}}^{6}a_j^{6}$$

$$+ 8744721099627909795672887753416351889485809029059518800a_{j+\frac{1}{2}}^{7}a_j^{5}$$

$$+ 63725509242974404237389260236210954684251780492952630125a_{j+\frac{1}{2}}^{8}a_j^{4}$$

$$- 23530211804947575547179775426357352139634642830010000a_{j+\frac{1}{2}}^{9}a_j^{3}$$

$$+ 2713350071991957938204503838551143251438027831688750a_{j+\frac{1}{2}}^{10}a_j^{2}$$

$$- 2361615064818495460702339761062741771125000000a_{j+\frac{1}{2}}^{11}a_j$$

$$+ 294268492226031378667969966077916478281250a_{j+\frac{1}{2}}^{12})\tau_{max}^{12}h^{16}$$

$$+ 6132341617459200(31138870323868746449369591656183703125a_j^{13}$$

$$- 11734483363541331378666623294564819685771875a_{j+\frac{1}{2}}a_j^{12}$$

$$- 33967905504115394998356045976621383135254595250a_{j+\frac{1}{2}}^{2}a_j^{11}$$

$$+ 660614725980244601792215805789889135943431963986550a_{j+\frac{1}{2}}^{3}a_j^{10}$$

$$+ 656921004685650654596836595699758385868303459796998655a_{j+\frac{1}{2}}^{4}a_j^{9}$$

$$+ 828238349517872069712092208307559854269603908735034523a_{j+\frac{1}{2}}^{5}a_j^{8}$$

$$+ 2907846615912738424611579188259045667564783472133520852a_{j+\frac{1}{2}}^{6}a_j^{7}$$

$$+ 2907846615912738424611579188259045667564783472133520852a_{j+\frac{1}{2}}^{7}a_j^{6}$$

$$+ 828238349517872069712092208307559854269603908735034523a_{j+\frac{1}{2}}^{8}a_j^{5}$$

$$+ 656921004685650654596836595699758385868303459796998655a_{j+\frac{1}{2}}^{9}a_j^{4}$$

$$+ 660614725980244601792215805789889135943431963986550a_{j+\frac{1}{2}}^{10}a_j^{3}$$

$$- 33967905504115394998356045976621383135254595250a_{j+\frac{1}{2}}^{11}a_j^2$$

$$- 1173448336354133137866662329456481968577 1875a_{j+\frac{1}{2}}^{12}a_j$$

$$+ 3113887032386874644936959165618370 3125a_{j+\frac{1}{2}}^{13})\tau_{max}^{13}h^{15}$$

$$+ 2690729902080(6577725399008957701334214758433687373750 00a_j^{14}$$

$$- 935792378625641846254166367499934604773750000a_{j+\frac{1}{2}}a_j^{13}$$

$$+ 21300219977070328338528245279899276449119296331851250a_{j+\frac{1}{2}}^2a_j^{12}$$

$$- 10550024991308305858221946061466191897071959169818 72500a_{j+\frac{1}{2}}^3a_j^{11}$$

$$+ 88133755357112771103762913510853830219018422864990793768 25a_{j+\frac{1}{2}}^4a_j^{10}$$

$$+ 128986464288274180933886367265395548927340084763695581557 0a_{j+\frac{1}{2}}^5a_j^9$$

$$+ 874445755686667646005649275951661988461358504213822420587 32a_{j+\frac{1}{2}}^6a_j^8$$

$$+ 610171092140274769413174683848325543963376538412478395715 0a_{j+\frac{1}{2}}^7a_j^7$$

$$+ 874445755686667646005649275951661988461358504213822420587 32a_{j+\frac{1}{2}}^8a_j^6$$

$$+ 128986464288274180933886367265395548927340084763695581557 0a_{j+\frac{1}{2}}^9a_j^5$$

$$+ 88133755357112771103762913510853830219018422864990793768 25a_{j+\frac{1}{2}}^{10}a_j^4$$

$$- 10550024991308305858221946061466191897071959169818 72500a_{j+\frac{1}{2}}^{11}a_j^3$$

$$+ 21300219977070328338528245279899276449119296331851250a_{j+\frac{1}{2}}^{12}a_j^2$$

$$- 935792378625641846254166367499934604773750000a_{j+\frac{1}{2}}^{13}a_j$$

$$+ 6577725399008957701334214758433687373750 00a_{j+\frac{1}{2}}^{14})\tau_{max}^{14}h^{14}$$

$$- 13082328783912960a_ja_{j+\frac{1}{2}}(2316518916288108165584169195913809747875000a_j^{13}$$

$$- 23009335833927844596786769820307182449456956250a_{j+\frac{1}{2}}a_j^{12}$$

$$+ 2234076898252230587581171210332767321591221886615000a_{j+\frac{1}{2}}^2a_j^{11}$$

$$- 256926269270031410547443250229941952961149029868424 3500a_{j+\frac{1}{2}}^3a_j^{10}$$

$$- 261010028917812380157495632765886013384948717516448133 5a_{j+\frac{1}{2}}^4a_j^9$$

$$- 312102136436876752370370198742048525374839679047326378 40a_{j+\frac{1}{2}}^5a_j^8$$

$$- 30882306708740486599466467604869866303812925806329378094a_{j+\frac{1}{2}}^{6}a_j^{7}$$

$$- 30882306708740486599466467604869866303812925806329378094a_{j+\frac{1}{2}}^{7}a_j^{6}$$

$$- 31210213643687675237037019874204852537483967904732637840a_{j+\frac{1}{2}}^{8}a_j^{5}$$

$$- 2610100289178123801574956327658860133849487175164481335a_{j+\frac{1}{2}}^{9}a_j^{4}$$

$$- 2569262692700314105474432502299419529611490298684243500a_{j+\frac{1}{2}}^{10}a_j^{3}$$

$$+ 223407689825223058758117121033276732159122188661 5000a_{j+\frac{1}{2}}^{11}a_j^{2}$$

$$- 2300933583392784459678676982030718244945695 6250a_{j+\frac{1}{2}}^{12}a_j$$

$$+ 23165189162881081655841691959138097478750 00a_{j+\frac{1}{2}}^{13})\tau_{max}^{15}h^{13}$$

$$+ 28701118955520a_j^2a_{j+\frac{1}{2}}^2$$

$$(3983005504880383406859973916353309923851753592100 00a_j^{12}$$

$$- 37180359759375534401449830283255044573383668595730000a_{j+\frac{1}{2}}a_j^{11}$$

$$+ 3573832402210526483256799138604423351086859453677682305375a_{j+\frac{1}{2}}^{2}a_j^{10}$$

$$+ 5172817615587619205423406997477824479160079001868745893 00a_{j+\frac{1}{2}}^{3}a_j^{9}$$

$$+ 108752310025750705460582991819978419235883663099732700371042a_{j+\frac{1}{2}}^{4}a_j^{8}$$

$$+ 866784114356566136888648108179882255974229200630327515298 0a_{j+\frac{1}{2}}^{5}a_j^{7}$$

$$+ 334416288697967943954453597407311821589951489108850085896415a_{j+\frac{1}{2}}^{6}a_j^{6}$$

$$+ 866784114356566136888648108179882255974229200630327515298 0a_{j+\frac{1}{2}}^{7}a_j^{5}$$

$$+ 108752310025750705460582991819978419235883663099732700371042a_{j+\frac{1}{2}}^{8}a_j^{4}$$

$$+ 5172817615587619205423406997477824479160079001868745893 00a_{j+\frac{1}{2}}^{9}a_j^{3}$$

$$+ 3573832402210526483256799138604423351086859453677682305375a_{j+\frac{1}{2}}^{10}a_j^{2}$$

$$- 37180359759375534401449830283255044573383668595730000a_{j+\frac{1}{2}}^{11}a_j$$

$$+ 3983005504880383406859973916353309923851753592100 00a_{j+\frac{1}{2}}^{12})\tau_{max}^{16}h^{12}$$

$$- 2439595111219200a_j^3a_{j+\frac{1}{2}}^3$$

$$(52168296053766639717368695100231986765024328240730 00a_j^{11}$$

$$- 387077549737790344907140855158645231232784224855736497 50 a_{j+\frac{1}{2}} a_j^{10}$$

$$- 2348766255795104080576530590205803196628728033902514 3620 a_{j+\frac{1}{2}}^2 a_j^9$$

$$- 154640112762569238917178292216970897872559510023911751 6145 a_{j+\frac{1}{2}}^3 a_j^8$$

$$- 138270111590546505953886748479432933718950018144452694 1897 a_{j+\frac{1}{2}}^4 a_j^7$$

$$- 477870829927002554218389958695922416590456672173581795 9751 a_{j+\frac{1}{2}}^5 a_j^6$$

$$- 477870829927002554218389958695922416590456672173581795 9751 a_{j+\frac{1}{2}}^6 a_j^5$$

$$- 138270111590546505953886748479432933718950018144452694 1897 a_{j+\frac{1}{2}}^7 a_j^4$$

$$- 154640112762569238917178292216970897872559510023911751 6145 a_{j+\frac{1}{2}}^8 a_j^3$$

$$- 2348766255795104080576530590205803196628728033902514 3620 a_{j+\frac{1}{2}}^9 a_j^2$$

$$- 387077549737790344907140855158645231232784224855736497 50 a_{j+\frac{1}{2}}^{10} a_j$$

$$+ 521682960537666397173686951002319867650243282407300 0 a_{j+\frac{1}{2}}^{11}) \tau_{max}^{17} h^{11}$$

$$+ 191340793036800 a_j^4 a_{j+\frac{1}{2}}^4$$

$$(19163246870925396534473323787807024988003349663010499990 0 a_j^{10}$$

$$+ 811637186570956030605958769970984273742176153736717884 0 a_{j+\frac{1}{2}} a_j^9$$

$$+ 716972291077673677412274986095514873214219526189935543 81338 a_{j+\frac{1}{2}}^2 a_j^8$$

$$+ 567556858963542238768951829755996516364655654862700637 6620 a_{j+\frac{1}{2}}^3 a_j^7$$

$$+ 678874571969404486593651361170436732182185553225373227 273135 a_{j+\frac{1}{2}}^4 a_j^6$$

$$+ 194154044344149866199675817405284723805438279074848175 68330 a_{j+\frac{1}{2}}^5 a_j^5$$

$$+ 678874571969404486593651361170436732182185553225373227 273135 a_{j+\frac{1}{2}}^6 a_j^4$$

$$+ 567556858963542238768951829755996516364655654862700637 6620 a_{j+\frac{1}{2}}^7 a_j^3$$

$$+ 716972291077673677412274986095514873214219526189935543 81338 a_{j+\frac{1}{2}}^8 a_j^2$$

$$+ 811637186570956030605958769970984273742176153736717884 0 a_{j+\frac{1}{2}}^9 a_j$$

$$+ 19163246870925396534473323787807024988003349663010499990 0 a_{j+\frac{1}{2}}^{10}) \tau_{max}^{18} h^{10}$$

$$- 6505586963251200 a_j^5 a_{j+\frac{1}{2}}^5$$

$$(7563227775089851309390916262351036341355712749517571940a_j^9$$
$$- 1894070184847691274286147261026921980758932210292314197170a_{j+\frac{1}{2}}a_j^8$$
$$- 1357157414949644435765512984388378233144690375592421558228a_{j+\frac{1}{2}}^2a_j^7$$
$$- 20213820140081346715907495724816240952055856286119940483478a_{j+\frac{1}{2}}^3a_j^6$$
$$- 19060018445278558110564613913870923066811341796082681421417a_{j+\frac{1}{2}}^4a_j^5$$
$$- 19060018445278558110564613913870923066811341796082681421417a_{j+\frac{1}{2}}^5a_j^4$$
$$- 20213820140081346715907495724816240952055856286119940483478a_{j+\frac{1}{2}}^6a_j^3$$
$$- 1357157414949644435765512984388378233144690375592421558228a_{j+\frac{1}{2}}^7a_j^2$$
$$- 1894070184847691274286147261026921980758932210292314197170a_{j+\frac{1}{2}}^8a_j$$
$$+ 7563227775089851309390916262351036341355712749517571940a_{j+\frac{1}{2}}^9)\tau_{max}^{19}h^9$$
$$+ 4081936918118400a_j^6a_{j+\frac{1}{2}}^6$$
$$(1203527824719757084737294693245305249155503269095733533992a_j^8$$
$$- 32007392047825031083599698065137591100139159789058029440a_{j+\frac{1}{2}}a_j^7$$
$$+ 141305873585123356093489161870497239419385588116437348159905a_{j+\frac{1}{2}}^2a_j^6$$
$$+ 3432170557744974325621663946640372968979056882320258136600a_{j+\frac{1}{2}}^3a_j^5$$
$$+ 434177154524320603888523162597427546054438914788557665283103a_{j+\frac{1}{2}}^4a_j^4$$
$$+ 3432170557744974325621663946640372968979056882320258136600a_{j+\frac{1}{2}}^5a_j^3$$
$$+ 141305873585123356093489161870497239419385588116437348159905a_{j+\frac{1}{2}}^6a_j^2$$
$$- 32007392047825031083599698065137591100139159789058029440a_{j+\frac{1}{2}}^7a_j$$
$$+ 1203527824719757084737294693245305249155503269095733533992a_{j+\frac{1}{2}}^8)\tau_{max}^{20}h^8$$
$$- 3469646380400064000a_j^7a_{j+\frac{1}{2}}^7$$
$$(17041150655545150324249372730994204430805308898697369160a_j^7$$
$$- 1448756723322317642969137006701205107298486867431840063050a_{j+\frac{1}{2}}a_j^6$$
$$- 1209932516900506352119038564861078883913741360316360114652a_{j+\frac{1}{2}}^2a_j^5$$

$$- 45911420096625622186611571093425377948871964124211422780 41a_{j+\frac{1}{2}}^3 a_j^4$$

$$- 4591142009662562218661157109342537794887196412421142278041a_{j+\frac{1}{2}}^4 a_j^3$$

$$- 1209932516900506352119038564861078883913741360316360114652a_{j+\frac{1}{2}}^5 a_j^2$$

$$- 1448756723322317642969137006701205107298486867431840063050a_{j+\frac{1}{2}}^6 a_j$$

$$+ 1704115065554515032424937273099420443080530889869736916 0a_{j+\frac{1}{2}}^7)\tau_{max}^{21} h^7$$

$$+ 4947802324992000a_j^8 a_{j+\frac{1}{2}}^8$$

$$(41815281802728355449000442728158149563514789136281327994372a_j^6$$

$$- 35212626749963099026239979350563182048866608448299429643 60a_{j+\frac{1}{2}} a_j^5$$

$$+ 159714634486820768620247569818221915269128467798463596995 8305a_{j+\frac{1}{2}}^2 a_j^4$$

$$+ 33192512324149975547300593955569327000811997103539782211 90a_{j+\frac{1}{2}}^3 a_j^3$$

$$+ 1597146344868207686202475698182219152691284677984635969958305a_{j+\frac{1}{2}}^4 a_j^2$$

$$- 3521262674996309902623997935056318204886660844829942964360a_{j+\frac{1}{2}}^5 a_j$$

$$+ 41815281802728355449000442728158149563514789136281327994372a_{j+\frac{1}{2}}^6)\tau_{max}^{22} h^6$$

$$- 98203150206255035842560 00a_j^9 a_{j+\frac{1}{2}}^9$$

$$(228145562728128118455174040366617025943380505850 04a_j^5$$

$$- 66829799656015559067631715273554952677483753295980 1a_{j+\frac{1}{2}} a_j^4$$

$$- 60366959792684158461477550045687178519646804457609 4a_{j+\frac{1}{2}}^2 a_j^3$$

$$- 603669597926841584614775500456871785196468044576094a_{j+\frac{1}{2}}^3 a_j^2$$

$$- 668297996560155590676317152735549526774837532959801a_{j+\frac{1}{2}}^4 a_j$$

$$+ 228145562728128118455174040366617025943380505850 04a_{j+\frac{1}{2}}^5)\tau_{max}^{23} h^5$$

$$+ 1583296743997440 00a_j^{10} a_{j+\frac{1}{2}}^{10}$$

$$(179480947453557901304446011468730570412263581018459590971 06a_j^4$$

$$- 189290932794157106124035858362602183651007419239470359402 0a_{j+\frac{1}{2}} a_j^3$$

$$+ 2230700929243243614748850426951478489671772681159327049453 75a_{j+\frac{1}{2}}^2 a_j^2$$

$- 18929093279415710612403585836260218365100741923947035940 20a_{j+\frac{1}{2}}^3 a_j$

$+ 17948094745355790130444601146873057041226358101845959097106a_{j+\frac{1}{2}}^4)\tau_{max}^{24}h^4$

$- 7935608097475154411520000a_j^{11}a_{j+\frac{1}{2}}^{11}$

$(36333181540214995065248629992687255517667385999167 8a_j^3$

$- 32207120310356711220937632443471878293100697937922 81a_{j+\frac{1}{2}}a_j^2$

$- 32207120310356711220937632443471878293100697937922 81a_{j+\frac{1}{2}}^2 a_j$

$+ 36333181540214995065248629992687255517667385999167 8a_{j+\frac{1}{2}}^3)\tau_{max}^{25}h^3$

$+ 330302228719588857039814656 0000a_j^{12}a_{j+\frac{1}{2}}^{12}$

$(3870943533213155837623585590140549971658824411a_j^2$

$- 4242445587269748615952849007183688310385835 30a_{j+\frac{1}{2}}a_j$

$+ 3870943533213155837623585590140549971658824411a_{j+\frac{1}{2}}^2)\tau_{max}^{26}h^2$

$- 1185120373162706375789310259833114080220634144672791066529230716410003456 0000$

$a_j^{13}a_{j+\frac{1}{2}}^{13}(a_j + a_{j+\frac{1}{2}})\tau_{max}^{27}h$

$+ 510660795390178602055849958688994971584182157491454754207872348059598848 0000$

$a_j^{14}a_{j+\frac{1}{2}}^{14}\tau_{max}^{28}))$

$/(10287711948800(10135125h^8 + 4380486227076960a_j^2\tau_{max}^2h^6$

$+ 294877585914354432a_j^4\tau_{max}^4h^4 + 2587519954603806720a_j^6\tau_{max}^6h^2$

$+ 10253930735078604 80a_j^8\tau_{max}^8)^2(10135125h^8 + 4380486227076960a_{j+\frac{1}{2}}^2\tau_{max}^2h^6$

$+ 294877585914354432a_{j+\frac{1}{2}}^4\tau_{max}^4h^4 + 2587519954603806720a_{j+\frac{1}{2}}^6\tau_{max}^6h^2$

$+ 10253930735078604 80a_{j+\frac{1}{2}}^8\tau_{max}^8)^2)$

$=O(h^{21})$

2 Proof of Lemma 3.2

Here we give the detail proof of Lemma 3.2, we will only give the formulas with $u = x^{k+1},\ k = 0, 1, ..., 8$ since for $u = y^{k+1}$ in two-dimensional case the formulas are symmetric to those of $u = x^{k+1}$ by switching x and y (i and j).

1. $k = 0, u = x$, by the definition of the projection,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dxdy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} xdxdy \tag{2.21}$$

$$\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dxdy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} xdxdy \tag{2.22}$$

then we have

$$u_I = x_i, \ \forall (x, y) \in K_{i,j} \tag{2.23}$$

$$v_I = x_{i+\frac{1}{2}}, \ \forall (x, y) \in K_{i+\frac{1}{2},j+\frac{1}{2}} \tag{2.24}$$

Hence, for $(x, y) \in (x_i, x_{i+\frac{1}{2}}) \times (y_j, y_{j+\frac{1}{2}})$,

$$v_I - x - u_I(x - \frac{h}{2}) + (x - \frac{h}{2}) = x_{i+\frac{1}{2}} - x_i - \frac{h}{2} = 0 \tag{2.25}$$

Similarly, we have

$$\begin{aligned}
v_I - x - u_I(x - \frac{h}{2}) + (x - \frac{h}{2}) &= 0, \ (x, y) \in (x_i, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_j) \\
v_I - x - u_I(x + \frac{h}{2}) + (x + \frac{h}{2}) &= 0, \ (x, y) \in (x_{i-\frac{1}{2}}, x_i) \times (y_j, y_{j+\frac{1}{2}}) \\
v_I - x - u_I(x + \frac{h}{2}) + (x + \frac{h}{2}) &= 0, \ (x, y) \in (x_{i-\frac{1}{2}}, x_i) \times (y_{j-\frac{1}{2}}, y_j)
\end{aligned}$$

2. $k = 1, u = x^2$, by the definition of the projection,

$$\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dxdy &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^2 dxdy \\
\tilde{P}_h(u_I; x; f, g, u)_{i,j} &= \tilde{P}_h(x^2; x; f, g, u)_{i,j} \\
\tilde{P}_h(u_I; y; f, g, u)_{i,j} &= \tilde{P}_h(x^2; y; f, g, u)_{i,j} \\
\tilde{P}_h(u_I; xy; f, g, u)_{i,j} &= \tilde{P}_h(x^2; xy; f, g, u)_{i,j} \\
\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dxdy &= \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^2 dxdy \\
\tilde{Q}_h(v_I; x; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} &= \tilde{Q}_h(x^2; x; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; y; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} &= \tilde{Q}_h(x^2; y; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}}
\end{aligned}$$

$$\tilde{Q}_h(v_I; xy; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} = \tilde{Q}_h(x^2; xy; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}$$

then by solving the above linear system we have

$$\begin{aligned} u_I(x, y) &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy \\ v_I(x, y) &= \beta_0 + \beta_1 x + \beta_2 y + \alpha_3 xy \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} \alpha_3 &= 0 \\ \alpha_2 &= 0 \\ \alpha_1 &= 2x_i - \frac{2\tau_{max} a_{i,j}}{3} \\ \alpha_0 &= \frac{1}{12}(8\tau_{max} x_i a_{i,j} + h^2 - 12x_i^2) \\ \beta_3 &= 0 \\ \beta_2 &= 0 \\ \beta_1 &= 2x_{i+\frac{1}{2}} - \frac{2\tau_{max} a_{i+\frac{1}{2}, j+\frac{1}{2}}}{3} \\ \beta_0 &= \frac{1}{12}(8\tau_{max} x_{i+\frac{1}{2}} a_{i+\frac{1}{2}, j+\frac{1}{2}} + h^2 - 12x_{i+\frac{1}{2}}^2) \end{aligned} \tag{2.27}$$

Hence,

$$\begin{aligned} &\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^2 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^2)^2 dxdy \\ =& \frac{1}{108} h^4 \tau_{max}^2 (a_{i+\frac{1}{2}, j+\frac{1}{2}} - a_{i,j})^2 \\ =& O(h^8) \end{aligned} \tag{2.28}$$

Similarly, we have

$$\begin{aligned} \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^2 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^2)^2 dxdy &= O(h^8), \\ \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^2 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^2)^2 dxdy &= O(h^8), \\ \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^2 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^2)^2 dxdy &= O(h^8). \end{aligned}$$

3. $k = 2, u = x^3$, by the definition of the projection,

$$\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dxdy =& \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^3 dxdy \\
\tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} =& \tilde{P}_h(x^3; x^k y^l; f, g, u)_{i,j}, \ \ k = 0, 1, 2, \ \ l = 0, 1, 2 \\
\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dxdy =& \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^3 dxdy \\
\tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} =& \tilde{Q}_h(x^3; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}}, \ \ k = 0, 1, 2, \ \ l = 0, 1, 2
\end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned}
u_I(x, y) =& \sum_{k=0}^{2} \sum_{l=0}^{2} \alpha_{k,l} x^k y^l \\
v_I(x, y) =& \sum_{k=0}^{2} \sum_{l=0}^{2} \beta_{k,l} x^k y^l
\end{aligned} \tag{2.29}$$

where

$$\begin{aligned}
\alpha_{k,l} =& 0, \ \ k = 0, 1, 2, \ \ l = 1, 2 \\
\alpha_{2,0} =& (-6h^2 \tau_{max} a_{i,j} + 240\tau_{max}^2 x_i a_{i,j}^2 + 45h^2 x_i)/(80\tau_{max}^2 a_{i,j}^2 + 15h^2) \\
\alpha_{1,0} =& (48\tau_{max}^2 a_{i,j}^2 (h^2 - 40x_i^2) + 96h^2 \tau_{max} x_i a_{i,j} + 15(h^4 - 24h^2 x_i^2)) \\
& /(40(16\tau_{max}^2 a_{i,j}^2 + 3h^2)) \\
\alpha_{0,0} =& (16\tau_{max}^2 x_i a_{i,j}^2 (40x_i^2 - 3h^2) + 4h^2 \tau_{max} a_{i,j} (h^2 - 12x_i^2) \\
& - 15h^2 x_i (h^2 - 8x_i^2))/(40(16\tau_{max}^2 a_{i,j}^2 + 3h^2)) \\
\beta_{k,l} =& 0, \ \ k = 0, 1, 2, \ \ l = 1, 2 \\
\beta_{2,0} =& (-6h^2 \tau_{max} a_{i+\frac{1}{2},j+\frac{1}{2}} + 240\tau_{max}^2 x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 45h^2 x_{i+\frac{1}{2}}) \\
& /(80\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 15h^2) \\
\beta_{1,0} =& (48\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 (h^2 - 40x_{i+\frac{1}{2}}^2) + 96h^2 \tau_{max} x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& + 15(h^4 - 24h^2 x_{i+\frac{1}{2}}^2))/(40(16\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 3h^2)) \\
\beta_{0,0} =& (16\tau_{max}^2 x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}}^2 (40x_{i+\frac{1}{2}}^2 - 3h^2) + 4h^2 \tau_{max} a_{i+\frac{1}{2},j+\frac{1}{2}} (h^2 - 12x_{i+\frac{1}{2}}^2) \\
& - 15h^2 x_{i+\frac{1}{2}} (h^2 - 8x_{i+\frac{1}{2}}^2))/(40(16\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 3h^2))
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^3-u_I(x-\frac{h}{2},y-\frac{h}{2})+(x-\frac{h}{2})^3)^2dxdy\\
=&(h^{10}\tau_{max}^2(a_{i+\frac{1}{2},j+\frac{1}{2}}-a_{i,j})^2(\tau_{max}(3ha_{i+\frac{1}{2},j+\frac{1}{2}}(16\tau_{max}a_{i,j}(5\tau_{max}a_{i,j}+h)\\
&-15h^2)+8\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}}^2(2\tau_{max}a_{i,j}(32\tau_{max}a_{i,j}+15h)+15h^2)\\
&+15h^2a_{i,j}(8\tau_{max}a_{i,j}-3h))+18h^4))\\
&/(1000(16\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2+3h^2)^2(16\tau_{max}^2a_{i,j}^2+3h^2)^2)\\
=&O(h^{10})
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}(v_I(x,y)-x^3-u_I(x-\frac{h}{2},y+\frac{h}{2})+(x-\frac{h}{2})^3)^2dxdy=&O(h^{10}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^3-u_I(x+\frac{h}{2},y-\frac{h}{2})+(x+\frac{h}{2})^3)^2dxdy=&O(h^{10}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^3-u_I(x+\frac{h}{2},y+\frac{h}{2})+(x+\frac{h}{2})^3)^2dxdy=&O(h^{10}).
\end{aligned}
$$

4. $k=3, u=x^4$, by the definition of the projection,

$$
\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}u_Idxdy=&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}x^4dxdy\\
\tilde{P}_h(u_I;x^ky^l;f,g,u)_{i,j}=&\tilde{P}_h(x^4;x^ky^l;f,g,u)_{i,j},\ \ k=0,1,2,3,\ \ l=0,1,2,3\\
\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}v_Idxdy=&\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}x^4dxdy\\
\tilde{Q}_h(v_I;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}}=&\tilde{Q}_h(x^4;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}},\ \ k=0,1,2,3,\ \ l=0,1,2,3
\end{aligned}
$$

then by solving the above linear system we have

$$
\begin{aligned}
u_I(x,y)=&\sum_{k=0}^{3}\sum_{l=0}^{3}\alpha_{k,l}x^ky^l\\
v_I(x,y)=&\sum_{k=0}^{3}\sum_{l=0}^{3}\beta_{k,l}x^ky^l
\end{aligned}
\tag{2.30}
$$

where

$$\alpha_{k,l}=0,\ \ k=0,1,2,3,\ \ l=1,2,3$$

$$\alpha_{3,0} =(76h^2\tau_{max}a_{i,j} + 704\tau_{max}^3 a_{i,j}^3 + 4032\tau_{max}^2 x_i a_{i,j}^2 + 420h^2 x_i)$$
$$/(1008\tau_{max}^2 a_{i,j}^2 + 105h^2)$$
$$\alpha_{2,0} =(4\tau_{max}^2 a_{i,j}^2(17h^2 - 504x_i^2) - 76h^2\tau_{max}x_i a_{i,j} - 704\tau_{max}^3 x_i a_{i,j}^3$$
$$+ 7(h^4 - 30h^2x_i^2))/(7(48\tau_{max}^2 a_{i,j}^2 + 5h^2))$$
$$\alpha_{1,0} = -(88\tau_{max}^3 a_{i,j}^3(h^2 - 40x_i^2) + 40\tau_{max}^2 x_i a_{i,j}^2(17h^2 - 168x_i^2)$$
$$+ 10h^2\tau_{max}a_{i,j}(h^2 - 38x_i^2) + 70h^2x_i(h^2 - 10x_i^2))/(35(48\tau_{max}^2 a_{i,j}^2 + 5h^2))$$
$$\alpha_{0,0} =(160h^2\tau_{max}a_{i,j}(3h^2x_i - 38x_i^3) + 1408\tau_{max}^3 x_i a_{i,j}^3(3h^2 - 40x_i^2)$$
$$- 32\tau_{max}^2 a_{i,j}^2(11h^4 - 510h^2x_i^2 + 2520x_i^4) - 35(h^6 - 48h^4x_i^2 + 240h^2x_i^4))$$
$$/(1680(48\tau_{max}^2 a_{i,j}^2 + 5h^2))$$
$$\beta_{k,l} =0, \ k = 0, 1, 2, 3, \ l = 1, 2, 3$$
$$\beta_{3,0} =(76h^2\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}} + 704\tau_{max}^3 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 4032\tau_{max}^2 x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 420h^2x_{i+\frac{1}{2}})$$
$$/(1008\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 105h^2)$$
$$\beta_{2,0} =(4\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2(17h^2 - 504x_{i+\frac{1}{2}}^2) - 76h^2\tau_{max}x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$- 704\tau_{max}^3 x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 7(h^4 - 30h^2x_{i+\frac{1}{2}}^2))$$
$$/(7(48\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 5h^2))$$
$$\beta_{1,0} = -(88\tau_{max}^3 a_{i+\frac{1}{2},j+\frac{1}{2}}^3(h^2 - 40x_{i+\frac{1}{2}}^2) + 40\tau_{max}^2 x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}}^2(17h^2 - 168x_{i+\frac{1}{2}}^2)$$
$$+ 10h^2\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}}(h^2 - 38x_{i+\frac{1}{2}}^2) + 70h^2x_{i+\frac{1}{2}}(h^2 - 10x_{i+\frac{1}{2}}^2))$$
$$/(35(48\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 5h^2))$$
$$\beta_{0,0} =(160h^2\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}}(3h^2x_{i+\frac{1}{2}} - 38x_{i+\frac{1}{2}}^3) + 1408\tau_{max}^3 x_{i+\frac{1}{2}} a_{i+\frac{1}{2},j+\frac{1}{2}}^3(3h^2 - 40x_{i+\frac{1}{2}}^2)$$
$$- 32\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2(11h^4 - 510h^2x_{i+\frac{1}{2}}^2 + 2520x_{i+\frac{1}{2}}^4)$$
$$- 35(h^6 - 48h^4x_{i+\frac{1}{2}}^2 + 240h^2x_{i+\frac{1}{2}}^4))$$
$$/(1680(48\tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 5h^2))$$

Hence,

$$\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^4 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^4)^2 dxdy$$

$$
\begin{aligned}
=&(h^8\tau_{max}^2(a_{i+\frac{1}{2},j+\frac{1}{2}}-a_{i,j})^2(286528\tau_{max}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4(48\tau_{max}^2a_{i,j}^2+5h^2)^2\\
&-88h^2\tau_{max}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3(880h^2\tau_{max}a_{i,j}+11760h\tau_{max}^2a_{i,j}^2+8448\tau_{max}^3a_{i,j}^3+1225h^3)\\
&-5h^4\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}}(1616h^2\tau_{max}a_{i,j}+21280h\tau_{max}^2a_{i,j}^2\\
&+15488\tau_{max}^3a_{i,j}^3+2275h^3)+160h^2\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2(-665h^3\tau_{max}a_{i,j}\\
&+179598h^2\tau_{max}^2a_{i,j}^2-6468h\tau_{max}^3a_{i,j}^3+859584\tau_{max}^4a_{i,j}^4+9381h^4)\\
&+5h^4(-2275h^3\tau_{max}a_{i,j}+300192h^2\tau_{max}^2a_{i,j}^2-21560h\tau_{max}^3a_{i,j}^3\\
&+1432640\tau_{max}^4a_{i,j}^4+15725h^4)))\\
&/(1234800(48\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2+5h^2)^2(48\tau_{max}^2a_{i,j}^2+5h^2)^2)\\
=&O(h^{12})
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}(v_I(x,y)-x^4-u_I(x-\frac{h}{2},y+\frac{h}{2})+(x-\frac{h}{2})^4)^2dxdy=&O(h^{12}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^4-u_I(x+\frac{h}{2},y-\frac{h}{2})+(x+\frac{h}{2})^4)^2dxdy=&O(h^{12}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^4-u_I(x+\frac{h}{2},y+\frac{h}{2})+(x+\frac{h}{2})^4)^2dxdy=&O(h^{12}).
\end{aligned}
$$

5. $k=4, u=x^5$, by the definition of the projection,

$$
\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}u_Idxdy=&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}x^5dxdy\\
\tilde{P}_h(u_I;x^ky^l;f,g,u)_{i,j}=&\tilde{P}_h(x^5;x^ky^l;f,g,u)_{i,j},\;\; k=0,1,2,3,4,\;\; l=0,1,2,3,4\\
\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}v_Idxdy=&\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}x^5dxdy\\
\tilde{Q}_h(v_I;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}}=&\tilde{Q}_h(x^5;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}},\;\; k=0,1,2,3,4,\;\; l=0,1,2,3,4
\end{aligned}
$$

then by solving the above linear system we have

$$
\begin{aligned}
u_I(x,y)=&\sum_{k=0}^{4}\sum_{l=0}^{4}\alpha_{k,l}x^ky^l\\
v_I(x,y)=&\sum_{k=0}^{4}\sum_{l=0}^{4}\beta_{k,l}x^ky^l
\end{aligned}
\tag{2.31}
$$

where

$$
\begin{aligned}
\alpha_{k,l} =&0,\ \ k=0,1,2,3,4,\ \ l=1,2,3,4\\
\alpha_{4,0} =&(464640a_{i,j}^4\tau_{max}^4x_i+80a_{i,j}^2h^2\tau_{max}^2(178a_{i,j}\tau_{max}+795x_i)\\
&+75h^4(20a_{i,j}\tau_{max}+21x_i))\\
&/(92928a_{i,j}^4\tau_{max}^4+12720a_{i,j}^2h^2\tau_{max}^2+315h^4)\\
\alpha_{3,0} =&(5(-2230272a_{i,j}^4\tau_{max}^4x_i^2-24h^4(-262a_{i,j}^2\tau_{max}^2+600a_{i,j}\tau_{max}x_i+315x_i^2)\\
&+128a_{i,j}^2h^2\tau_{max}^2(286a_{i,j}^2\tau_{max}^2-1068a_{i,j}\tau_{max}x_i-2385x_i^2)+225h^6))\\
&/(36(30976a_{i,j}^4\tau_{max}^4+4240a_{i,j}^2h^2\tau_{max}^2+105h^4))\\
\alpha_{2,0} =&-(5(-5203968a_{i,j}^4\tau_{max}^4x_i^3+896a_{i,j}^2h^2\tau_{max}^2x_i(286a_{i,j}^2\tau_{max}^2-534a_{i,j}\tau_{max}x_i-795x_i^2)\\
&-8h^4(-1856a_{i,j}^3\tau_{max}^3-5502a_{i,j}^2\tau_{max}^2x_i+6300a_{i,j}\tau_{max}x_i^2+2205x_i^3)\\
&+63h^6(26a_{i,j}\tau_{max}+25x_i)))\\
&/(84(30976a_{i,j}^4\tau_{max}^4+4240a_{i,j}^2h^2\tau_{max}^2+105h^4))\\
\alpha_{1,0} =&-(5(20815872a_{i,j}^4\tau_{max}^4x_i^4-56h^6(-86a_{i,j}^2\tau_{max}^2+468a_{i,j}\tau_{max}x_i+225x_i^2)\\
&-3584a_{i,j}^2h^2\tau_{max}^2x_i^2(572a_{i,j}^2\tau_{max}^2-712a_{i,j}\tau_{max}x_i-795x_i^2)\\
&+32h^4(440a_{i,j}^4\tau_{max}^4-7424a_{i,j}^3\tau_{max}^3x_i-11004a_{i,j}^2\tau_{max}^2x_i^2\\
&+8400a_{i,j}\tau_{max}x_i^3+2205x_i^4)+231h^8))\\
&/(672(30976a_{i,j}^4\tau_{max}^4+4240a_{i,j}^2h^2\tau_{max}^2+105h^4))\\
\alpha_{0,0} =&(62447616a_{i,j}^4\tau_{max}^4x_i^5-17920a_{i,j}^2h^2\tau_{max}^2x_i^3(572a_{i,j}^2\tau_{max}^2-534a_{i,j}\tau_{max}x_i-477x_i^2)\\
&-8h^6(-3608a_{i,j}^3\tau_{max}^3-9030a_{i,j}^2\tau_{max}^2x_i+24570a_{i,j}\tau_{max}x_i^2+7875x_i^3)\\
&+480h^4x_i(440a_{i,j}^4\tau_{max}^4-3712a_{i,j}^3\tau_{max}^3x_i-3668a_{i,j}^2\tau_{max}^2x_i^2\\
&+2100a_{i,j}\tau_{max}x_i^3+441x_i^4)+315h^8(12a_{i,j}\tau_{max}+11x_i))\\
&/(2016(30976a_{i,j}^4\tau_{max}^4+4240a_{i,j}^2h^2\tau_{max}^2+105h^4))\\
\beta_{k,l} =&0,\ \ k=0,1,2,3,4,\ \ l=1,2,3,4\\
\beta_{4,0} =&(464640a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}+80a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2(178a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}+795x_{i+\frac{1}{2}})
\end{aligned}
$$

$$+ 75h^4(20a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 21x_{i+\frac{1}{2}}))$$

$$/(92928a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max} + 12720a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max} + 315h^4)$$

$$\beta_{3,0} =(5(-2230272a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}x^2_{i+\frac{1}{2}} - 24h^4(-262a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}$$

$$+ 600a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 315x^2_{i+\frac{1}{2}})$$

$$+ 128a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max}(286a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}$$

$$- 1068a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 2385x^2_{i+\frac{1}{2}}) + 225h^6))$$

$$/(36(30976a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max} + 4240a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max} + 105h^4))$$

$$\beta_{2,0} = -(5(-5203968a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}x^3_{i+\frac{1}{2}} + 896a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max}x_{i+\frac{1}{2}}(286a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}$$

$$- 534a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 795x^2_{i+\frac{1}{2}})$$

$$- 8h^4(-1856a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max} - 5502a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}$$

$$+ 6300a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}} + 2205x^3_{i+\frac{1}{2}}) + 63h^6(26a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 25x_{i+\frac{1}{2}})))$$

$$/(84(30976a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max} + 4240a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max} + 105h^4))$$

$$\beta_{1,0} = -(5(20815872a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}x^4_{i+\frac{1}{2}} - 56h^6(-86a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}$$

$$+ 468a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 225x^2_{i+\frac{1}{2}})$$

$$- 3584a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max}x^2_{i+\frac{1}{2}}(572a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max} - 712a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 795x^2_{i+\frac{1}{2}})$$

$$+ 32h^4(440a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max} - 7424a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}x_{i+\frac{1}{2}} - 11004a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x^2_{i+\frac{1}{2}}$$

$$+ 8400a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^3_{i+\frac{1}{2}} + 2205x^4_{i+\frac{1}{2}}) + 231h^8))$$

$$/(672(30976a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max} + 4240a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max} + 105h^4))$$

$$\beta_{0,0} =(62447616a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}x^5_{i+\frac{1}{2}} - 17920a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^2_{max}x^3_{i+\frac{1}{2}}(572a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}$$

$$- 534a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 477x^2_{i+\frac{1}{2}})$$

$$- 8h^6(-3608a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max} - 9030a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}$$

$$+ 24570a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}} + 7875x^3_{i+\frac{1}{2}})$$

$$+ 480h^4x_{i+\frac{1}{2}}(440a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max} - 3712a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}x_{i+\frac{1}{2}} - 3668a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x^2_{i+\frac{1}{2}}$$

$$+ 2100a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^3_{i+\frac{1}{2}} + 441x^4_{i+\frac{1}{2}}) + 315h^8(12a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 11x_{i+\frac{1}{2}}))$$

$$/(2016(30976a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4+4240a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2+105h^4))$$

Hence, we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^5-u_I(x-\frac{h}{2},y-\frac{h}{2})+(x-\frac{h}{2})^5)^2dxdy$$
$$=(h^{14}\tau_{max}^2(a_{i,j}-a_{i+\frac{1}{2},j+\frac{1}{2}})^2(74297776099295232a_{i,j}^6a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^{12}$$
$$-17160990584995840a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}}^5h\tau_{max}^{11}(a_{i,j}+a_{i+\frac{1}{2},j+\frac{1}{2}})$$
$$+189000h^{10}\tau_{max}^2(1679899a_{i,j}^2-1244578a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}+1679899a_{i+\frac{1}{2},j+\frac{1}{2}}^2)$$
$$+7938000h^9\tau_{max}^3(a_{i,j}+a_{i+\frac{1}{2},j+\frac{1}{2}})(110956a_{i,j}^2-146113a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}+110956a_{i+\frac{1}{2},j+\frac{1}{2}}^2)$$
$$+317194240a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^{10}(51995327a_{i,j}^2-6974930a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}+51995327a_{i+\frac{1}{2},j+\frac{1}{2}}^2)$$
$$-7208960a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3h^3\tau_{max}^9(a_{i,j}+a_{i+\frac{1}{2},j+\frac{1}{2}})(411149046a_{i,j}^2-110513705a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$+411149046a_{i+\frac{1}{2},j+\frac{1}{2}}^2)+806400h^8\tau_{max}^4(4609881a_{i,j}^4-5666392a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$+21455275a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2-5666392a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3+4609881a_{i+\frac{1}{2},j+\frac{1}{2}}^4)$$
$$+403200h^7\tau_{max}^5(a_{i,j}+a_{i+\frac{1}{2},j+\frac{1}{2}})(11867779a_{i,j}^4-59803436a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$+53324708a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2-59803436a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3+11867779a_{i+\frac{1}{2},j+\frac{1}{2}}^4)$$
$$-47308800a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}h^5\tau_{max}^7(a_{i,j}+a_{i+\frac{1}{2},j+\frac{1}{2}})(2637558a_{i,j}^4-2676409a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$+11391631a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2-2676409a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3+2637558a_{i+\frac{1}{2},j+\frac{1}{2}}^4)$$
$$+1966080a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^8(532478408a_{i,j}^4-225653868a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$+1900203439a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2-225653868a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3+532478408a_{i+\frac{1}{2},j+\frac{1}{2}}^4)$$
$$+307200h^6\tau_{max}^6(45865050a_{i,j}^6-72215220a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}}+800405235a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^2$$
$$-293286178a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3+800405235a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^4$$
$$-72215220a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^5+45865050a_{i+\frac{1}{2},j+\frac{1}{2}}^6)$$
$$+40059613125h^{11}\tau_{max}(a_{i,j}+a_{i+\frac{1}{2},j+\frac{1}{2}})+8661350250h^{12}))$$
$$/(508032(30976a_{i,j}^4\tau_{max}^4+4240a_{i,j}^2h^2\tau_{max}^2+105h^4)^2(30976a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4$$
$$+4240a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2+105h^4)^2)$$
$$=O(h^{14})$$

Similarly, we have

$$
\begin{aligned}
\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}(v_I(x,y)-x^5-u_I(x-\frac{h}{2},y+\frac{h}{2})+(x-\frac{h}{2})^5)^2dxdy =&O(h^{14}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^5-u_I(x+\frac{h}{2},y-\frac{h}{2})+(x+\frac{h}{2})^5)^2dxdy =&O(h^{14}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^5-u_I(x+\frac{h}{2},y+\frac{h}{2})+(x+\frac{h}{2})^5)^2dxdy =&O(h^{14}).
\end{aligned}
$$

6. $k=5, u=x^6$, by the definition of the projection,

$$
\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}u_I dxdy =&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}x^6dxdy\\
\tilde{P}_h(u_I;x^ky^l;f,g,u)_{i,j} =&\tilde{P}_h(x^6;x^ky^l;f,g,u)_{i,j},\ \ k=0,1,...,5,\ \ l=0,1,...,5\\
\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}v_I dxdy =&\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}x^6dxdy\\
\tilde{Q}_h(v_I;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}} =&\tilde{Q}_h(x^6;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}},\ \ k=0,1,...,5,\ \ l=0,1,...,5
\end{aligned}
$$

then by solving the above linear system we have

$$
\begin{aligned}
u_I(x,y) &= \sum_{k=0}^{5}\sum_{l=0}^{5}\alpha_{k,l}x^ky^l\\
v_I(x,y) &= \sum_{k=0}^{5}\sum_{l=0}^{5}\beta_{k,l}x^ky^l
\end{aligned}
\tag{2.32}
$$

where

$$
\begin{aligned}
\alpha_{k,l} =&0,\ \ k=0,1,...,5,\ \ l=1,...,5\\
\alpha_{5,0} =&(630(33x_i+145a_{i,j}\tau_{max})h^4+288a_{i,j}^2\tau_{max}^2(70455x_i+481a_{i,j}\tau_{max})h^2\\
&+5632a_{i,j}^4\tau_{max}^4(35193x_i-1331a_{i,j}\tau_{max}))\\
&/(11(315h^4+307440a_{i,j}^2\tau_{max}^2h^2+3003136a_{i,j}^4\tau_{max}^4))\\
\alpha_{4,0} =&(5(3465h^6-12(10395x_i^2+91350a_{i,j}\tau_{max}x_i-191168a_{i,j}^2\tau_{max}^2)h^4\\
&+64a_{i,j}^2\tau_{max}^2(-1902285x_i^2-25974a_{i,j}\tau_{max}x_i\\
&+348082a_{i,j}^2\tau_{max}^2)h^2+33792a_{i,j}^4x_i\tau_{max}^4(2662a_{i,j}\tau_{max}-35193x_i)))\\
&/(132(315h^4+307440a_{i,j}^2\tau_{max}^2h^2+3003136a_{i,j}^4\tau_{max}^4))
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{3,0} =&(5(-9(1155x_i + 5531a_{i,j}\tau_{max})h^6 + 12(10395x_i^3 + 137025a_{i,j}\tau_{max}x_i^2 \\
&- 573504a_{i,j}^2\tau_{max}^2 x_i - 21856a_{i,j}^3\tau_{max}^3)h^4 + 64a_{i,j}^2\tau_{max}^2(1902285x_i^3 \\
&+ 38961a_{i,j}\tau_{max}x_i^2 - 1044246a_{i,j}^2\tau_{max}^2 x_i + 34606a_{i,j}^3\tau_{max}^3)h^2 \\
&- 101376a_{i,j}^4 x_i^2\tau_{max}^4(1331a_{i,j}\tau_{max} - 11731x_i))) \\
&/(99(315h^4 + 307440a_{i,j}^2\tau_{max}^2 h^2 + 3003136a_{i,j}^4\tau_{max}^4)) \\
\alpha_{2,0} =&-(5(2871h^8 - 48(3465x_i^2 + 33186a_{i,j}\tau_{max}x_i - 23006a_{i,j}^2\tau_{max}^2)h^6 \\
&+ 32(31185x_i^4 + 548100a_{i,j}\tau_{max}x_i^3 - 3441024a_{i,j}^2\tau_{max}^2 x_i^2 - 262272a_{i,j}^3\tau_{max}^3 x_i \\
&+ 329432a_{i,j}^4\tau_{max}^4)h^4 + 512a_{i,j}^2 x_i\tau_{max}^2(1902285x_i^3 + 51948a_{i,j}\tau_{max}x_i^2 \\
&- 2088492a_{i,j}^2\tau_{max}^2 x_i + 138424a_{i,j}^3\tau_{max}^3)h^2 \\
&- 270336a_{i,j}^4 x_i^3\tau_{max}^4(5324a_{i,j}\tau_{max} - 35193x_i))) \\
&/(1056(315h^4 + 307440a_{i,j}^2\tau_{max}^2 h^2 + 3003136a_{i,j}^4\tau_{max}^4)) \\
\alpha_{1,0} =&(3465(29x_i + 149a_{i,j}\tau_{max})h^8 - 336(5775x_i^3 + 82965a_{i,j}\tau_{max}x_i^2 \\
&- 115030a_{i,j}^2\tau_{max}^2 x_i - 12178a_{i,j}^3\tau_{max}^3)h^6 + 160(43659x_i^5 + 959175a_{i,j}\tau_{max}x_i^4 \\
&- 8029056a_{i,j}^2\tau_{max}^2 x_i^3 - 917952a_{i,j}^3\tau_{max}^3 x_i^2 + 2306024a_{i,j}^4\tau_{max}^4 x_i - 53240a_{i,j}^5\tau_{max}^5)h^4 \\
&+ 17920a_{i,j}^2 x_i^2\tau_{max}^2(380457x_i^3 + 12987a_{i,j}\tau_{max}x_i^2 - 696164a_{i,j}^2\tau_{max}^2 x_i \\
&+ 69212a_{i,j}^3\tau_{max}^3)h^2 - 1892352a_{i,j}^4 x_i^4\tau_{max}^4(6655a_{i,j}\tau_{max} - 35193x_i)) \\
&/(3696(315h^4 + 307440a_{i,j}^2\tau_{max}^2 h^2 + 3003136a_{i,j}^4\tau_{max}^4)) \\
\alpha_{0,0} =&(17325h^{10} - 84(14355x_i^2 + 147510a_{i,j}\tau_{max}x_i - 37792a_{i,j}^2\tau_{max}^2)h^8 + 32(363825x_i^4 \\
&+ 6969060a_{i,j}\tau_{max}x_i^3 - 14493780a_{i,j}^2\tau_{max}^2 x_i^2 - 3068856a_{i,j}^3\tau_{max}^3 x_i + 873136a_{i,j}^4\tau_{max}^4)h^6 \\
&- 1920x_i(14553x_i^5 + 383670a_{i,j}\tau_{max}x_i^4 - 4014528a_{i,j}^2\tau_{max}^2 x_i^3 - 611968a_{i,j}^3\tau_{max}^3 x_i^2 \\
&+ 2306024a_{i,j}^4\tau_{max}^4 x_i - 106480a_{i,j}^5\tau_{max}^5)h^4 - 14336a_{i,j}^2 x_i^3\tau_{max}^2(1902285x_i^3 \\
&+ 77922a_{i,j}\tau_{max}x_i^2 - 5221230a_{i,j}^2\tau_{max}^2 x_i + 692120a_{i,j}^3\tau_{max}^3)h^2 \\
&+ 22708224a_{i,j}^4 x_i^5\tau_{max}^4(2662a_{i,j}\tau_{max} - 11731x_i)) \\
&/(88704(315h^4 + 307440a_{i,j}^2\tau_{max}^2 h^2 + 3003136a_{i,j}^4\tau_{max}^4))
\end{aligned}
$$

$$
\begin{aligned}
\beta_{k,l} =&0,\ \ k=0,1,...,5,\ \ l=1,...,5\\
\beta_{5,0} =&(630(33x_{i+\frac{1}{2}}+145a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^4\\
&+288a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}(70455x_{i+\frac{1}{2}}+481a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^2\\
&+5632a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}(35193x_{i+\frac{1}{2}}-1331a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}))\\
&/(11(315h^4+307440a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}h^2+3003136a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}))\\
\beta_{4,0} =&(5(3465h^6-12(10395x^2_{i+\frac{1}{2}}+91350a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}-191168a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max})h^4\\
&+64a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}(-1902285x^2_{i+\frac{1}{2}}-25974a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}\\
&+348082a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max})h^2+33792a^4_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^4_{max}\\
&(2662a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}-35193x_{i+\frac{1}{2}})))\\
&/(132(315h^4+307440a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}h^2+3003136a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}))\\
\beta_{3,0} =&(5(-9(1155x_{i+\frac{1}{2}}+5531a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^6+12(10395x^3_{i+\frac{1}{2}}+137025a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}}\\
&-573504a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}-21856a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max})h^4+64a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}(1902285x^3_{i+\frac{1}{2}}\\
&+38961a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}}-1044246a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}+34606a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max})h^2\\
&-101376a^4_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^4_{max}(1331a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}-11731x_{i+\frac{1}{2}})))\\
&/(99(315h^4+307440a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}h^2+3003136a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}))\\
\beta_{2,0} =&-(5(2871h^8-48(3465x^2_{i+\frac{1}{2}}+33186a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}-23006a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max})h^6\\
&+32(31185x^4_{i+\frac{1}{2}}+548100a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^3_{i+\frac{1}{2}}-3441024a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x^2_{i+\frac{1}{2}}\\
&-262272a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}x_{i+\frac{1}{2}}+329432a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max})h^4\\
&+512a^2_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^2_{max}(1902285x^3_{i+\frac{1}{2}}\\
&+51948a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}}-2088492a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}+138424a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max})h^2\\
&-270336a^4_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^4_{max}(5324a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}-35193x_{i+\frac{1}{2}})))\\
&/(1056(315h^4+307440a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}h^2+3003136a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}))\\
\beta_{1,0} =&(3465(29x_{i+\frac{1}{2}}+149a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^8-336(5775x^3_{i+\frac{1}{2}}+82965a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}}\\
&-115030a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}-12178a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max})h^6+160(43659x^5_{i+\frac{1}{2}}
\end{aligned}
$$

$$+ 959175a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^4_{i+\frac{1}{2}} - 8029056a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x^3_{i+\frac{1}{2}} - 917952a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}x^2_{i+\frac{1}{2}}$$

$$+ 2306024a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}x_{i+\frac{1}{2}} - 53240a^5_{i+\frac{1}{2},j+\frac{1}{2}}\tau^5_{max})h^4$$

$$+ 17920a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^2_{max}(380457x^3_{i+\frac{1}{2}}$$

$$+ 12987a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}} - 696164a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}$$

$$+ 69212a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max})h^2 - 1892352a^4_{i+\frac{1}{2},j+\frac{1}{2}}x^4_{i+\frac{1}{2}}\tau^4_{max}$$

$$(6655a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 35193x_{i+\frac{1}{2}}))$$

$$/(3696(315h^4 + 307440a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}h^2 + 3003136a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}))$$

$$\beta_{0,0} =(17325h^{10} - 84(14355x^2_{i+\frac{1}{2}} + 147510a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 37792a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max})h^8$$

$$+ 32(363825x^4_{i+\frac{1}{2}} + 6969060a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^3_{i+\frac{1}{2}} - 14493780a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x^2_{i+\frac{1}{2}}$$

$$- 3068856a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}x_{i+\frac{1}{2}} + 873136a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max})h^6$$

$$- 1920x_{i+\frac{1}{2}}(14553x^5_{i+\frac{1}{2}} + 383670a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^4_{i+\frac{1}{2}} - 4014528a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x^3_{i+\frac{1}{2}}$$

$$- 611968a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}x^2_{i+\frac{1}{2}} + 2306024a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}x_{i+\frac{1}{2}} - 106480a^5_{i+\frac{1}{2},j+\frac{1}{2}}\tau^5_{max})h^4$$

$$- 14336a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^2_{max}(1902285x^3_{i+\frac{1}{2}} + 77922a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}}$$

$$- 5221230a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}} + 692120a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max})h^2$$

$$+ 22708224a^4_{i+\frac{1}{2},j+\frac{1}{2}}x^5_{i+\frac{1}{2}}\tau^4_{max}(2662a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 11731x_{i+\frac{1}{2}}))$$

$$/(88704(315h^4 + 307440a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}h^2 + 3003136a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}))$$

Hence, we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y) - x^6 - u_I(x-\frac{h}{2}, y-\frac{h}{2}) + (x-\frac{h}{2})^6)^2dxdy$$

$$= (h^{12}\tau^2_{max}(a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2(19501200013171323158958545 3056a^8_{i,j}a^8_{i+\frac{1}{2},j+\frac{1}{2}}\tau^{16}_{max}$$

$$- 2018919508523313339165573 12a^6_{i,j}a^6_{i+\frac{1}{2},j+\frac{1}{2}}h^3\tau^{13}_{max}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$+ 1800338400h^{14}\tau^2_{max}(13438284110a^2_{i,j} - 51320885931a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 13438284110a^2_{i+\frac{1}{2},j+\frac{1}{2}}) + 1004161414729071329 28a^6_{i,j}a^6_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^{14}_{max}$$

$$(397624535a^2_{i,j} - 18450573a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} + 397624535a^2_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$- 2200413600h^{13}\tau^3_{max}(10662687321a^3_{i,j} + 292187561882a^2_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 292187561882 a_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}} + 10662687321 a^3_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$+ 228614400 h^{12} \tau^4_{max} (2094049467654 a^4_{i,j} - 8864817048804 a^3_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 295334595970901 a^2_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 8864817048804 a_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}} + 2094049467654 a^4_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$- 23253221376 a^4_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}} h^5 \tau^{11}_{max} (2316625977228695 a^3_{i,j}$$

$$+ 3103819172036596 a^2_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}} + 3103819172036596 a_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 2316625977228695 a^3_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$+ 1056964608 a^4_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}} h^4 \tau^{12}_{max} (1972327189988974885 a^4_{i,j}$$

$$- 3577211711633213550 a^3_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}} + 8331528893607301417 a^2_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 3577211711633213550 a_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}} + 1972327189988974885 a^4_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$- 11735539200 h^{11} \tau^5_{max} (54815097606 a^5_{i,j} + 1107739544932 a^4_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 4065675623387 a^3_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 4065675623387 a^2_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}} + 1107739544932 a_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 54815097606 a^5_{i+\frac{1}{2},j+\frac{1}{2}}) - 19074908160 a^2_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$h^7 \tau^9_{max} (179294959298488 a^5_{i,j} + 308410865284400 a^4_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 770619223416953 a^3_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}} + 770619223416953 a^2_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 308410865284400 a_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 179294959298488 a^5_{i+\frac{1}{2},j+\frac{1}{2}}) + 487710720 h^{10} \tau^6_{max} (5441989209404 a^6_{i,j}$$

$$- 26664212415965 a^5_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}} + 2805981567667107 a^4_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 492603505952389 a^3_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}} + 2805981567667107 a^2_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 26664212415965 a_{i,j} a^5_{i+\frac{1}{2},j+\frac{1}{2}} + 5441989209404 a^6_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$+ 1189085184 a^2_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}} h^6 \tau^{10}_{max} (3522080594245675 a^6_{i,j}$$

$$- 16385982633633415 a^5_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}} + 466662656914525505 a^4_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 655805535133544431 a^3_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}} + 466662656914525505 a^2_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 16385982633633415 a_{i,j} a^5_{i+\frac{1}{2},j+\frac{1}{2}} + 3522080594245675 a^6_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$- 1788272640 h^9 \tau^7_{max} (1947374673365 a^7_{i,j} + 36401168398920 a^6_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 422084292605924 a^5_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}} + 595382564504327 a^4_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 595382564504327 a^3_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 422084292605924 a^2_{i,j} a^5_{i+\frac{1}{2},j+\frac{1}{2}} + 36401168398920 a_{i,j} a^6_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$+ 1947374673365 a^7_{i+\frac{1}{2},j+\frac{1}{2}}) + 185794560 h^8 \tau^8_{max} (11547805227035 a^8_{i,j}$$

$$- 106377383972860 a^7_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}} + 39707200376096864 a^6_{i,j} a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 22325282290601846 a^5_{i,j} a^3_{i+\frac{1}{2},j+\frac{1}{2}} + 259867644987323578 a^4_{i,j} a^4_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 22325282290601846 a^3_{i,j} a^5_{i+\frac{1}{2},j+\frac{1}{2}} + 39707200376096864 a^2_{i,j} a^6_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- 106377383972860 a_{i,j} a^7_{i+\frac{1}{2},j+\frac{1}{2}} + 11547805227035 a^8_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$+ 604417549765837500 h^{15} \tau_{max} (a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) + 71270165866433625 h^{16}))$$

$$/(21638098944 (3003136 a^4_{i,j} \tau^4_{max} + 307440 a^2_{i,j} h^2 \tau^2_{max} + 315 h^4)^2$$

$$(3003136 a^4_{i+\frac{1}{2},j+\frac{1}{2}} \tau^4_{max} + 307440 a^2_{i+\frac{1}{2},j+\frac{1}{2}} h^2 \tau^2_{max} + 315 h^4)^2) = O(h^{16})$$

Similarly, we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x,y) - x^6 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^6)^2 dxdy = O(h^{16}),$$
$$\int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x,y) - x^6 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^6)^2 dxdy = O(h^{16}),$$
$$\int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x,y) - x^6 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^6)^2 dxdy = O(h^{16}).$$

7. $k = 6, u = x^7$, by the definition of the projection,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dxdy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^7 dxdy$$
$$\tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} = \tilde{P}_h(x^7; x^k y^l; f, g, u)_{i,j}, \ k = 0, 1, ..., 6, \ l = 0, 1, ..., 6$$
$$\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dxdy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^7 dxdy$$
$$\tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} = \tilde{Q}_h(x^7; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}}, \ k = 0, 1, ..., 6, \ l = 0, 1, ..., 6$$

then by solving the above linear system we have

$$u_I(x,y) = \sum_{k=0}^{6}\sum_{l=0}^{6}\alpha_{k,l}x^k y^l$$
$$v_I(x,y) = \sum_{k=0}^{6}\sum_{l=0}^{6}\beta_{k,l}x^k y^l \qquad (2.33)$$

where

$$\alpha_{k,l} = 0,\ k=0,1,...,6,\ l=1,...,6$$

$$\begin{aligned}
\alpha_{6,0} =&(7(94332080128a_{i,j}^6\tau_{max}^6 x_i - 1280a_{i,j}^4 h^2\tau_{max}^4(4415546a_{i,j}\tau_{max} \\
&- 42028623x_i) - 672a_{i,j}^2 h^4\tau_{max}^2(789340a_{i,j}\tau_{max} - 6671691x_i) \\
&+ 63h^6(67718a_{i,j}\tau_{max} + 2145x_i))) \\
&/(13(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4 h^2\tau_{max}^4 \\
&+ 344875104a_{i,j}^2 h^4\tau_{max}^2 + 10395h^6))
\end{aligned}$$

$$\begin{aligned}
\alpha_{5,0} =&(21(-754656641024a_{i,j}^6\tau_{max}^6 x_i^2 - 504h^6(-1271288a_{i,j}^2\tau_{max}^2 \\
&+ 135436a_{i,j}\tau_{max}x_i + 2145x_i^2) + 256a_{i,j}^2 h^4\tau_{max}^2(28301533a_{i,j}^2\tau_{max}^2 \\
&+ 33152280a_{i,j}\tau_{max}x_i - 140105511x_i^2) \\
&+ 2048a_{i,j}^4 h^2\tau_{max}^4(4448202a_{i,j}^2\tau_{max}^2 \\
&+ 44155460a_{i,j}\tau_{max}x_i - 210143115x_i^2) + 28665h^8)) \\
&/(104(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4 h^2\tau_{max}^4 \\
&+ 344875104a_{i,j}^2 h^4\tau_{max}^2 + 10395h^6))
\end{aligned}$$

$$\begin{aligned}
\alpha_{4,0} =& -(35(-8301223051264a_{i,j}^6\tau_{max}^6 x_i^3 + 67584a_{i,j}^4 h^2\tau_{max}^4 x_i(4448202a_{i,j}^2\tau_{max}^2 \\
&+ 22077730a_{i,j}\tau_{max}x_i - 70047705x_i^2) - 24h^6(103797556a_{i,j}^3\tau_{max}^3 \\
&- 881002584a_{i,j}^2\tau_{max}^2 x_i + 46928574a_{i,j}\tau_{max}x_i^2 + 495495x_i^3) \\
&- 256a_{i,j}^2 h^4\tau_{max}^2(108151636a_{i,j}^3\tau_{max}^3 - 933950589a_{i,j}^2\tau_{max}^2 x_i \\
&- 547012620a_{i,j}\tau_{max}x_i^2 + 1541160621x_i^3) \\
&+ 2079h^8(15216a_{i,j}\tau_{max} + 455x_i)))
\end{aligned}$$

$$/(1144(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4$$

$$+ 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\alpha_{3,0} = -(35(16602446102528a_{i,j}^6\tau_{max}^6x_i^4 - 132h^8(-6457912a_{i,j}^2\tau_{max}^2$$

$$+ 1917216a_{i,j}\tau_{max}x_i + 28665x_i^2) - 45056a_{i,j}^4h^2\tau_{max}^4x_i^2(26689212a_{i,j}^2\tau_{max}^2$$

$$+ 88310920a_{i,j}\tau_{max}x_i - 210143115x_i^2) + 16h^6(569473600a_{i,j}^4\tau_{max}^4$$

$$+ 1245570672a_{i,j}^3\tau_{max}^3x_i - 5286015504a_{i,j}^2\tau_{max}^2x_i^2$$

$$+ 187714296a_{i,j}\tau_{max}x_i^3 + 1486485x_i^4) + 512a_{i,j}^2h^4\tau_{max}^2(13427128a_{i,j}^4\tau_{max}^4$$

$$+ 432606544a_{i,j}^3\tau_{max}^3x_i - 1867901178a_{i,j}^2\tau_{max}^2x_i^2$$

$$- 729350160a_{i,j}\tau_{max}x_i^3 + 1541160621x_i^4) + 63063h^{10}))$$

$$/(2288(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4$$

$$+ 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\alpha_{2,0} = (7(49807338307584a_{i,j}^6\tau_{max}^6x_i^5 - 675840a_{i,j}^4h^2\tau_{max}^4x_i^3(8896404a_{i,j}^2\tau_{max}^2$$

$$+ 22077730a_{i,j}\tau_{max}x_i - 42028623x_i^2) - 180h^8(7154768a_{i,j}^3\tau_{max}^3$$

$$- 71037032a_{i,j}^2\tau_{max}^2x_i + 10544688a_{i,j}\tau_{max}x_i^2 + 105105x_i^3)$$

$$+ 1536a_{i,j}^2h^4\tau_{max}^2x_i(67135640a_{i,j}^4\tau_{max}^4 + 1081516360a_{i,j}^3\tau_{max}^3x_i$$

$$- 3113168630a_{i,j}^2\tau_{max}^2x_i^2 - 911687700a_{i,j}\tau_{max}x_i^3$$

$$+ 1541160621x_i^4) + 16h^6(-999477488a_{i,j}^5\tau_{max}^5 + 8542104000a_{i,j}^4\tau_{max}^4x_i$$

$$+ 9341780040a_{i,j}^3\tau_{max}^3x_i^2 - 26430077520a_{i,j}^2\tau_{max}^2x_i^3$$

$$+ 703928610a_{i,j}\tau_{max}x_i^4 + 4459455x_i^5)$$

$$+ 99h^{10}(334172a_{i,j}\tau_{max} + 9555x_i)))$$

$$/(2288(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4$$

$$+ 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\alpha_{1,0} = (7(-796917412921344a_{i,j}^6\tau_{max}^6x_i^6 - 1584h^{10}(-2907158a_{i,j}^2\tau_{max}^2$$

$$+ 2005032a_{i,j}\tau_{max}x_i + 28665x_i^2) + 32440320a_{i,j}^4h^2\tau_{max}^4x_i^4$$

$$(4448202a_{i,j}^2\tau_{max}^2 + 8831092a_{i,j}\tau_{max}x_i - 14009541x_i^2)$$

$$+ 864h^8(54537656a_{i,j}^4\tau_{max}^4 + 143095360a_{i,j}^3\tau_{max}^3x_i$$

$$- 710370320a_{i,j}^2\tau_{max}^2x_i^2 + 70297920a_{i,j}\tau_{max}x_i^3 + 525525x_i^4)$$

$$- 24576a_{i,j}^2h^4\tau_{max}^2x_i^2(201406920a_{i,j}^4\tau_{max}^4 + 2163032720a_{i,j}^3\tau_{max}^3x_i$$

$$- 4669752945a_{i,j}^2\tau_{max}^2x_i^2 - 1094025240a_{i,j}\tau_{max}x_i^3 + 1541160621x_i^4)$$

$$- 256h^6(-55859408a_{i,j}^6\tau_{max}^6 - 5996864928a_{i,j}^5\tau_{max}^5x_i$$

$$+ 25626312000a_{i,j}^4\tau_{max}^4x_i^2 + 18683560080a_{i,j}^3\tau_{max}^3x_i^3 - 39645116280a_{i,j}^2\tau_{max}^2x_i^4$$

$$+ 844714332a_{i,j}\tau_{max}x_i^5 + 4459455x_i^6) + 637065h^{12}))$$

$$/(109824(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4$$

$$+ 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\alpha_{0,0} =(796917412921344a_{i,j}^6\tau_{max}^6x_i^7 - 15138816a_{i,j}^4h^2\tau_{max}^4x_i^5(13344606a_{i,j}^2\tau_{max}^2$$

$$+ 22077730a_{i,j}\tau_{max}x_i - 30020445x_i^2) + 1008h^{10}(1438204a_{i,j}^3\tau_{max}^3$$

$$- 31978738a_{i,j}^2\tau_{max}^2x_i + 11027676a_{i,j}\tau_{max}x_i^2 + 105105x_i^3)$$

$$+ 172032a_{i,j}^2h^4\tau_{max}^2x_i^3(67135640a_{i,j}^4\tau_{max}^4 + 540758180a_{i,j}^3\tau_{max}^3x_i$$

$$- 933950589a_{i,j}^2\tau_{max}^2x_i^2 - 182337540a_{i,j}\tau_{max}x_i^3 + 220165803x_i^4)$$

$$- 32h^8(-967818016a_{i,j}^5\tau_{max}^5 + 10307616984a_{i,j}^4\tau_{max}^4x_i$$

$$+ 13522511520a_{i,j}^3\tau_{max}^3x_i^2 - 44753330160a_{i,j}^2\tau_{max}^2x_i^3 + 3321576720a_{i,j}\tau_{max}x_i^4$$

$$+ 19864845x_i^5) + 1792h^6x_i(-55859408a_{i,j}^6\tau_{max}^6 - 2998432464a_{i,j}^5\tau_{max}^5x_i$$

$$+ 8542104000a_{i,j}^4\tau_{max}^4x_i^2 + 4670890020a_{i,j}^3\tau_{max}^3x_i^3 - 7929023256a_{i,j}^2\tau_{max}^2x_i^4$$

$$+ 140785722a_{i,j}\tau_{max}x_i^5 + 637065x_i^6) - 693h^{12}(232088a_{i,j}\tau_{max} + 6435x_i))$$

$$/(109824(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4$$

$$+ 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_{k,l} =0, \ k = 0, 1, ..., 6, \ l = 1, ..., 6$$

$$\beta_{6,0} =(7(94332080128a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}} - 1280a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4(4415546a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}$$

$$
\begin{aligned}
&- 42028623x_{i+\frac{1}{2}}) - 672a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2(789340a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} \\
&- 6671691x_{i+\frac{1}{2}}) + 63h^6(67718a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 2145x_{i+\frac{1}{2}}))) \\
&/(13(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4 \\
&+ 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6)) \\
\beta_{5,0} =&(21(-754656641024a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 x_{i+\frac{1}{2}}^2 - 504h^6(-1271288a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
&+ 135436a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 2145x_{i+\frac{1}{2}}^2) \\
&+ 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2(28301533a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 33152280a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} \\
&- 140105511x_{i+\frac{1}{2}}^2) + 2048a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4(4448202a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
&+ 44155460a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 210143115x_{i+\frac{1}{2}}^2) + 28665h^8)) \\
&/(104(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4 \\
&+ 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6)) \\
\beta_{4,0} =& - (35(-8301223051264a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 x_{i+\frac{1}{2}}^3 + 67584a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4 x_{i+\frac{1}{2}} \\
&(4448202a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 22077730a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} \\
&- 70047705x_{i+\frac{1}{2}}^2) - 24h^6(103797556a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 881002584a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 x_{i+\frac{1}{2}} \\
&+ 46928574a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 495495x_{i+\frac{1}{2}}^3) - 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2 \\
&(108151636a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 933950589a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 x_{i+\frac{1}{2}} \\
&- 547012620a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 1541160621x_{i+\frac{1}{2}}^3) \\
&+ 2079h^8(15216a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 455x_{i+\frac{1}{2}}))) \\
&/(1144(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4 \\
&+ 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2 + 10395h^6)) \\
\beta_{3,0} =& - (35(16602446102528a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 x_{i+\frac{1}{2}}^4 - 132h^8(-6457912a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
&+ 1917216a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 28665x_{i+\frac{1}{2}}^2) - 45056a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4 x_{i+\frac{1}{2}}^2 \\
&(26689212a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 88310920a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 210143115x_{i+\frac{1}{2}}^2) \\
&+ 16h^6(569473600a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 1245570672a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 x_{i+\frac{1}{2}}
\end{aligned}
$$

$$- 5286015504a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 + 187714296a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3$$

$$+ 1486485x_{i+\frac{1}{2}}^4) + 512a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2(13427128a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4$$

$$+ 432606544a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 1867901178a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2$$

$$- 729350160a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 1541160621x_{i+\frac{1}{2}}^4) + 63063h^{10}))$$

$$/(2288(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4$$

$$+ 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_{2,0} =(7(49807338307584a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}^5 - 675840a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4x_{i+\frac{1}{2}}^3$$

$$(8896404a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 22077730a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}$$

$$- 42028623x_{i+\frac{1}{2}}^2) - 180h^8(7154768a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 71037032a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}$$

$$+ 10544688a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 105105x_{i+\frac{1}{2}}^3) + 1536a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2x_{i+\frac{1}{2}}$$

$$(67135640a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 1081516360a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}$$

$$- 3113168630a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 - 911687700a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3$$

$$+ 1541160621x_{i+\frac{1}{2}}^4) + 16h^6(-999477488a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5$$

$$+ 8542104000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}} + 9341780040a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^2$$

$$- 26430077520a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^3 + 703928610a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^4$$

$$+ 4459455x_{i+\frac{1}{2}}^5) + 99h^{10}(334172a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 9555x_{i+\frac{1}{2}})))$$

$$/(2288(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4$$

$$+ 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_{1,0} =(7(-796917412921344a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}^6 - 1584h^{10}(-2907158a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2$$

$$+ 2005032a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 28665x_{i+\frac{1}{2}}^2) + 32440320a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4x_{i+\frac{1}{2}}^4$$

$$(4448202a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 8831092a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}$$

$$- 14009541x_{i+\frac{1}{2}}^2) + 864h^8(54537656a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4$$

$$+ 143095360a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 710370320a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2$$

$$+ 70297920a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 525525x_{i+\frac{1}{2}}^4) - 24576a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2x_{i+\frac{1}{2}}^2$$

$$(201406920a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 2163032720a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}$$
$$- 4669752945a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 - 1094025240a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3$$
$$+ 1541160621x_{i+\frac{1}{2}}^4) - 256h^6(-55859408a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6$$
$$- 5996864928a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5x_{i+\frac{1}{2}} + 25626312000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^2$$
$$+ 18683560080a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^3 - 39645116280a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^4$$
$$+ 844714332a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^5 + 4459455x_{i+\frac{1}{2}}^6) + 637065h^{12}))$$
$$/(109824(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4$$
$$+ 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))$$

$$\beta_{0,0} =(796917412921344a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}^7 - 15138816a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4x_{i+\frac{1}{2}}^5$$
$$(13344606a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 22077730a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 30020445x_{i+\frac{1}{2}}^2)$$
$$+ 1008h^{10}(1438204a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 31978738a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}$$
$$+ 11027676a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 105105x_{i+\frac{1}{2}}^3) + 172032a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2x_{i+\frac{1}{2}}^3$$
$$(67135640a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 540758180a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 933950589a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2$$
$$- 182337540a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 220165803x_{i+\frac{1}{2}}^4)$$
$$- 32h^8(-967818016a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5 + 10307616984a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}$$
$$+ 13522511520a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^2 - 44753330160a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^3$$
$$+ 3321576720a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^4 + 19864845x_{i+\frac{1}{2}}^5)$$
$$+ 1792h^6x_{i+\frac{1}{2}}(-55859408a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 - 2998432464a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5x_{i+\frac{1}{2}}$$
$$+ 8542104000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^2 + 4670890020a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^3$$
$$- 7929023256a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^4 + 140785722a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^5$$
$$+ 637065x_{i+\frac{1}{2}}^6) - 693h^{12}(232088a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 6435x_{i+\frac{1}{2}}))$$
$$/(109824(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4$$
$$+ 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))$$

Hence, we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x,y) - x^7 - u_I(x-\frac{h}{2}, y-\frac{h}{2}) + (x-\frac{h}{2})^7)^2 dxdy$$

$$= ((a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2 h^{18} \tau_{max}^2 (437403046994236795781925 h^{20}$$

$$+ 885987355549577049748980 0(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) \tau_{max} h^{19}$$

$$+ 1485279180(1699073037709623145 a_{i,j}^2$$

$$- 1615895394979739357 4 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}$$

$$+ 1699073037709623145 a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \tau_{max}^2 h^{18}$$

$$- 2973528918360(1414536046558650 a_{i,j}^3 + 101689335736344761 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^2$$

$$+ 101689335736344761 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j} + 1414536046558650 a_{i+\frac{1}{2},j+\frac{1}{2}}^3) \tau_{max}^3 h^{17}$$

$$+ 3168595584(20961537958383611375 a_{i,j}^4$$

$$- 3224254874401015446 00 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^3$$

$$+ 151880015288861843693378 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2$$

$$- 3224254874401015446 00 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}$$

$$+ 20961537958383611375 a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \tau_{max}^4 h^{16}$$

$$- 257448391200(420292950978969849 a_{i,j}^5$$

$$+ 17072570927322274705 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^4 + 58676880448834885225 6 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^3$$

$$+ 58676880448834885225 6 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^2 + 17072570927322274705 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}$$

$$+ 420292950978969849 a_{i+\frac{1}{2},j+\frac{1}{2}}^5) \tau_{max}^5 h^{15}$$

$$+ 3840721920(1565954360007405135 90 a_{i,j}^6$$

$$- 3699346840438621615628 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^5$$

$$+ 310992455849545045853146 1 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^4$$

$$- 3855737499325029678375014 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3$$

$$+ 310992455849545045853146 1 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^2$$

$$- 3699346840438621615628 a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}$$

$$
\begin{aligned}
&+ 156595436000740513590 a_{i+\frac{1}{2},j+\frac{1}{2}}^{6})\tau_{max}^{6} h^{14}\\
&- 549223234560(1593068096417356625 a_{i,j}^{7}\\
&- 11325611182537259267 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^{6}\\
&+ 744203387671104242992 2a_{i+\frac{1}{2},j+\frac{1}{2}}^{2} a_{i,j}^{5}\\
&- 1658182140872840834036 12a_{i+\frac{1}{2},j+\frac{1}{2}}^{3} a_{i,j}^{4}\\
&- 165818214087284083403612 a_{i+\frac{1}{2},j+\frac{1}{2}}^{4} a_{i,j}^{3}\\
&+ 74420338767110424299 22a_{i+\frac{1}{2},j+\frac{1}{2}}^{5} a_{i,j}^{2}\\
&- 11325611182537259267 a_{i+\frac{1}{2},j+\frac{1}{2}}^{6} a_{i,j}\\
&+ 1593068096417356625 a_{i+\frac{1}{2},j+\frac{1}{2}}^{7})\tau_{max}^{7} h^{13}\\
&+ 585252864(366796085696145646 5300a_{i,j}^{8}\\
&- 1267862006464011966029 00a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^{7}\\
&+ 16796014129162861711359 7954a_{i+\frac{1}{2},j+\frac{1}{2}}^{2} a_{i,j}^{6}\\
&- 552419651770920469669003840 a_{i+\frac{1}{2},j+\frac{1}{2}}^{3} a_{i,j}^{5}\\
&+ 1145773697291063997025997 9445a_{i+\frac{1}{2},j+\frac{1}{2}}^{4} a_{i,j}^{4}\\
&- 552419651770920469669003840 a_{i+\frac{1}{2},j+\frac{1}{2}}^{5} a_{i,j}^{3}\\
&+ 167960141291628617113597954 a_{i+\frac{1}{2},j+\frac{1}{2}}^{6} a_{i,j}^{2}\\
&- 126786200646401196602900 a_{i+\frac{1}{2},j+\frac{1}{2}}^{7} a_{i,j}\\
&+ 3667960856961456465300 a_{i+\frac{1}{2},j+\frac{1}{2}}^{8})\tau_{max}^{8} h^{12}\\
&- 19020718080(116883680195507193925 a_{i,j}^{9}\\
&- 1291306824893591501781 5a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^{8}\\
&+ 198616697559007636775438 0a_{i+\frac{1}{2},j+\frac{1}{2}}^{2} a_{i,j}^{7}\\
&- 10723798433951653276905773 6a_{i+\frac{1}{2},j+\frac{1}{2}}^{3} a_{i,j}^{6}\\
&+ 676505105051936160968017607 a_{i+\frac{1}{2},j+\frac{1}{2}}^{4} a_{i,j}^{5}\\
&+ 676505105051936160968017607 a_{i+\frac{1}{2},j+\frac{1}{2}}^{5} a_{i,j}^{4}
\end{aligned}
$$

$$- 1072379843395165327690577 36a^6_{i+\frac{1}{2},j+\frac{1}{2}}a^3_{i,j}$$

$$+ 1986166975590076367754380a^7_{i+\frac{1}{2},j+\frac{1}{2}}a^2_{i,j}$$

$$- 129130682489359150178 15a^8_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 116883680195507193925a^9_{i+\frac{1}{2},j+\frac{1}{2}})\tau^9_{max}h^{11}$$

$$+ 3901685760(65385477659831406 1485a^{10}_{i,j}$$

$$- 2634753769349014772274 2a_{i+\frac{1}{2},j+\frac{1}{2}}a^9_{i,j}$$

$$+ 77009004765064962422325163a^2_{i+\frac{1}{2},j+\frac{1}{2}}a^8_{i,j}$$

$$- 527477199135104510286360772a^3_{i+\frac{1}{2},j+\frac{1}{2}}a^7_{i,j}$$

$$+ 3710307312791050294485861 6329a^4_{i+\frac{1}{2},j+\frac{1}{2}}a^6_{i,j}$$

$$- 7135645235449669176830704 26a^5_{i+\frac{1}{2},j+\frac{1}{2}}a^5_{i,j}$$

$$+ 3710307312791050294485861 6329a^6_{i+\frac{1}{2},j+\frac{1}{2}}a^4_{i,j}$$

$$- 527477199135104510286360772a^7_{i+\frac{1}{2},j+\frac{1}{2}}a^3_{i,j}$$

$$+ 77009004765064962422325163a^8_{i+\frac{1}{2},j+\frac{1}{2}}a^2_{i,j}$$

$$- 2634753769349014772274 2a^9_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 65385477659831406 1485a^{10}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{10}_{max}h^{10} + 101443829760a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(455972135396566303436 9a^9_{i,j} - 115407910838718913842225 7a_{i+\frac{1}{2},j+\frac{1}{2}}a^8_{i,j}$$

$$+ 12937054306977211828846 4708a^2_{i+\frac{1}{2},j+\frac{1}{2}}a^7_{i,j}$$

$$- 276708424103437212320330 6878a^3_{i+\frac{1}{2},j+\frac{1}{2}}a^6_{i,j}$$

$$- 214206650610350958342976 3149a^4_{i+\frac{1}{2},j+\frac{1}{2}}a^5_{i,j}$$

$$- 214206650610350958342976 3149a^5_{i+\frac{1}{2},j+\frac{1}{2}}a^4_{i,j}$$

$$- 276708424103437212320330 6878a^6_{i+\frac{1}{2},j+\frac{1}{2}}a^3_{i,j}$$

$$+ 12937054306977211828846 4708a^7_{i+\frac{1}{2},j+\frac{1}{2}}a^2_{i,j}$$

$$- 115407910838718913842225 7a^8_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 455972135396566303436 9a^9_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{11}_{max}h^9 + 16647192576a^2_{i,j}a^2_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(18461283971311741166192542a_{i,j}^8$$

$$-1830110024921776710278962 40a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^7$$

$$+566971607311825828999428199 25a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^6$$

$$+2372628401241137086503337520a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^5$$

$$+19046956134104309269090887 0388a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^4$$

$$+2372628401241137086503337520a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^3$$

$$+566971607311825828999428199 25a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^2$$

$$-1830110024921776710278962 40a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}$$

$$+18461283971311741166192542a_{i+\frac{1}{2},j+\frac{1}{2}}^8)\tau_{max}^{12}h^8$$

$$+1082067517440a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3(18321852318992025713799428a_{i,j}^7$$

$$-1321003108505324002163493396a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^6$$

$$-498594225191753361165098406a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^5$$

$$-4512798124101130032009521037a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^4$$

$$-4512798124101130032009521037a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^3$$

$$-498594225191753361165098406a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^2$$

$$-1321003108505324002163493396a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}$$

$$+18321852318992025713799428a_{i+\frac{1}{2},j+\frac{1}{2}}^7)\tau_{max}^{13}h^7+31708938240a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4$$

$$(488021418006903921207659291 81a_{i,j}^6$$

$$+1104847816821693733114994379 8a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^5$$

$$+6515956381766708083032466815 84a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^4$$

$$+4355799964713717304319930169 8a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^3$$

$$+6515956381766708083032466815 84a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^2$$

$$+1104847816821693733114994379 8a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}$$

$$+488021418006903921207659291 81a_{i+\frac{1}{2},j+\frac{1}{2}}^6)\tau_{max}^{14}h^6$$

$$
\begin{aligned}
&+ 1097319516733440a_{i,j}^{5}a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}(3198843901481228518023451a_{i,j}^{5}\\
&- 2284376727359244090017896 4a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{4}\\
&- 13165800443519527815052054a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{3}\\
&- 13165800443519527815052054a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{2}\\
&- 22843767273592440900178964a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}\\
&+ 3198843901481228518023451a_{i+\frac{1}{2},j+\frac{1}{2}}^{5})\tau_{max}^{15}h^{5} + 67645734912a_{i,j}^{6}a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}\\
&(4922581587727992675717073522 58a_{i,j}^{4}\\
&+ 115400788701858795739154476200a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{3}\\
&+ 196859296663269477913837624359 5a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{2}\\
&+ 115400788701858795739154476200a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}\\
&+ 49225815877279926757170735225 8a_{i+\frac{1}{2},j+\frac{1}{2}}^{4})\tau_{max}^{16}h^{4}\\
&+ 5852370755911680a_{i,j}^{7}a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}(1086215260739592648717971 6a_{i,j}^{3}\\
&- 1322862400208928722719131 7a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{2}\\
&- 13228624002089287227191317a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}\\
&+ 10862152607395926487179716a_{i+\frac{1}{2},j+\frac{1}{2}}^{3})\ \tau_{max}^{17}h^{3}\\
&+ 798923638576250880a_{i,j}^{8}a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}(25816117200476989307718 7a_{i,j}^{2}\\
&+ 500572501284991495982 70a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}\\
&+ 258161172004769893077187a_{i+\frac{1}{2},j+\frac{1}{2}}^{2})\tau_{max}^{18}h^{2}\\
&+ 30453904949648581259039734418445472628736 0a_{i,j}^{9}a_{i+\frac{1}{2},j+\frac{1}{2}}^{9}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})\\
&\tau_{max}^{19}h + 2767744103750261678414902919166173296721 92a_{i,j}^{10}a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}\tau_{max}^{20}))\\
&/(12249768960(10395h^{6} + 344875104a_{i,j}^{2}\tau_{max}^{2}h^{4} + 4138202880a_{i,j}^{4}\tau_{max}^{4}h^{2}\\
&+ 7256313856a_{i,j}^{6}\tau_{max}^{6})^{2}(10395h^{6} + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}\tau_{max}^{2}h^{4}\\
&+ 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}\tau_{max}^{4}h^{2} + 7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}\tau_{max}^{6})^{2}) = O(h^{18})
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}
\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}(v_I(x,y)-x^7-u_I(x-\frac{h}{2},y+\frac{h}{2})+(x-\frac{h}{2})^7)^2dxdy =&O(h^{18}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^7-u_I(x+\frac{h}{2},y-\frac{h}{2})+(x+\frac{h}{2})^7)^2dxdy =&O(h^{18}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^7-u_I(x+\frac{h}{2},y+\frac{h}{2})+(x+\frac{h}{2})^7)^2dxdy =&O(h^{18}).
\end{aligned}$$

8. $k=7, u=x^8$, by the definition of the projection,

$$\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}u_Idxdy =&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}x^8dxdy\\
\tilde{P}_h(u_I;x^ky^l;f,g,u)_{i,j} =&\tilde{P}_h(x^8;x^ky^l;f,g,u)_{i,j},\ \ k=0,1,...,7,\ \ l=0,1,...,7\\
\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}v_Idxdy =&\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}x^8dxdy\\
\tilde{Q}_h(v_I;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}} =&\tilde{Q}_h(x^8;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}},\ \ k=0,1,...,7,\ \ l=0,1,...,7
\end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned}
u_I(x,y) =& \sum_{k=0}^{7}\sum_{l=0}^{7}\alpha_{k,l}x^ky^l\\
v_I(x,y) =& \sum_{k=0}^{7}\sum_{l=0}^{7}\beta_{k,l}x^ky^l
\end{aligned}\tag{2.34}$$

where

$$\begin{aligned}
\alpha_{k,l} =&0,\ \ k=0,1,...,7,\ \ l=1,...,7\\
\alpha_{7,0} =&(8(4096a_{i,j}^6\tau_{max}^6(2354497739a_{i,j}\tau_{max}+239935274463x_i)\\
&-6912a_{i,j}^4h^2\tau_{max}^4(879791707a_{i,j}\tau_{max}-21360056375x_i)\\
&-54432a_{i,j}^2h^4\tau_{max}^2(13002767a_{i,j}\tau_{max}-89468841x_i)\\
&+51975h^6(14369a_{i,j}\tau_{max}+39x_i)))\\
&/(3(327591628066816a_{i,j}^6\tau_{max}^6+49213569888000a_{i,j}^4h^2\tau_{max}^4\\
&+1623322651104a_{i,j}^2h^4\tau_{max}^2+675675h^6))\\
\alpha_{6,0} =&(28(-20480a_{i,j}^6\tau_{max}^6x_i(4708995478a_{i,j}\tau_{max}+239935274463x_i)
\end{aligned}$$

$$- 945h^6(-407585262a_{i,j}^2\tau_{max}^2 + 7902950a_{i,j}\tau_{max}x_i + 10725x_i^2)$$

$$+ 288a_{i,j}^2h^4\tau_{max}^2(36101338556a_{i,j}^2\tau_{max}^2 + 24575229630a_{i,j}\tau_{max}x_i$$

$$- 84548054745x_i^2) + 1280a_{i,j}^4h^2\tau_{max}^4(50856691162a_{i,j}^2\tau_{max}^2$$

$$+ 47508752178a_{i,j}\tau_{max}x_i - 576721522125x_i^2) + 259875h^8))$$

$$/(15(327591628066816a_{i,j}^6\tau_{max}^6 + 49213569888000a_{i,j}^4h^2\tau_{max}^4$$

$$+ 1623322651104a_{i,j}^2h^4\tau_{max}^2 + 675675h^6))$$

$$\alpha_{5,0} = -(56(-266240a_{i,j}^6\tau_{max}^6x_i^2(2354497739a_{i,j}\tau_{max} + 79978424821x_i)$$

$$- 9h^6(95613443942a_{i,j}^3\tau_{max}^3 - 556353882630a_{i,j}^2\tau_{max}^2x_i + 5393763375a_{i,j}\tau_{max}x_i^2$$

$$+ 4879875x_i^3) - 288a_{i,j}^2h^4\tau_{max}^2(27266771676a_{i,j}^3\tau_{max}^3 - 469317401228a_{i,j}^2\tau_{max}^2x_i$$

$$- 159738992595a_{i,j}\tau_{max}x_i^2 + 366374903895x_i^3) + 1280a_{i,j}^4h^2\tau_{max}^4(5911894398a_{i,j}^3\tau_{max}^3$$

$$+ 661136985106a_{i,j}^2\tau_{max}^2x_i + 308806889157a_{i,j}\tau_{max}x_i^2 - 2499126595875x_i^3)$$

$$+ 61425h^8(21053a_{i,j}\tau_{max} + 55x_i)))$$

$$/(65(327591628066816a_{i,j}^6\tau_{max}^6 + 49213569888000a_{i,j}^4h^2\tau_{max}^4$$

$$+ 1623322651104a_{i,j}^2h^4\tau_{max}^2 + 675675h^6))$$

$$\alpha_{4,0} = -((7(2641275h^{10} + 2129920a_{i,j}^6x_i^3\tau_{max}^6(239935274463x_i + 9417990956a_{i,j}\tau_{max})$$

$$- 5616h^8(28875x_i^2 + 22105650a_{i,j}x_i\tau_{max} - 383499197a_{i,j}^2\tau_{max}^2)$$

$$- 30720a_{i,j}^4h^2x_i\tau_{max}^4(-2499126595875x_i^3 + 411742518876a_{i,j}x_i^2\tau_{max}$$

$$+ 1322273970212a_{i,j}^2x_i\tau_{max}^2 + 23647577592a_{i,j}^3\tau_{max}^3)$$

$$+ 72h^6(14639625x_i^4 + 21575053500a_{i,j}x_i^3\tau_{max} - 3338123295780a_{i,j}^2x_i^2\tau_{max}^2$$

$$+ 1147361327304a_{i,j}^3x_i\tau_{max}^3 + 687203235848a_{i,j}^4\tau_{max}^4)$$

$$+ 256a_{i,j}^2h^4\tau_{max}^2(9892122405165x_i^4 - 5750603733420a_{i,j}x_i^3\tau_{max}$$

$$- 25343139666312a_{i,j}^2x_i^2\tau_{max}^2 + 2944811341008a_{i,j}^3x_i\tau_{max}^3$$

$$+ 1087949803400a_{i,j}^4\tau_{max}^4)))$$

$$/(156(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2$$

$$+ 49213569888000a_{i,j}^4 h^2 \tau_{max}^4 + 327591628066816a_{i,j}^6 \tau_{max}^6)))$$

$$\alpha_{3,0} =(7(96525h^{10}(301x_i + 118347a_{i,j}\tau_{max}) + 4685824a_{i,j}^6 x_i^4 \tau_{max}^6(239935274463x_i$$

$$+ 11772488695a_{i,j}\tau_{max}) - 4752h^8(125125x_i^3 + 143686725a_{i,j}x_i^2\tau_{max}$$

$$- 4985489561a_{i,j}^2 x_i \tau_{max}^2 + 985840581a_{i,j}^3\tau_{max}^3)$$

$$- 112640a_{i,j}^4 h^2 x_i^2 \tau_{max}^4(-1499475957525x_i^3$$

$$+ 308806889157a_{i,j}x_i^2\tau_{max} + 1322273970212a_{i,j}^2 x_i\tau_{max}^2 + 35471366388a_{i,j}^3\tau_{max}^3)$$

$$+ 72h^6(32207175x_i^5 + 59331397125a_{i,j}x_i^4\tau_{max} - 12239785417860a_{i,j}^2 x_i^3\tau_{max}^2$$

$$+ 6310487300172a_{i,j}^3 x_i^2\tau_{max}^3 + 7559235594328a_{i,j}^4 x_i \tau_{max}^4 - 625703362520a_{i,j}^5\tau_{max}^5)$$

$$+ 256a_{i,j}^2 h^4 \tau_{max}^2(21762669291363x_i^5$$

$$- 15814160266905a_{i,j}x_i^4\tau_{max} - 92924845443144a_{i,j}^2 x_i^3\tau_{max}^2$$

$$+ 16196462375544a_{i,j}^3 x_i^2\tau_{max}^3 + 11967447837400a_{i,j}^4 x_i\tau_{max}^4 + 89226796360a_{i,j}^5\tau_{max}^5)))$$

$$/(429(675675h^6 + 1623322651104a_{i,j}^2 h^4\tau_{max}^2$$

$$+ 49213569888000a_{i,j}^4 h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6))$$

$$\alpha_{2,0} = -((7(-2413125h^{12} + 28114944a_{i,j}^6 x_i^5\tau_{max}^6(79978424821x_i + 4708995478a_{i,j}\tau_{max})$$

$$+ 38610h^{10}(4515x_i^2 + 3550410a_{i,j}x_i\tau_{max} - 25259534a_{i,j}^2\tau_{max}^2)$$

$$- 135168a_{i,j}^4 h^2 x_i^3\tau_{max}^4(-2499126595875x_i^3 + 617613778314a_{i,j}x_i^2\tau_{max}$$

$$+ 3305684925530a_{i,j}^2 x_i\tau_{max}^2 + 118237887960a_{i,j}^3\tau_{max}^3) - 4752h^8(375375x_i^4$$

$$+ 574746900a_{i,j}x_i^3\tau_{max} - 29912937366a_{i,j}^2 x_i^2\tau_{max}^2 + 11830086972a_{i,j}^3 x_i\tau_{max}^3$$

$$+ 3794229608a_{i,j}^4\tau_{max}^4) + 1536a_{i,j}^2 h^4 x_i\tau_{max}^2$$

$$(7254223097121x_i^5 - 6325664106762a_{i,j}x_i^4\tau_{max}$$

$$- 46462422721572a_{i,j}^2 x_i^3\tau_{max}^2 + 10797641583696a_{i,j}^3 x_i^2\tau_{max}^3$$

$$+ 11967447837400a_{i,j}^4 x_i\tau_{max}^4$$

$$+ 178453592720a_{i,j}^5\tau_{max}^5) + 16h^6(289864575x_i^6 + 640779088950a_{i,j}x_i^5\tau_{max}$$

$$- 165237103141110a_{i,j}^2 x_i^4\tau_{max}^2 + 113588771403096a_{i,j}^3 x_i^3\tau_{max}^3$$

$$+ 204099361046856a_{i,j}^4x_i^2\tau_{max}^4$$

$$- 33787981576080a_{i,j}^5x_i\tau_{max}^5 - 5178530366512a_{i,j}^6\tau_{max}^6)))$$

$$/(1716(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2$$

$$+ 49213569888000a_{i,j}^4h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6)))$$

$$\alpha_{1,0} =(-675675h^{12}(375x_i + 149849a_{i,j}\tau_{max}) + 140574720a_{i,j}^6x_i^6\tau_{max}^6(239935274463x_i$$

$$+ 16481484173a_{i,j}\tau_{max}) + 4158h^{10}(1467375x_i^3 + 1730824875a_{i,j}x_i^2\tau_{max}$$

$$- 24628045650a_{i,j}^2x_i\tau_{max}^2 + 5177901778a_{i,j}^3\tau_{max}^3)$$

$$- 14192640a_{i,j}^4h^2x_i^4\tau_{max}^4(-357018085125x_i^3$$

$$+ 102935629719a_{i,j}x_i^2\tau_{max} + 661136985106a_{i,j}^2x_i\tau_{max}^2 + 29559471990a_{i,j}^3\tau_{max}^3)$$

$$- 720h^8(52026975x_i^5 + 99574900425a_{i,j}x_i^4\tau_{max} - 6909888531546a_{i,j}^2x_i^3\tau_{max}^2$$

$$+ 4099125135798a_{i,j}^3x_i^2\tau_{max}^3 + 2629401118344a_{i,j}^4x_i\tau_{max}^4 - 303683537672a_{i,j}^5\tau_{max}^5)$$

$$+ 10752a_{i,j}^2h^4x_i^2\tau_{max}^2(15544763779545x_i^5 - 15814160266905a_{i,j}x_i^4\tau_{max}$$

$$- 139387268164716a_{i,j}^2x_i^3\tau_{max}^2 + 40491155938860a_{i,j}^3x_i^2\tau_{max}^3$$

$$+ 59837239187000a_{i,j}^4x_i\tau_{max}^4 + 1338401945400a_{i,j}^5\tau_{max}^5)$$

$$+ 560h^6(124227675x_i^7 + 320389544475a_{i,j}x_i^6\tau_{max}$$

$$- 99142261884666a_{i,j}^2x_i^5\tau_{max}^2 + 85191578552322a_{i,j}^3x_i^4\tau_{max}^3$$

$$+ 204099361046856a_{i,j}^4x_i^3\tau_{max}^4 - 50681972364120a_{i,j}^5x_i^2\tau_{max}^5$$

$$- 15535591099536a_{i,j}^6x_i\tau_{max}^6 - 74240090992a_{i,j}^7\tau_{max}^7))$$

$$/(12870(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2$$

$$+ 49213569888000a_{i,j}^4h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6))$$

$$\alpha_{0,0} = -((211486275h^14 + 2249195520a_{i,j}^6x_i^7\tau_{max}^6$$

$$(239935274463x_i + 18835981912a_{i,j}\tau_{max})$$

$$- 1729728h^{12}(9375x_i^2 + 7492450a_{i,j}x_i\tau_{max} - 22675009a_{i,j}^2\tau_{max}^2)$$

$$- 10813440a_{i,j}^4h^2x_i^5\tau_{max}^4(-7497379787625x_i^3 + 2470455113256a_{i,j}x_i^2\tau_{max}$$

$$
\begin{aligned}
&+ 18511835582968 a_{i,j}^2 x_i \tau_{max}^2 + 993198258864 a_{i,j}^3 \tau_{max}^3) + 6336 h^{10} (30814875 x_i^4 \\
&+ 48463096500 a_{i,j} x_i^3 \tau_{max} - 1034377917300 a_{i,j}^2 x_i^2 \tau_{max}^2 + 434943749352 a_{i,j}^3 x_i \tau_{max}^3 \\
&+ 87945430928 a_{i,j}^4 \tau_{max}^4) + 172032 a_{i,j}^2 h^4 x_i^3 \tau_{max}^2 (15544763779545 x_i^5 \\
&- 18073326019320 a_{i,j} x_i^4 \tau_{max} - 185849690886288 a_{i,j}^2 x_i^3 \tau_{max}^2 \\
&+ 64785849502176 a_{i,j}^3 x_i^2 \tau_{max}^3 + 119674478374000 a_{i,j}^4 x_i \tau_{max}^4 \\
&+ 3569071854400 a_{i,j}^5 \tau_{max}^5) - 5120 h^8 (156080925 x_i^6 + 358469641530 a_{i,j} x_i^5 \tau_{max} \\
&- 31094498391957 a_{i,j}^2 x_i^4 \tau_{max}^2 + 24594750814788 a_{i,j}^3 x_i^3 \tau_{max}^3 \\
&+ 23664610065096 a_{i,j}^4 x_i^2 \tau_{max}^4 - 5466303678096 a_{i,j}^5 x_i \tau_{max}^5 - 321570260696 a_{i,j}^6 \tau_{max}^6) \\
&+ 1792 h^6 x_i (621138375 x_i^7 + 1830797397000 a_{i,j} x_i^6 \tau_{max} \\
&- 660948412564440 a_{i,j}^2 x_i^5 \tau_{max}^2 + 681532628418576 a_{i,j}^3 x_i^4 \tau_{max}^3 \\
&+ 2040993610468560 a_{i,j}^4 x_i^3 \tau_{max}^4 - 675759631521600 a_{i,j}^5 x_i^2 \tau_{max}^5 \\
&- 310711821990720 a_{i,j}^6 x_i \tau_{max}^6 - 2969603639680 a_{i,j}^7 \tau_{max}^7)) \\
&/(1647360 (675675 h^6 + 1623322651104 a_{i,j}^2 h^4 \tau_{max}^2 \\
&+ 49213569888000 a_{i,j}^4 h^2 \tau_{max}^4 + 327591628066816 a_{i,j}^6 \tau_{max}^6)))
\end{aligned}
$$

$$
\beta_{k,l} = 0, \; k = 0, 1, ..., 7, \; l = 1, ..., 7
$$

$$
\begin{aligned}
\beta_{7,0} =& (8 (4096 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 (2354497739 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} + 239935274463 x_{i+\frac{1}{2}}) \\
&- 6912 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 (879791707 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} - 21360056375 x_{i+\frac{1}{2}}) \\
&- 54432 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (13002767 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} - 89468841 x_{i+\frac{1}{2}}) \\
&+ 51975 h^6 (14369 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} + 39 x_{i+\frac{1}{2}}))) \\
&/(3 (327591628066816 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 + 49213569888000 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 \\
&+ 1623322651104 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 + 675675 h^6))
\end{aligned}
$$

$$
\begin{aligned}
\beta_{6,0} =& (28 (-20480 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 x_{i+\frac{1}{2}} (4708995478 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} + 239935274463 x_{i+\frac{1}{2}}) \\
&- 945 h^6 (-407585262 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 + 7902950 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}} + 10725 x_{i+\frac{1}{2}}^2) \\
&+ 288 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (36101338556 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2
\end{aligned}
$$

$$+ 24575229630a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 84548054745x^2_{i+\frac{1}{2}})$$

$$+ 1280a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^4_{max}(50856691162a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}$$

$$+ 47508752178a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 576721522125x^2_{i+\frac{1}{2}}) + 259875h^8))$$

$$/(15(327591628066816a^6_{i+\frac{1}{2},j+\frac{1}{2}}\tau^6_{max} + 49213569888000a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^4_{max}$$

$$+ 1623322651104a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^4\tau^2_{max} + 675675h^6))$$

$$\beta_{5,0} = -(56(-266240a^6_{i+\frac{1}{2},j+\frac{1}{2}}\tau^6_{max}x^2_{i+\frac{1}{2}}(2354497739a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 79978424821x_{i+\frac{1}{2}})$$

$$- 9h^6(95613443942a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max} - 556353882630a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}$$

$$+ 5393763375a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}} + 4879875x^3_{i+\frac{1}{2}}) - 288a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^4\tau^2_{max}$$

$$(27266771676a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max} - 469317401228a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}$$

$$- 159738992595a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}} + 366374903895x^3_{i+\frac{1}{2}})$$

$$+ 1280a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^4_{max}(5911894398a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}$$

$$+ 661136985106a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}x_{i+\frac{1}{2}}$$

$$+ 308806889157a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x^2_{i+\frac{1}{2}} - 2499126595875x^3_{i+\frac{1}{2}})$$

$$+ 61425h^8(21053a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 55x_{i+\frac{1}{2}})))$$

$$/(65(327591628066816a^6_{i+\frac{1}{2},j+\frac{1}{2}}\tau^6_{max} + 49213569888000a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^4_{max}$$

$$+ 1623322651104a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^4\tau^2_{max} + 675675h^6))$$

$$\beta_{4,0} = -((7(2641275h^{10} + 2129920a^6_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^6_{max}$$

$$(239935274463x_{i+\frac{1}{2}} + 9417990956a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})$$

$$- 5616h^8(28875x^2_{i+\frac{1}{2}} + 22105650a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau_{max} - 383499197a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max})$$

$$- 30720a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2x_{i+\frac{1}{2}}\tau^4_{max}(-2499126595875x^3_{i+\frac{1}{2}}$$

$$+ 411742518876a_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau_{max} + 1322273970212a^2_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^2_{max}$$

$$+ 23647577592a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}) + 72h^6(14639625x^4_{i+\frac{1}{2}}$$

$$+ 21575053500a_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau_{max} - 3338123295780a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^2_{max}$$

$$+ 1147361327304a^3_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^3_{max} + 687203235848a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max})$$

$$+ 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2(9892122405165x_{i+\frac{1}{2}}^4 - 5750603733420a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^3\tau_{max}$$

$$- 25343139666312a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^2\tau_{max}^2 + 2944811341008a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}\tau_{max}^3$$

$$+ 1087949803400a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)))$$

$$/(156(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2$$

$$+ 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6)))$$

$$\beta_{3,0} =(7(96525h^{10}(301x_{i+\frac{1}{2}} + 118347a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})$$

$$+ 4685824a_{i+\frac{1}{2},j+\frac{1}{2}}^6 x_{i+\frac{1}{2}}^4\tau_{max}^6(239935274463x_{i+\frac{1}{2}}$$

$$+ 11772488695a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) - 4752h^8(125125x_{i+\frac{1}{2}}^3 + 143686725a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2\tau_{max}$$

$$- 4985489561a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}\tau_{max}^2 + 985840581a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3)$$

$$- 112640a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 x_{i+\frac{1}{2}}^2\tau_{max}^4(-1499475957525x_{i+\frac{1}{2}}^3$$

$$+ 308806889157a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2\tau_{max} + 1322273970212a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}\tau_{max}^2$$

$$+ 35471366388a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3) + 72h^6(32207175x_{i+\frac{1}{2}}^5$$

$$+ 59331397125a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4\tau_{max} - 12239785417860a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^3\tau_{max}^2$$

$$+ 6310487300172a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}^2\tau_{max}^3 + 7559235594328a_{i+\frac{1}{2},j+\frac{1}{2}}^4 x_{i+\frac{1}{2}}\tau_{max}^4$$

$$- 625703362520a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5) + 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2(21762669291363x_{i+\frac{1}{2}}^5$$

$$- 15814160266905a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4\tau_{max} - 929248454443144a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^3\tau_{max}^2$$

$$+ 16196462375544a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}^2\tau_{max}^3 + 11967447837400a_{i+\frac{1}{2},j+\frac{1}{2}}^4 x_{i+\frac{1}{2}}\tau_{max}^4$$

$$+ 89226796360a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5)))$$

$$/(429(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4\tau_{max}^2$$

$$+ 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2\tau_{max}^4 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6))$$

$$\beta_{2,0} = - ((7(-2413125h^{12} + 28114944a_{i+\frac{1}{2},j+\frac{1}{2}}^6 x_{i+\frac{1}{2}}^5\tau_{max}^6$$

$$(79978424821x_{i+\frac{1}{2}} + 4708995478a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})$$

$$+ 38610h^{10}(4515x_{i+\frac{1}{2}}^2 + 3550410a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau_{max} - 25259534a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2)$$

$$- 135168a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 x_{i+\frac{1}{2}}^3\tau_{max}^4(-2499126595875x_{i+\frac{1}{2}}^3$$

$$
\begin{aligned}
&+ 617613778314 a_{i+\frac{1}{2},j+\frac{1}{2}} x_{i+\frac{1}{2}}^2 \tau_{max} + 3305684925530 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}} \tau_{max}^2 \\
&+ 118237887960 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3) - 4752 h^8 (375375 x_{i+\frac{1}{2}}^4 \\
&+ 574746900 a_{i+\frac{1}{2},j+\frac{1}{2}} x_{i+\frac{1}{2}}^3 \tau_{max} - 29912937366 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^2 \tau_{max}^2 \\
&+ 11830086972 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}} \tau_{max}^3 \\
&+ 3794229608 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \tau_{max}^4) + 1536 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 x_{i+\frac{1}{2}} \tau_{max}^2 \\
&(7254223097121 x_{i+\frac{1}{2}}^5 - 6325664106762 a_{i+\frac{1}{2},j+\frac{1}{2}} x_{i+\frac{1}{2}}^4 \tau_{max} \\
&- 46462422721572 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^3 \tau_{max}^2 + 10797641583696 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}^2 \tau_{max}^3 \\
&+ 11967447837400 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 x_{i+\frac{1}{2}} \tau_{max}^4 + 178453592720 a_{i+\frac{1}{2},j+\frac{1}{2}}^5 \tau_{max}^5) \\
&+ 16 h^6 (289864575 x_{i+\frac{1}{2}}^6 + 640779088950 a_{i+\frac{1}{2},j+\frac{1}{2}} x_{i+\frac{1}{2}}^5 \tau_{max} \\
&- 165237103141110 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^4 \tau_{max}^2 + 113588771403096 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}^3 \tau_{max}^3 \\
&+ 204099361046856 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 x_{i+\frac{1}{2}}^2 \tau_{max}^4 \\
&- 33787981576080 a_{i+\frac{1}{2},j+\frac{1}{2}}^5 x_{i+\frac{1}{2}} \tau_{max}^5 - 5178530366512 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6))) \\
&/(1716(675675 h^6 + 1623322651104 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 \\
&+ 49213569888000 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 327591628066816 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6)))
\end{aligned}
$$

$$
\begin{aligned}
\beta_{1,0} =&(-675675 h^{12} (375 x_{i+\frac{1}{2}} + 149849 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max}) \\
&+ 140574720 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 x_{i+\frac{1}{2}}^6 \tau_{max}^6 (239935274463 x_{i+\frac{1}{2}} \\
&+ 16481484173 a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max}) + 4158 h^{10} \\
&(1467375 x_{i+\frac{1}{2}}^3 + 1730824875 a_{i+\frac{1}{2},j+\frac{1}{2}} x_{i+\frac{1}{2}}^2 \tau_{max} \\
&- 24628045650 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}} \tau_{max}^2 + 5177901778 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3) \\
&- 14192640 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 x_{i+\frac{1}{2}}^4 \tau_{max}^4 (-357018085125 x_{i+\frac{1}{2}}^3 \\
&+ 102935629719 a_{i+\frac{1}{2},j+\frac{1}{2}} x_{i+\frac{1}{2}}^2 \tau_{max} + 661136985106 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}} \tau_{max}^2 \\
&+ 29559471990 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3) - 720 h^8 (52026975 x_{i+\frac{1}{2}}^5 \\
&+ 99574900425 a_{i+\frac{1}{2},j+\frac{1}{2}} x_{i+\frac{1}{2}}^4 \tau_{max} - 6909888531546 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^3 \tau_{max}^2 \\
&+ 4099125135798 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}^2 \tau_{max}^3 + 2629401118344 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 x_{i+\frac{1}{2}} \tau_{max}^4
\end{aligned}
$$

$$
\begin{aligned}
&- 303683537672a^5_{i+\frac{1}{2},j+\frac{1}{2}}\tau^5_{max} + 10752a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^4x^2_{i+\frac{1}{2}}\tau^2_{max} \\
&(155447637795 45x^5_{i+\frac{1}{2}} - 15814160266905a_{i+\frac{1}{2},j+\frac{1}{2}}x^4_{i+\frac{1}{2}}\tau_{max} \\
&- 139387268164716a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^2_{max} + 40491155938860a^3_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^3_{max} \\
&+ 59837239187000a^4_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^4_{max} + 1338401945400a^5_{i+\frac{1}{2},j+\frac{1}{2}}\tau^5_{max}) \\
&+ 560h^6(124227675x^7_{i+\frac{1}{2}} + 320389544475a_{i+\frac{1}{2},j+\frac{1}{2}}x^6_{i+\frac{1}{2}}\tau_{max} \\
&- 99142261884666a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^5_{i+\frac{1}{2}}\tau^2_{max} + 85191578552322a^3_{i+\frac{1}{2},j+\frac{1}{2}}x^4_{i+\frac{1}{2}}\tau^3_{max} \\
&+ 204099361046856a^4_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^4_{max} - 50681972364120a^5_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^5_{max} \\
&- 15535591099536a^6_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^6_{max} - 74240090992a^7_{i+\frac{1}{2},j+\frac{1}{2}}\tau^7_{max})) \\
&/(12870(675675h^6 + 1623322651104a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^4\tau^2_{max} \\
&+ 49213569888000a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^4_{max} + 327591628066816a^6_{i+\frac{1}{2},j+\frac{1}{2}}\tau^6_{max}))
\end{aligned}
$$

$$
\begin{aligned}
\beta_{0,0} = &- ((211486275h^14 + 2249195520a^6_{i+\frac{1}{2},j+\frac{1}{2}}x^7_{i+\frac{1}{2}}\tau^6_{max} \\
&(239935274463x_{i+\frac{1}{2}} + 18835981912a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) \\
&- 1729728h^{12}(9375x^2_{i+\frac{1}{2}} + 7492450a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau_{max} - 22675009a^2_{i+\frac{1}{2},j+\frac{1}{2}}\tau^2_{max}) \\
&- 10813440a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2x^5_{i+\frac{1}{2}}\tau^4_{max}(-7497379787625x^3_{i+\frac{1}{2}} \\
&+ 2470455113256a_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau_{max} + 18511835582968a^2_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^2_{max} \\
&+ 993198258864a^3_{i+\frac{1}{2},j+\frac{1}{2}}\tau^3_{max}) + 6336h^{10}(30814875x^4_{i+\frac{1}{2}} \\
&+ 48463096500a_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau_{max} - 1034377917300a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^2_{max} \\
&+ 434943749352a^3_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^3_{max} + 87945430928a^4_{i+\frac{1}{2},j+\frac{1}{2}}\tau^4_{max}) \\
&+ 172032a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^4x^3_{i+\frac{1}{2}}\tau^2_{max}(15544763779545x^5_{i+\frac{1}{2}} \\
&- 18073326019320a_{i+\frac{1}{2},j+\frac{1}{2}}x^4_{i+\frac{1}{2}}\tau_{max} - 185849690886288a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^2_{max} \\
&+ 64785849502176a^3_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^3_{max} + 119674478374000a^4_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^4_{max} \\
&+ 3569071854400a^5_{i+\frac{1}{2},j+\frac{1}{2}}\tau^5_{max}) - 5120h^8(156080925x^6_{i+\frac{1}{2}} \\
&+ 358469641530a_{i+\frac{1}{2},j+\frac{1}{2}}x^5_{i+\frac{1}{2}}\tau_{max} - 31094498391957a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^4_{i+\frac{1}{2}}\tau^2_{max} \\
&+ 24594750814788a^3_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^3_{max} + 23664610065096a^4_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^4_{max}
\end{aligned}
$$

$$- 5466303678096a^5_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^5_{max} - 321570260696a^6_{i+\frac{1}{2},j+\frac{1}{2}}\tau^6_{max})$$

$$+ 1792h^6x_{i+\frac{1}{2}}(621138375x^7_{i+\frac{1}{2}} + 1830797397000a_{i+\frac{1}{2},j+\frac{1}{2}}x^6_{i+\frac{1}{2}}\tau_{max}$$

$$- 660948412564440a^2_{i+\frac{1}{2},j+\frac{1}{2}}x^5_{i+\frac{1}{2}}\tau^2_{max} + 681532628418576a^3_{i+\frac{1}{2},j+\frac{1}{2}}x^4_{i+\frac{1}{2}}\tau^3_{max}$$

$$+ 2040993610468560a^4_{i+\frac{1}{2},j+\frac{1}{2}}x^3_{i+\frac{1}{2}}\tau^4_{max} - 675759631521600a^5_{i+\frac{1}{2},j+\frac{1}{2}}x^2_{i+\frac{1}{2}}\tau^5_{max}$$

$$- 310711821990720a^6_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau^6_{max} - 2969603639680a^7_{i+\frac{1}{2},j+\frac{1}{2}}\tau^7_{max}))$$

$$/(1647360(675675h^6 + 1623322651104a^2_{i+\frac{1}{2},j+\frac{1}{2}}h^4\tau^2_{max}$$

$$+ 49213569888000a^4_{i+\frac{1}{2},j+\frac{1}{2}}h^2\tau^4_{max} + 327591628066816a^6_{i+\frac{1}{2},j+\frac{1}{2}}\tau^6_{max})))$$

Hence, we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y) - x^8 - u_I(x-\frac{h}{2}, y-\frac{h}{2}) + (x-\frac{h}{2})^8)^2dxdy$$

$$= ((a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2h^{16}\tau^2_{max}(640761169721718712704112466015625 0h^{24}$$

$$+ 1496959745748072653734491363152343 75(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})\tau h^{23}$$

$$+ 141932907121005000000(102198405596032324406a^2_{i,j}$$

$$- 196491342642272020 9639a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 102198405596032324406a^2_{i+\frac{1}{2},j+\frac{1}{2}})\tau^2_{max}h^{22}$$

$$- 5125354979369625000(342475000791495976813a^3_{i,j}$$

$$+ 708542614972257019738 87a_{i+\frac{1}{2},j+\frac{1}{2}}a^2_{i,j}$$

$$+ 708542614972257019738 87a^2_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 342475000791495976813a^3_{i+\frac{1}{2},j+\frac{1}{2}})\tau^3_{max}h^{21}$$

$$+ 7646217218640000(116529188781377048742572 61a^4_{i,j}$$

$$- 356825898865465222780684354a_{i+\frac{1}{2},j+\frac{1}{2}}a^3_{i,j}$$

$$+ 4836984475588643523349889892863a^2_{i+\frac{1}{2},j+\frac{1}{2}}a^2_{i,j}$$

$$- 356825898865465222780684354a^3_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 116529188781377048742572 61a^4_{i+\frac{1}{2},j+\frac{1}{2}})\tau^4_{max}h^{20}$$

$$- 84108389405040000(915648732630103373510011a^5_{i,j}$$

$$- 9407889926703309649033127a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^4$$

$$+ 503900820016459858673021500063a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^3$$

$$+ 503900820016459858673021500063a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^2$$

$$- 9407889926703309649033127a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}$$

$$+ 9156487326301033735100 11a_{i+\frac{1}{2},j+\frac{1}{2}}^5)\tau_{max}^5h^{19}$$

$$+ 11950131916800(16488418522200533380531921030 0a_{i,j}^6$$

$$- 663034270661650159342180875667 5a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^5$$

$$+ 18807623995111092338895940205334469 8a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^4$$

$$- 20857515008184967964857714772452143 6a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^3$$

$$+ 18807623995111092338895940205334469 8a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^2$$

$$- 663034270661650159342180875667 5a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}$$

$$+ 16488418522200533380531921030 0a_{i+\frac{1}{2},j+\frac{1}{2}}^6)\tau_{max}^6h^{18}$$

$$- 116513786188800(903035554428090280024727087 5a_{i,j}^7$$

$$- 373970812972841082020185538180 0a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^6$$

$$+ 16396719622720606444403866151826 08a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^5$$

$$- 46810843593735883988285511555310999 9a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^4$$

$$- 46810843593735883988285511555310999 9a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^3$$

$$+ 16396719622720606444403866151826 08a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^2$$

$$- 373970812972841082020185538180 0a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}$$

$$+ 903035554428090280024727087 5a_{i+\frac{1}{2},j+\frac{1}{2}}^7)\tau_{max}^7h^{17}$$

$$+ 1042920603648(17693229844358932539826204876250a_{i,j}^8$$

$$- 85101692632460906138022351923000 0a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^7$$

$$+ 47125344545016585156950522109397583050a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^6$$

$$- 97529619282774141346719350823793132925a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^5$$

$$+ 5255857505572670614682746527402276260122a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{4}$$

$$- 9752961928277414134671935082379313 2925a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{3}$$

$$+ 471253445450165851569505221093975830 50a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{2}$$

$$- 85101692632460906138022351923000 0a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}$$

$$+ 17693229844358932539826204876250a_{i+\frac{1}{2},j+\frac{1}{2}}^{8})\tau_{max}^{8}h^{16}$$

$$- 49712548773888(1010407720237332326172862056 25a_{i,j}^{9}$$

$$- 14010521258252280584316069314 3750a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{8}$$

$$+ 515825307191800298232619112676452 50a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{7}$$

$$- 44163649657013283522468049574357842925a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{6}$$

$$+ 244708639996747076704243816052072844257a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{5}$$

$$+ 244708639996747076704243816052072844257a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{4}$$

$$- 44163649657013283522468049574357842925a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{3}$$

$$+ 515825307191800298232619112676452 50a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}^{2}$$

$$- 14010521258252280584316069314 3750a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}a_{i,j}$$

$$+ 1010407720237332326172862056 25a_{i+\frac{1}{2},j+\frac{1}{2}}^{9})\tau_{max}^{9}h^{15}$$

$$+ 1854081073152(3358433129156460430108185867 0000a_{i,j}^{10}$$

$$- 190618019491531259927290757569 3750a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{9}$$

$$+ 2457641393125471768600595379003016598 00a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{8}$$

$$- 7076639427975453291446518596318955126 00a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{7}$$

$$+ 1341630638163455189490745832475543477879 28a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{6}$$

$$+ 263786013929245432326130453052567724354 07a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{5}$$

$$+ 1341630638163455189490745832475543477879 28a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{4}$$

$$- 7076639427975453291446518596318955126 00a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}^{3}$$

$$+ 2457641393125471768600595379003016598 00a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}a_{i,j}^{2}$$

$$- 190618019491531259927290757569375\!0a^{9}_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 3358433129156460430108185867000\!0a^{10}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{10}_{max}h^{14}$$

$$- 927040536576(4806491069858999270093254178125a^{11}_{i,j}$$

$$- 32905119135827149176686917250697500a_{i+\frac{1}{2},j+\frac{1}{2}}a^{10}_{i,j}$$

$$+ 1194725551322270110832843511753630520\!0a^{2}_{i+\frac{1}{2},j+\frac{1}{2}}a^{9}_{i,j}$$

$$- 2951637273449536863450817484274942845322\!5a^{3}_{i+\frac{1}{2},j+\frac{1}{2}}a^{8}_{i,j}$$

$$+ 7164719114354112883509864925472857081115\!5a^{4}_{i+\frac{1}{2},j+\frac{1}{2}}a^{7}_{i,j}$$

$$- 5150272365447366947074934866999841210849\!66a^{5}_{i+\frac{1}{2},j+\frac{1}{2}}a^{6}_{i,j}$$

$$- 5150272365447366947074934866999841210849\!66a^{6}_{i+\frac{1}{2},j+\frac{1}{2}}a^{5}_{i,j}$$

$$+ 7164719114354112883509864925472857081115\!5a^{7}_{i+\frac{1}{2},j+\frac{1}{2}}a^{4}_{i,j}$$

$$- 2951637273449536863450817484274942845322\!5a^{8}_{i+\frac{1}{2},j+\frac{1}{2}}a^{3}_{i,j}$$

$$+ 1194725551322270110832843511753630520\!0a^{9}_{i+\frac{1}{2},j+\frac{1}{2}}a^{2}_{i,j}$$

$$- 32905119135827149176686917250697500a^{10}_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 4806491069858999270093254178125a^{11}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{11}_{max}h^{13}$$

$$+ 117719433216(1541604544580000163965512122812\!5a^{12}_{i,j}$$

$$- 11248959872916054394746123802068750a_{i+\frac{1}{2},j+\frac{1}{2}}a^{11}_{i,j}$$

$$+ 1291629292868388271317120016409027487277\!5a^{2}_{i+\frac{1}{2},j+\frac{1}{2}}a^{10}_{i,j}$$

$$- 4892988345819519147167133830738518226400\!0a^{3}_{i+\frac{1}{2},j+\frac{1}{2}}a^{9}_{i,j}$$

$$+ 2909007446697316403571950192573791017510984\!9a^{4}_{i+\frac{1}{2},j+\frac{1}{2}}a^{8}_{i,j}$$

$$+ 8725725200094031043177646404131246876282890a^{5}_{i+\frac{1}{2},j+\frac{1}{2}}a^{7}_{i,j}$$

$$+ 870059109294716574639376788643534427737056\!31a^{6}_{i+\frac{1}{2},j+\frac{1}{2}}a^{6}_{i,j}$$

$$+ 8725725200094031043177646404131246876282890a^{7}_{i+\frac{1}{2},j+\frac{1}{2}}a^{5}_{i,j}$$

$$+ 2909007446697316403571950192573791017510984\!9a^{8}_{i+\frac{1}{2},j+\frac{1}{2}}a^{4}_{i,j}$$

$$- 4892988345819519147167133830738518226400\!0a^{9}_{i+\frac{1}{2},j+\frac{1}{2}}a^{3}_{i,j}$$

$$+ 129162929286838827131712001640902748727775a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}a_{i,j}^{2}$$

$$- 1124895987291605439474612380206875\,0a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}a_{i,j}$$

$$+ 15416045445800001639655121228125a_{i+\frac{1}{2},j+\frac{1}{2}}^{12})\tau_{max}^{12}h^{12}$$

$$- 706316599296a_{i,j}^{2}a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}(42659429235751758490946707825732478 75a_{i,j}^{9}$$

$$- 153862613744907003545930019277911586979725a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{8}$$

$$- 347480729910164592617524605433603234333190 2a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{7}$$

$$- 183723519463544043805643252928810869726294 38a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{6}$$

$$- 232252209723026888431094930830620643566081 83a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{5}$$

$$- 232252209723026888431094930830620643566081 83a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{4}$$

$$- 183723519463544043805643252928810869726294 38a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{3}$$

$$- 347480729910164592617524605433603234333190 2a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}^{2}$$

$$- 153862613744907003545930019277911586979725a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}a_{i,j}$$

$$+ 42659429235751758490946707825732478 75a_{i+\frac{1}{2},j+\frac{1}{2}}^{9})\tau_{max}^{13}h^{11}$$

$$+ 2054739197952a_{i,j}^{2}a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}(424386375027401373691705641844566262 5a_{i,j}^{10}$$

$$- 154770398349271645497495693523684339462 5a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{9}$$

$$+ 772727966136016799800780756747698000629039 13a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{8}$$

$$+ 272171288433635497715496497439641861533398 0a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{7}$$

$$+ 662822030695161278420058089490347166696937 15a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{6}$$

$$+ 106991440730713581217230765429160223584481 99a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{5}$$

$$+ 662822030695161278420058089490347166696937 15a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{4}$$

$$+ 272171288433635497715496497439641861533398 0a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}^{3}$$

$$+ 772727966136016799800780756747698000629039 13a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}a_{i,j}^{2}$$

$$- 154770398349271645497495693523684339462 5a_{i+\frac{1}{2},j+\frac{1}{2}}^{9}a_{i,j}$$

$$+ 424386375027401373691705641844566262 5a_{i+\frac{1}{2},j+\frac{1}{2}}^{10})\tau_{max}^{14}h^{10}$$

$$+ 150680874516 48a_{i,j}^{4}a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}$$

$$(1226440005433813283100595708201779028363970a_{i,j}^{7}$$

$$+ 4637165397258385176176955537957113705171438a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{6}$$

$$+ 12794089326376582464931337782336108624614973a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{5}$$

$$+ 22775067494022410799181373159209801943311225a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{4}$$

$$+ 22775067494022410799181373159209801943311225a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{3}$$

$$+ 12794089326376582464931337782336108624614973a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{2}$$

$$+ 4637165397258385176176955537957113705171438a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}$$

$$+ 1226440005433813283100595708201779028363970a_{i+\frac{1}{2},j+\frac{1}{2}}^{7})\tau_{max}^{15}h^{9}$$

$$+ 1826434842624a_{i,j}^{4}a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}$$

$$(5735376305254471497397007185633322552332115a_{i,j}^{8}$$

$$+ 17095013521095120080625974785789599199148 00a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{7}$$

$$+ 35561454275647916127732710519937156478270 9331a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{6}$$

$$+ 70479522127743247319870522725960510761860785a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{5}$$

$$+ 100282966430715748542308251116403175935390 3459a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{4}$$

$$+ 70479522127743247319870522725960510761860785a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{3}$$

$$+ 35561454275647916127732710519937156478270 9331a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{2}$$

$$+ 17095013521095120080625974785789599199148 00a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}$$

$$+ 5735376305254471497397007185633322552332115a_{i+\frac{1}{2},j+\frac{1}{2}}^{8})\tau_{max}^{16}h^{8}$$

$$+ 3652869685248a_{i,j}^{6}a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}$$

$$(20338456565763686111942356374753108430016 3875a_{i,j}^{5}$$

$$+ 47230090390934741305563086295839555251375 0987a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{4}$$

$$+ 84543905969843570827932390017258357197140 2815a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{3}$$

$$+ 84543905969843570827932390017258357197140 2815a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{2}$$

$$+ 47230090390934741305563086295839555251375098 7a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}$$

$$+ 20338456565763686111942356374753108430016387 5a_{i+\frac{1}{2},j+\frac{1}{2}}^5)\tau_{max}^{17}h^7$$

$$+ 463856467968a_{i,j}^6 a_{i+\frac{1}{2},j+\frac{1}{2}}^6$$

$$(13692499820866095498627700397799349860111838 75a_{i,j}^6$$

$$+ 27813979900772321820200292146290258597596349 5a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^5$$

$$+ 20485983393544401265668693110581367857751110 947a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^4$$

$$+ 19069172497242839818462345512107389817751810 93a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3$$

$$+ 20485983393544401265668693110581367857751110 947a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^2$$

$$+ 27813979900772321820200292146290258597596349 5a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}$$

$$+ 13692499820866095498627700397799349860111838 75a_{i+\frac{1}{2},j+\frac{1}{2}}^6)\tau_{max}^{18}h^6$$

$$+ 5566277615616a_{i,j}^8 a_{i+\frac{1}{2},j+\frac{1}{2}}^8$$

$$(16649513466350982239618072929738693621401557 65a_{i,j}^3$$

$$+ 25261013500405677593125588074522931353183453 61a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^2$$

$$+ 25261013500405677593125588074522931353183453 61a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}$$

$$+ 16649513466350982239618072929738693621401557 65a_{i+\frac{1}{2},j+\frac{1}{2}}^3)\tau_{max}^{19}h^5$$

$$+ 11132555231232a_{i,j}^8 a_{i+\frac{1}{2},j+\frac{1}{2}}^8$$

$$(12445771201429779317578861870092034491371325 65a_{i,j}^4$$

$$+ 14570313547326560336726392600475065496500915 0a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^3$$

$$+ 52777829956761460541151299358724589034437017 03a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2$$

$$+ 14570313547326560336726392600475065496500915 0a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}$$

$$+ 12445771201429779317578861870092034491371325 65a_{i+\frac{1}{2},j+\frac{1}{2}}^4)\tau_{max}^{20}h^4$$

$$+ 36865115691022402100861417388552566459381060 05395288883 2000$$

$$a_{i,j}^{10} a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})\tau_{max}^{21}h^3$$

$$+ 49531972439734470923977749503756206080a_{i,j}^{10} a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}$$

$$(2587661596660313943875a_{i,j}^2+130692634564583157233a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+2587661596660313943875a_{i+\frac{1}{2},j+\frac{1}{2}}^2)\tau_{max}^{22}h^2$$

$$+426590355493017768656985138297061848222462989929694793564160$$

$$a_{i,j}^{12}a_{i+\frac{1}{2},j+\frac{1}{2}}^{12}\tau_{max}^{24}))/(254418278400(675675h^6+1623322651104a_{i,j}^2\tau_{max}^2h^4$$

$$+49213569888000a_{i,j}^4\tau_{max}^4h^2$$

$$+327591628066816a_{i,j}^6\tau_{max}^6)^2(675675h^6+1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^4$$

$$+49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4h^2+327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6)^2)$$

$$=O(h^{20})$$

Similarly, we have

$$\begin{aligned}
\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}(v_I(x,y)-x^8-u_I(x-\frac{h}{2},y+\frac{h}{2})+(x-\frac{h}{2})^8)^2dxdy =&O(h^{20}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^8-u_I(x+\frac{h}{2},y-\frac{h}{2})+(x+\frac{h}{2})^8)^2dxdy =&O(h^{20}),\\
\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y)-x^8-u_I(x+\frac{h}{2},y+\frac{h}{2})+(x+\frac{h}{2})^8)^2dxdy =&O(h^{20}).
\end{aligned}$$

9. $k=8, u=x^9$, by the definition of the projection,

$$\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}u_I dxdy =&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}x^9dxdy\\
\tilde{P}_h(u_I;x^ky^l;f,g,u)_{i,j} =&\tilde{P}_h(x^9;x^ky^l;f,g,u)_{i,j},\ \ k=0,1,...,8,\ \ l=0,1,...,8\\
\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}v_I dxdy =&\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}x^9dxdy\\
\tilde{Q}_h(v_I;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}} =&\tilde{Q}_h(x^9;x^ky^l;f,g,u)_{i+\frac{1}{2},j+\frac{1}{2}},\ \ k=0,1,...,8,\ \ l=0,1,...,8
\end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned}
u_I(x,y)=&\sum_{k=0}^{8}\sum_{l=0}^{8}\alpha_{k,l}x^ky^l\\
v_I(x,y)=&\sum_{k=0}^{8}\sum_{l=0}^{8}\beta_{k,l}x^ky^l
\end{aligned}\tag{2.35}$$

By similar calculation we have,

$$\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x,y) - x^9 - u_I(x-\frac{h}{2}, y-\frac{h}{2}) + (x-\frac{h}{2})^9)^2 dxdy$$

$$= (3(a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2 h^{22} \tau_{max}^2 (1100247162677562372226747959055445824218750 h^{28}$$

$$+ 362716434886413039714947456336468263183593\,75(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})\tau_{max} h^{27}$$

$$+ 4759258195128937500 0(60335853737847071707192205209 1 a_{i,j}^2$$

$$- 187769094267261767842501573683 70 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}$$

$$+ 60335853737847071707192205209 1 a_{i+\frac{1}{2},j+\frac{1}{2}}^2)\tau_{max}^2 h^{26}$$

$$- 1779962564978222625000 0(392265732763604422564840797 a_{i,j}^3$$

$$+ 881527885756504264593056283964 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^2$$

$$+ 881527885756504264593056283964 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}$$

$$+ 392265732763604422564840797 a_{i+\frac{1}{2},j+\frac{1}{2}}^3)\tau_{max}^3 h^{25}$$

$$+ 15620129460936000 00(2476844634069685910790681532329585 a_{i,j}^4$$

$$- 1012882469025845166697212823193072 40 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^3$$

$$+ 13158062015845444103510276118008854478317 4 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2$$

$$- 1012882469025845166697212823193072 40 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}$$

$$+ 2476844634069685910790681532329585 a_{i+\frac{1}{2},j+\frac{1}{2}}^4)\tau_{max}^4 h^{24}$$

$$- 863012152716714000 00(775084225556119253134715403330 5 a_{i,j}^5$$

$$- 240029847268612355026157091207839 35 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^4$$

$$+ 34968196700601573740765634573416756678 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^3$$

$$+ 34968196700601573740765634573416756678 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^2$$

$$- 240029847268612355026157091207839 35 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}$$

$$+ 775084225556119253134715403330 5 a_{i+\frac{1}{2},j+\frac{1}{2}}^5)\tau_{max}^5 h^{23}$$

$$+ 119010510178560000(138108827052908952875797920120787149 0 a_{i,j}^6$$

$$- 764578606837963075222467241054518337 00 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^5$$

$$+ 2325277956484856920559196160444930677861210 87a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^4$$

$$- 137642650758131781459680385519949473952301362a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3$$

$$+ 232527795648485692055919616044493067786121087a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^2$$

$$- 764578606837963075222467241054518337 00a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}$$

$$+ 1381088270529089528757979201207871490a_{i+\frac{1}{2},j+\frac{1}{2}}^6)\tau_{max}^6 h^{22}$$

$$- 2023178673035520000(90724231662342182038403286722213 25a_{i,j}^7$$

$$- 1090166767535681435363470110534277424 55a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^6$$

$$+ 1414713410849990014253934332175319788202 94a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^5$$

$$- 893618706668429901997509362349101515272669984a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^4$$

$$- 893618706668429901997509362349101515272669984a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^3$$

$$+ 141471341084999001425393433217531978820294a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^2$$

$$- 109016676753568143536347011053427742455a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}$$

$$+ 9072423166234218203840328672221325a_{i+\frac{1}{2},j+\frac{1}{2}}^7)\tau_{max}^7 h^{21}$$

$$+ 177544817049600(12998137365405447658144174189512481017500a_{i,j}^8$$

$$- 1135156802616869989367341847124668260937500a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^7$$

$$+ 6614772780157165536692795537288316997608604578870a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^6$$

$$- 97228885587140834448698154065587797215339662913 60a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^5$$

$$+ 231187924411062488407850240634118399063950362233 5887a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^4$$

$$- 9722888558714083444869815406558779721533966291360a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^3$$

$$+ 6614772780157165536692795537288316997608604578870a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^2$$

$$- 1135156802616869989367341847124668260937500a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}$$

$$+ 12998137365405447658144174189512481017500a_{i+\frac{1}{2},j+\frac{1}{2}}^8)\tau_{max}^8 h^{20}$$

$$- 215805725123788800(8884570650976351224377101695881781 25a_{i,j}^9$$

$$- 139584489748797144049151304944357848893 75a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^8$$

$$+ 34904708339364654174924652814938752680359150a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{7}$$

$$- 6211804114705462697323402159174107384990473593350a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{6}$$

$$- 6137736630581073285363696286829426419555158244462 3a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{5}$$

$$- 61377366305810732853636962868294264195551582444623a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{4}$$

$$- 6211804114705462697323402159174107384990473593350a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{3}$$

$$+ 34904708339364654174924652814938752680359150a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}^{2}$$

$$- 139584489748797144049151304944357848893 75a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}a_{i,j}$$

$$+ 888457065097635122437710169588178125a_{i+\frac{1}{2},j+\frac{1}{2}}^{9})\tau_{max}^{9}h^{19}$$

$$+ 6763612078080(163452863680517390647354939191764402504 0625a_{i,j}^{10}$$

$$- 2825524842635056936599350987799562105759687 50a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{9}$$

$$+ 24321603682503351532255895611599510542481080710823 75a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{8}$$

$$- 8310677190402830036475253936167190697765323270636500a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{7}$$

$$+ 7356127678092772811353630086301910924963391405438894565a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{6}$$

$$+ 7689971029108327923821124643338240856991632311163316 94a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{5}$$

$$+ 7356127678092772811353630086301910924963391405438894565a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{4}$$

$$- 8310677190402830036475253936167190697765323270636500a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}^{3}$$

$$+ 2432160368250335153225589561159951054248108071082375a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}a_{i,j}^{2}$$

$$- 282552484263505693659935098779956210575968750a_{i+\frac{1}{2},j+\frac{1}{2}}^{9}a_{i,j}$$

$$+ 1634528636805173906473549391917644025040625a_{i+\frac{1}{2},j+\frac{1}{2}}^{10})\tau_{max}^{10}h^{18}$$

$$- 229962810654720(2878123627484681181624264388736389250000a_{i,j}^{11}$$

$$- 1481202486925716958029636281810440272114 0625a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{10}$$

$$+ 3198794111926954704516221484918861772015530568 75a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{9}$$

$$- 8019635559103183879993033159378623968758372354886250a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{8}$$

$$- 7576428435205941580623299943584075584986104421329216 50a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{7}$$

$$- 1052041039947118762333138560053367878357723741168180491a^{5}_{i+\frac{1}{2},j+\frac{1}{2}}a^{6}_{i,j}$$

$$- 1052041039947118762333138560053367878357723741168180491a^{6}_{i+\frac{1}{2},j+\frac{1}{2}}a^{5}_{i,j}$$

$$- 757642843520594158062329994358407558498610442132921650a^{7}_{i+\frac{1}{2},j+\frac{1}{2}}a^{4}_{i,j}$$

$$- 80196355591031838799930331593786239687583723548862\,50a^{8}_{i+\frac{1}{2},j+\frac{1}{2}}a^{3}_{i,j}$$

$$+ 319879411192695470451622148491886177201553056875a^{9}_{i+\frac{1}{2},j+\frac{1}{2}}a^{2}_{i,j}$$

$$- 14812024869257169580296362818104402721140625a^{10}_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 2878123627484681181624264388736389250000a^{11}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{11}_{max}h^{17}$$

$$+ 28858078199808(2942684922260313786679699660779164782812\,50a^{12}_{i,j}$$

$$- 2361615064818495460702339761062741771125000000a_{i+\frac{1}{2},j+\frac{1}{2}}a^{11}_{i,j}$$

$$+ 271335007199195793820450383855114325143802783168875\,0a^{2}_{i+\frac{1}{2},j+\frac{1}{2}}a^{10}_{i,j}$$

$$- 235302118049475755471797754263573521396346428300100\,00a^{3}_{i+\frac{1}{2},j+\frac{1}{2}}a^{9}_{i,j}$$

$$+ 6372550924297440423738926023621095468425178049295263\,0125a^{4}_{i+\frac{1}{2},j+\frac{1}{2}}a^{8}_{i,j}$$

$$+ 8744721099627909795672887753416351889485809029059518800a^{5}_{i+\frac{1}{2},j+\frac{1}{2}}a^{7}_{i,j}$$

$$+ 20927376083364567357634157531346483090224681798990545\,1396a^{6}_{i+\frac{1}{2},j+\frac{1}{2}}a^{6}_{i,j}$$

$$+ 8744721099627909795672887753416351889485809029059518800a^{7}_{i+\frac{1}{2},j+\frac{1}{2}}a^{5}_{i,j}$$

$$+ 6372550924297440423738926023621095468425178049295263\,0125a^{8}_{i+\frac{1}{2},j+\frac{1}{2}}a^{4}_{i,j}$$

$$- 235302118049475755471797754263573521396346428300100\,00a^{9}_{i+\frac{1}{2},j+\frac{1}{2}}a^{3}_{i,j}$$

$$+ 271335007199195793820450383855114325143802783168875\,0a^{10}_{i+\frac{1}{2},j+\frac{1}{2}}a^{2}_{i,j}$$

$$- 2361615064818495460702339761062741771125000000a^{11}_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 2942684922260313786679699660779164782812\,50a^{12}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{12}_{max}h^{16}$$

$$+ 6132341617459200(311388703238687464493695916561837031\,25a^{13}_{i,j}$$

$$- 1173448336354133137866662329456481968577187\,5a_{i+\frac{1}{2},j+\frac{1}{2}}a^{12}_{i,j}$$

$$- 339679055041153949983560459766213831352545952\,50a^{2}_{i+\frac{1}{2},j+\frac{1}{2}}a^{11}_{i,j}$$

$$+ 6606147259802446017922158057898891359434319639865\,50a^{3}_{i+\frac{1}{2},j+\frac{1}{2}}a^{10}_{i,j}$$

$$+ 65692100468565065459683659569975838586830345979699865 5a^{4}_{i+\frac{1}{2},j+\frac{1}{2}} a^{9}_{i,j}$$

$$+ 828238349517872069712092208307559854269603908735034523a^{5}_{i+\frac{1}{2},j+\frac{1}{2}} a^{8}_{i,j}$$

$$+ 2907846615912738424611579188259045667564783472133520852a^{6}_{i+\frac{1}{2},j+\frac{1}{2}} a^{7}_{i,j}$$

$$+ 2907846615912738424611579188259045667564783472133520852a^{7}_{i+\frac{1}{2},j+\frac{1}{2}} a^{6}_{i,j}$$

$$+ 828238349517872069712092208307559854269603908735034523a^{8}_{i+\frac{1}{2},j+\frac{1}{2}} a^{5}_{i,j}$$

$$+ 656921004685650654596836595699758385868303459796998655a^{9}_{i+\frac{1}{2},j+\frac{1}{2}} a^{4}_{i,j}$$

$$+ 6606147259802446017922158057898891359434319639865 50a^{10}_{i+\frac{1}{2},j+\frac{1}{2}} a^{3}_{i,j}$$

$$- 33967905504115394998356045976621383135254595250a^{11}_{i+\frac{1}{2},j+\frac{1}{2}} a^{2}_{i,j}$$

$$- 11734483363541331378666623294564819685771875a^{12}_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}$$

$$+ 31138870323868746449369591656183703125a^{13}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{13}_{max}h^{15}$$

$$+ 2690729902080(65777253990089577013342147584336873737 5000a^{14}_{i,j}$$

$$- 93579237862564184625416636749993460477375 0000a_{i+\frac{1}{2},j+\frac{1}{2}} a^{13}_{i,j}$$

$$+ 21300219977070328338528245279899276449119296331851250a^{2}_{i+\frac{1}{2},j+\frac{1}{2}} a^{12}_{i,j}$$

$$- 105500249913083058582219460614661918970719591698187250 0a^{3}_{i+\frac{1}{2},j+\frac{1}{2}} a^{11}_{i,j}$$

$$+ 881337553571127711037629135108538302190184228649907937 6825a^{4}_{i+\frac{1}{2},j+\frac{1}{2}} a^{10}_{i,j}$$

$$+ 128986464288274180933886367265395548927340084763695581 5570a^{5}_{i+\frac{1}{2},j+\frac{1}{2}} a^{9}_{i,j}$$

$$+ 874445755686667646005649275951661988461358504213822420 58732a^{6}_{i+\frac{1}{2},j+\frac{1}{2}} a^{8}_{i,j}$$

$$+ 610171092140274769413174683848325543963376538412478395 7150a^{7}_{i+\frac{1}{2},j+\frac{1}{2}} a^{7}_{i,j}$$

$$+ 874445755686667646005649275951661988461358504213822420 58732a^{8}_{i+\frac{1}{2},j+\frac{1}{2}} a^{6}_{i,j}$$

$$+ 128986464288274180933886367265395548927340084763695581 5570a^{9}_{i+\frac{1}{2},j+\frac{1}{2}} a^{5}_{i,j}$$

$$+ 881337553571127711037629135108538302190184228649907937 6825a^{10}_{i+\frac{1}{2},j+\frac{1}{2}} a^{4}_{i,j}$$

$$- 105500249913083058582219460614661918970719591698187250 0a^{11}_{i+\frac{1}{2},j+\frac{1}{2}} a^{3}_{i,j}$$

$$+ 21300219977070328338528245279899276449119296331851250a^{12}_{i+\frac{1}{2},j+\frac{1}{2}} a^{2}_{i,j}$$

$$- 93579237862564184625416636749993460477375 0000a^{13}_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}$$

$$+ 6577725399008957701334214758433687373750000a_{i+\frac{1}{2},j+\frac{1}{2}}^{14})\tau_{max}^{14}h^{14}$$

$$- 13082328783912960a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(2316518916288108165584169195913809747875000a_{i,j}^{13}$$

$$- 23009335833927844596786769820307182449456956250a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{12}$$

$$+ 2234076898252230587581171210332767321591221886615000a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{11}$$

$$- 2569262692700314105474432502299419529611490298684243500a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{10}$$

$$- 2610100289178123801574956327658860133849487175164481335a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{9}$$

$$- 31210213643687675237037019874204852537483967904732637840a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{8}$$

$$- 30882306708740486599466467604869866303812925806329378094a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{7}$$

$$- 30882306708740486599466467604869866303812925806329378094a_{i+\frac{1}{2},j+\frac{1}{2}}^{7}a_{i,j}^{6}$$

$$- 31210213643687675237037019874204852537483967904732637840a_{i+\frac{1}{2},j+\frac{1}{2}}^{8}a_{i,j}^{5}$$

$$- 2610100289178123801574956327658860133849487175164481335a_{i+\frac{1}{2},j+\frac{1}{2}}^{9}a_{i,j}^{4}$$

$$- 2569262692700314105474432502299419529611490298684243500a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}a_{i,j}^{3}$$

$$+ 2234076898252230587581171210332767321591221886615000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}a_{i,j}^{2}$$

$$- 23009335833927844596786769820307182449456956250a_{i+\frac{1}{2},j+\frac{1}{2}}^{12}a_{i,j}$$

$$+ 2316518916288108165584169195913809747875000a_{i+\frac{1}{2},j+\frac{1}{2}}^{13})\tau_{max}^{15}h^{13}$$

$$+ 28701118955520a_{i,j}^{2}a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}$$

$$(3983005504880383406859973916353309923851753592100000a_{i,j}^{12}$$

$$- 371803597593755344014498302832550445733836685957300000a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^{11}$$

$$+ 3573832402210526483256799138604423351086859453677682305375a_{i+\frac{1}{2},j+\frac{1}{2}}^{2}a_{i,j}^{10}$$

$$+ 517281761558761920542340699747782447916007900186874589300a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}a_{i,j}^{9}$$

$$+ 108752310025750705460582991819978419235883663099732700371042a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}a_{i,j}^{8}$$

$$+ 8667841143565661368886481081798822559742292006303275152980a_{i+\frac{1}{2},j+\frac{1}{2}}^{5}a_{i,j}^{7}$$

$$+ 334416288697967943954453597407311821589951489108850085896415a_{i+\frac{1}{2},j+\frac{1}{2}}^{6}a_{i,j}^{6}$$

$$+ 8667841143565661368886481081798822559742292006303275152980 a_{i+\frac{1}{2},j+\frac{1}{2}}^{7} a_{i,j}^{5}$$

$$+ 108752310025750705460582991819978419235883663099732700371042 a_{i+\frac{1}{2},j+\frac{1}{2}}^{8} a_{i,j}^{4}$$

$$+ 517281761558761920542340699747782447916007900186874589300 a_{i+\frac{1}{2},j+\frac{1}{2}}^{9} a_{i,j}^{3}$$

$$+ 3573832402210526483256799138604423351086859453677682305375 a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^{2}$$

$$- 37180359759375534401449830283255044573383668595730000 a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} a_{i,j}$$

$$+ 398300550488038340685997391635330992385175359210000 a_{i+\frac{1}{2},j+\frac{1}{2}}^{12}) \tau_{max}^{16} h^{12}$$

$$- 2439595111219200 a_{i,j}^{3} a_{i+\frac{1}{2},j+\frac{1}{2}}^{3}$$

$$(5216829605376663971736869510023198676502432824073000 a_{i,j}^{11}$$

$$- 38707754973779034490714085515864523123278422485573649750 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^{10}$$

$$- 23487662557951040805765305902058031966287280339025143620 a_{i+\frac{1}{2},j+\frac{1}{2}}^{2} a_{i,j}^{9}$$

$$- 1546401127625692389171782922169708978725595100239117516145 a_{i+\frac{1}{2},j+\frac{1}{2}}^{3} a_{i,j}^{8}$$

$$- 1382701115905465059538867484794329337189500181444526941897 a_{i+\frac{1}{2},j+\frac{1}{2}}^{4} a_{i,j}^{7}$$

$$- 4778708299270025542183899586959224165904566721735817959751 a_{i+\frac{1}{2},j+\frac{1}{2}}^{5} a_{i,j}^{6}$$

$$- 4778708299270025542183899586959224165904566721735817959751 a_{i+\frac{1}{2},j+\frac{1}{2}}^{6} a_{i,j}^{5}$$

$$- 1382701115905465059538867484794329337189500181444526941897 a_{i+\frac{1}{2},j+\frac{1}{2}}^{7} a_{i,j}^{4}$$

$$- 1546401127625692389171782922169708978725595100239117516145 a_{i+\frac{1}{2},j+\frac{1}{2}}^{8} a_{i,j}^{3}$$

$$- 23487662557951040805765305902058031966287280339025143620 a_{i+\frac{1}{2},j+\frac{1}{2}}^{9} a_{i,j}^{2}$$

$$- 38707754973779034490714085515864523123278422485573649750 a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}$$

$$+ 5216829605376663971736869510023198676502432824073000 a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}) \tau_{max}^{17} h^{11}$$

$$+ 191340793036800 a_{i,j}^{4} a_{i+\frac{1}{2},j+\frac{1}{2}}^{4}$$

$$(191632468709253965344733237878070249880033496630104999900 a_{i,j}^{10}$$

$$+ 8116371865709560306059587699709842737421761537367178840 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^{9}$$

$$+ 71697229107767367741227498609551487321421952618993554381338 a_{i+\frac{1}{2},j+\frac{1}{2}}^{2} a_{i,j}^{8}$$

$$+ 5675568589635422387689518297559965163646556548627006376620 a_{i+\frac{1}{2},j+\frac{1}{2}}^{3} a_{i,j}^{7}$$

$$+ 678874571969404486593651361170436732182185553225373227273135a^{4}_{i+\frac{1}{2},j+\frac{1}{2}}a^{6}_{i,j}$$

$$+ 194154044344149866199675817405284723805438279074848175683330a^{5}_{i+\frac{1}{2},j+\frac{1}{2}}a^{5}_{i,j}$$

$$+ 678874571969404486593651361170436732182185553225373227273135a^{6}_{i+\frac{1}{2},j+\frac{1}{2}}a^{4}_{i,j}$$

$$+ 567556858963542238768951829755996516364655654862700637662 0a^{7}_{i+\frac{1}{2},j+\frac{1}{2}}a^{3}_{i,j}$$

$$+ 716972291077673677412274986095514873214219526189935543813 38a^{8}_{i+\frac{1}{2},j+\frac{1}{2}}a^{2}_{i,j}$$

$$+ 811637186570956030605958769970984273742176153736717884 0a^{9}_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 191632468709253965344733237878070249880033496630104999900a^{10}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{18}_{max}h^{10}$$

$$- 6505586963251200a^{5}_{i,j}a^{5}_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(756322777508985130939091626235103634135571274951757194 0a^{9}_{i,j}$$

$$- 189407018484769127428614726102692198075893221029231419717 0a_{i+\frac{1}{2},j+\frac{1}{2}}a^{8}_{i,j}$$

$$- 135715741494964443576551298438837823314469037559242155822 8a^{2}_{i+\frac{1}{2},j+\frac{1}{2}}a^{7}_{i,j}$$

$$- 202138201400813467159074957248162409520558562861199404834 78a^{3}_{i+\frac{1}{2},j+\frac{1}{2}}a^{6}_{i,j}$$

$$- 190600184452785581105646139138709230668113417960826814214 17a^{4}_{i+\frac{1}{2},j+\frac{1}{2}}a^{5}_{i,j}$$

$$- 190600184452785581105646139138709230668113417960826814214 17a^{5}_{i+\frac{1}{2},j+\frac{1}{2}}a^{4}_{i,j}$$

$$- 202138201400813467159074957248162409520558562861199404834 78a^{6}_{i+\frac{1}{2},j+\frac{1}{2}}a^{3}_{i,j}$$

$$- 135715741494964443576551298438837823314469037559242155822 8a^{7}_{i+\frac{1}{2},j+\frac{1}{2}}a^{2}_{i,j}$$

$$- 189407018484769127428614726102692198075893221029231419717 0a^{8}_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 756322777508985130939091626235103634135571274951757194 0a^{9}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{19}_{max}h^{9}$$

$$+ 4081936918118400a^{6}_{i,j}a^{6}_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(120352782471975708473729469324530524915550326909573353399 2a^{8}_{i,j}$$

$$- 320073920478250310835996980651375911001391597890580294 40a_{i+\frac{1}{2},j+\frac{1}{2}}a^{7}_{i,j}$$

$$+ 141305873585123356093489161870497239419385588116437348159905a^{2}_{i+\frac{1}{2},j+\frac{1}{2}}a^{6}_{i,j}$$

$$+ 343217055774497432562166394664037296897905688232025813660 0a^{3}_{i+\frac{1}{2},j+\frac{1}{2}}a^{5}_{i,j}$$

$$+ 434177154524320603888523162597427546054438914788557665283103a^{4}_{i+\frac{1}{2},j+\frac{1}{2}}a^{4}_{i,j}$$

$$+ 343217055774497432562166394664037296897905688232025813660 0a^{5}_{i+\frac{1}{2},j+\frac{1}{2}} a^{3}_{i,j}$$

$$+ 1413058735851233560934891618704972394193855881164373481599 05a^{6}_{i+\frac{1}{2},j+\frac{1}{2}} a^{2}_{i,j}$$

$$- 32007392047825031083599698065137591100139159789005802944 0a^{7}_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}$$

$$+ 1203527824719757084737294693245305249155503269095733533992a^{8}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{20}_{max}h^{8}$$

$$- 346964638040064000a^{7}_{i,j}a^{7}_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(17041150655545150324249372730994204430805308898697369160a^{7}_{i,j}$$

$$- 14487567233223176429691370067012051072984868674318400630 50a_{i+\frac{1}{2},j+\frac{1}{2}} a^{6}_{i,j}$$

$$- 120993251690050635211903856486107888391374136031636011465 2a^{2}_{i+\frac{1}{2},j+\frac{1}{2}} a^{5}_{i,j}$$

$$- 459114200966256221866115710934253779488719641242114227804 1a^{3}_{i+\frac{1}{2},j+\frac{1}{2}} a^{4}_{i,j}$$

$$- 459114200966256221866115710934253779488719641242114227804 1a^{4}_{i+\frac{1}{2},j+\frac{1}{2}} a^{3}_{i,j}$$

$$- 120993251690050635211903856486107888391374136031636011465 2a^{5}_{i+\frac{1}{2},j+\frac{1}{2}} a^{2}_{i,j}$$

$$- 14487567233223176429691370067012051072984868674318400630 50a^{6}_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}$$

$$+ 17041150655545150324249372730994204430805308898697369160a^{7}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{21}_{max}h^{7}$$

$$+ 4947802324992000a^{8}_{i,j}a^{8}_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(418152818027283554490004427281581495635147891362813279943 72a^{6}_{i,j}$$

$$- 35212626749963099026239979350563182048866608448299429643 60a_{i+\frac{1}{2},j+\frac{1}{2}} a^{5}_{i,j}$$

$$+ 159714634486820768620247569818221915269128467798463596995830 5a^{2}_{i+\frac{1}{2},j+\frac{1}{2}} a^{4}_{i,j}$$

$$+ 331925123241499755473005939555693270008119971035397822119 0a^{3}_{i+\frac{1}{2},j+\frac{1}{2}} a^{3}_{i,j}$$

$$+ 159714634486820768620247569818221915269128467798463596995830 5a^{4}_{i+\frac{1}{2},j+\frac{1}{2}} a^{2}_{i,j}$$

$$- 35212626749963099026239979350563182048866608448299429643 60a^{5}_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}$$

$$+ 418152818027283554490004427281581495635147891362813279943 72a^{6}_{i+\frac{1}{2},j+\frac{1}{2}})\tau^{22}_{max}h^{6}$$

$$- 98203150206255035842560 00a^{9}_{i,j}a^{9}_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(22814556272812811845517404036661702594338050585004a^{5}_{i,j}$$

$$- 66829799656015559067631715273554952677483753295980 1a_{i+\frac{1}{2},j+\frac{1}{2}} a^{4}_{i,j}$$

$$- 6036695979268415846147755004568717851964680445760 94a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^3$$

$$- 6036695979268415846147755004568717851964680445760 94a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^2$$

$$- 6682979965601555906763171527355495267748375329598 01a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}$$

$$+ 2281455627281281184551740403666170259433805058500 4a_{i+\frac{1}{2},j+\frac{1}{2}}^5)\tau_{max}^{23}h^5$$

$$+ 1583296743997440000a_{i,j}^{10}a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}$$

$$(17948094745355790130444601146873057041226358101845959097106a_{i,j}^4$$

$$- 18929093279415710612403585836260218365100741923947035940 20a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^3$$

$$+ 22307009292432436147488504269514784896717726811593270494537 5a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2$$

$$- 18929093279415710612403585836260218365100741923947035940 20a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}$$

$$+ 17948094745355790130444601146873057041226358101845959097106a_{i+\frac{1}{2},j+\frac{1}{2}}^4)\tau_{max}^{24}h^4$$

$$- 7935608097475154411520000a_{i,j}^{11}a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}$$

$$(3633318154021499506524862999268725551766738599916 78a_{i,j}^3$$

$$- 3220712031035671122093763244347187829310069793792281a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^2$$

$$- 3220712031035671122093763244347187829310069793792281a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}$$

$$+ 3633318154021499506524862999268725551766738599916 78a_{i+\frac{1}{2},j+\frac{1}{2}}^3)\tau_{max}^{25}h^3$$

$$+ 330302228719588857039814656000 0a_{i,j}^{12}a_{i+\frac{1}{2},j+\frac{1}{2}}^{12}$$

$$(387094353321315583762358559014054997165882441 1a_{i,j}^2$$

$$- 424244558726974861595284900718368831038583530a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}$$

$$+ 387094353321315583762358559014054997165882441 1a_{i+\frac{1}{2},j+\frac{1}{2}}^2)\tau_{max}^{26}h^2$$

$$- 118512037316270637578931025983311408022063414467 27$$

$$910665292307164100034560000a_{i,j}^{13}a_{i+\frac{1}{2},j+\frac{1}{2}}^{13}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})\tau_{max}^{27}h$$

$$+ 510660795390178602055849958688994971584182157491 45$$

$$4754207872348059598848000 0a_{i,j}^{14}a_{i+\frac{1}{2},j+\frac{1}{2}}^{14}\tau_{max}^{28}))$$

$$/(20575423897600(10135125h^8 + 4380486227076960a_{i,j}^2\tau_{max}^2 h^6$$

$$+ 294877585914354432a_{i,j}^4\tau_{max}^4 h^4$$

$$+ 2587519954603806720a_{i,j}^6\tau_{max}^6 h^2 + 10253930735078604804a_{i,j}^8$$

$$\tau_{max}^8)^2(10135125h^8 + 4380486227076960a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 h^6$$

$$+ 294877585914354432a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 h^4$$

$$+ 2587519954603806720a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 h^2$$

$$+ 10253930735078604804a_{i+\frac{1}{2},j+\frac{1}{2}}^8\tau_{max}^8)^2) = O(h^{22})$$

Similarly, we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}(v_I(x,y) - x^9 - u_I(x-\frac{h}{2}, y+\frac{h}{2}) + (x-\frac{h}{2})^9)^2 dxdy = O(h^{22}),$$
$$\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y) - x^9 - u_I(x+\frac{h}{2}, y-\frac{h}{2}) + (x+\frac{h}{2})^9)^2 dxdy = O(h^{22}),$$
$$\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}(v_I(x,y) - x^9 - u_I(x+\frac{h}{2}, y+\frac{h}{2}) + (x+\frac{h}{2})^9)^2 dxdy = O(h^{22}).$$