Stability of high order finite difference and local discontinuous Galerkin schemes with explicit-implicit-null time-marching for high order dissipative and dispersive equations

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Abstract

Time discretization is an important issue for time-dependent partial differential equations (PDEs). For the k-th ($k \geq 2$) order PDEs, the explicit time-marching method may suffer from a severe time step restriction $\tau = O(h^k)$ for stability. The implicit and implicit-explicit (IMEX) time-marching methods can overcome this constraint. However, for the equations with nonlinear high derivative terms, the IMEX methods are not good choices either, since a nonlinear algebraic system must be solved (e.g. by Newton iteration) at each time step. The explicit-implicit-null (EIN) time-marching method is designed to cope with the above mentioned shortcomings. The basic idea of the EIN method discussed in this paper is to add and subtract a sufficiently large linear highest derivative term on one side of the considered equation, and then apply the IMEX time-marching method to the equivalent equation. The EIN method so designed does not need any nonlinear iterative solver, and the severe time step restriction for explicit methods can be removed. Coupled with the EIN time-marching method, we will discuss the high order finite difference and local discontinuous Galerkin schemes for solving high order dissipative and dispersive equations, respectively. By the aid of the Fourier method, we perform stability analysis for the schemes on the simplified equations with periodic boundary conditions, which demonstrates the stability criteria for the resulting schemes. Even though the analysis is only performed on the simplified equations, numerical experiments show that the proposed schemes are stable and can achieve optimal orders of accuracy for both one-dimensional and two-dimensional linear and nonlinear equations.

Keywords: Dissipative equation, dispersive equation, stability, explicit-implicit-null time discretization, finite difference, local discontinuous Galerkin, Fourier method.

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1 Introduction

In this paper, we consider a sequence of time-dependent partial differential equations (PDEs) with high order spatial derivatives. To simplify the presentation, below we will show only one-dimensional equations, although the conclusions are verified to hold also for two-dimensional equations in the numerical experiment section.

The second order diffusion equation

$$U_t - (a(U)U_x)_x = 0, \tag{1.1}$$

where $a(U) \geq 0$ is smooth and bounded, is a partial differential equation with second derivatives. The above equation, which has been widely used to model various processes in physics and engineering, usually involves the computation of nonlinear diffusion terms, such as the carburizing and nitriding models [8] for the thermo-chemical diffusion process, the models for the miscible displacement of one incompressible fluid by another in a porous media [15], and the augmented drift-diffusion model in semiconductor device simulations [9], etc. Here and below, we use the capital letter U to denote the exact solution of the considered PDE.

The dispersive equation

$$U_t + (r^{'}(U)g(r(U)_x)_x)_x = 0 \tag{1.2}$$

is a special KdV-type equation with third derivatives. The KdV-type equations, whose travelling-wave solutions called solitary waves play an important role in the long-term evolution of initial data [5], are often used to model the propagation of waves in a variety of nonlinear and dispersive media [4]. The best-known example may be the Korteweg-de Vries (KdV) equation [19], which is widely studied in fluid dynamics and plasma physics.

The fourth order diffusion equation

$$U_t + (a(U_x)U_{xx})_{xx} = 0 \tag{1.3}$$

is a special biharmonic-type equation, where the nonlinear term could be more general but we just present (1.3) as an example. The biharmonic-type equations have wide applications in thin bending theory problems, strain gradient theory problems, phase-field modelling and mathematical biology.

Time discretization is a very important issue for time-dependent partial differential equations. For the k-th ($k \geq 2$) order PDEs, the explicit time-marching method may suffer from a severe time step restriction $\tau = O(h^k)$ for stability, where τ is the time step and h is the mesh size. This time step is too small, resulting in excessive computational cost and rendering the explicit schemes impractical. Implicit time-marching method can overcome the constraint of small time step, however, it is cumbersome for nonlinear equations, since a nonlinear algebraic system must be solved (e.g. by Newton iteration) at each time step. The implicit-explicit (IMEX) time-marching methods, which treat different derivative terms differently, e.g., the higher order derivative terms are treated implicitly, whereas the rest of the terms are treated explicitly, have been proposed and studied by many authors [1, 2, 6, 7]. The IMEX method can not only alleviate the stringent time step restriction, but also reduce the difficulty of solving the algebraic equations when the higher order derivative terms of the equations are linear. However, for equations with nonlinear high derivative terms, this method may be much more expensive than explicit methods, because an iterative solution of the nonlinear algebraic equations is needed. As we will see later, the explicit-implicit-null (EIN) time-marching methods could cope with these shortcomings.

The basic idea of the EIN methods is to add and subtract a sufficiently large linear term on one side of the equation, and then apply the IMEX time-marching methods to the equivalent equation. The crucial step of the EIN method consists in adding and subtracting the right term, which needs to have the same scaling in wave number as the most stiff term in the equation [14]. In this paper, two equal highest derivative terms with constant coefficients are added to and subtracted from one side of the equation. In the following, we take the diffusion equation (1.1) as an example to introduce this method in detail. We add and subtract a term with constant diffusion coefficient $a_1 U_{xx}$ on the left-hand side of the considered PDE

$$U_t + \underbrace{a_1 U_{xx} - (a(U)U_x)_x}_{T_1} - \underbrace{a_1 U_{xx}}_{T_2} = 0, \qquad a_1 = a_0 \times \max_U a(U), \tag{1.4}$$

and then treat the damping term T_2 implicitly and the remaining term T_1 explicitly. Here, a_0 is a constant to be determined. Being implicit-explicit in time, the severe time step restriction for explicit methods will be removed. Besides, the EIN methods so designed can be very efficient for equations with nonlinear highest derivative terms, since the nonlinear terms are treated explicitly and the inverse matrix is only needed to be solved once.

As far as we know, the EIN methods were first proposed and adopted by Douglas and Dupont [13] to assure the stability for a nonlinear diffusion equation with an alternating-direction Galerkin spatial discretization on a rectangular domain. Subsequently, the methods were also applied with success to the level set equation for motion by mean curvature [24], the Boltzmann kinetic equations with stiff sources [16], the Kuramoto-Sivashinsky equation and the Rayleigh-Taylor instability in a Hele-Shaw cell [14], the Cahn-Hilliard equation [23] and so on. However, the discussion of the methods in these papers is limited to the lower order ones. Recently, in [26] the authors applied pairs of EIN methods up to order three with local discontinuous Galerkin (LDG) spatial discretization to solve the diffusion equations. However, stability analysis was given only for the first and second order schemes. In addition, to the best of our knowledge, there are few applications and analyses of the EIN methods with high order finite difference spatial discretization, and to PDEs with higher than second order spatial derivatives.

In this paper, we discuss high order finite difference and local discontinuous Galerkin schemes coupled with a specific EIN time discretization for solving high order diffusion and dispersive equations, respectively. The finite difference method has been used in a wide range of practical applications because of its simplicity in design and coding. More details about the method can be found in [17]. The LDG method has been first introduced by Cockburn and Shu in [11] for nonlinear convection-diffusion equations. Their work was motivated by the successful numerical experiments of Bassi and Rebay [3] for the compressible Navier-Stokes equations. The main idea of the LDG method is to rewrite the equation with higher order derivatives into an equivalent first order system, and then apply the discontinuous Galerkin method [10] to the system. The LDG techniques have already been developed for various high order PDEs [20, 23, 25–27, 29, 30].

For simplicity, we will give stability analysis for the schemes on the following simplified equations with periodic boundary conditions:

- The linear diffusion equation

$$U_t - aU_{xx} = 0; \tag{1.5}$$

- The linear dispersive equation

$$U_t + aU_{xxx} = 0; \tag{1.6}$$

- The linear biharmonic equation

$$U_t + aU_{xxxx} = 0, \tag{1.7}$$

where $a > 0$ is a constant. By the aid of the Fourier method, we show that the finite difference and the LDG schemes coupled with the EIN time discretization are stable for these simplified equations provided that $a_0 \geq 0.54$. Recall that a_0 is a constant to stabilize the scheme, as shown in (1.4). Even though the analysis is only performed on the linear equations containing the highest derivatives, numerical experiments show that the stability criterion appears to be also valid for both nonlinear equations as shown in (1.1)-(1.3) and equations containing lower order derivatives, such as the following convection-diffusion equations

$$U_t + f(U)_x = (a(U)U_x)_x$$

and other cases as demonstrated by the numerical examples. We have also studied other EIN time-marching methods, obtaining similar results, but we will not present those results here to save space.

The organization of this paper is as follows. In Section 2, we first present the semi-discrete finite difference scheme, the semi-discrete LDG scheme and the EIN time-marching method for the diffusion equation. Then the standard Fourier techniques are used to analyze the stability of the schemes in the linear case. Section 3 and Section 4 are devoted to the dispersive and biharmonic-type equations, respectively. Section 5 shows a series of numerical tests to examine the performance of the proposed schemes for both one-dimensional and two-dimensional linear and nonlinear problems. Finally, the concluding remarks are given in Section 6.

2 The second order diffusion equations

In this section, we will explore the high order finite difference and LDG schemes coupled with a specific high order EIN time-marching method for solving the diffusion equations. In this paper, the EIN time-marching method with an LDG spatial discretization will be referred to as the EIN-LDG scheme. Similarly, the EIN time-marching method with a finite difference spatial discretization will be referred to as the EIN finite difference scheme. The stability analysis of these schemes is performed on the simplified equation (1.5) by the aid of the Fourier method. Even though the analysis for the general nonlinear model is not available at present, the numerical experiments in Section 5 show that the schemes for the nonlinear diffusion equations (1.1) with and without first derivative convection terms are also stable with the same stability criterion.

2.1 The semi-discrete finite difference scheme

Assume that the computational domain $\Omega = [x_L, x_R]$ is uniformly partitioned into N cells, with the spatial mesh size $h = (x_R - x_L)/N$. In the following, we take the fourth order finite difference scheme as an example. We begin with the equation (1.4) to present the finite difference spatial discretization. We introduce the notation:

$$\mathscr{D}_x u_j = \frac{-(u_{j+2} - u_{j-2}) + 8(u_{j+1} - u_{j-1})}{12h}, \tag{2.1}$$

where u_j denotes the numerical solution at the grid point x_j. It can be easily checked that $\mathscr{D}_x u_j$ is a fourth order central difference approximation for $u_x(x_j)$. Then the fourth order semi-discrete finite difference scheme for (1.4) can be given in the following form:

$$(u_t)_j = \underbrace{\mathscr{D}_x(a(u)_j \mathscr{D}_x u_j) - a_1 \mathscr{D}_x \mathscr{D}_x u_j}_{T_1} + \underbrace{a_1 \mathscr{D}_x \mathscr{D}_x u_j}_{T_2}.$$

It is worth pointing out that when $a(U) = 1$, the formula for $\mathscr{D}_x \mathscr{D}_x u_j$ is equivalent to that for $\mathscr{D}_x(a(u)_j \mathscr{D}_x u_j)$, and is given by

$$\frac{(u_{j-4} + u_{j+4}) - 16(u_{j-3} + u_{j+3}) + 64(u_{j-2} + u_{j+2}) + 16(u_{j-1} + u_{j+1}) - 130u_j}{144h^2}.$$

2.2 The semi-discrete local discontinuous Galerkin scheme

Let $\mathcal{T}_h = \{I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]\}_{j=1}^N$ be a uniform partition of Ω, where $x_{\frac{1}{2}} = x_L$ and $x_{N+\frac{1}{2}} = x_R$ are the two boundary endpoints. Denote the center of each cell I_j by $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$. We begin with the equation (1.4) to give a preview of the LDG scheme. For a detailed introduction of the scheme, we refer the readers to [11,26]. By introducing the new variables

$$b(U) = \sqrt{a(U)}, \qquad B(U) = \int^U b(s)ds, \qquad P = B(U)_x, \qquad Q = U_x,$$

we can rewrite (1.4) into the following first order system:

$$U_t + (a_1 Q - b(U)P)_x = a_1 Q_x, \qquad P - B(U)_x = 0, \qquad Q - U_x = 0.$$

Then we seek piecewise polynomial solutions u, p, q from V_h such that for all the test functions ϕ_1, ϕ_2, $\phi_3 \in V_h$ and $1 \le j \le N$, we have

$$\int_{I_j} u_t \phi_1 dx - \int_{I_j} (a_1 q - b(u)p)(\phi_1)_x dx + (a_1 \hat{q} - \hat{b}\hat{p})_{j+\frac{1}{2}} (\phi_1)^-_{j+\frac{1}{2}} -$$
$$(a_1 \hat{q} - \hat{b}\hat{p})_{j-\frac{1}{2}} (\phi_1)^+_{j-\frac{1}{2}} = -a_1 \Big(\int_{I_j} q (\phi_1)_x dx - \hat{q}_{j+\frac{1}{2}} (\phi_1)^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}} (\phi_1)^+_{j-\frac{1}{2}} \Big),$$

$$\int_{I_j} p\phi_2 dx + \int_{I_j} B(u)(\phi_2)_x dx - \hat{B}_{j+\frac{1}{2}}(\phi_2)^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}}(\phi_2)^+_{j-\frac{1}{2}} = 0, \tag{2.2}$$
$$\int_{I_j} q\phi_3 dx + \int_{I_j} u(\phi_3)_x dx - \hat{u}_{j+\frac{1}{2}}(\phi_3)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}(\phi_3)^+_{j-\frac{1}{2}} = 0,$$

where

$$V_h = \{v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_m(I_j), \forall j = 1, ..., N\}. \tag{2.3}$$

Here, $\mathcal{P}_m(I_j)$ is the space of polynomials in I_j of degree less than or equal to m. The functions in V_h are allowed to have discontinuities across element interfaces. For any piecewise function p in V_h, we denote by $p^+_{j+\frac{1}{2}}$ and $p^-_{j+\frac{1}{2}}$ the values of p at the interface $x_{j+\frac{1}{2}}$, from the right cell, I_{j+1}, and the left cell, I_j, respectively. Now, the only ambiguity in the above algorithms is the definition of the so-called "numerical fluxes" (the terms with the "hat"), which are yet to be determined carefully to ensure stability. It was shown in [11] that we can prove a strong L^2-stability result and obtain error estimates if we adopt the following numerical fluxes (for simplicity, we omit the subscripts $j \pm \frac{1}{2}$ in the definition of the fluxes, as all quantities are evaluated at the interfaces $x_{j\pm\frac{1}{2}}$):

$$\hat{p} = p^+, \qquad \hat{q} = q^+, \qquad \hat{B} = B(u^-), \qquad u = u^-,$$
$$\hat{b} = \begin{cases} [B(u)]/[u], & \text{if } [u] \neq 0, \\ b((u^+ + u^-)/2), & \text{otherwise,} \end{cases} \tag{2.4}$$

where $[u]$ denotes $u^+ - u^-$.

We concentrate on the piecewise quadratic polynomial ($m = 2$) case for the LDG schemes discussed in this paper. For the LDG schemes based on the other order piecewise polynomials, the design and analysis are similar.

Define the Lagrangian nodal basis polynomials

$$L^s_j(x) = \prod_{\substack{l=0 \\ l\neq s}}^{2} \frac{(x - x^l_j)}{(x^s_j - x^l_j)}, \quad 0 \leq s \leq 2; \quad 1 \leq j \leq N, \tag{2.5}$$

and then the solution $u(x, t)$ inside each cell I_j can be represented as

$$u(x, t)|_{I_j} = \sum_{s=0}^{2} u^s_j(t) L^s_j(x). \tag{2.6}$$

Here we use $u_j^s(t)$ to represent $u(x_j^s, t)$, where x_j^s is given by

$$x_j^s = x_j + \frac{s-1}{3}h, \quad 0 \le s \le 2; \quad 1 \le j \le N. \tag{2.7}$$

With this representation, taking the test functions also as the Lagrangian nodal basis polynomials, respectively, and inverting the small 3×3 mass matrix, we can obtain easily the block finite difference schemes corresponding to the LDG method. In the following, we take the linear diffusion equation (1.5) as an example. We add and subtract a term with constant diffusion coefficient $a_1 U_{xx}$ on the left-hand side of the considered equation

$$U_t + \underbrace{(a_1 - a)U_{xx}}_{T_1} - \underbrace{a_1 U_{xx}}_{T_2} = 0, \qquad a_1 = a_0 \times a. \tag{2.8}$$

For this equation, we find that the LDG scheme (2.2) can be simplified to

$$\begin{aligned}
\int_{I_j} u_t \phi_1 dx - (a_1 - a)\Big(\int_{I_j} q(\phi_1)_x dx - \hat{q}_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}}\Big) = \\
-a_1\Big(\int_{I_j} q(\phi_1)_x dx - \hat{q}_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}}\Big), \\
\int_{I_j} q\phi_2 dx + \int_{I_j} u(\phi_2)_x dx - \hat{u}_{j+\frac{1}{2}}(\phi_2)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}(\phi_2)^+_{j-\frac{1}{2}} = 0.
\end{aligned}$$

With the choice of the numerical fluxes (2.4), we take the test functions ϕ_l, $1 \le l \le 2$ also as the Lagrangian nodal basis polynomials, respectively, and obtain easily the block finite difference schemes corresponding to the LDG method

$$\begin{aligned}
\vec{q}_j &= -\tilde{A}^{-1}(\tilde{B}\vec{u}_j + \tilde{C}\vec{u}_{j-1}), \\
(\vec{u}_j)_t &= \tilde{A}^{-1}\Big((a_1 - a)(\bar{B}\vec{q}_j + \bar{C}\vec{q}_{j+1}) - a_1(\bar{B}\vec{q}_j + \bar{C}\vec{q}_{j+1})\Big),
\end{aligned} \tag{2.9}$$

where

$$\begin{cases}
\tilde{A} = (\tilde{a}_{sl})_{3\times 3}, & \tilde{a}_{sl} = \int_{I_j} L_j^s L_j^l dx \\
\tilde{B} = (\tilde{b}_{sl})_{3\times 3}, & \tilde{b}_{sl} = \int_{I_j} (L_j^s)_x L_j^l dx - L_j^s(x_{j+\frac{1}{2}}) L_j^l(x_{j+\frac{1}{2}}) \\
\tilde{C} = (\tilde{c}_{sl})_{3\times 3}, & \tilde{c}_{sl} = L_j^s(x_{j-\frac{1}{2}}) L_{j-1}^l(x_{j-\frac{1}{2}}) \\
\bar{B} = (\bar{b}_{sl})_{3\times 3}, & \bar{b}_{sl} = \int_{I_j} (L_j^s)_x L_j^l dx + L_j^s(x_{j-\frac{1}{2}}) L_j^l(x_{j-\frac{1}{2}}) \\
\bar{C} = (\bar{c}_{sl})_{3\times 3}, & \bar{c}_{sl} = -L_j^s(x_{j+\frac{1}{2}}) L_{j+1}^l(x_{j+\frac{1}{2}})
\end{cases} \tag{2.10}$$

We denote the vectors $\vec{q}_j$, $\vec{u}_j$ to be the values of q and u at the grid points x_j^s, $0 \le s \le 2$ within the cell I_j. Now, we get the semi-discrete LDG scheme for (2.8).

2.3 The explicit-implicit-null time discretization

Let $\{t^n = n\tau \in [0,T]\}_{n=0}^{M}$ be the time at the n-th time step, in which τ is the time step and T is the final computing time. To give a brief introduction of the time-marching method discussed in this paper, let us consider the following system of ordinary differential equations

$$\frac{du}{dt} = \mathcal{L}(t,u) + \mathcal{N}(t,u),$$

where $\mathcal{L}(t,u)$ and $\mathcal{N}(t,u)$ are derived from the spatial discretization of the two parts T_2 and T_1 shown in (1.4), respectively. Given u^n, we would like to find the numerical solution at the next time level t^{n+1}. Then the time-marching method [26] can be given by

$$\begin{cases} u^{(1)} = u^n \\ u^{(s)} = u^n + \tau \displaystyle\sum_{l=2}^{s} a_{sl}\mathcal{L}(t_l^n, u^{(l)}) + \tau \sum_{l=1}^{s-1} \hat{a}_{sl}\mathcal{N}(t_l^n, u^{(l)}) \qquad 2 \le s \le 5\,, \\ u^{n+1} = u^n + \tau \displaystyle\sum_{l=2}^{5} b_l\mathcal{L}(t_l^n, u^{(l)}) + \tau \sum_{l=1}^{4} \hat{b}_l\mathcal{N}(t_l^n, u^{(l)}) \end{cases}$$

where $u^{(s)}$ approximates $u(x, t^n + c_s\tau)$, $c_s = \sum_{l=2}^{s} a_{sl} = \sum_{l=1}^{s-1} \hat{a}_{sl}$, and $t_l^n = t^n + c_l\tau$. Obviously, the damping term $\mathcal{L}(t,u)$ is treated implicitly and the remaining term $\mathcal{N}(t,u)$ is treated explicitly. Denote $A = (a_{sl})$, $\hat{A} = (\hat{a}_{sl}) \in \mathbb{R}^{5\times 5}$, $b^T = [b_1, ..., b_5]$ and $\hat{b}^T = [\hat{b}_1, ..., \hat{b}_5]$. Then we can express the time-marching method as the following Butcher tableau:

$$\begin{array}{|c|ccccc||ccccc|c|}
\hline
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \\
a_{sl} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{11}{18} & \frac{1}{18} & 0 & 0 & 0 & \hat{a}_{sl} \\
 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{5}{6} & -\frac{5}{6} & \frac{1}{2} & 0 & 0 & \\
 & 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{7}{4} & \frac{3}{4} & -\frac{7}{4} & 0 & \\
\hline
b_l & 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{7}{4} & \frac{3}{4} & -\frac{7}{4} & 0 & \hat{b}_l \\
\hline
\end{array} \tag{2.11}$$

The left half of the table lists a_{sl} and b_l, with the five rows from top to bottom corresponding to $s = 1, ..., 5$, and the columns from left to right corresponding to $l = 1, ..., 5$. Similarly, the right half lists $\hat{a}_{sl}$ and $\hat{b}_l$. With the above Butcher coefficients, we then get a third order EIN time-marching method. We only discuss the above-mentioned time discretization in this paper. We have also carried out analysis for other EIN time-marching methods and have obtained similar stability conclusions according to the Fourier analysis, but we will not state them here to save space.

2.4 Stability analysis

In this subsection, we will give stability analysis for the proposed EIN finite difference and EIN-LDG schemes by the aid of the Fourier method. For simplicity of analysis, we consider the simplified linear equation (1.5). Adding and subtracting a term with constant diffusion coefficient $a_1 U_{xx}$ on the left-hand side of the considered PDE, we then get (2.8). The semi-discrete finite difference scheme for (2.8) reads

$$(u_t)_j = (a - a_1)\mathscr{D}_x\mathscr{D}_x u_j + a_1\mathscr{D}_x\mathscr{D}_x u_j,$$

and the semi-discrete LDG scheme is given by (2.9). Coupled with the EIN time-marching method (2.11) where the term $a_1\mathscr{D}_x\mathscr{D}_x u_j$ is taken as $\mathcal{L}$ and the term $(a-a_1)\mathscr{D}_x\mathscr{D}_x u_j$ is taken as $\mathcal{N}$, we then obtain the fully discrete EIN finite difference and EIN-LDG schemes.

The Fourier method, which depends heavily on the assumption of uniform mesh and periodic boundary condition, is a powerful tool for stability analysis. Let us now perform the following standard Fourier analysis. Take the EIN finite difference scheme first. We substitute the Fourier mode

$$u_j^n = v^n e^{ikx_j}, \qquad i^2 = -1 \tag{2.12}$$

into the scheme to find the evolution equation as

$$v^{n+1} = Gv^n,$$

where the amplification factor G is a scalar function of variables ξ, λ, a, a_0, and $\xi = kh, \lambda = \frac{\tau}{h^2}$. Because the L^2 norm of the exact solution to the equation (1.5) does not increase in time, the necessary and sufficient stability condition of the EIN finite difference scheme is given by the following lemma.

Lemma 2.1. *(***von Neumann condition***) If $|G| \leq 1$ holds for all $\xi \in [-\pi, \pi]$, then the EIN finite difference scheme is stable.*

Clearly, the stability can always be achieved, if the damping term T_2 is sufficiently large. However, it does not mean that a_0 should be as large as possible. We can intuitively conclude that larger a_0 would cause larger error, because the two identical terms we add to and subtract from the equation are treated in different ways, i.e., one is treated explicitly and the other is treated implicitly. On the other hand, the value of a_0 cannot be too small, otherwise the stability of the scheme may not be guaranteed. To sum up, the key problem is to find the minimum value of a_0 to assure that the EIN finite difference scheme is stable under a relaxed time step restriction. Of course, it is preferable that the stability is assured regardless of the time step. Since the specific formula for the amplification factor G is very complex, we will try to get the minimum value of a_0 numerically. The specific procedure to obtain it is as follows.

According to the von Neumann condition, the values of a_0 should lead to $|G| \leq 1$. However, due to the round-off error in the numerical verification, the magnitude of the amplification factor could be a bit larger than 1. In this case the chosen values of a_0 have been those satisfying $|G| \leq 1 + 10^{-10}$. During the search for a_0 we take a series of discrete point values ξ in the interval $[-\pi, \pi]$. For any fixed λ and a, the value of $|G|$ is computed. By checking whether the inequality $|G| \leq 1 + 10^{-10}$ is satisfied for all discrete values of ξ, we can get a range of a_0. In the code, we take $\lambda = 10^{-10}, 10^{-9}, \cdots, 10^{9}, 10^{10}$, respectively, and find that the minimum value of the intersection of a_0 under different λ is 0.54, which is recorded as δ. This verifies that the scheme is unconditionally stable if $a_0 \geq \delta$ and $\delta = 0.54$ is independent of λ. The code to determine the stability condition for the scheme is implemented in Matlab. We summarize the stability result in the following theorem.

Theorem 2.1. *The EIN finite difference scheme is unconditionally stable for the simplified* *equation* (1.5) *provided that*

$$a_0 \geq \delta,$$

where $\delta = 0.54$ *is obtained by the Fourier analysis numerically.*

Next, we consider the EIN-LDG scheme. We make an ansatz of the form

$$\begin{pmatrix} u_j^0(t) \\ u_j^1(t) \\ u_j^2(t) \end{pmatrix} = \begin{pmatrix} \hat{u}_k^0(t) \\ \hat{u}_k^1(t) \\ \hat{u}_k^2(t) \end{pmatrix} e^{ikx_j}, \quad i^2 = -1, \tag{2.13}$$

and substitute this into the scheme to find the evolution equation for the coefficient vector as

$$\frac{d}{dt} \begin{pmatrix} \hat{u}_k^0(t) \\ \hat{u}_k^1(t) \\ \hat{u}_k^2(t) \end{pmatrix} = G \begin{pmatrix} \hat{u}_k^0(t) \\ \hat{u}_k^1(t) \\ \hat{u}_k^2(t) \end{pmatrix}, \tag{2.14}$$

where the amplification matrix G is a function of the variables ξ, λ, a, a_0. Similarly, because the L^2 norm of the exact solution to the equation (1.5) does not increase in time, the necessary and sufficient stability condition of the EIN-LDG scheme can be given in the following lemma.

Lemma 2.2. *If* G *is uniformly diagonalizable and* $|\lambda_G| \leq 1$ *holds for all* $\xi \in [-\pi, \pi]$, *where* λ_G *is the spectral radius of* G, *then the EIN-LDG scheme is stable.*

Similarly, the stability can always be achieved, if the damping term T_2 is sufficiently large. The only question is the determination of the lower bound for a_0 to stabilize the scheme. Of course, it is best to ensure stability regardless of the time step. The specific formula for the amplification matrix is very complex. Thus we will again try to obtain a_0 numerically. The specific procedure to obtain it is similar to that for the EIN finite difference scheme. So we omit it and summarize the stability result in the following theorem.

Theorem 2.2. *The EIN-LDG scheme is unconditionally stable for the simplified equation* (1.5) *provided that*

$$a_0 \geq \delta,$$

where $\delta = 0.54$ *is obtained by the Fourier analysis numerically.*

Now, we have finished the research of a_0 for stability. As a result, we find that the smallest a_0 to assure unconditional stability of the EIN finite difference scheme and the EIN-LDG scheme are the same, both of which are 0.54.

3 The third order dispersive equations

In this section, we will discuss the high order finite difference and LDG schemes coupled with the EIN time discretization (2.11) for solving the third order dispersive equations. By the aid of the Fourier method, we will perform stability analysis for the proposed schemes on the simplified equation (1.6). Even though the analysis is only performed on the linear equations containing the highest derivatives, the numerical experiments in Section 5 show that the stability criteria for the linear model are also valid for nonlinear equations with and without lower order derivative terms.

3.1 The semi-discrete finite difference scheme

Consider the third order dispersive equation (1.2). For such equation, in order to guarantee stability and convergence, the sign of the additional term $a_1 U_{xxx}$ we add to both sides of the equation needs to be adjusted according to the wind direction which depends on the solution. For example, if the solution to (1.2) moves from right to left within the cell I_j, then we should add two equal term with negative dispersion coefficient $-a_1 U_{xxx}, a_1 > 0$ to both sides of the considered equation. Otherwise, the sign of the additional term $a_1 U_{xxx}$ needs to be positive. If the wind direction changes inside the cell I_j, taking the LDG spatial discretization as an example, then we may need to make a_1 vary in space and use a central flux rather than the upwind flux at least for the additional terms in order to make the EIN method work. For simplicity of analysis, we only consider the case of a fixed wind direction. Assume that the solution to (1.2) moves from right to left, we add two equal term with constant dispersion coefficient $-a_1 U_{xxx}$ to both sides of the considered equation and get

$$U_t + \underbrace{(r'(U)g(r(U)_x)_x)_x - a_1 U_{xxx}}_{T_1} = \underbrace{-a_1 U_{xxx}}_{T_2}, \tag{3.1}$$

where $a_1 > 0$ is equal to a_0 times the maximum coefficient of the equation's highest derivative. Take the following $K(3,3)$ equation [20] as an example,

$$U_t + (U^3)_x + (U^3)_{xxx} = 0$$

we should take $a_1 = a_0 \times \max_U \{3U^2\}$. We still quote the difference quotient symbol $\mathscr{D}_x u_j$, as shown in (2.1). Then the fourth order semi-discrete finite difference scheme for (3.1) can be given by:

$$(u_t)_j = \underbrace{-\mathscr{D}_x(r'(u)_j \mathscr{D}_x g(\mathscr{D}_x r(u)_j)_j) + a_1 \mathscr{D}_x \mathscr{D}_x \mathscr{D}_x u_j}_{T_1} - \underbrace{a_1 \mathscr{D}_x \mathscr{D}_x \mathscr{D}_x u_j}_{T_2}.$$

Obviously, $\mathscr{D}_x \mathscr{D}_x \mathscr{D}_x u$ is a central difference approximation for u_{xxx}. We can easily check that the formula for $\mathscr{D}_x \mathscr{D}_x \mathscr{D}_x u_j$ is

$$\frac{1}{1728h^3}\Big((u_{j-6} - u_{j+6}) - 24(u_{j-5} - u_{j+5}) + 192(u_{j-4} - u_{j+4}) - 488(u_{j-3} - u_{j+3}) \\ - 387(u_{j-2} - u_{j+2}) + 1584(u_{j-1} - u_{j+1})\Big).$$

We have also carried out analysis for similar finite difference schemes with different order of accuracy, which can obtain the same stability result, but we will not introduce those schemes here to save space.

3.2 The semi-discrete local discontinuous Galerkin scheme

In this subsection, we present a LDG method for the dispersive equation (3.1). For a detailed introduction of the method, we refer the readers to [29]. By introducing the new variables

$$Q = r(U)_x, \qquad P = g(Q)_x, \qquad V = U_x, \qquad W = V_x,$$

we can rewrite the equation (3.1) into the following first order system:

$$U_t + (r'(U)P - a_1 W)_x = -a_1 W_x, \qquad P - g(Q)_x = 0, \qquad W - V_x = 0,$$

$$Q - r(U)_x = 0, \qquad V - U_x = 0.$$

Then we search for piecewise polynomial solutions u, p, w, q, v from V_h such that for all the test functions ϕ_1, ϕ_2, ϕ_3, ϕ_4, $\phi_5 \in V_h$ and $1 \le j \le N$, we have

$$\begin{aligned}
\int_{I_j} u_t \phi_1 dx - \int_{I_j} (r'(u)p - a_1 w)(\phi_1)_x dx + (\hat{r'}\hat{p} - a_1\hat{w})_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} - \\
(\hat{r'}\hat{p} - a_1\hat{w})_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}} = a_1\Big(\int_{I_j} w(\phi_1)_x dx - \hat{w}_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} + \hat{w}_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}}\Big), \\
\int_{I_j} p\phi_2 dx + \int_{I_j} g(q)(\phi_2)_x dx - \hat{g}_{j+\frac{1}{2}}(\phi_2)^-_{j+\frac{1}{2}} + \hat{g}_{j-\frac{1}{2}}(\phi_2)^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} w\phi_3 dx + \int_{I_j} v(\phi_3)_x dx - \hat{v}_{j+\frac{1}{2}}(\phi_3)^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}}(\phi_3)^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} q\phi_4 dx + \int_{I_j} r(u)(\phi_4)_x dx - \hat{r}_{j+\frac{1}{2}}(\phi_4)^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}}(\phi_4)^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} v\phi_5 dx + \int_{I_j} u(\phi_5)_x dx - \hat{u}_{j+\frac{1}{2}}(\phi_5)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}(\phi_5)^+_{j-\frac{1}{2}} = 0,
\end{aligned} \tag{3.2}$$

where V_h is defined in (2.3) and $(\phi_1)^-_{j+\frac{1}{2}} = \phi_1(x^-_{j+\frac{1}{2}})$, $(\phi_1)^+_{j+\frac{1}{2}} = \phi_1(x^+_{j+\frac{1}{2}})$. Now, the only ambiguity in the above algorithms is the definition of the numerical fluxes (the terms with the "hat"), which should be designed carefully to ensure stability. As shown in [28, 29], we can prove an L^2-stability result, a cell entropy inequality for the general case and obtain optimal error estimates for the linear case, if the numerical fluxes are taken as:

$$\begin{aligned}
\hat{p} = p^+, \qquad \hat{r} = r(u^-), \qquad \hat{g} = \hat{g}(q^-, q^+), \\
\hat{r'} = \frac{r(u^+) - r(u^-)}{u^+ - u^-}, \qquad \hat{w} = w^+, \qquad \hat{v} = v^+, \qquad \hat{u} = u^-,
\end{aligned} \tag{3.3}$$

where $-\hat{g}(q^-, q^+)$ is a monotone flux for $-g(q)$, i.e., $\hat{g}(q^-, q^+)$ is Lipschitz continuous in both arguments q^- and q^+, is consistent with $g(q)$ in the sense that $\hat{g}(q, q) = g(q)$, and is a non-increasing function in q^- and a non-decreasing function in q^+. For more details about the monotone fluxes which are suitable for discontinuous Galerkin methods, we refer the readers to [10]. In this paper, we will use the simple Lax-Friedrichs flux

$$\hat{f}(u^-, u^+) = \frac{1}{2}(f(u^-) + f(u^+) - \alpha(u^+ - u^-)), \quad \alpha = \max_u |f'(u)|. \tag{3.4}$$

We also concentrate on the piecewise quadratic polynomial ($m = 2$) case. With the representation (2.6), we take the test functions ϕ_l, $1 \le l \le 5$ also as the Lagrangian nodal basis polynomials (2.5), respectively, and, by inverting the small 3×3 mass matrix, we can obtain easily the block finite difference schemes corresponding to the LDG method. In the following, we take the linear dispersive equation (1.6) as an example. We add and subtract a term with constant dispersion coefficient $-a_1 U_{xxx}$ on the left-hand side of the considered equation

$$U_t + \underbrace{(a - a_1)U_{xxx}}_{T_1} + \underbrace{a_1 U_{xxx}}_{T_2} = 0, \qquad a_1 = a_0 \times a. \tag{3.5}$$

For this equation, we find that the LDG scheme (3.2) can be simplified to

$$\begin{gathered}
\int_{I_j} u_t \phi_1 dx - (a - a_1)\Big(\int_{I_j} w(\phi_1)_x dx - \hat{w}_{j+\frac{1}{2}} (\phi_1)^-_{j+\frac{1}{2}} + \hat{w}_{j-\frac{1}{2}} (\phi_1)^+_{j-\frac{1}{2}} \Big) = \\
a_1 \Big(\int_{I_j} w(\phi_1)_x dx - \hat{w}_{j+\frac{1}{2}} (\phi_1)^-_{j+\frac{1}{2}} + \hat{w}_{j-\frac{1}{2}} (\phi_1)^+_{j-\frac{1}{2}} \Big), \\
\int_{I_j} w\phi_2 dx + \int_{I_j} v(\phi_2)_x dx - \hat{v}_{j+\frac{1}{2}} (\phi_2)^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}} (\phi_2)^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} v\phi_3 dx + \int_{I_j} u(\phi_3)_x dx - \hat{u}_{j+\frac{1}{2}} (\phi_3)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} (\phi_3)^+_{j-\frac{1}{2}} = 0.
\end{gathered}$$

With the choice of the fluxes (3.3), we take the test functions ϕ_l, $1 \le l \le 3$ above also as the Lagrangian nodal basis polynomials (2.5), respectively, and obtain easily the block finite difference schemes corresponding to the LDG method

$$\begin{aligned}
\vec{v}_j &= -\tilde{A}^{-1}(\tilde{B}\vec{u}_j + \tilde{C}\vec{u}_{j-1}), \\
\vec{w}_j &= -\tilde{A}^{-1}(\bar{B}\vec{v}_j + \bar{C}\vec{v}_{j+1}), \\
(\vec{u}_j)_t &= \tilde{A}^{-1}\big[(a - a_1)(\bar{B}\vec{w}_j + \bar{C}\vec{w}_{j+1}) + a_1(\bar{B}\vec{w}_j + \bar{C}\vec{w}_{j+1})\big],
\end{aligned} \tag{3.6}$$

where the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\bar{B}$, $\bar{C}$ are defined in (2.10) and the vectors $\vec{v}_j$, $\vec{w}_j$, $\vec{u}_j$ denote the values of v, w, u at the grid points x_j^s, $0 \le s \le 2$ defined in (2.7). Now, we get the semi-discrete LDG scheme for (3.5).

3.3 Stability analysis

In this subsection, we will present stability analysis for the proposed EIN finite difference and EIN-LDG schemes by the aid of the Fourier method. We would like to investigate how to choose a_0 such that the schemes are stable. Similarly, it is preferable to get the values of a_0 to assure unconditional stability of the schemes. For simplicity of analysis, we consider the simplified linear equation (1.6). Adding and subtracting a term with constant dispersion coefficient $-a_1 U_{xxx}$ on the left-hand side of the considered PDE, we then get (3.5). The semi-discrete finite difference scheme for (3.5) becomes

$$(u_t)_j = (a_1 - a)\mathscr{D}_x\mathscr{D}_x\mathscr{D}_x u_j - a_1\mathscr{D}_x\mathscr{D}_x\mathscr{D}_x u_j,$$

and the semi-discrete LDG scheme is given by (3.6). Coupled with the EIN time-marching method (2.11) where the term $a_1\mathscr{D}_x\mathscr{D}_x\mathscr{D}_x u_j$ is taken as $\mathcal{L}$ and the term $(a_1 - a)\mathscr{D}_x\mathscr{D}_x\mathscr{D}_x u_j$ is taken as $\mathcal{N}$, we then obtain the fully discrete EIN finite difference and EIN-LDG schemes.

Let us now perform the standard Fourier analysis. For the EIN finite difference scheme, we substitute the Fourier mode (2.12) into the scheme to yield the single-valued amplification factor, which is a scalar function of variables ξ, λ, a, a_0, where $\lambda = \frac{\tau}{h^3}$. For the EIN-LDG scheme, we make the same ansatz as in (2.13) and substitute it into the scheme to obtain the evolution equation for the coefficient vector (2.14) with the amplification matrix. In order to ensure stability, the amplification factor (or its spectral radius if it is a matrix) of the evolution equation for the considered scheme needs to be bounded by 1 uniformly with respect to the two parameters ξ and λ, see Lemmas 2.1 and 2.2 for details. The specific formula for the amplification factor (or matrix) is very complex. Thus we will try to numerically obtain the values of a_0 for stability. The specific procedure to obtain them is similar to that shown in Subsection 2.4. We omit the details and state the stability result in the following theorem.

Theorem 3.1. *The EIN finite difference and EIN-LDG schemes are unconditionally stable for the third order dispersive equation* (1.6) *provided that*

$$a_0 \geq \delta,$$

where $\delta = 0.54$ *is obtained by the Fourier analysis numerically.*

4 The fourth order biharmonic-type equations

In this section, we will discuss the high order finite difference and LDG schemes coupled with the EIN time discretization (2.11) for solving the fourth order biharmonic-type equations. The stability analysis of the proposed schemes is performed on the simplified equation (1.7) by the aid of the Fourier method. Even though the analysis is only performed on the linear equations, through the numerical experiments in Section 5 we can see that the stability criteria for the linear model are also valid for the nonlinear biharmonic-type equations with and without lower order derivative terms.

4.1 The semi-discrete finite difference scheme

Considering the fourth order biharmonic-type equation (1.3), we add and subtract a term with constant dissipation coefficient $-a_1 U_{xxxx}$ on the left-hand side of the equation, and obtain

$$U_t + \underbrace{(a(U_x)U_{xx})_{xx} - a_1 U_{xxxx}}_{T_1} + \underbrace{a_1 U_{xxxx}}_{T_2} = 0, \qquad a_1 = a_0 \times \max_{U_x} a(U_x), \tag{4.1}$$

where $a(U_x) \geq 0$ is bounded and smooth. In the following, we take the fourth order finite difference scheme as an example. We begin with the equation (4.1) to present the finite difference spatial discretization. Introducing the following notation of difference quotient:

$$\mathscr{D}_{xx} u_j = \frac{-(u_{j-2} + u_{j+2}) + 16(u_{j-1} + u_{j+1}) - 30u_j}{12h^2},$$

we can easily check that $\mathscr{D}_{xx} u_j$ is a fourth order central difference approximation for u_{xx} at the grid point x_j. Then the fourth order semi-discrete finite difference scheme for (4.1) can be given by

$$(u_t)_j = \underbrace{-\mathscr{D}_{xx}(a(\mathscr{D}_x u_j)\mathscr{D}_{xx} u_j) + a_1 \mathscr{D}_{xx}\mathscr{D}_{xx} u_j}_{T_1} - \underbrace{a_1 \mathscr{D}_{xx}\mathscr{D}_{xx} u_j}_{T_2},$$

with the formula for $\mathscr{D}_x u_j$ given by (2.1). We can check that when $a(U_x) = 1$, the formula for $\mathscr{D}_{xx}\mathscr{D}_{xx} u_j$ is equivalent to that for $\mathscr{D}_{xx}(a(\mathscr{D}_x u_j)\mathscr{D}_{xx} u_j)$, and is given by

$$\frac{\big(u_{i-4} + u_{i+4}\big) - 32\big(u_{i-3} + u_{i+3}\big) + 316\big(u_{i-2} + u_{i+2}\big) - 992\big(u_{i-1} + u_{i+1}\big) + 1414u_i}{144h^4}.$$

4.2 The semi-discrete local discontinuous Galerkin scheme

In this subsection, we present a LDG method for the biharmonic-type equation. For a detailed introduction of the method, we refer the readers to [30]. To define the LDG method, we introduce the new variables

$$R = U_x, \qquad Q = B(R)_x, \qquad P = (b(R)Q)_x, \qquad V = R_x, \qquad W = V_x,$$

where $b(R) = \sqrt{a(R)}$ and $B(R) = \int^R b(R)dR$, and rewrite the equation (4.1) as the following first order system:

$$U_t + (P - a_1 W)_x = -a_1 W_x, \qquad P - (b(R)Q)_x = 0, \qquad W - V_x = 0,$$
$$Q - B(R)_x = 0, \qquad V - R_x = 0, \qquad R - U_x = 0.$$

The semi-discrete version of the local discontinuous Galerkin scheme can be described as follows: find piecewise polynomial functions u, p, w, q, v, $r \in V_h$, where V_h is defined in (2.3), such that for all the test functions ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, $\phi_6 \in V_h$ and $1 \le j \le N$, we have

$$\begin{aligned}
&\int_{I_j} u_t \phi_1 dx - \int_{I_j} (p - a_1 w)(\phi_1)_x dx + (\hat{p} - a_1 \hat{w})_{j+\frac{1}{2}} (\phi_1)^-_{j+\frac{1}{2}} - \\
&(\hat{p} - a_1 \hat{w})_{j-\frac{1}{2}} (\phi_1)^+_{j-\frac{1}{2}} = a_1 \Big(\int_{I_j} w(\phi_1)_x dx - \hat{w}_{j+\frac{1}{2}} (\phi_1)^-_{j+\frac{1}{2}} + \hat{w}_{j-\frac{1}{2}} (\phi_1)^+_{j-\frac{1}{2}} \Big), \\
&\int_{I_j} p\phi_2 dx + \int_{I_j} b(r) q (\phi_2)_x dx - (\hat{b}\hat{q})_{j+\frac{1}{2}} (\phi_2)^-_{j+\frac{1}{2}} + (\hat{b}\hat{q})_{j-\frac{1}{2}} (\phi_2)^+_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} w\phi_3 dx + \int_{I_j} v (\phi_3)_x dx - \hat{v}_{j+\frac{1}{2}} (\phi_3)^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}} (\phi_3)^+_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} q\phi_4 dx + \int_{I_j} B(r) (\phi_4)_x dx - \hat{B}_{j+\frac{1}{2}} (\phi_4)^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} (\phi_4)^+_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} v\phi_5 dx + \int_{I_j} r (\phi_5)_x dx - \hat{r}_{j+\frac{1}{2}} (\phi_5)^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}} (\phi_5)^+_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} r\phi_6 dx + \int_{I_j} u (\phi_6)_x dx - \hat{u}_{j+\frac{1}{2}} (\phi_6)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} (\phi_6)^+_{j-\frac{1}{2}} = 0,
\end{aligned} \tag{4.2}$$

where $(\phi_1)^-_{j+\frac{1}{2}} = \phi_1(x^-_{j+\frac{1}{2}})$, $(\phi_1)^+_{j+\frac{1}{2}} = \phi_1(x^+_{j+\frac{1}{2}})$. A crucial ingredient for this method to be stable is the correct choice of the numerical fluxes (the terms with the "hat"). It is found out in [12, 30] that one can take the following simple choice of fluxes to guarantee stability and convergence

$$\hat{p} = p^-, \qquad \hat{q} = q^+, \qquad \hat{B} = B(r^-), \qquad \hat{b} = \frac{B(r^+) - B(r^-)}{r^+ - r^-},$$
$$\hat{u} = u^+, \qquad \hat{w} = w^-, \qquad \hat{v} = v^+, \qquad \hat{r} = r^-. \tag{4.3}$$

Again, we concentrate on the piecewise quadratic polynomial ($m = 2$) case. With the representation (2.6), we take the test functions ϕ_l, $1 \le l \le 6$ also as the Lagrangian nodal basis polynomials (2.5), respectively. By inverting the small 3×3 mass matrix, we can obtain easily the block finite difference schemes corresponding to the LDG method. In the following, take the linear fourth order dissipative equation (1.7) as an example. We add and subtract a term with constant dissipation coefficient $-a_1 U_{xxxx}$ on the left-hand side of the considered PDE, and get

$$U_t + \underbrace{(a - a_1)U_{xxxx}}_{T_1} + \underbrace{a_1 U_{xxxx}}_{T_2} = 0, \qquad a_1 = a_0 \times a. \tag{4.4}$$

For this equation, we find that the LDG scheme (4.2) can be simplified to

$$\int_{I_j} u_t \phi_1 dx - (a - a_1)\Big(\int_{I_j} w(\phi_1)_x dx - \hat{w}_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} + \hat{w}_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}}\Big) =$$
$$a_1\Big(\int_{I_j} w(\phi_1)_x dx - \hat{w}_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} + \hat{w}_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}}\Big),$$
$$\int_{I_j} w\phi_2 dx + \int_{I_j} v(\phi_2)_x dx - \hat{v}_{j+\frac{1}{2}}(\phi_2)^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}}(\phi_2)^+_{j-\frac{1}{2}} = 0,$$
$$\int_{I_j} v\phi_3 dx + \int_{I_j} r(\phi_3)_x dx - \hat{r}_{j+\frac{1}{2}}(\phi_3)^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}}(\phi_3)^+_{j-\frac{1}{2}} = 0,$$
$$\int_{I_j} r\phi_4 dx + \int_{I_j} u(\phi_4)_x dx - \hat{u}_{j+\frac{1}{2}}(\phi_4)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}(\phi_4)^+_{j-\frac{1}{2}} = 0.$$

With the choice of the fluxes (4.3), we take the test functions ϕ_l, $1 \le l \le 4$ above also as the Lagrangian nodal basis polynomials, respectively, and obtain the block finite difference schemes corresponding to the LDG method

$$\vec{r}_j = -\tilde{A}^{-1}(\tilde{B}\vec{u}_j + \tilde{C}\vec{u}_{j+1}),$$
$$\vec{v}_j = -\tilde{A}^{-1}(\bar{B}\vec{r}_j + \bar{C}\vec{r}_{j-1}),$$
$$\vec{w}_j = -\tilde{A}^{-1}(\tilde{B}\vec{v}_j + \tilde{C}\vec{v}_{j+1}), \tag{4.5}$$

$$(\vec{u}_j)_t = \tilde{A}^{-1}\big[(a-a_1)(\bar{B}\vec{w}_j + \bar{C}\vec{w}_{j-1}) + a_1(\bar{B}\vec{w}_j + \bar{C}\vec{w}_{j-1})\big],$$

where

$$\begin{cases} \tilde{A} = (\tilde{a}_{sl})_{3\times 3}, & \tilde{a}_{sl} = \int_{I_j} L_j^s L_j^l dx \\ \tilde{B} = (\tilde{b}_{sl})_{3\times 3}, & \tilde{b}_{sl} = \int_{I_j} (L_j^s)_x L_j^l dx + L_j^s(x_{j-\frac{1}{2}}) L_j^l(x_{j-\frac{1}{2}}) \\ \tilde{C} = (\tilde{c}_{sl})_{3\times 3}, & \tilde{c}_{sl} = -L_j^s(x_{j+\frac{1}{2}}) L_{j+1}^l(x_{j+\frac{1}{2}}) \\ \bar{B} = (\bar{b}_{sl})_{3\times 3}, & \bar{b}_{sl} = \int_{I_j} (L_j^s)_x L_j^l dx - L_j^s(x_{j+\frac{1}{2}}) L_j^l(x_{j+\frac{1}{2}}) \\ \bar{C} = (\bar{c}_{sl})_{3\times 3}, & \bar{c}_{sl} = L_j^s(x_{j-\frac{1}{2}}) L_{j-1}^l(x_{j-\frac{1}{2}}) \end{cases}$$

and the vectors $\vec{r}_j$, $\vec{v}_j$, $\vec{w}_j$, $\vec{u}_j$ denote the values of r, v, w, u at the grid points x_j^s, $0 \le s \le 2$ defined in (2.7). Now, we get the semi-discrete LDG scheme for (4.4).

4.3 Stability analysis

In this subsection, we would like to analyze the stability of the proposed EIN finite difference and EIN-LDG schemes for solving the biharmonic-type equations (1.3). We would like to investigate how to choose a_0 such that the schemes are stable. Similarly, it is preferable to get the values of a_0 to assure unconditional stability of the schemes. For simplicity of analysis, we consider the simplified equation (1.7). Adding and subtracting a term with constant dissipation coefficient $-a_1 U_{xxxx}$ on the left-hand side of the considered PDE, we then get (4.4). The semi-discrete finite difference scheme for (4.4) becomes

$$(u_t)_j = (a_1 - a)\mathscr{D}_{xx}\mathscr{D}_{xx}u_j - a_1\mathscr{D}_{xx}\mathscr{D}_{xx}u_j, \tag{4.6}$$

and the semi-discrete LDG scheme is given by (4.5). Coupled with the EIN time-marching method (2.11) where the term $a_1\mathscr{D}_{xx}\mathscr{D}_{xx}u_j$ is taken as $\mathcal{L}$ and the term $(a_1 - a)\mathscr{D}_{xx}\mathscr{D}_{xx}u_j$ is taken as $\mathcal{N}$, we then obtain the fully discrete EIN finite difference and EIN-LDG schemes.

Here, we follow the Fourier type analysis in the previous subsection to analyze the stability of the schemes. We make the same ansatz as in (2.12) and (2.13) and substitute them into the EIN finite difference and EIN-LDG schemes, respectively, to yield the amplification factor and the amplification matrix. In order to ensure stability, the amplification factor (or its spectral radius if it is a matrix) of the evolution equation for the considered scheme needs to be uniformly bounded by 1, see Lemmas 2.1 and 2.2 for details. The specific formula for the amplification factor (or matrix) is very complex. Therefore, we will try to numerically obtain the values of a_0 for stability. The specific procedure to obtain them is similar to that shown in Subsection 2.4. We omit the details and summarize the stability result in the following theorem.

Theorem 4.1. *The EIN finite difference and EIN-LDG schemes are unconditionally stable for the fourth order dissipative equation* (1.7) *provided that*

$$a_0 \geq \delta,$$

where $\delta = 0.54$ *is obtained by the Fourier analysis numerically.*

5 Numerical experiments

In this section, we will numerically validate the orders of accuracy and performance of the proposed EIN finite difference and EIN-LDG schemes for high order dissipative and dispersive equations in one and two space dimensions. In addition, we would like to illustrate the sharpness of $\delta = 0.54$ for stability. The equations with periodic boundary conditions will be considered, unless otherwise stated.

5.1 The second order dissipative equations

In this subsection we consider the diffusion equations in one and two space dimensions. The generalization of the finite difference scheme given in Subsection 2.1 to the two-dimensional diffusion equation is straightforward. Following the lines in [11], we can also generalize the LDG scheme (2.2) to the two-dimensional case. From the experiments we can find that the smallest a_0 to assure the stability of the schemes is 0.54. The result is consistent with that in [26], in which the local discontinuous Galerkin methods coupled with the EIN time discretization (2.11) have been implemented to solve the diffusion equations in one-dimension. Special attention has been paid to large a_0, of which the numerical results are significantly worse than those of $a_0 = 0.54$. In addition, we also note that the stability criterion appears to be also valid for convection-diffusion equations.

5.1.1 One-dimensional numerical tests

First, we consider the linear diffusion equation

$$U_t = U_{xx}, \qquad x \in (-\pi, \pi) \tag{5.1}$$

augmented with the initial condition $U(x,0) = \sin(x)$. The problem has an exact solution

$$U(x,t) = e^{-t}\sin(x). \tag{5.2}$$

We compute to $T = 1$ with the time step $\tau = h$. The L^1, L^∞ and L^2 errors and orders of accuracy for this problem are listed in Tables 5.1-5.2. In each table, we display the numerical results of the schemes with different a_0. From the experiment we can see that the EIN finite difference and EIN-LDG schemes are stable and can achieve optimal orders of accuracy if $a_0 \geq 0.54$, while the simulation deteriorates significantly with the refinement of the mesh if $a_0 = 0.53$. We also note that larger a_0 causes larger error. The simulation results coincide with the theory.

Next, we consider the convection-diffusion equation

$$U_t + \frac{1}{2}(U^2)_x = (a(U)U_x)_x + f(x,t), \qquad x \in (-\pi, \pi) \tag{5.3}$$

augmented with the diffusion coefficient $a(U) = U^2 + 2$, the initial condition $U(x,0) = \sin(x)$ and the source term

$$f(x,t) = \frac{1}{4}\Big(4\cos(x+t) + 9\sin(x+t) + 2\sin(2(x+t)) - 3\sin\big(3(x+t)\big)\Big).$$

The problem has an exact solution

$$U(x,t) = \sin(x+t). \tag{5.4}$$

Table 5.1: The errors and orders of the EIN finite difference scheme for Example (5.1).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	6.81E-07		1.07E-06		7.56E-07	
	160	1.27E-07	2.45	1.99E-07	2.45	1.41E-07	2.45
	320	1.85E-08	2.79	2.91E-08	2.79	2.06E-08	2.79
	640	2.47E-09	2.91	3.87E-09	2.91	2.74E-09	2.91
	1280	1.18E-06	-8.91	6.33E-06	-10.69	1.64E-06	-9.23
0.54	80	6.72E-07		1.06E-06		7.46E-07	
	160	1.26E-07	2.44	1.97E-07	2.44	1.40E-07	2.44
	320	1.84E-08	2.78	2.89E-08	2.78	2.04E-08	2.78
	640	2.45E-09	2.91	3.85E-09	2.91	2.72E-09	2.91
	1280	3.16E-10	2.96	4.97E-10	2.96	3.51E-10	2.96
10	80	4.02E-03		6.32E-03		4.47E-03	
	160	7.90E-04	2.37	1.24E-03	2.37	8.78E-04	2.37
	320	1.29E-04	2.63	2.02E-04	2.63	1.43E-04	2.63
	640	1.84E-05	2.81	2.89E-05	2.81	2.04E-05	2.81
	1280	2.47E-06	2.90	3.88E-06	2.90	2.74E-06	2.90

Table 5.2: The errors and orders of the EIN-LDG scheme for Example (5.1).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	5.24E-06		1.91E-05		7.15E-06	
	160	3.59E-06	0.55	1.47E-05	0.38	5.26E-06	0.45
	320	1.55E-05	-2.12	6.30E-05	-2.11	2.27E-05	-2.12
	640	2.54E-03	-7.38	1.03E-02	-7.38	3.72E-03	-7.38
	1280	5.49E+02	-17.74	2.23E+03	-17.74	8.03E+02	-17.74
0.54	80	1.39E-06		2.46E-06		1.55E-06	
	160	1.69E-07	3.07	2.83E-07	3.14	1.87E-07	3.07
	320	2.16E-08	2.98	3.67E-08	2.96	2.40E-08	2.98
	640	2.70E-09	3.00	4.59E-09	3.01	3.00E-09	3.00
	1280	3.38E-10	3.00	5.74E-10	3.00	3.76E-10	3.00
10	80	4.02E-03		6.32E-03		4.47E-03	
	160	7.90E-04	2.37	1.24E-03	2.37	8.78E-04	2.37
	320	1.29E-04	2.63	2.02E-04	2.63	1.43E-04	2.63
	640	1.84E-05	2.81	2.89E-05	2.81	2.04E-05	2.81
	1280	2.47E-06	2.90	3.88E-06	2.90	2.74E-06	2.90

For the finite difference type spatial approximation, the third order upwind biased finite difference scheme is used for the convection term, which is the standard third order weighted essentially non-oscillatory (WENO) scheme [18, 21] with linear weights. In addition, in order to ensure correct upwind biasing and stability, a simple Lax-Friedrichs flux splitting is used. As for the LDG spatial approximation of the convection term, we refer the readers to [11, 26]. We compute to $T = 1$ with the time step $\tau = h$ and the stabilization parameter $a_1 = a_0 \max_{u^n}\{(u^n)^2 + 2\}$. Here, u^n is the numerical solution at the time level t^n. The numerical results of the schemes with different a_0 are listed in Tables 5.3-5.4. From the experiment we can see that the EIN finite difference and EIN-LDG schemes are stable and can achieve optimal orders of accuracy if $a_0 = 0.54$, while the simulation in the EIN-LDG scheme deteriorates significantly with the refinement of the mesh if $a_0 = 0.53$. When $a_0 = 10$, the errors are larger and the numerical orders of accuracy settle down towards the asymptotic value slower with mesh refinements, in comparison with the results of $a_0 = 0.54$. For this reason, large a_0 is not recommended.

Table 5.3: The errors and orders of the EIN finite difference scheme for Example (5.3).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	3.10E-05		6.24E-05		3.52E-05	
	160	4.33E-06	2.87	8.88E-06	2.84	4.85E-06	2.88
	320	5.84E-07	2.90	1.30E-06	2.79	6.58E-07	2.89
	640	7.43E-08	2.98	1.61E-07	3.01	8.32E-08	2.99
	1280	9.40E-09	2.99	2.03E-08	2.99	1.05E-08	2.99
0.54	80	3.19E-05		6.39E-05		3.60E-05	
	160	4.45E-06	2.87	9.12E-06	2.83	4.97E-06	2.88
	320	6.01E-07	2.90	1.34E-06	2.78	6.77E-07	2.89
	640	7.64E-08	2.98	1.67E-07	3.01	8.57E-08	2.99
	1280	9.67E-09	2.99	2.10E-08	2.99	1.08E-08	2.99
10	80	3.66E-02		7.22E-02		4.23E-02	
	160	1.09E-02	1.76	2.18E-02	1.74	1.26E-02	1.76
	320	2.50E-03	2.13	5.03E-03	2.12	2.89E-03	2.13
	640	4.43E-04	2.50	8.99E-04	2.49	5.14E-04	2.50
	1280	6.74E-05	2.72	1.38E-04	2.71	7.83E-05	2.72

Table 5.4: The errors and orders of the EIN-LDG scheme for Example (5.3).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	2.85E-05		7.38E-05		3.42E-05	
	160	3.94E-06	2.88	1.01E-05	2.90	4.66E-06	2.90
	320	5.29E-07	2.91	1.44E-06	2.82	6.32E-07	2.89
	640	8.14E-08	2.70	1.67E-05	-3.55	3.51E-07	0.85
	1280	1.11E-07	-0.44	6.19E-05	-1.89	1.14E-06	-1.70
0.54	80	2.94E-05		7.54E-05		3.52E-05	
	160	4.07E-06	2.88	1.03E-05	2.89	4.80E-06	2.90
	320	5.46E-07	2.91	1.48E-06	2.82	6.53E-07	2.89
	640	6.95E-08	2.98	1.84E-07	3.02	8.23E-08	2.99
	1280	8.71E-09	3.00	2.30E-08	3.00	1.03E-08	3.00
10	80	3.66E-02		7.22E-02		4.23E-02	
	160	1.09E-02	1.77	2.18E-02	1.74	1.26E-02	1.76
	320	2.50E-03	2.13	5.03E-03	2.12	2.89E-03	2.13
	640	4.43E-04	2.50	8.99E-04	2.49	5.14E-04	2.50
	1280	6.74E-05	2.72	1.38E-04	2.71	7.83E-05	2.72

5.1.2 Two-dimensional numerical tests

First, we compute the two-dimensional diffusion equation

$$U_t = \Delta U, \qquad (x, y) \in (-\pi, \pi)^2 \tag{5.5}$$

with the initial condition $U(x, y, 0) = \sin(x + y)$. This problem has an exact solution

$$U(x, y, t) = e^{-2t} \sin(x + y). \tag{5.6}$$

We compute to $T = 1$ with the time step $\tau = h$. The errors and numerical orders of accuracy are contained in Tables 5.5 and 5.6. This time, since the meshes we have used are not refined enough, the results of $a_0 = 0.53$ have not shown signs of stability deterioration, except that the EIN-LDG scheme has a slight order loss with the refinement of the mesh. Also, as expected, when $a_0 = 10$, the errors are larger and the numerical orders of accuracy settle down towards the asymptotic value slower with mesh refinements, in comparison with the results of $a_0 = 0.54$. In Tables 5.7 and 5.8, we have shown a more extensive mesh refinement study from 120^2 to 180^2 grid points to verify the instability of the schemes for the case of

$a_0 = 0.53$, and the slow progress of the numerical orders of accuracy towards the asymptotic value for the case of $a_0 = 10$.

Table 5.5: The errors and orders of the EIN finite difference scheme for Example (5.5).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	20	3.18E-04		5.03E-04		3.56E-04	
	30	1.05E-04	2.85	1.64E-04	2.88	1.17E-04	2.87
	40	4.92E-05	2.70	7.74E-05	2.68	5.47E-05	2.70
	50	2.74E-05	2.67	4.31E-05	2.68	3.05E-05	2.68
	60	1.57E-05	3.12	2.47E-05	3.11	1.74E-05	3.12
0.54	20	3.09E-04		4.89E-04		3.46E-04	
	30	1.03E-04	2.83	1.61E-04	2.86	1.14E-04	2.84
	40	4.84E-05	2.68	7.62E-05	2.67	5.39E-05	2.69
	50	2.71E-05	2.66	4.25E-05	2.67	3.01E-05	2.67
	60	1.55E-05	3.11	2.44E-05	3.10	1.73E-05	3.12
10	20	9.04E-02		1.43E-01		1.01E-01	
	30	5.62E-02	1.22	8.81E-02	1.25	6.26E-02	1.23
	40	3.74E-02	1.46	5.89E-02	1.44	4.16E-02	1.46
	50	2.79E-02	1.35	4.38E-02	1.36	3.10E-02	1.35
	60	2.00E-02	1.86	3.14E-02	1.85	2.22E-02	1.86

Next, we consider the nonlinear convection-diffusion equation in two-dimension

$$U_t + \frac{1}{2}\Big(\big(U^2\big)_x + \big(U^2\big)_y\Big) - \nabla\cdot(a(U)\nabla U) = f(x,y,t), \qquad (x,y)\in(-\pi,\pi)^2 \tag{5.7}$$

augmented with the diffusion coefficient $a(U) = U^2 + 1$, the initial condition $U(x,y,0) = \sin(x+y)$ and the source term

$$f(x,y,t) = e^{-6t}\Big(-1 + 2e^{2t}\cos(x+y) - 3\cos\big(2(x+y)\big)\Big)\sin(x+y).$$

The exact solution to the problem is given by (5.6). Similarly, for the finite difference type spatial discretization, the third order upwind biased finite difference scheme coupled with the Lax-Friedrichs flux splitting is used for the convection term. As for the LDG spatial approximation of the convection term, we refer the readers to [11]. We compute to $T = 1$ with the time step $\tau = h$ and the stabilization parameter $a_1 = a_0 \max\limits_{u^n}\{(u^n)^2 + 1\}$. The numerical results of the proposed schemes are listed in Tables 5.9 and 5.10, from which we

Table 5.6: The errors and orders of the EIN-LDG scheme for Example (5.5).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	20	4.82E-04		1.10E-03		5.40E-04	
	30	1.90E-04	2.39	9.19E-04	0.54	2.37E-04	2.12
	40	9.05E-05	2.65	4.57E-04	2.50	1.15E-04	2.59
	50	4.85E-05	2.86	2.19E-04	3.37	6.02E-05	2.96
	60	3.26E-05	2.22	1.65E-04	1.59	4.22E-05	1.99
0.54	20	4.27E-04		6.84E-04		4.75E-04	
	30	1.40E-04	2.86	4.34E-04	1.17	1.59E-04	2.80
	40	6.03E-05	3.02	1.68E-04	3.40	6.80E-05	3.05
	50	3.02E-05	3.16	4.80E-05	5.74	3.36E-05	3.23
	60	1.70E-05	3.21	2.69E-05	3.24	1.89E-05	3.21
10	20	9.11E-02		1.43E-01		1.01E-01	
	30	5.64E-02	1.23	8.85E-02	1.24	6.26E-02	1.23
	40	3.75E-02	1.46	5.89E-02	1.46	4.16E-02	1.46
	50	2.79E-02	1.35	4.38E-02	1.35	3.10E-02	1.35
	60	2.00E-02	1.86	3.14E-02	1.86	2.22E-02	1.86

Table 5.7: The errors and orders of the EIN finite difference scheme for Example (5.5) with a denser mesh.

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	100	3.71E-06		5.82E-06		4.12E-06	
	120	2.19E-06	2.91	3.44E-06	2.91	2.43E-06	2.91
	140	1.38E-06	3.01	2.17E-06	3.01	1.54E-06	3.01
	160	9.28E-07	3.00	1.46E-06	3.00	1.03E-06	3.00
	180	6.55E-07	2.97	1.03E-06	2.97	7.28E-07	2.97
0.54	100	3.68E-06		5.78E-06		4.08E-06	
	120	2.17E-06	2.91	3.42E-06	2.91	2.42E-06	2.91
	140	1.37E-06	3.01	2.15E-06	3.01	1.52E-06	3.01
	160	9.21E-07	3.00	1.45E-06	3.00	1.02E-06	3.00
	180	6.51E-07	2.97	1.02E-06	2.97	7.23E-07	2.97
10	100	8.13E-03		1.28E-02		9.04E-03	
	120	5.66E-03	2.01	8.89E-03	2.01	6.29E-03	2.01
	140	4.06E-03	2.17	6.38E-03	2.17	4.51E-03	2.17
	160	3.03E-03	2.22	4.75E-03	2.22	3.36E-03	2.22
	180	2.33E-03	2.25	3.65E-03	2.25	2.58E-03	2.25

Table 5.8: The errors and orders of the EIN-LDG scheme for Example (5.5) with a denser mesh.

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	100	1.48E-05		9.18E-05		2.11E-05	
	120	1.27E-05	0.83	8.04E-05	0.74	1.85E-05	0.75
	140	1.37E-05	-0.50	9.20E-05	-0.88	2.05E-05	-0.68
	160	1.40E-05	-0.14	9.33E-05	-0.11	2.09E-05	-0.15
	180	1.53E-05	-0.76	1.03E-04	-0.87	2.30E-05	-0.81
0.54	100	3.88E-06		6.63E-06		4.31E-06	
	120	2.28E-06	2.93	4.15E-06	2.60	2.54E-06	2.93
	140	1.46E-06	2.92	3.17E-06	1.75	1.63E-06	2.90
	160	9.62E-07	3.16	1.81E-06	4.22	1.07E-06	3.18
	180	6.86E-07	2.89	1.45E-06	1.94	7.64E-07	2.88
10	100	8.13E-03		1.28E-02		9.04E-03	
	120	5.66E-03	2.01	8.89E-03	2.01	6.29E-03	2.01
	140	4.06E-03	2.17	6.38E-03	2.17	4.51E-03	2.17
	160	3.03E-03	2.22	4.75E-03	2.22	3.36E-03	2.22
	180	2.33E-03	2.25	3.65E-03	2.25	2.58E-03	2.25

can clearly observe optimal orders of accuracy when $a_0 = 0.54$. When $a_0 = 10$, the errors are larger and the numerical orders of accuracy is inferior to the results of $a_0 = 0.54$. Just as Example 5.5 shows, we believe that the numerical orders of accuracy will settle down towards the asymptotic value slowly with the refinement of the mesh. The schemes still show stable performance for this problem for the meshes under consideration when $a_0 = 0.53$, the results are not shown here to save space.

5.2 The third order dispersive equations

In this subsection, we validate the stability and orders of accuracy of the EIN finite difference and EIN-LDG schemes for the third order dispersive equations in one and two space dimensions. The generalization of the finite difference scheme given in Subsection 3.1 to the two-dimensional dispersive equation is straightforward. To generalize the LDG method (3.2) to the two-dimensional equations, we refer the readers to [29]. From the numerical results we can find that the smallest a_0 to ensure the stability of the schemes is 0.54, which validates

Table 5.9: The errors and orders of the EIN finite difference scheme for Example (5.7).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	6.08E-04		9.85E-04		6.80E-04	
	30	1.91E-04	2.97	3.02E-04	3.04	2.12E-04	2.99
	40	8.67E-05	2.82	1.36E-04	2.85	9.64E-05	2.83
	50	4.69E-05	2.82	7.33E-05	2.83	5.21E-05	2.82
	60	2.71E-05	3.06	4.23E-05	3.07	3.01E-05	3.06
10	20	1.37E-01		2.07E-01		1.52E-01	
	30	8.45E-02	1.23	1.28E-01	1.24	9.33E-02	1.25
	40	5.60E-02	1.47	8.56E-02	1.43	6.19E-02	1.47
	50	4.13E-02	1.39	6.33E-02	1.38	4.57E-02	1.39
	60	2.99E-02	1.81	4.60E-02	1.78	3.30E-02	1.81

Table 5.10: The errors and orders of the EIN-LDG scheme for Example (5.7).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	5.73E-04		9.67E-04		6.36E-04	
	30	1.69E-04	3.13	3.02E-04	2.98	1.88E-04	3.13
	40	7.46E-05	2.93	1.34E-04	2.92	8.29E-05	2.93
	50	4.00E-05	2.86	7.21E-05	2.83	4.45E-05	2.85
	60	2.28E-05	3.13	4.14E-05	3.10	2.54E-05	3.14
10	20	1.38E-01		2.09E-01		1.52E-01	
	30	8.52E-02	1.24	1.29E-01	1.24	9.37E-02	1.24
	40	5.62E-02	1.49	8.57E-02	1.45	6.19E-02	1.48
	50	4.15E-02	1.39	6.33E-02	1.39	4.57E-02	1.39
	60	2.99E-02	1.82	4.60E-02	1.78	3.31E-02	1.81

our stability result stated in previous subsection. In addition, we find that the stability result is also valid for the nonlinear dispersive equations containing lower order derivatives.

5.2.1 One-dimensional numerical tests

First we compute the linear dispersive equation

$$U_t + U_{xxx} = 0, \qquad x \in (-\pi, \pi) \tag{5.8}$$

with the initial condition $U(x, 0) = \sin(x)$. The exact solution is given by (5.4). The numerical errors and orders of accuracy are measured at $T = 1$ with the time step $\tau = h$. In Tables 5.11 and 5.12, we list the numerical results of the EIN finite difference and EIN-LDG schemes with different a_0. From these two tables we can see that the schemes are stable and can achieve optimal orders of accuracy if $a_0 \geq 0.54$. If we take $a_0 = 0.53$, the errors of the schemes will explode. The observation serves to confirm our theory result, i.e., the smallest a_0 to ensure the stability of the schemes is 0.54. In addition, we find that larger a_0 brings larger errors.

Table 5.11: The errors and orders of the EIN finite difference scheme for Example (5.8).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	4.31E-06		6.77E-06		4.79E-06	
	160	4.75E-07	3.21	7.47E-07	3.21	5.28E-07	3.21
	320	5.77E-08	3.06	9.09E-08	3.05	6.41E-08	3.06
	640	1.28E-06	-4.49	8.14E-06	-6.50	1.77E-06	-4.80
	1280	1.01E+01	-22.94	5.56E+01	-22.73	1.38E+01	-22.92
0.54	80	4.30E-06		6.75E-06		4.78E-06	
	160	4.73E-07	3.21	7.44E-07	3.21	5.26E-07	3.21
	320	5.74E-08	3.06	9.02E-08	3.06	6.38E-08	3.06
	640	7.11E-09	3.02	1.12E-08	3.02	7.89E-09	3.02
	1280	8.87E-10	3.01	1.39E-09	3.01	9.85E-10	3.01
10	80	2.31E-02		3.62E-02		2.56E-02	
	160	3.44E-03	2.77	5.40E-03	2.77	3.82E-03	2.77
	320	4.54E-04	2.93	7.14E-04	2.93	5.05E-04	2.93
	640	5.74E-05	2.99	9.02E-05	2.99	6.38E-05	2.99
	1280	7.21E-06	3.00	1.13E-05	3.00	8.00E-06	3.00

Table 5.12: The errors and orders of the EIN-LDG scheme for Example (5.8).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	8.23E-06		3.43E-05		1.19E-05	
	160	4.78E-06	0.79	1.85E-05	0.90	7.13E-06	0.74
	320	2.07E-05	-2.12	8.30E-05	-2.18	3.09E-05	-2.13
	640	3.40E-03	-7.38	1.36E-02	-7.37	5.08E-03	-7.38
	1280	7.33E+02	-17.74	2.94E+03	-17.74	1.10E+03	-17.74
0.54	80	4.26E-06		1.08E-05		4.96E-06	
	160	5.29E-07	3.04	1.19E-06	3.22	6.01E-07	3.07
	320	6.71E-08	2.99	1.55E-07	2.95	7.66E-08	2.99
	640	8.39E-09	3.01	1.93E-08	3.01	9.57E-09	3.01
	1280	1.05E-09	3.00	2.41E-09	3.00	1.20E-09	3.00
10	80	2.31E-02		3.62E-02		2.56E-02	
	160	3.44E-03	2.77	5.40E-03	2.77	3.82E-03	2.77
	320	4.54E-04	2.93	7.14E-04	2.93	5.05E-04	2.93
	640	5.74E-05	2.99	9.02E-05	2.99	6.38E-05	2.99
	1280	7.21E-06	3.00	1.13E-05	3.00	8.00E-06	3.00

Next, we consider the general KdV equation [20]

$$U_t + (U^3)_x + (U(U^2)_{xx})_x = 0, \qquad x \in (-\frac{3}{2}\pi, \frac{5}{2}\pi) \tag{5.9}$$

with the initial condition $U(x,0) = \sqrt{2\alpha}\cos\left(\frac{x}{2}\right)$. The problem has an exact solution

$$U(x,t) = \sqrt{2\alpha}\cos\left(\frac{x-\alpha t}{2}\right).$$

For the finite difference type spatial discretization, we use the third order upwind biased finite difference scheme coupled with the Lax-Friedrichs flux splitting to discretize the convection term. The complete LDG spatial discretization of the equation (5.9) can be found in [20,29]. We compute to $T = \pi$ with the time step $\tau = h$ and the stabilization parameter $a_1 = a_0 \max_{u^n}\{2(u^n)^2\}$. In Tables 5.13 and 5.14, we list the numerical errors and orders of accuracy for this example with the parameter $\alpha = 0.1$. As expected, the EIN finite difference and EIN-LDG schemes are stable and can achieve optimal orders of accuracy if $a_0 \geq 0.54$, while the results of the schemes deteriorate significantly with mesh refinements if $a_0 = 0.53$. This verifies that the smallest a_0 to ensure the stability of the schemes is 0.54. For this example, the deterioration of the errors for the case of $a_0 = 10$ versus that for $a_0 = 0.54$ is less severe.

Table 5.13: The errors and orders of the EIN finite difference scheme for Example (5.9).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	1.18E-05		1.62E-05		1.25E-05	
	160	1.43E-06	3.08	2.04E-06	3.02	1.53E-06	3.05
	320	1.75E-07	3.04	2.61E-07	2.98	1.90E-07	3.02
	640	4.03E-07	-1.21	1.40E-05	-5.76	1.57E-06	-3.05
	1280	1.76E-03	-12.10	1.24E-02	-9.81	2.90E-03	-10.86
0.54	80	1.18E-05		1.62E-05		1.25E-05	
	160	1.43E-06	3.08	2.04E-06	3.02	1.53E-06	3.05
	320	1.75E-07	3.04	2.61E-07	2.98	1.90E-07	3.02
	640	2.16E-08	3.02	3.31E-08	2.98	2.38E-08	3.01
	1280	2.69E-09	3.01	4.17E-09	2.99	2.98E-09	3.00
10	80	1.56E-05		2.12E-05		1.65E-05	
	160	1.90E-06	3.07	2.76E-06	2.97	2.05E-06	3.04
	320	2.34E-07	3.03	3.54E-07	2.98	2.56E-07	3.01
	640	2.90E-08	3.02	4.48E-08	2.99	3.20E-08	3.01
	1280	3.62E-09	3.01	5.64E-09	2.99	4.00E-09	3.00

Table 5.14: The errors and orders of the EIN-LDG scheme for Example (5.9).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	8.23E-07		3.62E-06		1.21E-06	
	160	1.20E-07	2.80	1.58E-06	1.21	1.97E-07	2.64
	320	3.58E-07	-1.59	2.16E-05	-3.79	1.39E-06	-2.83
	640	1.32E-03	-11.88	4.89E-03	-7.84	1.87E-03	-10.42
	1280	3.86E-03	-1.55	2.26E-02	-2.21	6.01E-03	-1.69
0.54	80	8.04E-07		3.61E-06		1.20E-06	
	160	1.02E-07	3.00	4.51E-07	3.03	1.53E-07	3.00
	320	1.29E-08	3.00	5.67E-08	3.00	1.94E-08	2.99
	640	1.61E-09	3.00	7.14E-09	3.00	2.44E-09	3.00
	1280	2.00E-10	3.01	8.81E-10	3.02	3.05E-10	3.00
10	80	3.80E-06		6.79E-06		4.26E-06	
	160	4.80E-07	3.01	8.31E-07	3.06	5.38E-07	3.01
	320	6.06E-08	3.00	1.03E-07	3.03	6.78E-08	3.00
	640	7.48E-09	3.02	1.29E-08	3.01	8.39E-09	3.02
	1280	9.33E-10	3.01	1.60E-09	3.01	1.04E-09	3.01

5.2.2 Two-dimensional numerical tests

First, we solve the Zakharov-Kuznetsov (ZK) equation [22] in two-dimension

$$U_t + (\frac{U^2}{2})_x + U_{xxx} + U_{xyy} = f(x, y, t), \qquad (x, y) \in (\pi, \pi)^2 \tag{5.10}$$

with the initial condition $U(x, y, 0) = \sin(x + y)$ and the source term

$$f(x, y, t) = \cos(x + y + t)(-1 + \sin(x + y + t)).$$

The exact solution is given by

$$U(x, y, t) = \sin(x + y + t).$$

Similarly, for the finite difference type spatial discretization of the convection term, the standard third order WENO scheme with linear weights (when the smoothness indicators and nonlinear weights are turned off) is used. As for the complete description of the LDG spatial approximation, we refer to [29]. We compute to $T = \pi$ with the time step $\tau = h$. The errors and numerical orders of accuracy of the EIN finite difference and EIN-LDG schemes can be found in Tables 5.15 and 5.16, respectively. In each table, we present the numerical results of the schemes with different a_0. As one can see, the schemes are stable and can achieve optimal orders of accuracy if $a_0 = 0.54$, and the orders of the EIN-LDG scheme deteriorates significantly if $a_0 = 0.53$. Once again, when $a_0 = 10$, the errors are much larger and the numerical orders of accuracy settle down towards the asymptotic value much slower with mesh refinements, in comparison with the results of $a_0 = 0.54$.

Next, we consider the general KdV equation in two-dimension

$$U_t + (U^3)_x + (U^3)_y + (U(U^2)_{xx})_x + (U(U^2)_{yy})_y = 0, \qquad (x, y) \in (-\frac{3}{2}\pi, \frac{5}{2}\pi)^2 \tag{5.11}$$

with the initial condition $U(x, y, 0) = \sqrt{2\alpha}\cos\left(\frac{x+y}{2}\right)$. The exact solution is given by

$$U(x, y, t) = \sqrt{2\alpha}\cos\left(\frac{x + y - 2\alpha t}{2}\right).$$

Table 5.15: The errors and orders of the EIN finite difference scheme for Example (5.10).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	20	6.90E-03		1.22E-02		7.80E-03	
	30	2.10E-03	3.10	3.40E-03	3.25	2.30E-03	3.13
	40	8.69E-04	3.09	1.40E-03	3.12	9.72E-04	3.10
	50	4.47E-04	3.05	7.37E-04	3.08	4.97E-04	3.07
	60	2.58E-04	3.07	4.20E-04	3.14	2.87E-04	3.07
0.54	20	7.10E-03		1.21E-02		7.90E-03	
	30	2.10E-03	3.12	3.50E-03	3.16	2.30E-03	3.12
	40	8.87E-04	3.10	1.50E-03	3.14	9.89E-04	3.09
	50	4.55E-04	3.06	7.42E-04	3.12	5.06E-04	3.07
	60	2.63E-04	3.06	4.26E-04	3.10	2.92E-04	3.06
10	20	3.74E-01		5.90E-01		4.16E-01	
	30	2.80E-01	0.74	4.41E-01	0.75	3.11E-01	0.75
	40	2.18E-01	0.89	3.49E-01	0.84	2.43E-01	0.89
	50	1.76E-01	0.99	2.84E-01	0.95	1.96E-01	0.98
	60	1.51E-01	0.87	2.44E-01	0.85	1.68E-01	0.87

Table 5.16: The errors and orders of the EIN-LDG scheme for Example (5.10).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	20	3.50E-03		1.78E-02		4.60E-03	
	30	1.10E-03	3.03	3.60E-03	4.10	1.30E-03	3.21
	40	7.05E-04	1.45	4.80E-03	-1.01	1.10E-03	0.70
	50	6.04E-04	0.71	2.90E-03	2.31	8.04E-04	1.32
	60	6.79E-04	-0.65	4.40E-03	-2.41	1.00E-03	-1.30
0.54	20	3.30E-03		1.14E-02		3.90E-03	
	30	1.10E-03	2.96	3.10E-03	3.34	1.20E-03	3.00
	40	4.58E-04	2.97	1.40E-03	2.86	5.24E-04	2.96
	50	2.37E-04	3.01	6.83E-04	3.25	2.70E-04	3.04
	60	1.38E-04	3.01	4.03E-04	2.95	1.57E-04	3.01
10	20	3.74E-01		5.95E-01		4.16E-01	
	30	2.80E-01	0.75	4.42E-01	0.76	3.11E-01	0.75
	40	2.18E-01	0.89	3.49E-01	0.85	2.43E-01	0.89
	50	1.76E-01	0.98	2.84E-01	0.95	1.96E-01	0.98
	60	1.51E-01	0.87	2.44E-01	0.85	1.68E-01	0.87

For the finite difference type spatial discretization, the third order upwind biased finite difference scheme coupled with the Lax-Friedrichs flux splitting is used for the convection term. For the complete LDG spatial approximation, please refer to [29]. We compute to $T = 1$ with the time step $\tau = h$ and the stabilization parameter $a_1 = a_0 \max_{u^n}\{2(u^n)^2\}$. In Tables 5.17 and 5.18, we display the numerical errors and orders of accuracy for this example with the parameter $\alpha = 0.5$. Similarly as before, since the meshes we have used are not refined enough, the results with $a_0 = 0.53$ have not shown signs of instability and hence are not shown here to save space. When $a_0 = 0.54$, the schemes remain stable as always. Once again, the errors are significantly larger for this example when $a_0 = 10$, in comparison with those when $a_0 = 0.54$.

Table 5.17: The errors and orders of the EIN finite difference scheme for Example (5.11).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	9.77E-03		1.54E-02		1.01E-02	
	30	3.03E-03	3.01	3.83E-03	3.58	3.12E-03	3.03
	40	1.33E-03	2.94	1.78E-03	2.75	1.41E-03	2.85
	50	6.95E-04	2.98	9.54E-04	2.85	7.45E-04	2.91
	60	3.92E-04	3.20	5.55E-04	3.02	4.24E-04	3.15
10	20	4.63E-01		7.33E-01		5.18E-01	
	30	2.93E-01	1.18	4.59E-01	1.20	3.26E-01	1.19
	40	1.67E-01	2.02	2.61E-01	2.02	1.85E-01	2.03
	50	8.60E-02	3.03	1.35E-01	3.02	9.55E-02	3.03
	60	4.37E-02	3.78	6.88E-02	3.77	4.86E-02	3.77

5.3 The fourth order biharmonic-type equations

In this subsection we test the stability and orders of accuracy of the EIN finite difference and EIN-LDG schemes for the fourth order biharmonic-type equations in one and two space dimensions. The generalization of the finite difference scheme given in Subsection 4.1 to the two-dimensional equation is straightforward. For a detailed introduction of the LDG method for the two-dimensional biharmonic-type equations, we refer the readers to [30]. The numerical results further confirm the sharpness of $a_0 = 0.54$ for stability. Besides, we

Table 5.18: The errors and orders of the EIN-LDG scheme for Example (5.11).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	3.70E-03		8.10E-03		4.10E-03	
	30	1.30E-03	2.65	2.40E-03	3.11	1.40E-03	2.68
	40	6.21E-04	2.65	1.10E-03	2.97	6.85E-04	2.63
	50	3.37E-04	2.81	5.52E-04	2.98	3.72E-04	2.79
	60	1.85E-04	3.36	2.95E-04	3.51	2.05E-04	3.34
10	20	4.75E-01		7.48E-01		5.27E-01	
	30	3.02E-01	1.16	4.74E-01	1.17	3.35E-01	1.16
	40	1.68E-01	2.09	2.64E-01	2.09	1.87E-01	2.09
	50	8.68E-02	3.03	1.37E-01	3.03	9.64E-02	3.03
	60	4.40E-02	3.80	6.92E-02	3.79	4.89E-02	3.80

can find that the schemes for the nonlinear biharmonic-type equations with and without lower order derivative terms are also stable under the condition $a_0 \geq 0.54$.

5.3.1 One-dimensional numerical tests

We consider the following linear biharmonic equation

$$U_t + U_{xxxx} = 0, \qquad x \in (-\pi, \pi) \tag{5.12}$$

with the initial condition $U(x, 0) = \sin(x)$. The exact solution is given by (5.2). We compute to $T = 1$ with the time step $\tau = h$. The numerical errors and orders of accuracy are listed in Tables 5.19 and 5.20. In each table, we display the numerical results of the two schemes with different a_0. If $a_0 = 0.53$, we can clearly see the explosive growth of the errors with mesh refinements, while the schemes are stable and give optimal orders of accuracy if $a_0 \geq 0.54$. This verifies that the smallest a_0 to stabilize the schemes is 0.54. Also, larger a_0 causes significantly larger errors. The numerical results are in good agreement with the theory.

Next, we consider the following nonlinear biharmonic-type equation [25]

$$U_t + ((U^2 + 2)U_{xx})_{xx} = f(x, t), \qquad x \in (-\pi, \pi) \tag{5.13}$$

with the initial condition $U(x, 0) = \sin(x)$ and the source term

$$f(x, t) = e^{-3t}\big(e^{2t} - 6\cos^2(x) + 3\sin^2(x)\big)\sin(x).$$

Table 5.19: The errors and orders of the EIN finite difference scheme for Example (5.12).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	1.08E-06		1.69E-06		1.20E-06	
	160	1.51E-07	2.86	2.39E-07	2.85	1.68E-07	2.86
	320	8.65E-08	0.81	4.51E-07	-0.92	1.21E-07	0.48
	640	9.57E-04	-13.46	4.84E-03	-13.42	1.31E-03	-13.44
	1280	1.30E+04	-23.73	6.67E+04	-23.74	1.76E+04	-23.70
0.54	80	1.07E-06		1.68E-06		1.19E-06	
	160	1.50E-07	2.85	2.36E-07	2.85	1.67E-07	2.85
	320	1.99E-08	2.93	3.13E-08	2.93	2.22E-08	2.93
	640	2.55E-09	2.97	4.00E-09	2.97	2.83E-09	2.97
	1280	3.22E-10	2.99	5.06E-10	2.99	3.58E-10	2.99
10	80	4.02E-03		6.32E-03		4.47E-03	
	160	7.90E-04	2.37	1.24E-03	2.37	8.78E-04	2.37
	320	1.29E-04	2.63	2.02E-04	2.63	1.43E-04	2.63
	640	1.84E-05	2.81	2.89E-05	2.81	2.04E-05	2.81
	1280	2.47E-06	2.90	3.88E-06	2.90	2.74E-06	2.90

Table 5.20: The errors and orders of the EIN-LDG scheme for Example (5.12).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	6.85E-06		2.53E-05		9.67E-06	
	160	4.94E-06	0.48	2.01E-05	0.34	7.35E-06	0.40
	320	9.24E-05	-4.25	9.15E-04	-5.53	1.49E-04	-4.36
	640	9.55E-01	-13.37	1.42E+01	-13.96	1.63E+00	-13.45
	1280	1.36E+07	-23.79	1.92E+08	-23.71	2.29E+07	-23.77
0.54	80	1.42E-06		2.56E-06		1.59E-06	
	160	1.69E-07	3.10	2.81E-07	3.22	1.87E-07	3.11
	320	2.16E-08	2.98	3.67E-08	2.95	2.40E-08	2.98
	640	2.70E-09	3.00	4.59E-09	3.01	3.00E-09	3.00
	1280	3.38E-10	3.00	5.74E-10	3.00	3.76E-10	3.00
10	80	4.02E-03		6.32E-03		4.47E-03	
	160	7.90E-04	2.37	1.24E-03	2.37	8.78E-04	2.37
	320	1.29E-04	2.63	2.02E-04	2.63	1.43E-04	2.63
	640	1.84E-05	2.81	2.89E-05	2.81	2.04E-05	2.81
	1280	2.47E-06	2.90	3.88E-06	2.90	2.74E-06	2.90

The exact solution is given by (5.2). Obviously, the above equation cannot be described by the model shown in (1.3). Therefore, the LDG scheme we give in previous subsection is not suitable for it. In order to discretize the above equation, we modify the LDG scheme slightly, and give stability analysis for the modified scheme by the aid of the energy analysis in the appendix. The generalization of our new LDG scheme to the two-dimensional equations is straightforward. It is worth pointing out that when $U^2 + 2 = 1$, the modified LDG scheme is equivalent to that we give in Subsection 4.2. We compute to $T = 1$ with the time step $\tau = h$ and the stabilization parameter $a_1 = a_0 \max_{u^n}\{(u^n)^2 + 2\}$. In Tables 5.21 and 5.22, we can observe that our schemes give the optimal orders of accuracy for this nonlinear problem if $a_0 \geq 0.54$, while the numerical results deteriorate significantly if $a_0 = 0.53$. Once more, when $a_0 = 10$, the errors are much larger and the numerical orders of accuracy settle down towards the asymptotic value at a slower speed with mesh refinements, in comparison with the results of $a_0 = 0.54$.

Table 5.21: The errors and orders of the EIN finite difference scheme for Example (5.13).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	3.91E-06		6.03E-06		4.29E-06	
	160	4.78E-07	3.06	1.09E-06	2.48	5.35E-07	3.03
	320	8.42E-08	2.52	4.14E-07	1.41	1.17E-07	2.20
	640	1.10E-06	-3.71	1.69E-05	-5.36	2.96E-06	-4.67
	1280	3.90E-03	-11.81	2.92E-02	-10.77	7.60E-03	-11.34
0.54	80	4.12E-06		6.05E-06		4.52E-06	
	160	5.09E-07	3.05	7.84E-07	2.98	5.62E-07	3.03
	320	5.84E-08	3.14	7.94E-08	3.32	6.32E-08	3.17
	640	6.32E-09	3.22	8.97E-09	3.15	6.89E-09	3.21
	1280	7.31E-10	3.12	1.07E-09	3.07	8.01E-10	3.11
10	80	1.15E-02		1.79E-02		1.27E-02	
	160	3.00E-03	1.96	4.70E-03	1.95	3.30E-03	1.96
	320	6.15E-04	2.30	9.57E-04	2.30	6.82E-04	2.30
	640	1.02E-04	2.60	1.59E-04	2.60	1.13E-04	2.60
	1280	1.49E-05	2.78	2.32E-05	2.78	1.65E-05	2.78

Table 5.22: The errors and orders of the EIN-LDG scheme for Example (5.13).

a_0	N	L^1 error	order	L^∞ error	order	L^2 error	order
0.53	80	4.14E-06		6.74E-06		4.55E-06	
	160	4.89E-07	3.11	4.08E-06	0.73	5.79E-07	3.00
	320	5.99E-05	-6.97	8.13E-04	-7.67	1.02E-04	-7.49
	640	1.56E-02	-8.04	4.40E-02	-5.77	1.95E-02	-7.60
	1280	2.17E-02	-0.48	6.61E-02	-0.59	2.62E-02	-0.43
0.54	80	4.35E-06		6.60E-06		4.77E-06	
	160	5.23E-07	3.08	8.17E-07	3.04	5.79E-07	3.07
	320	5.95E-08	3.15	8.72E-08	3.24	6.45E-08	3.18
	640	6.41E-09	3.22	9.63E-09	3.19	6.99E-09	3.21
	1280	7.38E-10	3.12	1.13E-09	3.10	8.09E-10	3.11
10	80	1.15E-02		1.79E-02		1.27E-02	
	160	3.00E-03	1.96	4.66E-03	1.95	3.32E-03	1.96
	320	6.15E-04	2.30	9.57E-04	2.30	6.82E-04	2.30
	640	1.02E-04	2.60	1.59E-04	2.60	1.13E-04	2.60
	1280	1.49E-05	2.78	2.32E-05	2.78	1.65E-05	2.78

5.3.2 Two-dimensional numerical tests

First, we show an accuracy test for the Kuramoto-Sivashinsky (KS) equation [27] in two-dimension

$$U_t + \frac{1}{2}(U^2)_x - \sigma\Delta U + \Delta^2 U = f(x,y,t), \qquad (x,y) \in (-\pi,\pi)^2 \tag{5.14}$$

with the initial condition $U(x,y,0) = \sin(x+y)$ and the source term

$$f(x,y,t) = e^{-4t}\sin(x+y)\big(2e^{2t}(1+\sigma) + \cos(x+y)\big).$$

The exact solution is given by (5.6). A complete LDG scheme for discretizing one-dimensional KS equations of the above form is given in [27]. If we remove the convection term, it is worth mentioning that the scheme is just a simple combination of the LDG methods given in this paper to discretize the second order and the fourth order dissipative terms. We can easily extend the LDG scheme given in [27] to the two-dimensional case. The numerical errors and orders of accuracy are measured at $T = 1$ with the time step $\tau = h$ and the diffusion coefficient $\sigma = 2$. In order to obtain nicer-looking numerical orders of accuracy, we take an extensive mesh refinement from 20^2 to 100^2 grid points in the tests. In Tables 5.23 and 5.24, we display the numerical results of the EIN finite difference and EIN-LDG schemes with different a_0. Similarly as before, since the meshes we have used are not refined enough, the results with $a_0 = 0.53$ have not shown signs of instability and hence are not shown here to save space. Our observation is that the schemes remain stable as always when $a_0 = 0.54$. Once again, when $a_0 = 10$, the errors are much larger and the numerical orders of accuracy settle down towards the asymptotic value much slower with mesh refinements, in comparison with the results of $a_0 = 0.54$.

Table 5.23: The errors and orders of the EIN finite difference scheme for Example (5.14).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	3.84E-03		6.27E-03		4.32E-03	
	40	6.64E-04	2.62	1.47E-03	2.17	7.95E-04	2.53
	60	1.65E-04	3.50	2.82E-04	4.16	1.85E-04	3.67
	80	5.42E-05	3.93	9.24E-05	3.93	6.05E-05	3.95
	100	1.07E-05	7.34	2.20E-05	6.51	1.23E-05	7.21
10	20	3.43E-02		5.44E-02		3.85E-02	
	40	1.41E-02	1.34	2.21E-02	1.34	1.56E-02	1.34
	60	8.22E-03	1.35	1.29E-02	1.35	9.14E-03	1.35
	80	5.62E-03	1.34	8.83E-03	1.34	6.24E-03	1.34
	100	4.20E-03	1.32	6.60E-03	1.32	4.67E-03	1.32

Table 5.24: The errors and orders of the EIN-LDG scheme for Example (5.14).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	3.94E-03		7.90E-03		4.39E-03	
	40	6.66E-04	2.66	1.54E-03	2.44	7.93E-04	2.56
	60	1.66E-04	3.49	3.10E-04	4.04	1.85E-04	3.66
	80	5.42E-05	3.95	9.36E-05	4.23	6.04E-05	3.95
	100	1.08E-05	7.32	2.61E-05	5.78	1.24E-05	7.19
10	20	3.45E-02		5.43E-02		3.84E-02	
	40	1.41E-02	1.34	2.21E-02	1.34	1.56E-02	1.34
	60	8.23E-03	1.35	1.29E-02	1.35	9.14E-03	1.35
	80	5.62E-03	1.34	8.83E-03	1.34	6.24E-03	1.34
	100	4.20E-03	1.32	6.60E-03	1.32	4.67E-03	1.32

Next, we consider the general biharmonic-type equation in two-dimension

$$U_t + (U^3)_x + \Delta(U^2 \Delta U) = f(x,y,t), \qquad (x,y) \in (-\pi,\pi)^2 \tag{5.15}$$

augmented with the initial condition $U(x,y,0) = \sin(x+y)$ and the source term

$$f(x,y,t) = -\frac{1}{2}e^{-6t}\sin(x+y)\big(4e^{4t} + 12 + 36\cos(2(x+y)) - 3\sin(2(x+y))\big).$$

The exact solution is given by (5.6). The numerical errors and orders of accuracy are measured at $T = 1$ with the time step $\tau = h$ and the stabilization parameter $a_1 = a_0 \max_{u^n}\{(u^n)^2\}$. In Tables 5.25 and 5.26, we present the numerical results of the EIN finite difference and EIN-LDG schemes with different a_0. Owing to the relatively coarse meshes, the results of $a_0 = 0.53$ are not showing signs of instability yet, and we omit them here to save space. When $a_0 = 0.54$, the numerical results are in good agreement with our expectations. When $a_0 = 10$, the errors are larger and the numerical orders of accuracy settle down to the asymptotic value slower, in comparison with the case for $a_0 = 0.54$.

Table 5.25: The errors and orders of the EIN finite difference scheme for Example (5.15).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	5.39E-03		7.32E-03		5.70E-03	
	30	1.23E-03	3.79	1.49E-03	4.09	1.28E-03	3.84
	40	4.37E-04	3.71	5.55E-04	3.53	4.53E-04	3.71
	50	1.93E-04	3.75	2.64E-04	3.41	2.01E-04	3.73
	60	1.06E-04	3.32	1.54E-04	2.99	1.12E-04	3.25
10	20	1.08E-01		1.57E-01		1.17E-01	
	30	6.16E-02	1.43	9.05E-02	1.41	6.70E-02	1.44
	40	4.04E-02	1.51	6.03E-02	1.45	4.41E-02	1.50
	50	2.83E-02	1.63	4.31E-02	1.55	3.09E-02	1.62
	60	2.11E-02	1.65	3.25E-02	1.57	2.30E-02	1.65

6 Concluding remarks

We have considered the high order finite difference and local discontinuous Galerkin schemes coupled with a specific third order explicit-implicit-null time-marching method for solving

Table 5.26: The errors and orders of the EIN-LDG scheme for Example (5.15).

a_0	N_x, N_y	L^1 error	order	L^∞ error	order	L^2 error	order
0.54	20	5.92E-03		7.95E-03		6.25E-03	
	30	1.43E-03	3.65	1.75E-03	3.89	1.48E-03	3.69
	40	4.99E-04	3.77	6.51E-04	3.53	5.19E-04	3.76
	50	2.27E-04	3.61	3.17E-04	3.30	2.38E-04	3.57
	60	1.24E-04	3.37	1.82E-04	3.09	1.31E-04	3.33
10	20	1.08E-01		1.59E-01		1.18E-01	
	30	6.23E-02	1.42	9.18E-02	1.41	6.78E-02	1.43
	40	4.05E-02	1.53	6.05E-02	1.49	4.41E-02	1.53
	50	2.85E-02	1.62	4.34E-02	1.52	3.11E-02	1.61
	60	2.11E-02	1.68	3.26E-02	1.60	2.30E-02	1.67

high order dissipative and dispersive equations in one and two space dimensions. We have presented the stability analysis of the proposed schemes for the one-dimensional simplified models, and through the analysis we show that the schemes are stable provided that a crucial parameter a_0 satisfies $a_0 \geq 0.54$. To verify the correctness of the result, ample numerical tests including one-dimensional and two-dimensional linear and nonlinear problems have been considered. Numerical experiments show that the schemes are stable and can achieve optimal orders of accuracy if $a_0 \geq 0.54$, while the simulation results for almost the one-dimensional tests and a few two-dimensional examples deteriorate significantly if $a_0 = 0.53$. Even though the analysis is only performed on the linear equations containing the highest derivatives, numerical experiments show that the stability criterion can be extended to the nonlinear equations containing lower order derivatives. In the future, we would like to consider spatially varying linear terms for the implicit part of the EIN scheme, and explore stability analysis for the schemes with non-periodic boundary conditions and non-uniform meshes through the aid of energy analysis.

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Appendix

Stability of an alternative LDG scheme for the biharmonic-type equations

Consider the following nonlinear equation

$$U_t + (a(U)U_{xx})_{xx} = 0$$

with periodic boundary condition. Here $a(U) \geq 0$ is bounded and smooth. Obviously, the above equation cannot be discretized by the LDG scheme we give in Subsection 4.2. Here we will modify the LDG scheme slightly, and give a stability analysis for the modified scheme by the aid of the energy analysis. Before we present the new LDG method, we introduce some variables

$$R = U_x, \qquad V = R_x, \qquad Q = a(U)V, \qquad P = Q_x,$$

and rewrite the equation above as a first order system:

$$U_t + P_x = 0, \qquad P - Q_x = 0, \qquad Q = a(U)V,$$
$$V - R_x = 0, \qquad R - U_x = 0.$$

Here the LDG method is defined as follows: find u, r, v, q, $p \in V_h$ such that, for all the test functions ϕ_1, ϕ_2, ϕ_3, ϕ_4, $\phi_5 \in V_h$ and $1 \leq j \leq N$

$$\int_{I_j} u_t\phi_1 dx - \int_{I_j} p(\phi_1)_x dx + \hat{p}_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} - \hat{p}_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}} = 0, \tag{A.1}$$

$$\int_{I_j} p\phi_2 dx + \int_{I_j} q(\phi_2)_x dx - \hat{q}_{j+\frac{1}{2}}(\phi_2)^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}}(\phi_2)^+_{j-\frac{1}{2}} = 0, \tag{A.2}$$

$$\int_{I_j} q\phi_3 dx - \int_{I_j} a(u)v\phi_3 dx = 0, \tag{A.3}$$

$$\int_{I_j} v\phi_4 dx + \int_{I_j} r(\phi_4)_x dx - \hat{r}_{j+\frac{1}{2}}(\phi_4)^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}}(\phi_4)^+_{j-\frac{1}{2}} = 0, \tag{A.4}$$

$$\int_{I_j} r\phi_5 dx + \int_{I_j} u(\phi_5)_x dx - \hat{u}_{j+\frac{1}{2}}(\phi_5)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}(\phi_5)^+_{j-\frac{1}{2}} = 0. \tag{A.5}$$

Here $\hat{p}_{j+\frac{1}{2}}$, $\hat{q}_{j+\frac{1}{2}}$, $\hat{r}_{j+\frac{1}{2}}$, $\hat{u}_{j+\frac{1}{2}}$ are single-valued numerical fluxes at the cell interfaces of $I_j; I_{j+1}$, depending on the values of numerical solution from both sides $x^-_{j+\frac{1}{2}}, x^+_{j+\frac{1}{2}}$ etc. We can take the following simple choice of fluxes for stability

$$\hat{p}_{j+\frac{1}{2}} = p(x^-_{j+\frac{1}{2}}), \qquad \hat{q}_{j+\frac{1}{2}} = q(x^+_{j+\frac{1}{2}}), \qquad \hat{r}_{j+\frac{1}{2}} = r(x^-_{j+\frac{1}{2}}), \qquad \hat{u}_{j+\frac{1}{2}} = u(x^+_{j+\frac{1}{2}}). \tag{A.6}$$

The generalization of our new LDG method to the two-dimensional equations is straightforward. It is worth pointing out that when $a(U) = 1$, the modified LDG scheme is equivalent to that we give in Subsection 4.2. Next, we will show the stability property of the scheme with the choice of fluxes above.

Theorem: *The numerical scheme* (A.1)-(A.5) *with the choice of fluxes* (A.6) *is L^2 stable, i.e.*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega a(u)v^2 dx = 0. \tag{A.7}$$

Proof. We sum up the five equalities (A.1)-(A.5) and introduce the notation

$$\begin{aligned} B_j(u, r, v, q, p; \phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = &\int_{I_j} u_t\phi_1 dx - \int_{I_j} p(\phi_1)_x dx + \hat{p}_{j+\frac{1}{2}}(\phi_1)^-_{j+\frac{1}{2}} - \hat{p}_{j-\frac{1}{2}}(\phi_1)^+_{j-\frac{1}{2}} + \\ &\int_{I_j} p\phi_2 dx + \int_{I_j} q(\phi_2)_x dx - \hat{q}_{j+\frac{1}{2}}(\phi_2)^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}}(\phi_2)^+_{j-\frac{1}{2}} + \\ &\int_{I_j} q\phi_3 dx - \int_{I_j} a(u)v\phi_3 dx + \\ &\int_{I_j} v\phi_4 dx + \int_{I_j} r(\phi_4)_x dx - \hat{r}_{j+\frac{1}{2}}(\phi_4)^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}}(\phi_4)^+_{j-\frac{1}{2}} + \\ &\int_{I_j} r\phi_5 dx + \int_{I_j} u(\phi_5)_x dx - \hat{u}_{j+\frac{1}{2}}(\phi_5)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}(\phi_5)^+_{j-\frac{1}{2}}. \end{aligned}$$

Obviously, the solutions u, r, v, q, p of the scheme satisfy

$$B_j(u, r, v, q, p; \phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = 0$$

for all $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \in V_h$. We then take

$$\phi_1 = u, \qquad \phi_2 = r, \qquad \phi_3 = -v, \qquad \phi_4 = q, \qquad \phi_5 = -p$$

to obtain, after some algebraic manipulations,

$$0 = B_j(u, r, v, q, p;\, u, r, -v, q, -p) = \frac{1}{2}\frac{d}{dt}\int_{I_j} u^2 dx + \int_{I_j} a(u) v^2 dx + (\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}) + \Theta_{j-\frac{1}{2}},$$

where

$$\hat{H} = (rq)^- - (pu)^- + \hat{p}u^- + \hat{u}p^- - \hat{q}r^- - \hat{r}q^-,$$

$$\Theta = -[rq] + [pu] - \hat{p}[u] - \hat{u}[p] + \hat{q}[r] + \hat{r}[q],$$

where $[p]$ denotes $p^+ - p^-$. To this end, we notice that, with the definition (A.6) of the numerical fluxes, we can easily obtain $\Theta_{j-\frac{1}{2}} = 0$. Then we sum over j to obtain (A.7). $\square$