Third order maximum-principle-satisfying and positivity-preserving Lax-Wendroff discontinuous Galerkin methods for hyperbolic conservation laws*

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Abstract

There have been intensive studies on maximum-principle-satisfying and positivity-preserving methods for hyperbolic conservation laws. Most of them are based on the method of lines type time marching approaches, e.g. the Runge-Kutta methods, multi-step methods and backward Euler method. As an alternative, the Lax-Wendroff time marching approach utilizes the information of PDEs in the Taylor expansion of the solution in time, hence it is a high order and single-stage method. In this work, we propose third order maximum-principle-satisfying and positivity-preserving schemes for scalar conservation laws and the Euler equations based on the Lax-Wendroff time discretization and discontinuous Galerkin spatial discretization. The accuracy and effectiveness of the maximum-principle-satisfying and positivity-preserving techniques are demonstrated by ample numerical tests.

Key Words: maximum-principle-satisfying, positivity-preserving, Lax-Wendroff discontinuous Galerkin methods (LWDG), scalar conservation laws, Euler equations.

1 Introduction

Hyperbolic conservation laws are basic tools to characterize the phenomena of flow and transport, e.g. the Burgers' equation for traffic flow and the Buckley-Leverett equation for two phase flow as the scalar cases, and the Euler equations for compressible gas dynamics and shallow water equations for water with shallow depth as the system cases.

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The scalar conservation laws are known to satisfy the maximum-principle, e.g. for the one dimensional scalar equation

$$u_t + f(u)_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},$$

the entropy solution satisfies $m \le u(x,t) \le M, \forall x \in \mathbb{R}, t > 0$, where $m = \min_{x\in\mathbb{R}} u_0(x)$ and $M = \max_{x\in\mathbb{R}} u_0(x)$. Same results also hold for periodic boundary conditions, bounded domain with compactly supported solution, and higher dimensions.

Similarly, the positivity of certain important physical quantities are satisfied by some hyperbolic systems, e.g. for the Euler equations

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}, \quad x \in \mathbb{R}, t > 0 \tag{1.2}$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} m \\ \rho u^2 + p \\ (E+p)u \end{pmatrix},$$

with

$$m = \rho u, \quad E = \frac{1}{2}\rho u^2 + \rho e, \quad p = (\gamma - 1)\rho e,$$

in which ρ is the density of fluid, m is the momentum, u is the velocity, E is the total energy, p is the pressure, e is the specific internal energy, and $\gamma > 1$ is the ratio of specific heats, it is well-known that the physical solution $\mathbf{u} \in G$ for all $t > 0$ if it holds at $t = 0$, where G is the admissible set of solutions defined as

$$G = \{\mathbf{u} : \rho \ge 0, p(\mathbf{u}) \ge 0\}. \tag{1.3}$$

Rigorously preserving these physical bounds of solutions is of great importance for the robustness of numerical algorithms, in that once the quantities were out of their physical range, the hyperbolicity of equations is lost, which often leads to the simulation failure. There have been intensive studies on the maximum-principle-satisfying and positivity-preserving numerical methods for hyperbolic conservation laws. In 2010, the genuinely maximum-principle satisfying high-order discontinuous Galerkin (DG) and finite volume methods for scalar conservation laws were proposed by Zhang and Shu in [38]. The algorithm is composed of two steps under the DG framework. The first step is to prove desired physical bounds for the cell averages of numerical solutions are automatically satisfied by the unmodulated high order DG scheme with appropriate CFL conditions and numerical fluxes. Then a scaling limiter, which does not destroy accuracy and mass conservation, are adopted to modify the solution such that the physical bounds satisfied by cell averages are extended to the entire solution. Based on this simple and general framework, the high order maximum-principle-satisfying and positivity-preserving numerical schemes have been rapidly developed for different problems ever since, for instance, the Euler equations [37, 39, 32], the Navier-Stokes equations [36], the shallow water equations [34, 33, 19], convection diffusion equations [40, 17, 2], and hyperbolic equations involving $\delta-$singularities [42, 35], etc. For convenience, we call both maximum-principle-satisfying and positivity-preserving techniques the bound-preserving methods in this paper.

It should be noted that, in order to gain high order accuracy, the bound-preserving schemes also need to combine with temporal discretization whose order is consistent with the order of spatial discretization. Almost all time discretizations in the aforementioned bound-preserving methods are based on the method of lines, which treats the spatially discretized equation as ODE systems and use appropriate time marching approaches to evolve in time. In particular, the strong stability preserving Runge-Kutta (SSP-RK) methods or the SSP multi-step methods [13, 14, 30] are preferable because they are convex combinations of forward Euler time discretization, which greatly simplifies the proof of the bound-preserving since all analysis only need to be carried out on a single forward Euler time step. Besides the explicit methods, there are also studies on backward Euler time discretization [23, 16].

As an alternative to method of lines, the Lax-Wendroff methods are also widely used in the computation of time-dependent partial differential equations, for instance, the combination of Lax-Wendroff type time discretization with DG (LWDG) methods [26, 24, 15] or with the WENO schemes [27, 25], the two-stage fourth-order methods [22, 18], the arbitrary high order derivative Riemann problem (ADER) approach [31, 12, 11], and its variant based on the Galerkin space-time predictor [9, 1, 10], etc. The Lax-Wendroff methods utilizes the information of the partial differential equations to replace temporal derivatives by spatial derivatives in the Taylor expansion of the solution in time. Therefore, the Lax-Wendroff methods are one-stage, explicit, high order methods, and only need the stabilizing scaling limiters once per time step.

Regarding to the situation that there are very limited researches on bound-preserving techniques for Lax-Wendroff schemes, we study the LWDG to construct third order maximum-principle-satisfying and positivity-preserving LWDG schemes for scalar conservation laws and the Euler equations in one and two space dimensions. Different to the previous works [21, 29] on positivity-preserving Lax-Wendroff type methods, our algorithm does not rely on the flux limiter that needs to combine low order positivity-preserving flux and high order flux together, hence the high order accuracy of our approach is easier to guarantee.

The construction of our numerical schemes is based on the third order Taylor expansion of solution in time

$$u(x,t^{n+1}) = u(x,t^n) + \Delta t u_t(x,t^n) + \frac{\Delta t^2}{2} u_{tt}(x,t^n) + \frac{\Delta t^3}{6} u_{ttt}(x,t^n) + O(\Delta t^4), \tag{1.4}$$

where $\Delta t = t^{n+1} - t^n$. Due to the Lax-wendroff procedure, there will be many spatial derivatives to replace the original time derivatives in (1.4), especially for the system case in high dimensions. In this paper, we adopt the discontinuous Galerkin methods for the spatial discretization of the derivatives. In 1970, Reed et al. [28] proposed the first discontinuous Galerkin method to solve the steady linear transport problem. It was developed into Runge-Kutta discontinuous galerkin methods (RKDG) by Cockburn et al. in a series papers [7, 6, 4, 3, 8] to solve nonlinear hyperbolic conservation laws. Limiters such as the total variation bounded (TVB) limiter [8] are usually applied to stabilize the solution near shocks after each Runge-Kutta stage. Discontinuous Galerkin methods have been widely used in computational fluid dynamics due to their advantages in high order accuracy, flexibility in complex geometry and easiness to be parallelized, and is one of the most common choices in developing bound-preserving schemes.

In our work, we develop the idea of bound-preserving direct discontinuous Galerkin (DDG) method from [2] to resolve the difficulty caused by high order spatial derivatives produced by the Lax-Wendroff procedure. When it extends to multi-dimensions, we avoid the appearance of mixed derivatives in our numerical schemes based on carefully designed expansions of high order temporal derivatives in the Lax-Wendroff procedure, which is the key for the success of bound-preserving in high dimensions. We only demonstrate the treatments in two dimensions but the technique can be generalized into three dimensions directly.

The rest of the paper is organized as follows. In Section 2, we first introduce the notations to be used throughout the paper, and then construct the maximum-principle-satisfying LWDG methods for scalar conservation laws in one and two space dimensions. In Section 3, we establish the positivity-preserving LWDG schemes for the Euler equations in one and two dimensional spaces. The scaling limiters are introduced in Section 4 to ensure the boundedness and stability of the numerical solution. In Section 5, we give extensive numerical examples to demonstrate the effectiveness of our algorithm. We end up with some concluding remarks in Section 6. The discussion in the above sections are based on uniform meshes. In the appendices, we give illustrations on how to extend the algorithms to nonuniform meshes and take the one dimensional scalar conservation law as an example.

2 Maximum-principle-preserving for scalar conservation laws

In this section, we study the maximum-principle-satisfying LWDG methods for scalar conservation laws. Based on the framework of [38], we only need to put our effort on attaining the maximum-principle for cell averages of the solution, i.e. $m \le \bar{u}^{n+1} \le M$, provided $m \le u^n \le M$, where the superscripts n and $n+1$ denote the time level t^n and t^{n+1}, respectively. The slope limiters introduced in Section 4 will make up the gap between the maximum-principles of $\bar{u}^{n+1}$ and u^{n+1}.

For simplicity, we only discuss the one and two dimensional problems with periodic boundary conditions on uniform meshes, but the results can be directly extended to three space dimensions and non-periodic cases. However, the extension from uniform meshes to nonuniform meshes is not trivial, which will be demonstrated in the appendices with one dimensional space as an example.

We first introduce the notations to be used throughout the paper, then construct and prove the maximum-principle-satisfying LWDG schemes.

2.1 Notations

In the one dimensional space, we assume the domain $\Omega = [a, b]$ is discretized by $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots x_{N+\frac{1}{2}} = b$, and denote by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ the cells on Ω for $j = 1, 2, \dots N$. Moreover, we denote the length and center of the cell I_j by $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ and $x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right)$, respectively, and let $u_j = u(x_j)$

Similarly, in the two dimensional space, we assume $\Omega = [a, b] \times [c, d]$ is discretized by $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x+\frac{1}{2}} = b$ and $c = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y+\frac{1}{2}} = d$ in the x and y directions, respectively. We denote by $K_{i,j} = I_i \times J_j = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ the cells in Ω for $i = 1, \dots, N_x, j = 1, \dots, N_y$, and by $\Delta x_i \Delta y_j = (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})$, $(x_i, y_j) = (\frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}))$ the area and center of the cell $K_{i,j}$, respectively, and let $u_{i,j} = u(x_i, y_j)$.

We only consider the uniform meshes in this section and the next section to simplify the discussion, i.e. $\Delta x_i \equiv \Delta x$ and $\Delta y_j \equiv \Delta y$, for $i = 1, \dots, N_x, j = 1, \dots, N_y$. The case of nonuniform meshes will be discussed in the appendices.

The finite element spaces in the DG schemes are taken as $V = \{v \in L^2 : v|_{I_j} \in P^2(I_j),\ j = 1, 2, \dots, N\}$ and $W = \{v \in L^2 : v|_{K_{i,j}} \in Q^2(K_{i,j}),\ i = 1, \dots, N_x, j = 1, \dots, N_y\}$ in one and two dimensional spaces, respectively, where $P^2(I)$ is the space of quadratic polynomials on interval I and $Q^2(K)$ is the tensor product space of quadratic polynomials on rectangle K.

Due to discontinuities, functions in the schemes may have double values on cell interfaces. In one space dimension, we denote by $v^-_{j+\frac{1}{2}}$ and $v^+_{j+\frac{1}{2}}$ the left and right limits of v at $x_{j+\frac{1}{2}}$, respectively, i.e. $v^{\pm}_{j+\frac{1}{2}} = v(x_{j+\frac{1}{2}} \pm 0)$. Moreover, we denote the average and jump of v at $x_{j+\frac{1}{2}}$ by $\{v\}_{j+\frac{1}{2}} = \frac{1}{2}\left(v^-_{j+\frac{1}{2}} + v^+_{j+\frac{1}{2}}\right)$ and $[v]_{j+\frac{1}{2}} = v^+_{j+\frac{1}{2}} - v^-_{j+\frac{1}{2}}$, respectively. Similarly, in two space dimensions, we denote the left/right and lower/upper limits of v on vertical and horizontal cell interfaces by $v(x^{\pm}_{i+\frac{1}{2}}, y) = v(x_{i+\frac{1}{2}} \pm 0, y)$ and $v(x, y^{\pm}_{j+\frac{1}{2}}) = v(x, y_{j+\frac{1}{2}} \pm 0)$, respectively. The averages and jumps of v on vertical and horizontal cell interfaces are defined as $\{v\}(x_{i+\frac{1}{2}}, y) = \frac{1}{2}\left(v(x^-_{i+\frac{1}{2}}, y) + v(x^+_{i+\frac{1}{2}}, y)\right)$, $[v](x_{i+\frac{1}{2}}, y) = v(x^+_{i+\frac{1}{2}}, y) - v(x^-_{i+\frac{1}{2}}, y)$ and $\{v\}(x, y_{j+\frac{1}{2}}) = \frac{1}{2}\left(v(x, y^-_{j+\frac{1}{2}}) + v(x, y^+_{j+\frac{1}{2}})\right)$, $[v](x, y_{j+\frac{1}{2}}) = v(x, y^+_{j+\frac{1}{2}}) - v(x, y^-_{j+\frac{1}{2}})$, respectively. For simplicity, these notations will be abbreviated as $v^{\pm}$, $\{v\}$ and $[v]$ when the cell interface is clear from the context.

We denote the L^2 inner product on cell I_j in one space dimension as

$$(u, v)_{I_j} = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x)v(x)dx,$$

and on $K_{i,j}$ in two space dimensions as

$$(u, v)_{K_{i,j}} = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x, y)v(x, y)dxdy,$$

for $u, v \in L^2(\Omega)$.

We use the Gauss-Lobatto quadrature of $2N_q - 1$ points to evaluate integrals in one dimensional cells, where N_q is taken such that the third order accuracy is attained in the scheme, e.g. $N_q = 3$. We denote the quadrature points on I_j as $\{\hat{x}_\gamma, \gamma = 1, \ldots, 2N_q - 1\}$, and let $\{\hat{\omega}_\gamma, \gamma = 1, \ldots, 2N_q - 1\}$ be the corresponding quadrature weights satisfying $\sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma = 1$. In particular, $\hat{x}_1 = x_{j-\frac{1}{2}}, \hat{x}_{N_q} = x_j$ and $\hat{x}_{2N_q-1} = x_{j+\frac{1}{2}}$. We denote $\hat{u}^\gamma = u(\hat{x}_\gamma)$, for $\gamma = 1, \ldots, 2N_q - 1$. The quadrature rule adopted in two dimensional cells follows from tensor product and we denote $\hat{u}^{\beta,\gamma} = u(\hat{x}_\beta, \hat{y}_\gamma)$, for $\beta, \gamma = 1, \ldots, 2N_q - 1$, on the cell $K_{i,j}$.

2.2 Scalar conservation laws in one dimension

Consider the scalar conservation law (1.1). Direct computation gives the expressions of u_t, u_{tt} and u_{ttt} as follows:

$$u_t = -f(u)_x, \tag{2.1}$$

$$u_{tt} = ((f')^2 u_x)_x \tag{2.2}$$

$$u_{ttt} = -\left(3f''(f')^2 u_x^2 + (f')^3 u_{xx}\right)_x \tag{2.3}$$

Based on the expansions (2.1), (2.2) and (2.3), the third order maximum-principle-satisfying LWDG scheme of (1.1) at time level t^n is to find $u^{n+1} \in V$, s.t. $\forall \xi \in V$, the equation

$$\begin{aligned}(u^{n+1}, \xi)_{I_j} =& (u, \xi)_{I_j} + \Delta t (f(u), \xi_x)_{I_j} - \frac{\Delta t^2}{2}((f')^2 u_x, \xi_x)_{I_j} + \frac{\Delta t^3}{6}(3f''(f')^2 u_x^2 + (f')^3 u_{xx}, \xi_x)_{I_j} \\ & - \Delta t \hat{F}_{j+\frac{1}{2}} \xi^-_{j+\frac{1}{2}} + \Delta t \hat{F}_{j-\frac{1}{2}} \xi^+_{j-\frac{1}{2}},\end{aligned} \tag{2.4}$$

holds for $j = 1, 2, \ldots, N$, where the superscript n denoting time level t^n on the right hand side is omitted.

In the scheme (2.4), $\hat{F}_{j+\frac{1}{2}}$ is the numerical flux at $x_{j+\frac{1}{2}}$ defined as

$$\hat{F}_{j+\frac{1}{2}} = \hat{f}^{\text{LF}}_{j+\frac{1}{2}} - \frac{\Delta t}{2}\{f'^2\}_{j+\frac{1}{2}} \widehat{u_x}^{\text{DDG}}_{j+\frac{1}{2}} + \frac{\Delta t^2}{6}\{3f'^2 f'' u_x^2 + f'^3 u_{xx}\}_{j+\frac{1}{2}}, \tag{2.5}$$

where

$$\hat{f}^{\text{LF}}_{j+\frac{1}{2}} = \{f\}_{j+\frac{1}{2}} - \frac{\alpha}{2}[u]_{j+\frac{1}{2}}, \quad \alpha = \max_u |f'(u)| \tag{2.6}$$

is the Lax-Friedriches flux as used in [38], and

$$\widehat{u_x}^{\text{DDG}}_{j+\frac{1}{2}} = \beta_0 \frac{[u]_{j+\frac{1}{2}}}{\Delta x} + \{u_x\}_{j+\frac{1}{2}} + \beta_1 \Delta x [u_{xx}]_{j+\frac{1}{2}} \tag{2.7}$$

is the bound-preserving direct discontinuous Galerkin (DDG) flux [20, 2], with β_0, β_1 satisfying

$$\frac{1}{8} < \beta_1 < \frac{1}{4}, \quad \beta_0 > \frac{3}{2} - 4\beta_1 \tag{2.8}$$

The following lemmas are useful in the proofs of maximum-principle-satisfying and positivity-preserving in this section and the next section.

Lemma 2.1. *For $u \in V$, the DDG flux $\widehat{u_x}^{DDG}_{j+\frac{1}{2}}$ defined in* (2.7) *can be expanded on uniform meshes as*

$$\begin{aligned}\widehat{u_x}^{DDG}_{j+\frac{1}{2}} =& \frac{1}{\Delta x}\Big((\frac{1}{2} - 4\beta_1) u^+_{j-\frac{1}{2}} + (-2 + 8\beta_1) u_j + (-\beta_0 + \frac{3}{2} - 4\beta_1) u^-_{j+\frac{1}{2}} \\ & + (\beta_0 - \frac{3}{2} + 4\beta_1) u^+_{j+\frac{1}{2}} + (2 - 8\beta_1) u_{j+1} + (-\frac{1}{2} + 4\beta_1) u^-_{j+\frac{3}{2}} \Big)\end{aligned} \tag{2.9}$$

Proof. Since the mesh is uniform and u is piecewise quadratic, it follows from direct calculations. □

Lemma 2.2. *If $u \in V$ and $m \le u \le M$, then*

$$|\frac{du}{dx}| \le \frac{5(M-m)}{\Delta x_j}, \quad \forall x \in I_j. \tag{2.10}$$

Proof. We first consider $v \in P^2([-1,1])$ with $-\frac{R}{2} \le v \le \frac{R}{2}$. The Lagrange interpolation gives

$$v(r) = v(-1)L_{-1}(r) + v(0)L_0(r) + v(1)L_1(r), \quad r \in [-1,1], \tag{2.11}$$

where $L_{-1}(r) = \frac{1}{2}r(r-1), L_0(r) = -(r+1)(r-1), L_1(r) = \frac{1}{2}r(r+1)$.

Therefore, $|v'(r)| \le |v(-1)| \cdot |L'_{-1}(r)| + |v(0)| \cdot |L'_0(r)| + |v(1)| \cdot |L'_1(r)| \le \frac{R}{2} \times \frac{3}{2} + \frac{R}{2} \times 2 + \frac{R}{2} \times \frac{3}{2} = \frac{5R}{2}, \forall r \in [-1,1]$. Then (2.10) follows from changing of variables and the chain rule. ☐

We now state our main result for the LWDG scheme (2.4).

Theorem 2.3. *Given $m \le u^n \le M$, the cell averages $\bar{u}_j^{n+1}, j = 1, \ldots, N$ of the solution of scheme (2.4) are bounded between m and M under the CFL condition (2.12).*

$$\lambda \le \min\{q_1, q_2, \ldots, q_6\}, \tag{2.12}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $q_1 = \frac{\hat{\omega}_1}{2M_1}, q_2 = \frac{4\beta_1 - \frac{1}{2}}{5(M-m)M_2 + \frac{4}{3}M_1}, q_3 = \frac{2-8\beta_1}{20(M-m)M_2 + \frac{8}{3}M_1}, q_4 = \frac{\beta_0 - \frac{3}{2} + 4\beta_1}{15(M-m)M_2 + \frac{4}{3}M_1}, q_5 = \frac{\hat{\omega}_1^{1/2}}{M_1(\beta_0 - 1 + 4\beta_1)^{1/2}}, q_6 = \frac{\hat{\omega}_{N_q}^{1/2}}{M_1(6-24\beta_1)^{1/2}}$, *and* $M_1 = \max_{m \le u \le M} |f'(u)|, M_2 = \max_{m \le u \le M} |f''(u)|$

Proof. Take the test function $\xi = 1$ on I_j and zero anywhere else in the scheme (2.4) and denote $\lambda = \frac{\Delta t}{\Delta x}$, we obtain the equation satisfied by cell average of u^{n+1} on cell I_j,

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda \hat{F}_{j+\frac{1}{2}} + \lambda \hat{F}_{j-\frac{1}{2}} = \mathrm{I} + \mathrm{II}, \tag{2.13}$$

where

$$\mathrm{I} = \frac{1}{2}\left(\bar{u}_j^n - 2\lambda \hat{f}^{\mathrm{LF}}_{j+\frac{1}{2}} + 2\lambda \hat{f}^{\mathrm{LF}}_{j-\frac{1}{2}}\right), \tag{2.14}$$

and

$$\begin{aligned}
\mathrm{II} =& \frac{1}{2}\sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \hat{u}^\gamma \\
& - \lambda\Bigg(-\frac{\Delta t}{4}({f'}^{2}{}^{-}_{j+\frac{1}{2}} + {f'}^{2}{}^{+}_{j+\frac{1}{2}})\widehat{u_x}^{\mathrm{DDG}}_{j+\frac{1}{2}} \\
& \quad + \frac{\Delta t^2}{12}(3{f'}^{2}{}^{-}_{j+\frac{1}{2}}{f''}^{-}_{j+\frac{1}{2}}u_x^2{}^{-}_{j+\frac{1}{2}} + 3{f'}^{2}{}^{+}_{j+\frac{1}{2}}{f''}^{+}_{j+\frac{1}{2}}u_x^2{}^{+}_{j+\frac{1}{2}} + {f'}^{3}{}^{-}_{j+\frac{1}{2}}u_{xx}{}^{-}_{j+\frac{1}{2}} + {f'}^{3}{}^{+}_{j+\frac{1}{2}}u_{xx}{}^{+}_{j+\frac{1}{2}})\Bigg) \\
& + \lambda\Bigg(-\frac{\Delta t}{4}({f'}^{2}{}^{-}_{j-\frac{1}{2}} + {f'}^{2}{}^{+}_{j-\frac{1}{2}})\widehat{u_x}^{\mathrm{DDG}}_{j-\frac{1}{2}} \\
& \quad + \frac{\Delta t^2}{12}(3{f'}^{2}{}^{-}_{j-\frac{1}{2}}{f''}^{-}_{j-\frac{1}{2}}u_x^2{}^{-}_{j-\frac{1}{2}} + 3{f'}^{2}{}^{+}_{j-\frac{1}{2}}{f''}^{+}_{j-\frac{1}{2}}u_x^2{}^{+}_{j-\frac{1}{2}} + {f'}^{3}{}^{-}_{j-\frac{1}{2}}u_{xx}{}^{-}_{j-\frac{1}{2}} + {f'}^{3}{}^{+}_{j-\frac{1}{2}}u_{xx}{}^{+}_{j-\frac{1}{2}})\Bigg)
\end{aligned}$$

Note that the cell average $\bar{u}_j^n$ is split equally in I and II just for the ease of written, rather than to obtain an optimal CFL condition, which is the same case for all other proofs in this paper.

Since I has exactly the same form as in [38], we have $\frac{1}{2}m \le \mathrm{I} \le \frac{1}{2}M$, under the condition $\lambda \le q_1$ based on the conclusion therein. One can refer to [38] for more details.

As for II, it can be expanded as follows:

$$\begin{aligned} \mathrm{II} =& \frac{1}{2}\sum_{\gamma=2}^{N_q-1}\hat{\omega}_\gamma \hat{u}^\gamma + \frac{1}{2}\sum_{\gamma=N_q+1}^{2N_q-2}\hat{\omega}_\gamma \hat{u}^\gamma \\ &+ z_1 u^+_{j-\frac{3}{2}} + z_2 u_{j-1} + z_3 u^-_{j-\frac{1}{2}} + z_4 u^+_{j-\frac{1}{2}} + z_5 u_j + z_6 u^-_{j+\frac{1}{2}} + z_7 u^+_{j+\frac{1}{2}} + z_8 u_{j+1} + z_9 u^-_{j+\frac{3}{2}}, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
z_1 &= \frac{\lambda^2}{4}{f'}^{2-}_{j-\frac{1}{2}}\left((4\beta_1-\frac{1}{2}) + \Delta t {f''}^-_{j-\frac{1}{2}}{u_x}^-_{j-\frac{1}{2}} + \frac{4\lambda}{3}{f'}^-_{j-\frac{1}{2}}\right) + \frac{\lambda^2}{4}{f'}^{2+}_{j-\frac{1}{2}}(4\beta_1-\frac{1}{2}),\\
z_2 &= \frac{\lambda^2}{4}{f'}^{2-}_{j-\frac{1}{2}}\left((2-8\beta_1) - 4\Delta t {f''}^-_{j-\frac{1}{2}}{u_x}^-_{j-\frac{1}{2}} - \frac{8\lambda}{3}{f'}^-_{j-\frac{1}{2}}\right) + \frac{\lambda^2}{4}{f'}^{2+}_{j-\frac{1}{2}}(2-8\beta_1)\\
z_3 &= \frac{\lambda^2}{4}{f'}^{2-}_{j-\frac{1}{2}}\left((\beta_0-\frac{3}{2}+4\beta_1) + 3\Delta t {f''}^-_{j-\frac{1}{2}}{u_x}^-_{j-\frac{1}{2}} + \frac{4\lambda}{3}{f'}^-_{j-\frac{1}{2}}\right) + \frac{\lambda^2}{4}{f'}^{2+}_{j-\frac{1}{2}}(\beta_0-\frac{3}{2}+4\beta_1)\\
z_4 =& \frac{1}{2}\hat{\omega}_1 - \frac{\lambda^2}{4}{f'}^{2-}_{j-\frac{1}{2}}(\beta_0-\frac{3}{2}+4\beta_1) - \frac{\lambda^2}{4}{f'}^{2+}_{j-\frac{1}{2}}\left((\beta_0-\frac{3}{2}+4\beta_1) + 3\Delta t {f''}^+_{j-\frac{1}{2}}{u_x}^+_{j-\frac{1}{2}} - \frac{4\lambda}{3}{f'}^+_{j-\frac{1}{2}}\right)\\
&- \frac{\lambda^2}{4}{f'}^{2-}_{j+\frac{1}{2}}\left((4\beta_1-\frac{1}{2}) + \Delta t {f''}^-_{j+\frac{1}{2}}{u_x}^-_{j+\frac{1}{2}} + \frac{4\lambda}{3}{f'}^-_{j+\frac{1}{2}}\right) - \frac{\lambda^2}{4}{f'}^{2+}_{j+\frac{1}{2}}(4\beta_1-\frac{1}{2})\\
z_5 =& \frac{1}{2}\hat{\omega}_{N_q} - \frac{\lambda^2}{4}{f'}^{2-}_{j-\frac{1}{2}}(2-8\beta_1) - \frac{\lambda^2}{4}{f'}^{2+}_{j-\frac{1}{2}}\left((2-8\beta_1) - 4\Delta t {f''}^+_{j-\frac{1}{2}}{u_x}^+_{j-\frac{1}{2}} + \frac{8\lambda}{3}{f'}^+_{j-\frac{1}{2}}\right)\\
&- \frac{\lambda^2}{4}{f'}^{2-}_{j+\frac{1}{2}}\left((2-8\beta_1) - 4\Delta t {f''}^-_{j+\frac{1}{2}}{u_x}^-_{j+\frac{1}{2}} - \frac{8\lambda}{3}{f'}^-_{j+\frac{1}{2}}\right) - \frac{\lambda^2}{4}{f'}^{2+}_{j+\frac{1}{2}}(2-8\beta_1)\\
z_6 =& \frac{1}{2}\hat{\omega}_{2N_q-1} - \frac{\lambda^2}{4}{f'}^{2-}_{j-\frac{1}{2}}(4\beta_1-\frac{1}{2}) - \frac{\lambda^2}{4}{f'}^{2+}_{j-\frac{1}{2}}\left((4\beta_1-\frac{1}{2}) + \Delta t {f''}^+_{j-\frac{1}{2}}{u_x}^+_{j-\frac{1}{2}} - \frac{4\lambda}{3}{f'}^+_{j-\frac{1}{2}}\right)\\
&- \frac{\lambda^2}{4}{f'}^{2-}_{j+\frac{1}{2}}\left((\beta_0-\frac{3}{2}+4\beta_1) + 3\Delta t {f''}^-_{j+\frac{1}{2}}{u_x}^-_{j+\frac{1}{2}} + \frac{4\lambda}{3}{f'}^-_{j+\frac{1}{2}}\right) - \frac{\lambda^2}{4}{f'}^{2+}_{j+\frac{1}{2}}(\beta_0-\frac{3}{2}+4\beta_1)\\
z_7 &= \frac{\lambda^2}{4}{f'}^{2-}_{j+\frac{1}{2}}(\beta_0-\frac{3}{2}+4\beta_1) + \frac{\lambda^2}{4}{f'}^{2+}_{j+\frac{1}{2}}\left((\beta_0-\frac{3}{2}+4\beta_1) + 3\Delta t {f''}^+_{j+\frac{1}{2}}{u_x}^+_{j+\frac{1}{2}} - \frac{4\lambda}{3}{f'}^+_{j+\frac{1}{2}}\right)\\
z_8 &= \frac{\lambda^2}{4}{f'}^{2-}_{j+\frac{1}{2}}(2-8\beta_1) + \frac{\lambda^2}{4}{f'}^{2+}_{j+\frac{1}{2}}\left((2-8\beta_1) - 4\Delta t {f''}^+_{j+\frac{1}{2}}{u_x}^+_{j+\frac{1}{2}} + \frac{8\lambda}{3}{f'}^+_{j+\frac{1}{2}}\right)\\
z_9 &= \frac{\lambda^2}{4}{f'}^{2-}_{j+\frac{1}{2}}(4\beta_1-\frac{1}{2}) + \frac{\lambda^2}{4}{f'}^{2+}_{j+\frac{1}{2}}\left((4\beta_1-\frac{1}{2}) + \Delta t {f''}^+_{j+\frac{1}{2}}{u_x}^+_{j+\frac{1}{2}} - \frac{4\lambda}{3}{f'}^+_{j+\frac{1}{2}}\right)
\end{aligned}$$

It is not difficult to verify that

$$\frac{1}{2}\sum_{\gamma=2}^{N_q-1}\hat{\omega}_\gamma + \frac{1}{2}\sum_{\gamma=N_q+1}^{2N_q-2}\hat{\omega}_\gamma + z_1 + z_2 + \cdots + z_9 = \frac{1}{2},$$

Moreover, we claim that $z_1, z_2, \ldots, z_9 \geq 0$ under the CFL conditions (2.12). In fact, the following estimates can be made under the CFL conditions,

$$z_1 \geq \frac{\lambda^2}{4} f'^{2-}_{j-\frac{1}{2}} \left((4\beta_1 - \frac{1}{2}) - 5\lambda(M-m)M_2 - \frac{4\lambda}{3}M_1 \right) + \frac{\lambda^2}{4} f'^{2+}_{j-\frac{1}{2}} (4\beta_1 - \frac{1}{2}) \geq 0,$$

$$z_2 \geq \frac{\lambda^2}{4} f'^{2-}_{j-\frac{1}{2}} \left((2 - 8\beta_1) - 20\lambda(M-m)M_2 - \frac{8\lambda}{3}M_1 \right) + \frac{\lambda^2}{4} f'^{2+}_{j-\frac{1}{2}} (2 - 8\beta_1) \geq 0,$$

$$z_3 \geq \frac{\lambda^2}{4} f'^{2-}_{j-\frac{1}{2}} \left((\beta_0 - \frac{3}{2} + 4\beta_1) - 15\lambda(M-m)M_2 - \frac{4\lambda}{3}M_1 \right) + \frac{\lambda^2}{4} f'^{2+}_{j-\frac{1}{2}} (\beta_0 - \frac{3}{2} + 4\beta_1) \geq 0,$$

$$\begin{aligned} z_4 \geq & \frac{1}{2}\hat{\omega}_1 - \frac{\lambda^2}{4} M_1^2 (\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\lambda^2}{4} M_1^2 \left((\beta_0 - \frac{3}{2} + 4\beta_1) + 15\lambda(M-m)M_2 + \frac{4\lambda}{3}M_1 \right) \\ & - \frac{\lambda^2}{4} M_1^2 \left((4\beta_1 - \frac{1}{2}) + 5\lambda(M-m)M_2 + \frac{4\lambda}{3}M_1 \right) - \frac{\lambda^2}{4} M_1^2 (4\beta_1 - \frac{1}{2}) \geq 0, \end{aligned}$$

$$\begin{aligned} z_5 \geq & \frac{1}{2}\hat{\omega}_N - \frac{\lambda^2}{4} M_1^2 (2 - 8\beta_1) - \frac{\lambda^2}{4} M_1^2 \left((2 - 8\beta_1) + 20\lambda(M-m)M_2 + \frac{8\lambda}{3}M_1 \right) \\ & - \frac{\lambda^2}{4} M_1^2 \left((2 - 8\beta_1) + 20\lambda(M-m)M_2 + \frac{8\lambda}{3}M_1 \right) - \frac{\lambda^2}{4} M_1^2 (2 - 8\beta_1) \geq 0, \end{aligned}$$

$$\begin{aligned} z_6 \geq & \frac{1}{2}\hat{\omega}_{2N_q-1} - \frac{\lambda^2}{4} M_1^2 (4\beta_1 - \frac{1}{2}) - \frac{\lambda^2}{4} M_1^2 \left((4\beta_1 - \frac{1}{2}) + 5\lambda(M-m)M_2 + \frac{4\lambda}{3}M_1 \right) \\ & - \frac{\lambda^2}{4} M_1^2 \left((\beta_0 - \frac{3}{2} + 4\beta_1) + 15\lambda(M-m)M_2 + \frac{4\lambda}{3}M_1 \right) - \frac{\lambda^2}{4} M_1^2 (\beta_0 - \frac{3}{2} + 4\beta_1) \geq 0, \end{aligned}$$

$$z_7 \geq \frac{\lambda^2}{4} f'^{2-}_{j+\frac{1}{2}} (\beta_0 - \frac{3}{2} + 4\beta_1) + \frac{\lambda^2}{4} f'^{2+}_{j+\frac{1}{2}} \left((\beta_0 - \frac{3}{2} + 4\beta_1) - 15\lambda(M-m)M_2 - \frac{4\lambda}{3}M_1 \right) \geq 0,$$

$$z_8 \geq \frac{\lambda^2}{4} f'^{2-}_{j+\frac{1}{2}} (2 - 8\beta_1) + \frac{\lambda^2}{4} f'^{2+}_{j+\frac{1}{2}} \left((2 - 8\beta_1) - 20\lambda(M-m)M_2 - \frac{8\lambda}{3}M_1 \right) \geq 0,$$

$$z_9 \geq \frac{\lambda^2}{4} f'^{2-}_{j+\frac{1}{2}} (4\beta_1 - \frac{1}{2}) + \frac{\lambda^2}{4} f'^{2+}_{j+\frac{1}{2}} \left((4\beta_1 - \frac{1}{2}) - 5\lambda(M-m)M_2 - \frac{4\lambda}{3}M_1 \right) \geq 0.$$

Therefore, II is one half of a convex combination of values of u^n at different quadrature points, which implies $\frac{1}{2}m \leq \text{II} \leq \frac{1}{2}M$ since we assume $m \leq u^n \leq M$.

Since $\bar{u}_j^{n+1} = \text{I} + \text{II}$, we finish the proof by summing up the inequalities of I and II . □

2.3 Scalar conservation laws in two dimensions

Consider the scalar conservation law in two space dimensions

$$u_t + f(u)_x + g(u)_y = 0. \tag{2.16}$$

Direct computation gives the expressions of u_t, u_{tt}, u_{ttt} as follows:

$$u_t = -f(u)_x - g(u)_y, \tag{2.17}$$

$$u_{tt} = \left(f'^2 u_x\right)_x + (f'g'u_y)_x + (f'g'u_x)_y + \left(g'^2 u_y\right)_y, \tag{2.18}$$

$$\begin{aligned} u_{ttt} = &-\left(3f'^2 f'' u_x^2 + 6f'g'g''u_y^2 + 3g'^2 f'' u_y^2 + f'^3 u_{xx} + 3f'g'^2 u_{yy}\right)_x \\ &-\left(6f'g'f''u_x^2 + 3f'^2 g'' u_x^2 + 3g'^2 g'' u_y^2 + 3f'^2 g' u_{xx} + g'^3 u_{yy}\right)_y \end{aligned} \tag{2.19}$$

Note that there are different ways to expand u_{ttt}, among which we choose the one that avoids the appearance of mixed derivatives in the numerical scheme.

Based on the expansions (2.17), (2.18) and (2.19), the third order maximum-principle-preserving LWDG scheme of (2.16) at time level t^n is to find $u^{n+1} \in W$, s.t. $\forall \xi \in W$, the equation

$$\begin{aligned} (u^{n+1}, \xi)_{K_{i,j}} =& (u, \xi)_{K_{i,j}} + \Delta t (f(u), \xi_x)_{K_{i,j}} + \Delta t (g(u), \xi_y)_{K_{i,j}} \\ &- \frac{\Delta t^2}{2}(f'^2 u_x + f'g'u_y, \xi_x)_{K_{i,j}} - \frac{\Delta t^2}{2}(f'g'u_x + g'^2 u_y, \xi_y)_{K_{i,j}} \\ &+ \frac{\Delta t^3}{6}\left(3f'^2 f'' u_x^2 + 6f'g'g''u_y^2 + 3g'^2 f'' u_y^2 + f'^3 u_{xx} + 3f'g'^2 u_{yy}, \xi_x\right)_{K_{i,j}} \\ &+ \frac{\Delta t^3}{6}\left(6f'g'f''u_x^2 + 3f'^2 g'' u_x^2 + 3g'^2 g'' u_y^2 + 3f'^2 g' u_{xx} + g'^3 u_{yy}, \xi_y\right)_{K_{i,j}} \\ &- \Delta t \int_{y_{j-\frac12}}^{y_{j+\frac12}} \hat{F}_{i+\frac12,j}\ \xi(x^-_{i+\frac12}, y) dy + \Delta t \int_{y_{j-\frac12}}^{y_{j+\frac12}} \hat{F}_{i-\frac12,j}\ \xi(x^+_{i-\frac12}, y) dy \\ &- \Delta t \int_{x_{i-\frac12}}^{x_{i+\frac12}} \hat{G}_{i,j+\frac12}\ \xi(x, y^-_{j+\frac12}) dx + \Delta t \int_{x_{i-\frac12}}^{x_{i+\frac12}} \hat{G}_{i,j-\frac12}\ \xi(x, y^+_{j-\frac12}) dx \end{aligned} \tag{2.20}$$

holds for $i = 1, \dots, N_x, j = 1, \dots, N_y$. In the scheme, $\hat{F}_{i+\frac12,j}$ and $\hat{G}_{i,j+\frac12}$ are numerical fluxes defined as

$$\hat{F}_{i+\frac12,j} = \hat{F}^0_{i+\frac12,j} + \hat{F}^1_{i+\frac12,j}, \quad \hat{G}_{i,j+\frac12} = \hat{G}^0_{i,j+\frac12} + \hat{G}^1_{i,j+\frac12},$$

where

$$\hat{F}^0_{i+\frac12,j} = \hat{f}^{\mathrm{LF}}_{i+\frac12,j} - \frac{\Delta t}{2}\{f'^2\}_{i+\frac12,j}\widehat{u_x}^{\mathrm{DDG}}_{i+\frac12,j} + \frac{\Delta t^2}{6}\{3f'^2 f'' u_x^2 + f'^3 u_{xx}\}_{i+\frac12,j}, \tag{2.21}$$

$$\hat{F}^1_{i+\frac12,j} = -\frac12 \alpha^1_x [u]_{i+\frac12,j} - \frac{\Delta t}{2}\{f'g'u_y\}_{i+\frac12,j} + \frac{\Delta t^2}{6}\{6f'g'g''u_y^2 + 3g'^2 f'' u_y^2 + 3f'g'^2 u_{yy}\}_{i+\frac12,j}, \tag{2.22}$$

$$\hat{G}^0_{i,j+\frac12} = \hat{g}^{\mathrm{LF}}_{i,j+\frac12} - \frac{\Delta t}{2}\{g'^2\}_{i,j+\frac12}\widehat{u_y}^{\mathrm{DDG}}_{i,j+\frac12} + \frac{\Delta t^2}{6}\{3g'^2 g'' u_y^2 + g'^3 u_{yy}\}_{i,j+\frac12}, \tag{2.23}$$

$$\hat{G}^1_{i,j+\frac12} = -\frac12 \alpha^1_y [u]_{i,j+\frac12} - \frac{\Delta t}{2}\{f'g'u_x\}_{i,j+\frac12} + \frac{\Delta t^2}{6}\{6f'g'f''u_x^2 + 3f'^2 g'' u_x^2 + 3f'^2 g' u_{xx}\}_{i,j+\frac12}, \tag{2.24}$$

in which the Lax-Friedrichs fluxes and DDG fluxes are defined the same way as before, and α^1_x, α^1_y are positive viscosity constants that can be taken as $0.05 \max_u |f'(u)|$ and $0.05 \max_u |g'(u)|$ for instance.

We now state the main result for the LWDG scheme (2.20).

Theorem 2.4. *Given $m \le u^n \le M$, the cell averages $\bar{u}_{i,j}^{n+1}$, $i = 1, \dots, N_x, j = 1, \dots, N_y$ of the solution of scheme* (2.20) *are bounded between m and M under the CFL condition* (2.25)*:*

$$\lambda_x \le \min\{Q_1, Q_3\}, \quad \lambda_y \le \min\{Q_2, Q_4\}, \tag{2.25}$$

where $\lambda_x = \frac{\Delta t}{\Delta x}$, $\lambda_y = \frac{\Delta t}{\Delta y}$, and the definitions of Q_1, Q_2, Q_3, Q_4 are given in Appendix A.1.

The proof is very similar to that of the one dimensional case, except that the expansions are much more tedious, which results in much more complicated CFL conditions.

Proof. Take the test function $\xi = 1$ on $K_{i,j}$ and zero anywhere else in the scheme (2.20) and denote by $\lambda_x = \frac{\Delta t}{\Delta x}$, $\lambda_y = \frac{\Delta t}{\Delta y}$, we obtain

$$\bar{u}_{i,j}^{n+1} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

where

$$\begin{aligned}
\mathrm{I} &= \frac{1}{4}\bar{u}_{i,j}^n - \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^0_{i+\frac{1}{2},j} dy + \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^0_{i-\frac{1}{2},j} dy, \\
\mathrm{II} &= \frac{1}{4}\bar{u}_{i,j}^n - \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^1_{i+\frac{1}{2},j} dy + \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^1_{i-\frac{1}{2},j} dy, \\
\mathrm{III} &= \frac{1}{4}\bar{u}_{i,j}^n - \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^0_{i,j+\frac{1}{2}} dx + \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^0_{i,j-\frac{1}{2}} dx \\
\mathrm{IV} &= \frac{1}{4}\bar{u}_{i,j}^n - \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^1_{i,j+\frac{1}{2}} dx + \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^1_{i,j-\frac{1}{2}} dx
\end{aligned}$$

It suffices to shown $\frac{1}{4}m \le \mathrm{I}, \mathrm{II} \le \frac{1}{4}M$ under the CFL condition (2.25), due to the symmetry in x and y directions.

It is clear that I can be decomposed in the form of convex combination

$$\mathrm{I} = \frac{1}{4}\sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma H_\gamma$$

where

$$H_\gamma = \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \hat{u}^{\beta,\gamma} - 4\lambda_x \hat{F}^0_{i+\frac{1}{2},j}(x_{i+\frac{1}{2}}, \hat{y}_\gamma) + 4\lambda_x \hat{F}^0_{i-\frac{1}{2},j}(x_{i-\frac{1}{2}}, \hat{y}_\gamma),$$

Notice that H_γ has exactly the same structure as (2.13). Therefore, $\mathrm{I} \in [\frac{1}{4}m, \frac{1}{4}M]$, under the CFL condition (2.12) for one dimensional scalar case with λ replaced by $4\lambda_x$, i.e. $\lambda_x \le Q_1$.

As for the term II, it can be expanded as follows,

$$
\begin{aligned}
\mathrm{II} =& \frac{1}{4}\sum_{\alpha=2}^{2N_q-2}\sum_{\beta=1}^{2N_q-1}\hat{\omega}_\alpha\hat{\omega}_\beta\hat{u}^{\alpha,\beta} + \sum_{\beta=2}^{N_q-1}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 u(x^-_{i-\frac{1}{2}},\hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 u(x^-_{i-\frac{1}{2}},\hat{y}_\beta) \\
&+ \sum_{\beta=2}^{N_q-1}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 u(x^+_{i+\frac{1}{2}},\hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 u(x^+_{i+\frac{1}{2}},\hat{y}_\beta) \\
&+ z_1 u(x^-_{i-\frac{1}{2}},y^+_{j-\frac{1}{2}}) + z_2 u(x^-_{i-\frac{1}{2}},y_j) + z_3 u(x^-_{i-\frac{1}{2}},y^-_{j+\frac{1}{2}}) + z_4 u(x^+_{i-\frac{1}{2}},y^+_{j-\frac{1}{2}}) + z_5 u(x^+_{i-\frac{1}{2}},y_j) + z_6 u(x^+_{i-\frac{1}{2}},y^-_{j+\frac{1}{2}}) \\
&+ z_7 u(x^-_{i+\frac{1}{2}},y^+_{j-\frac{1}{2}}) + z_8 u(x^-_{i+\frac{1}{2}},y_j) + z_9 u(x^-_{i+\frac{1}{2}},y^-_{j+\frac{1}{2}}) + z_{10} u(x^+_{i+\frac{1}{2}},y^+_{j-\frac{1}{2}}) + z_{11} u(x^+_{i+\frac{1}{2}},y_j) + z_{12} u(x^+_{i+\frac{1}{2}},y^-_{j+\frac{1}{2}}), \\
&+ \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{13,\beta} u(x^-_{i+\frac{1}{2}},\hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{13,\beta} u(x^-_{i+\frac{1}{2}},\hat{y}_\beta) + \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{14,\beta} u(x^+_{i-\frac{1}{2}},\hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{14,\beta} u(x^+_{i-\frac{1}{2}},\hat{y}_\beta),
\end{aligned}
\tag{2.26}
$$

where the expressions of $z_1, \ldots, z_{14,\beta}$ are given in Appendix A.2.

It can be verified that the following equality holds,

$$
\begin{aligned}
&\frac{1}{4}\sum_{\alpha=2}^{2N_q-2}\sum_{\beta=1}^{2N_q-1}\hat{\omega}_\alpha\hat{\omega}_\beta + \sum_{\beta=2}^{N_q-1}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 + \sum_{\beta=N_q+1}^{2N_q-2}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 + \sum_{\beta=2}^{N_q-1}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 + \sum_{\beta=N_q+1}^{2N_q-2}\frac{\lambda_x}{2}\hat{\omega}_\beta\alpha_x^1 \\
&+ z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9 + z_{10} + z_{11} + z_{12} \\
&+ \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{13,\beta} + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{13,\beta} + \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{14,\beta} + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{14,\beta} = \frac{1}{4},
\end{aligned}
$$

Moreover, all z's are nonnegative under the CFL condition (2.25). The detailed estimates can be found in Appendix A.2

To sum up, II can be written as one fourth of a convex combination of point values of u^n under the CFL condition (2.25), which implies $\frac{1}{4}m \le \mathrm{II} \le \frac{1}{4}M$ since $m \le u^n \le M$. The similar arguments apply to III and IV

Since $\bar{u}^{n+1}_{i,j} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$, we finish the proof by summing up the inequalities of I, II, III and IV. □

3 Positivity-preserving for the Euler equations

3.1 The Euler equations in one dimension

Consider the Euler equations (1.2). Direct computation gives the expressions of ρ_t, ρ_{tt} and ρ_{ttt} as follows:

$$\rho_t = -(\rho u)_x, \tag{3.1}$$

$$\rho_{tt} = \left((\rho u^2)_x + \hat{\gamma}(\rho e)_x\right)_x, \tag{3.2}$$

$$\rho_{ttt} = -\left(u_{xx}(\rho u^2) + 2u_x(\rho u^2)_x + u(\rho u^2)_{xx} + \hat{\gamma}\gamma u_{xx}(\rho e) + \hat{\gamma}(3+\gamma)u_x(\rho e)_x + 3\hat{\gamma}u(\rho e)_{xx}\right)_x \tag{3.3}$$

where $\hat{\gamma} = \gamma - 1$. Moreover,

$$m_t = A^1_x, \quad m_{tt} = A^2_x, \quad m_{ttt} = A^3_x,$$

and

$$E_t = B^1_x, \quad E_{tt} = B^2_x, \quad E_{ttt} = B^3_x,$$

where $A^1, A^2, A^3, B^1, B^2, B^3$ are shorthand notations introduced for convenience of later discussion. For the full expressions of m_t, m_{tt}, m_{ttt}, and E_t, E_{tt}, E_{ttt}, see Appendix B.1.

The positivity-preserving LWDG scheme of (1.2) for ρ at time level t^n is to find $\rho^{n+1} \in V$, s.t. $\forall \xi \in V$, the equation

$$\begin{aligned}(\rho^{n+1}, \xi)_{I_j} =& (\rho, \xi)_{I_j} + \Delta t(\rho u, \xi_x)_{I_j} - \frac{\Delta t^2}{2}((\rho u^2)_x + \hat{\gamma}(\rho e)_x, \xi_x)_{I_j} \\ &+ \frac{\Delta t^3}{6}\left(u_{xx}(\rho u^2) + 2u_x(\rho u^2)_x + u(\rho u^2)_{xx} + \hat{\gamma}\gamma u_{xx}(\rho e) + \hat{\gamma}(3+\gamma)u_x(\rho e)_x + 3\hat{\gamma}u(\rho e)_{xx}, \xi_x\right)_{I_j} \\ &- \Delta t \hat{F}_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}} + \Delta t \hat{F}_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}},\end{aligned} \tag{3.4}$$

holds for $j = 1, 2, \ldots, N$. In the scheme, $\hat{F}_{j+\frac{1}{2}}$ is the numerical flux of ρ at $x_{j+\frac{1}{2}}$ defined as

$$\begin{aligned}\hat{F}_{j+\frac{1}{2}} =& \hat{f}^{\mathrm{LF}}_{j+\frac{1}{2}} - \frac{\Delta t}{2}\widehat{(\mathcal{I}(\rho u^2))}^{\mathrm{DDG}}_{x\,j+\frac{1}{2}} - \frac{\Delta t}{2}\hat{\gamma}\widehat{(\mathcal{I}(\rho e))}^{\mathrm{DDG}}_{x\,j+\frac{1}{2}} \\ &+ \frac{\Delta t^2}{6}\{u_{xx}(\rho u^2) + 2u_x\left(\mathcal{I}(\rho u^2)\right)_x + u\left(\mathcal{I}(\rho u^2)\right)_{xx}\}_{j+\frac{1}{2}} \\ &+ \frac{\Delta t^2}{6}\{\hat{\gamma}\gamma u_{xx}(\rho e) + \hat{\gamma}(3+\gamma)u_x\left(\mathcal{I}(\rho e)\right)_x + 3\hat{\gamma}u\left(\mathcal{I}(\rho e)\right)_{xx}\}_{j+\frac{1}{2}}\end{aligned}, \tag{3.5}$$

where

$$\hat{f}^{\mathrm{LF}}_{j+\frac{1}{2}} = \{\rho u\}_{j+\frac{1}{2}} - \frac{1}{2}\alpha[\rho]_{j+\frac{1}{2}}, \quad \alpha = ||(|u| + c)||_\infty, \tag{3.6}$$

is the Lax-Friedriches flux used in the positivity-preserving for the Euler equations in [37], $c = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed, $\widehat{(\mathcal{I}(\rho u^2))}^{\mathrm{DDG}}_{x\,j+\frac{1}{2}}$ and $\widehat{(\mathcal{I}(\rho e))}^{\mathrm{DDG}}_{x\,j+\frac{1}{2}}$ are the DDG fluxes defined in (2.7), with u replaced by $\mathcal{I}(\rho u^2)$ and $\mathcal{I}(\rho e)$, respectively, where $\mathcal{I}$ is the quadratic interpolation operator with interpolation points at $x^+_{j-\frac{1}{2}}, x_j$, and $x^-_{j+\frac{1}{2}}$ on I_j.

The variables m and E are discretized by the standard discontinuous Galerkin method with the first order flux terms adopting the Lax-Friedriches flux and high-order flux terms adopting the average flux, i.e.

$$
\begin{aligned}
(m^{n+1},\xi)_{I_j} =& (m,\xi)_{I_j} - \Delta t(A^1,\xi_x)_{I_j} \\
&+ \Delta t\{A^1\}_{j+\frac{1}{2}} + \Delta t\alpha[m]_{j+\frac{1}{2}} \\
&- \Delta t\{A^1\}_{j-\frac{1}{2}} - \Delta t\alpha[m]_{j-\frac{1}{2}} \\
&- \frac{\Delta t^2}{2}(A^2,\xi_x)_{I_j} + \frac{\Delta t^2}{2}\{A^2\}_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}} - \frac{\Delta t^2}{2}\{A^2\}_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}} \\
&- \frac{\Delta t^3}{6}(A^3,\xi_x)_{I_j} + \frac{\Delta t^3}{6}\{A^3\}_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}} - \frac{\Delta t^3}{6}\{A^3\}_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}}
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
(E^{n+1},\xi)_{I_j} =& (E,\xi)_{I_j} - \Delta t(B^1,\xi_x)_{I_j} \\
&+ \Delta t\{B^1\}_{j+\frac{1}{2}} + \Delta t\alpha[E]_{j+\frac{1}{2}} \\
&- \Delta t\{B^1\}_{j-\frac{1}{2}} - \Delta t\alpha[E]_{j-\frac{1}{2}} \\
&- \frac{\Delta t^2}{2}(B^2,\xi_x)_{I_j} + \frac{\Delta t^2}{2}\{B^2\}_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}} - \frac{\Delta t^2}{2}\{B^2\}_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}} \\
&- \frac{\Delta t^3}{6}(B^3,\xi_x)_{I_j} + \frac{\Delta t^3}{6}\{B^3\}_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}} - \frac{\Delta t^3}{6}\{B^3\}_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}}
\end{aligned}
\tag{3.8}
$$

We now state the result for the positivity-preserving of $\bar{\rho}_j^{n+1}$.

Theorem 3.1. *Given $\mathbf{u}^n \in G$, the cell averages $\bar{\rho}_j^{n+1}, j = 1,\dots,N$ of the solution of scheme (3.4) are nonnegative under the CFL condition (3.9):*

$$\lambda \le \min\{q_1, q_2, \dots, q_{11}\}, \tag{3.9}$$

where $q_1 = \frac{\hat{\omega}_1}{2||(|u|+c)||_\infty}$, $q_2 = \frac{6(\beta_0-\frac{3}{2}+4\beta_1)}{\Delta x^2||u_{xx}||_\infty+6\Delta x||u_x||_\infty+4||u||_\infty}$, $q_3 = \frac{3(2-8\beta_1)}{4(\Delta x||u_x||_\infty+||u||_\infty)}$, $q_4 = \frac{3(4\beta_1-\frac{1}{2})}{\Delta x||u_x||_\infty+2||u||_\infty}$, $q_5 = \frac{1}{2||u||_\infty}\left(\frac{\omega_1}{\beta_0-2+8\beta_1}\right)^{\frac{1}{2}}$, $q_6 = \frac{1}{2||u||_\infty}\left(\frac{\omega_{N_q}}{2(2-8\beta_1)}\right)^{\frac{1}{2}}$, $q_7 = \frac{6(4\beta_1-\frac{1}{2})}{(3+\gamma)\Delta x||u_x||_\infty+12||u||_\infty}$, $q_8 = \frac{3(2-8\beta_1)}{2(3+\gamma)\Delta x||u_x||_\infty+12||u||_\infty}$, $q_9 = \frac{6(\beta_0-\frac{3}{2}+4\beta_1)}{\gamma\Delta x^2||u_{xx}||_\infty+3(3+\gamma)\Delta x||u_x||_\infty+12||u||_\infty}$, $q_{10} = \left(\frac{\omega_1}{4\hat{\gamma}(\beta_0-2+8\beta_1)||e||_\infty}\right)^{\frac{1}{2}}$, $q_{11} = \left(\frac{\omega_{N_q}}{8\hat{\gamma}(2-8\beta_1)||e||_\infty}\right)^{\frac{1}{2}}$.

Proof. Take $\xi = 1$ on I_j and zero on other cells in the scheme (3.4), we obtain

$$\bar{\rho}_j^{n+1} = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{3.10}$$

where

$$\mathrm{I} = \frac{1}{2}\left(\bar{\rho}_j^n - 2\lambda\hat{f}^{\mathrm{LF}}_{j+\frac{1}{2}} + 2\lambda\hat{f}^{\mathrm{LF}}_{j-\frac{1}{2}}\right)$$

$$\text{II} = \frac{1}{4}\bar{\rho}^n - \lambda\left(-\frac{\Delta t}{2}\widehat{(\mathcal{I}(\rho u^2))}_{x\,j+\frac{1}{2}}^{\text{DDG}} + \frac{\Delta t^2}{6}\left\{u_{xx}(\rho u^2) + 2u_x\left(\mathcal{I}(\rho u^2)\right)_x + u\left(\mathcal{I}(\rho u^2)\right)_{xx}\right\}_{j+\frac{1}{2}}\right)$$
$$+ \lambda\left(-\frac{\Delta t}{2}\widehat{(\mathcal{I}(\rho u^2))}_{x\,j-\frac{1}{2}}^{\text{DDG}} + \frac{\Delta t^2}{6}\left\{u_{xx}(\rho u^2) + 2u_x\left(\mathcal{I}(\rho u^2)\right)_x + u\left(\mathcal{I}(\rho u^2)\right)_{xx}\right\}_{j-\frac{1}{2}}\right)$$

$$\text{III} = \frac{1}{4}\bar{\rho}^n - \lambda\left(-\frac{\Delta t}{2}\hat{\gamma}\widehat{(\mathcal{I}(\rho e))}_{x\,j+\frac{1}{2}}^{\text{DDG}} + \frac{\Delta t^2}{6}\left\{\hat{\gamma}\gamma u_{xx}(\rho e) + \hat{\gamma}(3+\gamma)u_x\left(\mathcal{I}(\rho e)\right)_x + 3\hat{\gamma}u\left(\mathcal{I}(\rho e)\right)_{xx}\right\}_{j+\frac{1}{2}}\right)$$
$$+ \lambda\left(-\frac{\Delta t}{2}\hat{\gamma}\widehat{(\mathcal{I}(\rho e))}_{x\,j-\frac{1}{2}}^{\text{DDG}} + \frac{\Delta t^2}{6}\left\{\hat{\gamma}\gamma u_{xx}(\rho e) + \hat{\gamma}(3+\gamma)u_x\left(\mathcal{I}(\rho e)\right)_x + 3\hat{\gamma}u\left(\mathcal{I}(\rho e)\right)_{xx}\right\}_{j-\frac{1}{2}}\right)$$

Since I has exactly the same form as in [37], $I \geq 0$ is guaranteed under the condition $\lambda \leq q_1$ from the conclusion therein. Now we expand II as follows,

$$\begin{aligned}\text{II} =& \frac{1}{4}\sum_{\gamma=2}^{N_q-1}\hat{\omega}_\gamma\hat{\rho}^\gamma + \frac{1}{4}\sum_{\gamma=N_q+1}^{2N_q-2}\hat{\omega}_\gamma\hat{\rho}^\gamma \\ &+ z_1\rho^+_{j-\frac{3}{2}} + z_2\rho_{j-1} + z_3\rho^-_{j-\frac{1}{2}} + z_4\rho^+_{j-\frac{1}{2}} + z_5\rho_j + z_6\rho^-_{j+\frac{1}{2}} + z_7\rho^+_{j+\frac{1}{2}} + z_8\rho_{j+1} + z_9\rho^-_{j+\frac{3}{2}},\end{aligned}$$

where

$$z_1 = \lambda^2\left(\frac{1}{2}(4\beta_1 - \frac{1}{2}) + \frac{\Delta t}{6}\left(u_x\right)^-_{j-\frac{1}{2}} + \frac{\lambda}{3}u^-_{j-\frac{1}{2}}\right)\left(u^+_{j-\frac{3}{2}}\right)^2$$
$$z_2 = \lambda^2\left(\frac{1}{2}(2 - 8\beta_1) - \frac{2\Delta t}{3}\left(u_x\right)^-_{j-\frac{1}{2}} - \frac{2\lambda}{3}u^-_{j-\frac{1}{2}}\right)\left(u_{j-1}\right)^2$$
$$z_3 = \lambda^2\left(\frac{1}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) + \frac{\Delta t^2}{12\lambda}\left(u_{xx}\right)^-_{j-\frac{1}{2}} + \frac{\Delta t}{2}\left(u_x\right)^-_{j-\frac{1}{2}} + \frac{\lambda}{3}\left(u^-_{j-\frac{1}{2}}\right)\right)\left(u^-_{j-\frac{1}{2}}\right)^2$$
$$z_4 = \frac{1}{4}\omega_1 - \lambda^2\left(\frac{1}{2}(4\beta_1 - \frac{1}{2}) + \frac{\Delta t}{6}\left(u_x\right)^-_{j+\frac{1}{2}} + \frac{\lambda}{3}u^-_{j+\frac{1}{2}} + \frac{1}{2}(\beta_0 - \frac{3}{2} + 4\beta_1)\ - \frac{\Delta t^2}{12\lambda}\left(u_{xx}\right)^+_{j-\frac{1}{2}} + \frac{\Delta t}{2}\left(u_x\right)^+_{j-\frac{1}{2}} - \frac{\lambda}{3}\left(u^+_{j-\frac{1}{2}}\right)\right)\left(u^+_{j-\frac{1}{2}}\right)^2$$
$$z_5 = \frac{1}{4}\omega_{N_q} - \lambda^2\left(\frac{1}{2}(2 - 8\beta_1) - \frac{2\Delta t}{3}\left(u_x\right)^-_{j+\frac{1}{2}} - \frac{2}{3}\lambda u^-_{j+\frac{1}{2}} + \frac{1}{2}(2 - 8\beta_1)\ - \frac{2\Delta t}{3}\left(u_x\right)^+_{j-\frac{1}{2}} + \frac{2}{3}\lambda u^+_{j-\frac{1}{2}}\right)\left(u_j\right)^2$$
$$z_6 = \frac{1}{4}\omega_{2N_q-1} - \lambda^2\left(\frac{1}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) + \frac{1}{2}(4\beta_1 - \frac{1}{2}) + \frac{\Delta t^2}{12\lambda}\left(u_{xx}\right)^-_{j+\frac{1}{2}}\ + \frac{\Delta t}{2}\left(u_x\right)^-_{j+\frac{1}{2}} + \frac{\Delta t}{6}\left(u_x\right)^+_{j-\frac{1}{2}} + \frac{\lambda}{3}\left(u^-_{j+\frac{1}{2}}\right) - \frac{\lambda}{3}u^+_{j-\frac{1}{2}}\right)\left(u^-_{j+\frac{1}{2}}\right)^2$$
$$z_7 = \lambda^2\left(\frac{1}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\Delta t^2}{12\lambda}\left(u_{xx}\right)^+_{j+\frac{1}{2}} + \frac{\Delta t}{2}\left(u_x\right)^+_{j+\frac{1}{2}} - \frac{\lambda}{3}\left(u^+_{j+\frac{1}{2}}\right)\right)\left(u^+_{j+\frac{1}{2}}\right)^2$$
$$z_8 = \lambda^2\left(\frac{1}{2}(2 - 8\beta_1) - \frac{2\Delta t}{3}\left(u_x\right)^+_{j+\frac{1}{2}} + \frac{2\lambda}{3}u^+_{j+\frac{1}{2}}\right)\left(u_{j+1}\right)^2$$
$$z_9 = \lambda^2\left(\frac{1}{2}(4\beta_1 - \frac{1}{2}) + \frac{\Delta t}{6}\left(u_x\right)^+_{j+\frac{1}{2}} - \frac{\lambda}{3}u^+_{j+\frac{1}{2}}\right)\left(u^-_{j+\frac{3}{2}}\right)^2$$

We claim that $z_1, z_2, \ldots, z_9 \geq 0$ under the CFL condition $\lambda \leq \min\{q_2, q_3, \ldots, q_6\}$. In fact, we have the following estimates

$$z_1 \geq \lambda^2\left(\frac{1}{2}(4\beta_1 - \frac{1}{2}) - \frac{\Delta t}{6}||u_x||_\infty - \frac{\lambda}{3}||u||_\infty\right)\left(u^+_{j-\frac{3}{2}}\right)^2 \geq 0,$$

$$z_2 \geq \lambda^2 \left(\frac{1}{2}(2-8\beta_1) - \frac{2\Delta t}{3}||u_x||_\infty - \frac{2\lambda}{3}||u||_\infty \right) (u_{j-1})^2 \geq 0,$$

$$z_3 \geq \lambda^2 \left(\frac{1}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\Delta t^2}{12\lambda}||u_{xx}||_\infty - \frac{\Delta t}{2}||u_x||_\infty - \frac{\lambda}{3}||u||_\infty \right) \left(u^-_{j-\frac{1}{2}}\right)^2 \geq 0,$$

$$z_4 \geq \frac{1}{4}\omega_1 - \lambda^2 \left(\frac{1}{2}(\beta_0 - 2 + 8\beta_1) + \frac{\Delta t^2}{12\lambda}||u_{xx}||_\infty + \frac{2\Delta t}{3}||u_x||_\infty + \frac{2\lambda}{3}||u||_\infty \right) ||u||^2_\infty \geq 0,$$

$$z_5 \geq \frac{1}{4}\omega_{N_q} - \lambda^2 \left((2-8\beta_1) + \frac{4\Delta t}{3}||u_x||_\infty + \frac{4}{3}\lambda||u||_\infty \right) ||u||^2_\infty \geq 0,$$

$$z_6 \geq \frac{1}{4}\omega_{2N_q-1} - \lambda^2 \left(\frac{1}{2}(\beta_0 - 2 + 8\beta_1) + \frac{\Delta t^2}{12\lambda}||u_{xx}||_\infty + \frac{2\Delta t}{3}||u_x||_\infty + \frac{2\lambda}{3}||u||_\infty \right) ||u||^2_\infty \geq 0,$$

$$z_7 \geq \lambda^2 \left(\frac{1}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\Delta t^2}{12\lambda}||u_{xx}||_\infty - \frac{\Delta t}{2}||u_x||_\infty - \frac{\lambda}{3}||u||_\infty \right) \left(u^+_{j+\frac{1}{2}}\right)^2 \geq 0,$$

$$z_8 \geq \lambda^2 \left(\frac{1}{2}(2-8\beta_1) - \frac{2\Delta t}{3}||u_x||_\infty - \frac{2\lambda}{3}||u||_\infty \right) (u_{j+1})^2 \geq 0,$$

$$z_9 \geq \lambda^2 \left(\frac{1}{2}(4\beta_1 - \frac{1}{2}) - \frac{\Delta t}{6}||u_x||_\infty - \frac{\lambda}{3}||u||_\infty \right) \left(u^-_{j+\frac{3}{2}}\right)^2 \geq 0,$$

Similarly, we can expand III as

$$\begin{aligned} \text{III} =& \frac{1}{4}\sum_{\gamma=2}^{N_q-1} \hat{\omega}_\gamma \hat{\rho}^\gamma + \frac{1}{4}\sum_{\gamma=N_q+1}^{2N_q-2} \hat{\omega}_\gamma \hat{\rho}^\gamma \\ &+ z_{10}\rho^+_{j-\frac{3}{2}} + z_{11}\rho_{j-1} + z_{12}\rho^-_{j-\frac{1}{2}} + z_{13}\rho^+_{j-\frac{1}{2}} + z_{14}\rho_j + z_{15}\rho^-_{j+\frac{1}{2}} + z_{16}\rho^+_{j+\frac{1}{2}} + z_{17}\rho_{j+1} + z_{18}\rho^-_{j+\frac{3}{2}}, \end{aligned} \tag{3.11}$$

and $z_{10}, \dots, z_{18} \geq 0$ under the condition $\lambda \leq \min\{q_7, q_8, q_9, q_{10}, q_{11}\}$. The expressions and estimates of $z_{10}, \dots, z_{18}$ are similar to those of $z_1, \dots, z_9$, thus are given in Appendix A.3.

By the same arguments as in the scalar cases, we have $\text{II}, \text{III} \geq 0$, provided the positivity of ρ^n. Since $\bar{\rho}_j^{n+1} = \text{I} + \text{II} + \text{III}$, we finish the proof by collecting the results for I, II and III. □

The remaining task is to preserve the positivity of internal energy of cell averages of the solution, i.e. $e(\bar{\mathbf{u}}_j^{n+1}) \geq 0$. We have the results as follows.

Theorem 3.2. *Given $\mathbf{u}^n \in G$, the specific internal energy of the cell averages $e(\bar{\mathbf{u}}_j^{n+1}), j = 1, \dots, N$ of scheme* (3.4), (3.7) *and* (3.8) *are nonnegative under the CFL condition* (3.12)*:*

$$\lambda \leq \frac{\gamma+1}{2\alpha^2(\gamma-1)} \min_j \left\{ \frac{(p^-_{j+\frac{1}{2}})^2}{C^-_{j+\frac{1}{2}}}, \frac{(p^+_{j+\frac{1}{2}})^2}{C^+_{j+\frac{1}{2}}} \right\}, \tag{3.12}$$

where

$$
\begin{aligned}
C^-_{j+\frac{1}{2}} =& \frac{\Delta x}{\alpha}\left((2E^-_{j+\frac{1}{2}}+p^-_{j+\frac{1}{2}})\left(|\tilde{f}^1_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^1_{j+\frac{1}{2}}|\right)+2\rho^-_{j+\frac{1}{2}}\left(|\tilde{f}^3_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^3_{j+\frac{1}{2}}|\right)\right.\\
&+Q_1\frac{\Delta x}{\alpha}\left(|\tilde{f}^1_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^1_{j+\frac{1}{2}}|\right)\left(|\tilde{f}^3_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^3_{j+\frac{1}{2}}|\right)\\
&+\frac{1}{2}Q_1\frac{\Delta x}{\alpha}\left(|\tilde{f}^2_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^2_{j+\frac{1}{2}}|\right)^2\\
&\left.+(2|m^-_{j+\frac{1}{2}}|+\frac{p^-_{j+\frac{1}{2}}}{\alpha})\left(|\tilde{f}^2_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^2_{j+\frac{1}{2}}|\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
C^+_{j+\frac{1}{2}} =& \frac{\Delta x}{\alpha}\left((2E^+_{j+\frac{1}{2}}+p^+_{j+\frac{1}{2}})\left(|\tilde{f}^1_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^1_{j+\frac{1}{2}}|\right)+2\rho^+_{j+\frac{1}{2}}\left(|\tilde{f}^3_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^3_{j+\frac{1}{2}}|\right)\right.\\
&+Q_1\frac{\Delta x}{\alpha}\left(|\tilde{f}^1_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^1_{j+\frac{1}{2}}|\right)\left(|\tilde{f}^3_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^3_{j+\frac{1}{2}}|\right)\\
&+\frac{1}{2}Q_1\frac{\Delta x}{\alpha}\left(|\tilde{f}^2_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^2_{j+\frac{1}{2}}|\right)^2\\
&\left.+(2|m^+_{j+\frac{1}{2}}|+\frac{p^+_{j+\frac{1}{2}}}{\alpha})\left(|\tilde{f}^2_{j+\frac{1}{2}}|+Q_1\Delta x|\check{f}^2_{j+\frac{1}{2}}|\right)\right)
\end{aligned}
$$

Proof. Take $\xi = 1$ on I_j and zero anywhere else in the scheme (3.4),(3.7) and (3.8), we can obtain the following vector equation satisfied by the cell average of $\mathbf{u}^{n+1}$ on I_j,

$$
\bar{\mathbf{u}}^{n+1}_j = \bar{\mathbf{u}}^n_j - \lambda\left(\hat{\mathbf{f}}^{\mathrm{LF}}_{j+\frac{1}{2}}+\Delta t\tilde{\mathbf{f}}_{j+\frac{1}{2}}+\Delta t^2\check{\mathbf{f}}_{j+\frac{1}{2}}\right)+\lambda\left(\hat{\mathbf{f}}^{\mathrm{LF}}_{j-\frac{1}{2}}+\Delta t\tilde{\mathbf{f}}_{j-\frac{1}{2}}+\Delta t^2\check{\mathbf{f}}_{j-\frac{1}{2}}\right),
$$

where $\hat{\mathbf{f}}^{\mathrm{LF}}_{j+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{f}(\mathbf{u}^-_{j+\frac{1}{2}})+\mathbf{f}(\mathbf{u}^+_{j+\frac{1}{2}})-\alpha\left(\mathbf{u}^+_{j+\frac{1}{2}}-\mathbf{u}^-_{j+\frac{1}{2}}\right)\right)$, $\alpha = ||(|u|+c)||_\infty$, is the standard Lax-Friedriches flux, which is the leading term in the total flux constructed in the LWDG scheme (3.4)-(3.8), $\tilde{\mathbf{f}}_{j+\frac{1}{2}} = (\tilde{f}^1_{j+\frac{1}{2}},\tilde{f}^2_{j+\frac{1}{2}},\tilde{f}^3_{j+\frac{1}{2}})$ and $\check{\mathbf{f}}_{j+\frac{1}{2}} = (\check{f}^1_{j+\frac{1}{2}},\check{f}^2_{j+\frac{1}{2}},\check{f}^3_{j+\frac{1}{2}})$ are the second and third order terms contained in the total flux, respectively.

Similar to [37], we have the decomposition

$$
\begin{aligned}
\bar{\mathbf{u}}^{n+1}_j =& \sum_{\gamma=2}^{2N_q-2}\hat{\omega}_\gamma\mathbf{u}^\gamma+\hat{\omega}_1\left(1-\frac{\alpha\lambda}{\hat{\omega}_1}\right)\mathbf{u}^+_{j-\frac{1}{2}}+\hat{\omega}_{2N_q-1}\left(1-\frac{\alpha\lambda}{\hat{\omega}_{2N_q-1}}\right)\mathbf{u}^-_{j+\frac{1}{2}}\\
&+\frac{\alpha\lambda}{2}\left(\mathbf{u}^-_{j+\frac{1}{2}}-\frac{1}{\alpha}\mathbf{f}(\mathbf{u}^-_{j+\frac{1}{2}})-\frac{\Delta t}{\alpha}\left(\tilde{\mathbf{f}}_{j+\frac{1}{2}}+\Delta t\check{\mathbf{f}}_{j+\frac{1}{2}}\right)\right)\\
&+\frac{\alpha\lambda}{2}\left(\mathbf{u}^+_{j+\frac{1}{2}}-\frac{1}{\alpha}\mathbf{f}(\mathbf{u}^+_{j+\frac{1}{2}})-\frac{\Delta t}{\alpha}\left(\tilde{\mathbf{f}}_{j+\frac{1}{2}}+\Delta t\check{\mathbf{f}}_{j+\frac{1}{2}}\right)\right)\\
&+\frac{\alpha\lambda}{2}\left(\mathbf{u}^-_{j-\frac{1}{2}}+\frac{1}{\alpha}\mathbf{f}(\mathbf{u}^-_{j-\frac{1}{2}})+\frac{\Delta t}{\alpha}\left(\tilde{\mathbf{f}}_{j-\frac{1}{2}}+\Delta t\check{\mathbf{f}}_{j-\frac{1}{2}}\right)\right)\\
&+\frac{\alpha\lambda}{2}\left(\mathbf{u}^+_{j-\frac{1}{2}}+\frac{1}{\alpha}\mathbf{f}(\mathbf{u}^+_{j-\frac{1}{2}})+\frac{\Delta t}{\alpha}\left(\tilde{\mathbf{f}}_{j-\frac{1}{2}}+\Delta t\check{\mathbf{f}}_{j-\frac{1}{2}}\right)\right)
\end{aligned}
$$

Since $\hat{\omega}_\gamma \geq 0, \gamma = 1, \ldots, 2N_q - 1$ and $(1 - \frac{\alpha\lambda}{\hat{\omega}_1}), (1 - \frac{\alpha\lambda}{\hat{\omega}_{2N_q-1}}) \geq 0$ from the CFL condition (3.9), by convexity of G, it suffices to show

$$\mathbf{u}^+_{j+\frac{1}{2}} \pm \frac{1}{\alpha}\mathbf{f}(\mathbf{u}^+_{j+\frac{1}{2}}) \pm \frac{\Delta t}{\alpha}\left(\tilde{\mathbf{f}}_{j+\frac{1}{2}} + \Delta t\check{\mathbf{f}}_{j+\frac{1}{2}}\right) \in G,$$

provided $\mathbf{u}^+_{j+\frac{1}{2}} \in G$. For simplicity, we omit the superscripts and subscripts in the following proof.

One can calculate that

$$\begin{aligned}
&\rho^2 e\left(\mathbf{u} \pm \frac{1}{\alpha}\mathbf{f}(\mathbf{u}) \pm \frac{\Delta t}{\alpha}\left(\tilde{\mathbf{f}} + \Delta t\check{\mathbf{f}}\right)\right)\\
=&\frac{p\rho}{\alpha^2(\gamma-1)}\left((\alpha \pm u)^2 - \frac{\gamma-1}{2\gamma}c^2\right) \pm \frac{\Delta t}{\alpha}(\tilde{f}^1 + \Delta t\check{f}^1)\left((1 \pm \frac{u}{\alpha})E \pm \frac{u}{\alpha}p\right)\\
&\pm \frac{\Delta t}{\alpha}(\tilde{f}^3 + \Delta t\check{f}^3)\left((1 \pm \frac{u}{\alpha})\rho\right) + \frac{\Delta t^2}{\alpha^2}(\tilde{f}^1 + \Delta t\check{f}^1)(\tilde{f}^3 + \Delta t\check{f}^3) - \frac{1}{2}\frac{\Delta t^2}{\alpha^2}(\tilde{f}^2 + \Delta t\check{f}^2)^2\\
&\mp \frac{\Delta t}{\alpha}(\tilde{f}^2 + \Delta t\check{f}^2)\left((1 \pm \frac{u}{\alpha})m \pm \frac{1}{\alpha}p\right)\\
\geq&\frac{\gamma+1}{2\alpha^2(\gamma-1)}p^2 - C\lambda,
\end{aligned}$$

where

$$\begin{aligned}
C =&\frac{\Delta x}{\alpha}\left((2E+p)\left(|\tilde{f}^1| + Q_1\Delta x|\check{f}^1|\right) + 2\rho\left(|\tilde{f}^3| + Q_1\Delta x|\check{f}^3|\right)\right.\\
&+Q_1\frac{\Delta x}{\alpha}\left(|\tilde{f}^1| + Q_1\Delta x|\check{f}^1|\right)\left(|\tilde{f}^3| + Q_1\Delta x|\check{f}^3|\right)\\
&+\frac{1}{2}Q_1\frac{\Delta x}{\alpha}\left(|\tilde{f}^2| + Q_1\Delta x|\check{f}^2|\right)^2\\
&\left.+(2|m| + \frac{p}{\alpha})\left(|\tilde{f}^2| + Q_1\Delta x|\check{f}^2|\right)\right).
\end{aligned}$$

Under the CFL condition (3.12), we can get the positivity of $\rho^2 e$, which finishes the proof. □

Collecting the above two theorems, we reach our final result.

Theorem 3.3. *Given* $\mathbf{u}^n \in G$*, we have* $\bar{\mathbf{u}}_j^{n+1} \in G, j = 1, \ldots, N$ *for scheme* (3.4)*,* (3.7) *and* (3.8)*, under the CFL conditions* (3.9) *and* (3.12)*.*

3.2 The Euler equations in two dimensions

Consider the Euler equations in two space dimensions

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x + \mathbf{g}(\mathbf{u})_y = \mathbf{0}, \tag{3.13}$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{pmatrix}, \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{pmatrix},$$

with

$$m = \rho u, \quad n = \rho v, \quad E = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2 + \rho e, \quad p = (\gamma - 1)\rho e,$$

in which u and v are velocities in x and y directions, respectively, and m and n are momentums in x and y directions, respectively.

Direct computation gives the expressions of ρ_t, ρ_{tt} and ρ_{ttt} as follows:

$$\rho_t = -(\rho u)_x - (\rho v)_y, \tag{3.14}$$

$$\rho_{tt} = \left((\rho u^2)_x + \hat{\gamma}(\rho e)_x\right)_x + 2\,(\rho uv)_{xy} + \left((\rho v^2)_y + \hat{\gamma}(\rho e)_y\right)_y, \tag{3.15}$$

$$\begin{aligned}
\rho_{ttt} = &- \big(u_{xx}(\rho u^2) + 2u_x(\rho u^2)_x + u(\rho u^2)_{xx} \\
&\quad +\hat{\gamma}\gamma u_{xx}(\rho e) + (\hat{\gamma}(3+\gamma)u_x + \hat{\gamma}^2 v_y)(\rho e)_x + 3\hat{\gamma}u(\rho e)_{xx}\big)_x \\
&- \big(v_{yy}(\rho v^2) + 2v_y(\rho v^2)_y + v(\rho v^2)_{yy} \\
&\quad +\hat{\gamma}\gamma v_{yy}(\rho e) + (\hat{\gamma}(3+\gamma)v_y + \hat{\gamma}^2 u_x)(\rho e)_y + 3\hat{\gamma}v(\rho e)_{yy}\big)_y \\
&- \left(\left(\gamma\hat{\gamma}ev_{xx} + \hat{\gamma}(\gamma+3)e_xv_x + 6vu_x^2 + 12uu_xv_x + 3\hat{\gamma}ve_{xx} + 3u^2v_{xx} + 6uvu_{xx} - \hat{\gamma}^2u_ye_x\right)\rho\right)_y \\
&- \left(\left(6\hat{\gamma}ve_x + \hat{\gamma}(\gamma+3)ev_x + 6u(uv_x + 2vu_x) - \hat{\gamma}^2u_ye\right)\rho_x\right)_y \\
&- \left((3(\hat{\gamma}e + u^2)v)\rho_{xx}\right)_y \\
&- \left(\left(\gamma\hat{\gamma}eu_{yy} + \hat{\gamma}(\gamma+3)e_yu_y + 6uv_y^2 + 12vu_yv_y + 3\hat{\gamma}ue_{yy} + 3v^2u_{yy} + 6uvv_{yy} - \hat{\gamma}^2v_xe_y\right)\rho\right)_x \\
&- \left(\left(6\hat{\gamma}ue_y + \hat{\gamma}(\gamma+3)eu_y + 6v(vu_y + 2uv_y) - \hat{\gamma}^2v_xe\right)\rho_y\right)_x \\
&- \left(\left(3(\hat{\gamma}e + v^2)u\right)\rho_{yy}\right)_x
\end{aligned} \tag{3.16}$$

where $\hat{\gamma} = \gamma - 1$. Note that there are a lot of ways to expand ρ_{ttt}, among which we choose the one that avoids the appearance of mixed derivatives in the LWDG scheme.

Moreover,

$$m_t = B_x^1 + B_y^2, \quad m_{tt} = B_x^3 + B_y^4, \quad m_{ttt} = B_x^5 + B_y^6,$$

$$n_t = C_x^1 + C_y^2, \quad n_{tt} = C_x^3 + C_y^4, \quad n_{ttt} = C_x^5 + C_y^6,$$

and

$$E_t = D^1_x + D^2_y, \quad E_{tt} = D^3_x + D^4_y, \quad E_{ttt} = D^5_x + D^6_y,$$

where $B^1, B^2, B^3, B^4, B^5, B^6, C^1, C^2, C^3, C^4, C^5, C^6, D^1, D^2, D^3, D^4, D^5, D^6$, are shorthand notations introduced for convenience of later discussion. For the full expressions of $m_t, m_{tt}, m_{ttt}, n_t, n_{tt}, n_{ttt}$,and E_t, E_{tt}, E_{ttt}, see Appendix B.2.

The positivity-preserving LWDG of ρ at time level t^n is to find $\rho^{n+1} \in W$, s.t. $\forall \xi \in W$, the equation

$$\begin{aligned}
(\rho^{n+1}, \xi)_{K_{i,j}} =&(\rho, \xi)_{K_{i,j}} + \Delta t(\rho u, \xi_x)_{K_{i,j}} + \Delta t(\rho v, \xi_y)_{K_{i,j}} \\
&- \frac{\Delta t^2}{2}((\rho u^2)_x + \hat{\gamma}(\rho e)_x + (\rho uv)_y, \xi_x)_{K_{i,j}} \\
&- \frac{\Delta t^2}{2}((\rho v^2)_y + \hat{\gamma}(\rho e)_y + (\rho uv)_x, \xi_y)_{K_{i,j}} \\
&- \Delta t \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}_{i+\frac{1}{2},j}\ \xi(x^-_{i+\frac{1}{2}}, y)dy + \Delta t \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}_{i-\frac{1}{2},j}\ \xi(x^+_{i-\frac{1}{2}}, y)dy \\
&- \Delta t \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}_{i,j+\frac{1}{2}}\ \xi(x, y^-_{j+\frac{1}{2}})dx + \Delta t \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}_{i,j-\frac{1}{2}}\ \xi(x, y^+_{j-\frac{1}{2}})dx
\end{aligned} \tag{3.17}$$

holds for $i = 1, 2, \ldots, N_x, j = 1, 2, \ldots, N_y$. $\hat{F}_{i+\frac{1}{2},j}$ and $\hat{G}_{i,j+\frac{1}{2}}$ are numerical fluxes defined as

$$\hat{F}_{i+\frac{1}{2},j} =\hat{F}^0_{i+\frac{1}{2},j} + \hat{F}^1_{i+\frac{1}{2},j}, \tag{3.18}$$

and

$$\hat{G}_{i,j+\frac{1}{2}} =\hat{G}^0_{i,j+\frac{1}{2}} + \hat{G}^1_{i,j+\frac{1}{2}}, \tag{3.19}$$

where

$$\begin{aligned}
\hat{F}^0_{i+\frac{1}{2},j} =&\{\rho u\}_{i+\frac{1}{2},j} - \frac{1}{2}\alpha^0_x[\rho]_{i+\frac{1}{2},j} - \frac{\Delta t}{2}(\widehat{\mathcal{I}(\rho u^2)})^{\text{DDG}}_{x i+\frac{1}{2},j} - \frac{\Delta t}{2}\hat{\gamma}(\widehat{\mathcal{I}(\rho e)})^{\text{DDG}}_{x i+\frac{1}{2},j} \\
&+ \frac{\Delta t^2}{6}\{u_{xx}(\rho u^2) + 2u_x \left(\mathcal{I}(\rho u^2)\right)_x + u \left(\mathcal{I}(\rho u^2)\right)_{xx}\}_{i+\frac{1}{2},j} \\
&+ \frac{\Delta t^2}{6}\{\hat{\gamma}\gamma u_{xx}(\rho e) + \left(\hat{\gamma}(3+\gamma)u_x + \hat{\gamma}^2 v_y\right) (\mathcal{I}(\rho e))_x + 3\hat{\gamma} u \left(\mathcal{I}(\rho e)\right)_{xx}\}_{i+\frac{1}{2},j}
\end{aligned} , \tag{3.20}$$

$$\hat{F}^1_{i+\frac{1}{2},j} = -\frac{1}{2}\alpha^1_x[\rho]_{i+\frac{1}{2},j} - \frac{\Delta t}{2}\{\rho_y uv + \rho(u_y v + uv_y)\} + \frac{\Delta t^2}{6}\left\{A^1\rho + A^2\rho_y + A^3\rho_{yy}\right\}$$

$$
\begin{aligned}
\hat{G}^0_{i,j+\frac{1}{2}} =& \{\rho v\}_{i,j+\frac{1}{2}} - \frac{1}{2}\alpha^0_y[\rho]_{i,j+\frac{1}{2}} - \frac{\Delta t}{2}\widehat{(\mathcal{I}(\rho v^2))}^{\text{DDG}}_{y_{i,j+\frac{1}{2}}} - \frac{\Delta t}{2}\hat{\gamma}\widehat{(\mathcal{I}(\rho e))}^{\text{DDG}}_{y_{i,j+\frac{1}{2}}} \\
&+ \frac{\Delta t^2}{6}\{v_{yy}(\rho v^2) + 2v_y\left(\mathcal{I}(\rho v^2)\right)_y + v\left(\mathcal{I}(\rho v^2)\right)_{yy}\}_{i,j+\frac{1}{2}} \\
&+ \frac{\Delta t^2}{6}\{\hat{\gamma}\gamma v_{yy}(\rho e) + \left(\hat{\gamma}(3+\gamma)v_y + \hat{\gamma}^2 u_x\right)\left(\mathcal{I}(\rho e)\right)_y + 3\hat{\gamma}v\left(\mathcal{I}(\rho e)\right)_{yy}\}_{i,j+\frac{1}{2}}
\end{aligned} \quad , \tag{3.21}
$$

$$
\hat{G}^1_{i,j+\frac{1}{2}} = -\frac{1}{2}\alpha^1_y[\rho]_{i,j+\frac{1}{2}} - \frac{\Delta t}{2}\{\rho_x uv + \rho(uv_x + u_x v)\} + \frac{\Delta t^2}{6}\left\{A^4\rho + A^5\rho_x + A^6\rho_{xx}\right\}
$$

in which $\alpha^0_x = ||(|u| + c)||_\infty$, $\alpha^0_y = ||(|v| + c)||_\infty$, $\alpha^1_x, \alpha^1_y > 0$, and

$$
\begin{aligned}
A^1 &= \left(\gamma\hat{\gamma}eu_{yy} + \hat{\gamma}(\gamma+3)e_y u_y + 6uv_y^2 + 12vu_y v_y + 3\hat{\gamma}ue_{yy} + 3v^2u_{yy} + 6uvv_{yy} - \hat{\gamma}^2 v_x e_y\right) \\
A^2 &= \left(6\hat{\gamma}ue_y + \hat{\gamma}(\gamma+3)eu_y + 6v(vu_y + 2uv_y) - \hat{\gamma}^2 v_x e\right) \\
A^3 &= \left(3(\hat{\gamma}e + v^2)u\right) \\
A^4 &= \left(\gamma\hat{\gamma}ev_{xx} + \hat{\gamma}(\gamma+3)e_x v_x + 6vu_x^2 + 12uu_x v_x + 3\hat{\gamma}ve_{xx} + 3u^2v_{xx} + 6uvu_{xx} - \hat{\gamma}^2 u_y e_x\right) \\
A^5 &= \left(6\hat{\gamma}ve_x + \hat{\gamma}(\gamma+3)ev_x + 6u(uv_x + 2vu_x) - \hat{\gamma}^2 u_y e\right) \\
A^6 &= (3(\hat{\gamma}e + u^2)v)
\end{aligned}
$$

The variables m, n and E are discretized by the standard discontinuous Galerkin method with the first order flux terms adopting the Lax-Friedriches flux, in which the viscosity constant $\alpha_x = \alpha^0_x + \alpha^1_x$ for the vertical cell interfaces and $\alpha_y = \alpha^0_y + \alpha^1_y$ for the horizontal cell interfaces, and high-order flux terms adopting the average flux, i.e.

$$
\begin{aligned}
(m^{n+1}, \xi)_{K_{i,j}} =& (m, \xi)_{K_{i,j}} - \Delta t(B^1, \xi_x)_{K_{i,j}} - \Delta t(B^2, \xi_y)_{K_{i,j}} \\
&- \frac{\Delta t^2}{2}(B^3, \xi_x)_{K_{i,j}} - \frac{\Delta t^2}{2}(B^4, \xi_y)_{K_{i,j}} \\
&- \frac{\Delta t^3}{6}(B^5, \xi_x)_{K_{i,j}} - \frac{\Delta t^3}{6}(B^6, \xi_y)_{K_{i,j}} \\
&+ \Delta t\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left(\{B^1\} + \alpha_x[m] + \frac{\Delta t}{2}\{B^3\} + \frac{\Delta t^2}{6}\{B^5\}\right)\ \xi(x^-_{i+\frac{1}{2}}, y)dy \\
&- \Delta t\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left(\{B^1\} + \alpha_x[m] + \frac{\Delta t}{2}\{B^3\} + \frac{\Delta t^2}{6}\{B^5\}\right)\ \xi(x^+_{i-\frac{1}{2}}, y)dy \\
&+ \Delta t\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(\{B^2\} + \alpha_y[m] + \frac{\Delta t}{2}\{B^4\} + \frac{\Delta t^2}{6}\{B^6\}\right)\ \xi(x, y^-_{j+\frac{1}{2}})dx \\
&- \Delta t\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(\{B^2\} + \alpha_y[m] + \frac{\Delta t}{2}\{B^4\} + \frac{\Delta t^2}{6}\{B^6\}\right)\ \xi(x, y^+_{j-\frac{1}{2}})dx
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
(n^{n+1},\xi)_{K_{i,j}} =&(n,\xi)_{K_{i,j}} - \Delta t(C^1,\xi_x)_{K_{i,j}} - \Delta t(C^2,\xi_y)_{K_{i,j}} \\
&- \frac{\Delta t^2}{2}(C^3,\xi_x)_{K_{i,j}} - \frac{\Delta t^2}{2}(C^4,\xi_y)_{K_{i,j}} \\
&- \frac{\Delta t^3}{6}(C^5,\xi_x)_{K_{i,j}} - \frac{\Delta t^3}{6}(C^6,\xi_y)_{K_{i,j}} \\
&+ \Delta t\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\{C^1\} + \alpha_x[n] + \frac{\Delta t}{2}\{C^3\} + \frac{\Delta t^2}{6}\{C^5\}\right)\ \xi(x^-_{i+\frac{1}{2}},y)dy \\
&- \Delta t\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\{C^1\} + \alpha_x[n] + \frac{\Delta t}{2}\{C^3\} + \frac{\Delta t^2}{6}\{C^5\}\right)\ \xi(x^+_{i-\frac{1}{2}},y)dy \\
&+ \Delta t\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\{C^2\} + \alpha_y[n] + \frac{\Delta t}{2}\{C^4\} + \frac{\Delta t^2}{6}\{C^6\}\right)\ \xi(x,y^-_{j+\frac{1}{2}})dx \\
&- \Delta t\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\{C^2\} + \alpha_y[n] + \frac{\Delta t}{2}\{C^4\} + \frac{\Delta t^2}{6}\{C^6\}\right)\ \xi(x,y^+_{j-\frac{1}{2}})dx
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
(E^{n+1},\xi)_{K_{i,j}} =&(E,\xi)_{K_{i,j}} - \Delta t(D^1,\xi_x)_{K_{i,j}} - \Delta t(D^2,\xi_y)_{K_{i,j}} \\
&- \frac{\Delta t^2}{2}(D^3,\xi_x)_{K_{i,j}} - \frac{\Delta t^2}{2}(D^4,\xi_y)_{K_{i,j}} \\
&- \frac{\Delta t^3}{6}(D^5,\xi_x)_{K_{i,j}} - \frac{\Delta t^3}{6}(D^6,\xi_y)_{K_{i,j}} \\
&+ \Delta t\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\{D^1\} + \alpha_x[E] + \frac{\Delta t}{2}\{D^3\} + \frac{\Delta t^2}{6}\{D^5\}\right)\ \xi(x^-_{i+\frac{1}{2}},y)dy \\
&- \Delta t\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\{D^1\} + \alpha_x[E] + \frac{\Delta t}{2}\{D^3\} + \frac{\Delta t^2}{6}\{D^5\}\right)\ \xi(x^+_{i-\frac{1}{2}},y)dy \\
&+ \Delta t\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\{D^2\} + \alpha_y[E] + \frac{\Delta t}{2}\{D^4\} + \frac{\Delta t^2}{6}\{D^6\}\right)\ \xi(x,y^-_{j+\frac{1}{2}})dx \\
&- \Delta t\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\{D^2\} + \alpha_y[E] + \frac{\Delta t}{2}\{D^4\} + \frac{\Delta t^2}{6}\{D^6\}\right)\ \xi(x,y^+_{j-\frac{1}{2}})dx
\end{aligned}
\tag{3.24}
$$

Similar to the one dimensional Euler equations, we have the results for positivity of $\bar{\rho}^{n+1}$ as follows.

Theorem 3.4. *Given $\mathbf{u}^n \in G$, the cell averages $\bar{\rho}^{n+1}_{i,j}, i = 1,\ldots,N_x, j = 1,\ldots,N_y$ of the solution of scheme* (3.17) *are nonnegative under the CFL condition* (3.25)*:*

$$
\lambda_x \le \min\{Q_1, Q_3\}, \quad \lambda_y \le \min\{Q_2, Q_4\} \tag{3.25}
$$

where the definitions of $Q_1,\ldots,Q_4$ are given in Appendix A.4.

Proof. Take $\xi = 1$ in $K_{i,j}$ and zero on other cells in (3.17), we obtain

$$\bar{\rho}_{i,j}^{n+1} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \tag{3.26}$$

where

$$\mathrm{I} = \frac{1}{4}\bar{\rho}_{i,j}^{n} - \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^0_{i+\frac{1}{2},j} dy + \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^0_{i-\frac{1}{2},j} dy,$$

$$\mathrm{II} = \frac{1}{4}\bar{\rho}_{i,j}^{n} - \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^1_{i+\frac{1}{2},j} dy + \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{F}^1_{i-\frac{1}{2},j} dy,$$

$$\mathrm{III} = \frac{1}{4}\bar{\rho}_{i,j}^{n} - \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^0_{i,j+\frac{1}{2}} dx + \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^0_{i,j-\frac{1}{2}} dx$$

$$\mathrm{IV} = \frac{1}{4}\bar{\rho}_{i,j}^{n} - \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^1_{i,j+\frac{1}{2}} dx + \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{G}^1_{i,j-\frac{1}{2}} dx$$

It suffices to show $\mathrm{I}, \mathrm{II} \geq 0$ under the CFL condition (3.25), due to the symmetry in the x and y directions.

One can observe that I can be decomposed in the form of convex combination

$$\mathrm{I} = \frac{1}{4}\sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma H_\gamma,$$

where

$$H_\gamma = \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \hat{\rho}^{\beta,\gamma} - 4\lambda_x \hat{F}^0_{i+\frac{1}{2},j}(x_{i+\frac{1}{2}}, \hat{y}_\gamma) + 4\lambda_x \hat{F}^0_{i-\frac{1}{2},j}(x_{i-\frac{1}{2}}, \hat{y}_\gamma),$$

Notice that H_γ has the same structure as (3.10). Thus $I \geq 0$ provided $\lambda_x \leq Q_1$. We omit the proof since it is almost the same with that of the one dimensional Euler equations.

As for II, we have the expansion as follows.

$$
\begin{aligned}
\mathrm{II} =& \frac{1}{4}\sum_{\alpha=2}^{2N_q-2}\sum_{\beta=1}^{2N_q-1}\hat{\omega}_\alpha\hat{\omega}_\beta\hat{\rho}^{\alpha,\beta} \\
&+ z_1\rho(x^-_{i-\frac{1}{2}}, y^+_{j-\frac{1}{2}}) + z_2\rho(x^-_{i-\frac{1}{2}}, y_j) + z_3\rho(x^-_{i-\frac{1}{2}}, y^-_{j+\frac{1}{2}}) \\
&+ z_4\rho(x^+_{i-\frac{1}{2}}, y^+_{j-\frac{1}{2}}) + z_5\rho(x^+_{i-\frac{1}{2}}, y_j) + z_6\rho(x^+_{i-\frac{1}{2}}, y^-_{j+\frac{1}{2}}) \\
&+ z_7\rho(x^-_{i+\frac{1}{2}}, y^+_{j-\frac{1}{2}}) + z_8\rho(x^-_{i+\frac{1}{2}}, y_j) + z_9\rho(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) \\
&+ z_{10}\rho(x^+_{i+\frac{1}{2}}, y^+_{j-\frac{1}{2}}) + z_{11}\rho(x^+_{i+\frac{1}{2}}, y_j) + z_{12}\rho(x^+_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) \\
&+ \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{13,\beta}\rho(x^-_{i-\frac{1}{2}}, \hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{13,\beta}\rho(x^-_{i-\frac{1}{2}}, \hat{y}_\beta) \\
&+ \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{14,\beta}\rho(x^+_{i+\frac{1}{2}}, \hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{14,\beta}\rho(x^+_{i+\frac{1}{2}}, \hat{y}_\beta) \\
&+ \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{15,\beta}\rho(x^-_{i+\frac{1}{2}}, \hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{15,\beta}\rho(x^-_{i+\frac{1}{2}}, \hat{y}_\beta) \\
&+ \sum_{\beta=2}^{N_q-1}\hat{\omega}_\beta z_{16,\beta}\rho(x^+_{i-\frac{1}{2}}, \hat{y}_\beta) + \sum_{\beta=N_q+1}^{2N_q-2}\hat{\omega}_\beta z_{16,\beta}\rho(x^+_{i-\frac{1}{2}}, \hat{y}_\beta),
\end{aligned}
\tag{3.27}
$$

The expressions of $z_1, \ldots, z_{16,\beta}$ and their estimates can be found in Appendix A.5. The conclusion is that all coefficients of point values of ρ^n appearing in (3.27) are nonnegative under the CFL condition (3.25), which implies the nonnegativity of II. Similar arguments also apply to III and IV.

Since $\bar{\rho}^{n+1}_{i,j} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$, we finish the proof of positivity of $\bar{\rho}^{n+1}_{i,j}$ by summing up the inequalities of $\mathrm{I}, \mathrm{II}, \mathrm{III}$ and IV. □

It remains to show the positivity of specific internal energy of cell averages. Similar to Theorem 3.2, we have the result as follows.

Theorem 3.5. *Given* $\mathbf{u}^n \in G$*, the specific internal energy of the cell averages* $e(\bar{\mathbf{u}}^{n+1}_{i,j}), i = 1, 2, \ldots N_x, j = 1, 2, \ldots, N_y$ *of scheme* (3.17),(3.22) ,(3.23) *and* (3.24) *are nonnegative under the CFL condition* (3.28)*.*

$$
\begin{aligned}
\lambda_x \leq& \frac{\gamma+1}{4\alpha_x^2(\gamma-1)}\min_{i,\beta}\left\{\frac{p(x^-_{i+\frac{1}{2}}, \hat{y}_\beta)^2}{C(x^-_{i+\frac{1}{2}}, \hat{y}_\beta)}, \frac{p(x^+_{i+\frac{1}{2}}, \hat{y}_\beta)^2}{C(x^+_{i+\frac{1}{2}}, \hat{y}_\beta)}\right\}, \\
\lambda_y \leq& \frac{\gamma+1}{4\alpha_y^2(\gamma-1)}\min_{\alpha,j}\left\{\frac{p(\hat{x}_\alpha, y^-_{j+\frac{1}{2}})^2}{D(\hat{x}_\alpha, y^-_{j+\frac{1}{2}})}, \frac{p(\hat{x}_\alpha, y^+_{j+\frac{1}{2}})^2}{D(\hat{x}_\alpha, y^+_{j+\frac{1}{2}})}\right\},
\end{aligned}
\tag{3.28}
$$

where the definitions of the constants are given in Appendix A.6.

Proof. By taking $\xi = 1$ on $K_{i,j}$ and zero anywhere else in (3.17), (3.22), (3.23) and (3.24), we have the decomposition of $\bar{\mathbf{u}}_{i,j}^{n+1}$ in x and y directions:

$$\bar{\mathbf{u}}_{i,j}^{n+1} = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \frac{1}{2}\bar{\mathbf{u}}_{i,j}^{n} - \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{\mathbf{F}}_{i+\frac{1}{2},j}dy + \lambda_x \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \hat{\mathbf{F}}_{i-\frac{1}{2},j}dy,$$

$$\mathrm{II} = \frac{1}{2}\bar{\mathbf{u}}_{i,j}^{n} - \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{\mathbf{G}}_{i,j+\frac{1}{2}}dx + \lambda_y \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{\mathbf{G}}_{i,j-\frac{1}{2}}dx,$$

where $\hat{\mathbf{F}}_{i+\frac{1}{2},j} = \hat{\mathbf{f}}^{\mathrm{LF}}_{i+\frac{1}{2},j} + \Delta t\tilde{\mathbf{f}}_{i+\frac{1}{2},j} + \Delta t^2\check{\mathbf{f}}_{i+\frac{1}{2},j}$, $\hat{\mathbf{G}}_{i,j+\frac{1}{2}} = \hat{\mathbf{g}}^{\mathrm{LF}}_{i,j+\frac{1}{2}} + \Delta t\tilde{\mathbf{g}}_{i,j+\frac{1}{2}} + \Delta t^2\check{\mathbf{g}}_{i,j+\frac{1}{2}}$ are the total fluxes of LWDG defined before, $\hat{\mathbf{f}}^{\mathrm{LF}}_{i+\frac{1}{2},j}, \hat{\mathbf{g}}^{\mathrm{LF}}_{i,j+\frac{1}{2}}$ are Lax-Friedriches fluxes, $\tilde{\mathbf{f}}_{i+\frac{1}{2},j}, \tilde{\mathbf{g}}_{i,j+\frac{1}{2}}$ and $\check{\mathbf{f}}_{i+\frac{1}{2},j}, \check{\mathbf{g}}_{i,j+\frac{1}{2}}$ are the second and third order terms in the total flux.

By symmetry and concaveness of the internal energy ρe, it suffices to show $\rho e(\mathrm{I}) \geq 0$. We can decompose the term I as

$$\begin{aligned}
\mathrm{I} =& \frac{1}{2}\sum_{\alpha=1}^{2N_q-1}\sum_{\beta=1}^{2N_q-1} \hat{\omega}_\alpha\hat{\omega}_\beta \mathbf{u}^{\alpha,\beta} - \lambda_x \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \hat{\mathbf{F}}(x_{i+\frac{1}{2}}, \hat{y}_\beta) + \lambda_x \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \hat{\mathbf{F}}(x_{i-\frac{1}{2}}, \hat{y}_\beta) \\
=& \frac{1}{2}\sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \mathbf{H}_\beta,
\end{aligned}$$

where $\mathbf{H}_\beta = \sum_{\alpha=1}^{2N_q-1} \hat{\omega}_\alpha \mathbf{u}^{\alpha,\beta} - 2\lambda_x \left(\hat{\mathbf{f}}^{\mathrm{LF}}_{j+\frac{1}{2}} + \Delta t\tilde{\mathbf{f}}_{j+\frac{1}{2}} + \Delta t^2\check{\mathbf{f}}_{j+\frac{1}{2}}\right) + 2\lambda_x \left(\hat{\mathbf{f}}^{\mathrm{LF}}_{j-\frac{1}{2}} + \Delta t\tilde{\mathbf{f}}_{j-\frac{1}{2}} + \Delta t^2\check{\mathbf{f}}_{j-\frac{1}{2}}\right)$

Following the same lines as the proof of (3.2), we can show $\rho e(\mathbf{H}_\beta) \geq 0$, which implies $\rho e(\mathrm{I}) \geq 0$ □

Collecting the above two theorems, we reach our final result.

Theorem 3.6. *Given* $\mathbf{u}^n \in G$, *we have* $\bar{\mathbf{u}}_{i,j}^{n+1} \in G, i = 1, \ldots, N_x, j = 1, \ldots, N_y$ *for the schemes* (3.17), (3.22), (3.23) *and* (3.24), *under the CFL conditions* (3.25) *and* (3.28).

Remark 3.1. *To this end, we would like to comment on the CFL conditions obtained in this paper. These conditions are not optimal for bound-preserving since the splitting of cell averages in the proofs are just for the ease of writing and the bounds may not be sharp in some of the estimates. Moreover, the expressions of the CFL conditions are too tedious to be coded up in practice. Therefore, we actually take the CFL conditions of the bound-preserving Euler forward DG schemes derived in [38, 37] as the initial guess in practice, since the Euler forward methods are the first order approximation of the LWDG in our work. Once the initial step* *size is not small enough to obtain boundedness of the cell averages, we rewind the computation back to the beginning of the time step with a halved step-size of time. The value of the theoretical proofs in this paper is that we can be guaranteed to obtain bound-preserving cell averages with finitely many halvings of the time step-size.*

We also want to note that, for simplicity, we take the viscosity parameter in the Lax-Friedrichs flux to be global in all proofs. However, the local Lax-Friedrichs flux can be used in the bound-preserving technique as well. In practice, the global Lax-Friedrichs flux is more dissipative, thus it may preserve the bounds of target variables more easily, but may result in a more smeared solution.

4 Scaling limiters

In the Sections 2 and 3, we have constructed the maximum-principle-satisfying and positivity-preserving LWDG schemes for hyperbolic equations of scalar and system cases. The cell averages of the target variables fall into their physical bounds under appropriate CFL conditions, provided these bounds are satisfied by the entire solution at the previous time level. To close the cycle of the algorithm, it remains to use appropriate scaling limiters to achieve the bound-preserving for the entire solution.

We adopt the following maximum-principle-satisfying limiter for scalar conservation laws. Given $u \in V$ with $m \leq \bar{u}_j \leq M, j = 1, 2, \ldots, N$, define the modified solution $\tilde{u} \in V$ as follows:

$$\tilde{u}_j(x) = \theta_j \left(u_j(x) - \bar{u}_j\right) + \bar{u}_j, \quad \theta_j = \min\left\{1, \frac{M - \bar{u}_j}{M_j - \bar{u}_j}, \frac{\bar{u}_j - m}{\bar{u}_j - m_j}\right\},$$

$$M_j = \max_{x \in I_j} u_j(x), \quad m_j = \min_{x \in I_j} u_j(x), \quad j = 1, 2, \ldots, N.$$

It is clear that the modified solution $\tilde{u}_j(x) \in [m, M], j = 1, \ldots, N$ and it preserves the cell average. Moreover, it was proved in [36] that such a limiter does not destroy the order of convergence, i.e. $||u - \tilde{u}||_\infty = O(\Delta x^{k+1})$, where k is the order of polynomial space V, which is 2 in this paper. In practice, one usually take the max and min in the definition of M_j and m_j only over the quadrature points, i.e. $M_j = \max_{1 \leq \gamma \leq 2N_q - 1} u_j(\hat{x}_\gamma), m_j = \min_{1 \leq \gamma \leq 2N_q - 1} u_j(\hat{x}_\gamma)$, as we only need to control the values at quadrature points. Such a treatment does not affect the accuracy and cell average of the modified solution, as indicated in [38], and we shall use this definition in the numerical section.

For the solution $\mathbf{u} = (\rho, m, E)^T \in V \times V \times V$ of the Euler equations with $\bar{\mathbf{u}}_j \in G, j = 1, 2, \ldots, N$, we adopt the following limiting process which is introduced in [37] and modified in [32]

First, enforce the positivity of the density function ρ by,

$$\hat{\rho}_j(x) = \theta_j^\rho \left(\rho_j(x) - \bar{\rho}_j\right) + \bar{\rho}_j, \quad \theta_j^\rho = \min\left\{1, \frac{\bar{\rho}_j}{\bar{\rho}_j - \min_{1\le\gamma\le 2N_q-1} \rho(\hat{x}_\gamma)}\right\}, \quad j = 1, 2, \ldots, N.$$

Then let $\hat{\mathbf{u}}_j = (\hat{\rho}_j, m_j, E_j)^T$ and define

$$\tilde{\mathbf{u}}_j(x) = \theta_j^e \left(\hat{\mathbf{u}}_j(x) - \bar{\mathbf{u}}_j\right) + \bar{\mathbf{u}}_j, \quad \theta_j^e = \min\left\{1, \frac{\rho e(\bar{\mathbf{u}}_j)}{\rho e(\bar{\mathbf{u}}_j) - \min_{1\le\gamma\le 2N_q-1} \rho e(\hat{\mathbf{u}}_j(\hat{x}_\gamma))}\right\}, \quad j = 1, 2, \ldots, N.$$

It follows from the concaveness of the function $\rho e(\mathbf{u})$ that $\tilde{\mathbf{u}}_j(\hat{x}_\gamma) \in G, \gamma = 1, 2, \ldots, 2N_q - 1$, and also it does not destroy accuracy of the solution, see the detailed proof in [37] and [32].

The above limiters are demonstrated based on one space dimension but can be directly extended to multi-dimensions. In implementation, to enhance the stability of algorithms, we can set a threshold $\epsilon = 10^{-10}$ and let $\tilde{u}_j = \bar{u}_j$ if $M - \bar{u}_j < \epsilon$ or $\bar{u}_j - m < \epsilon$ for scalar conservation law, and $\tilde{\mathbf{u}}_j = \bar{\mathbf{u}}_j$ if $\bar{\rho}_j < \epsilon$ or $\rho e(\bar{\mathbf{u}}_j) < \epsilon$ for the Euler equations.

5 Numerical tests

In this section, we demonstrate the accuracy and effectiveness of the third order maximum-principle-satisfying and positivity-preserving LWDG schemes by ample numerical tests. The tests are presented from scalar to systems and from one space dimension to two space dimensions with an order of increasing complexity. Most of them can be found in [38, 37, 36, 32].

We have tried both global Lax-Friedrichs and local Lax-Friedrichs fluxes in simulations. The plots of their solutions are very close. However, the accuracy and order of convergence of the global one may be not as good as the local one for some nonlinear problems when the order of DG polynomial space is even, see [5], which is our case. We only present the results computed using the local Lax-Friedrichs flux to save space.

5.1 Scalar conservation laws

Example 5.1. *We solve the linear equation $u_t + u_x = 0$ in the domain $\Omega = [-1, 1]$ with periodic boundary conditions.*

To test the accuracy, we take the smooth initial condition $u_0(x) = \sin(\pi x)$ and the terminal time $T = 1$.

To show the effect of maximum-principle-preserving, we adopt the discontinuous initial condition

$$u_0(x) = \begin{cases} 1, & -1 \le x \le 0, \\ -1, & 0 \le x \le 1, \end{cases}$$

and take the terminal time $T = 100$*.*

The errors and order of convergence of the problem with the smooth initial condition are given in Table 1, from which the third order accuracy can be clearly observed.

The results of the problem with the discontinuous initial condition is shown in Figure 1, where a comparison with the exact solution and the result of the unlimited LWDG solution are given. The effect of maximum-principle-preserving is obvious by comparison.

N	L^1 error	order	L^∞ error	order
20	2.06E-04	–	5.09E-04	–
40	2.48E-05	3.05	6.38E-05	3.00
80	3.08E-06	3.01	7.97E-06	3.00
160	3.85E-07	3.00	9.97E-07	3.00
320	4.81E-08	3.00	1.25E-07	3.00
640	6.01E-09	3.00	1.56E-08	3.00

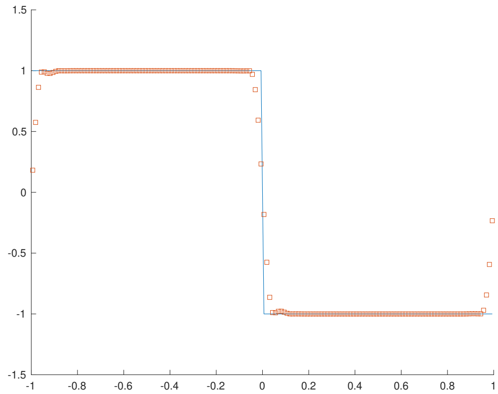
Table 1: Results of Example 5.1 with smooth initial condition

Example 5.2. *We solve the Burgers' equation* $u_t + \left(\frac{u^2}{2}\right)_x = 0$ *in the domain* $\Omega = [0, 2\pi]$ *with initial condition* $u_0(x) = \frac{1}{2} + \sin(x)$ *and periodic boundary conditions.*

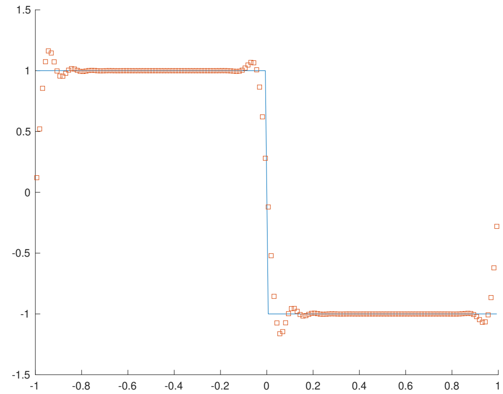
The solution is smooth up to $t = 1$*, when shock appears. We list the errors and order of convergence at* $T = 0.3$ *in Table 2, which shows third order accuracy, and plot the comparison of the numerical solution with the exact solution at* $T = 2.0$ *in Figure 2.*

Example 5.3. *We solve the two dimensional linear equation* $u_t + u_x + u_y = 0$ *in the domain* $\Omega = [-1, 1] \times [-1, 1]$ *with periodic boundary conditions.*

To show the accuracy, we take the smooth initial condition $u_0(x, y) = \sin(\pi(x + y))$ *and the terminal time* $T = 1$*.*



(a) with limiter



(b) without limiter

Figure 1: Results of Example 5.1 for discontinuous initial condition. $N = 160$. Solid line: exact solution; Squares: numerical solution (cell averages).

N	L^1 error	order	L^∞ error	order
20	9.05E-04	–	1.40E-03	–
40	1.13E-04	3.00	2.35E-04	2.58
80	1.37E-05	3.05	3.23E-05	2.87
160	1.66E-06	3.04	4.23E-06	2.93
320	2.04E-07	3.03	5.38E-07	2.98
640	2.52E-08	3.02	6.78E-08	2.99

Table 2: Results of Example 5.2 at $T = 0.3$

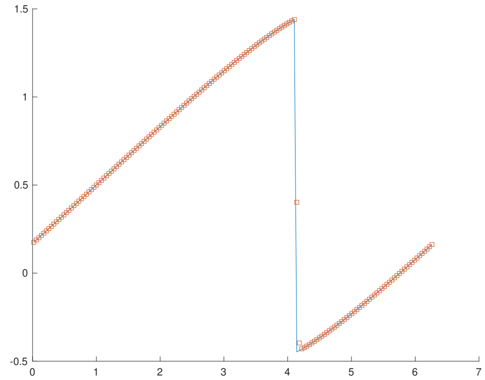


Figure 2: Results of Example 5.2 at $T = 2.0$. $N = 160$. Solid line: exact solution; Squares: numerical solution (cell averages).

To test the effect of maximum-principle-preserving, we adopt a discontinuous initial condition

$$u_0(x) = \begin{cases} 1, & (x,y) \in [-\frac{1}{2}, \frac{1}{2}]^2 \\ -1, & \textit{elsewhere}, \end{cases}$$

and take the terminal time $T = 100$.

The errors and order of convergence for the smooth initial condition are given in Table 3, from which the third order accuracy can be observed.

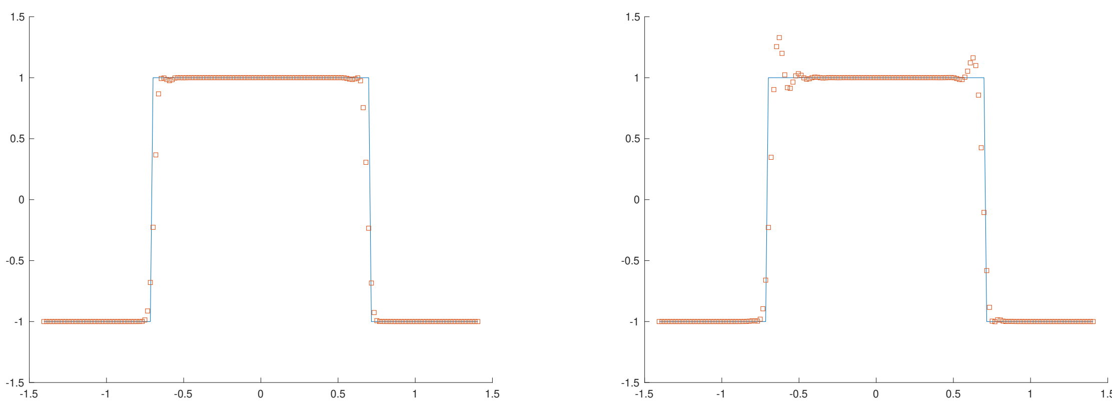
The results of the problem with the discontinuous initial condition is shown in Figure 3, where a comparison with the exact solution and the result of the unlimited LWDG solution are given, from which we can see the maximum-principle-preserving limiter works effectively.

Example 5.4. *We solve the two dimensional Burgers' equation* $u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0$ *in the domain* $\Omega = [0, 2\pi] \times [0, 2\pi]$ *with the initial condition* $u_0(x,y) = \frac{1}{2} + \sin(x+y)$ *and periodic boundary conditions.*

The solution is smooth up to $t = 0.5$*, when shock appears. We list the errors and order of convergence at* $T = 0.2$ *under the* L^1 *and* L^∞ *norms in Table 4, and plot the comparison of the numerical solution with the exact solution at* $T = 1.0$ *along the diagonal of* Ω *in Figure 4.*

$N_x \times N_y$	L^1 error	order	L^∞ error	order
20×20	7.49E-04	–	1.11E-03	–
40×40	7.99E-05	3.23	1.29E-04	3.11
80×80	9.71E-06	3.04	1.61E-05	3.00
160×160	1.21E-06	3.01	2.01E-06	3.00
320×320	1.51E-07	3.00	2.51E-07	3.00
640×640	1.89E-08	3.00	3.14E-08	3.00

Table 3: Results of Example 5.3 with smooth initial condition



(a) with limiter

(b) without limiter

Figure 3: Results of Example 5.3 with discontinuous initial condition cut along the diagonal of Ω. $N_x = 160, N_y = 160$. Solid line: exact solution; Squares: numerical solution (cell averages).

$N_x \times N_y$	L^1 error	order	L^∞ error	order
20×20	1.06E-02	–	5.33E-03	–
40×40	1.33E-03	2.99	7.67E-04	2.80
80×80	1.67E-04	3.00	1.12E-04	2.77
160×160	2.09E-05	3.00	1.52E-05	2.89
320×320	2.59E-06	3.01	1.95E-06	2.97
640×640	3.20E-07	3.01	2.45E-07	2.99

Table 4: Results of Example 5.4 at $T = 0.2$

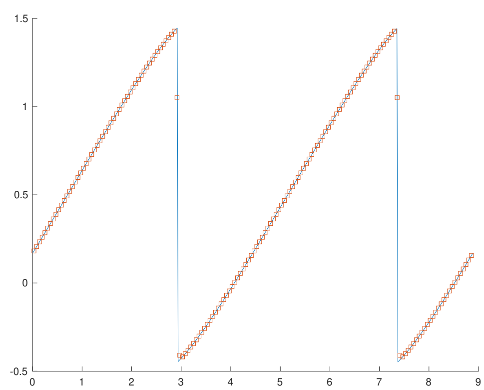


Figure 4: Results of Example 5.4 cut along the diagonal of Ω at $T = 1.0$. $N_x = 160, N_y = 160$. Solid line: exact solution; Squares: numerical solution (cell averages).

5.2 The Euler equations

Example 5.5. *We solve the one dimensional problem in the domain* $\Omega = [0, 2\pi]$ *with the initial condition*

$$\rho_0(x) = 1 + 0.999\sin(x), \quad u_0(x) = 1, \quad p_0(x) = 1$$

and periodic boundary conditions. The ratio of specific heat is $\gamma = 1.4$*.*

The exact solution of the problem is

$$\rho(x,t) = 1 + 0.999\sin(x-t), \quad u(x,t) = 1, \quad p(x,t) = 1. \tag{5.1}$$

This is a low density problem with the minimum density 0.001*. The positivity of density is preserved during simulation and the third order convergence of density at time* $T = 1$ *is shown in Table* (5)*.*

N	L^1 error	order	L^∞ error	order
20	1.13E-03	–	8.60E-04	–
40	1.40E-04	3.01	1.07E-04	3.01
80	1.72E-05	3.02	1.34E-05	3.00
160	2.14E-06	3.01	1.65E-06	3.02
320	2.67E-07	3.00	2.04E-07	3.01
640	3.33E-08	3.00	2.55E-08	3.01

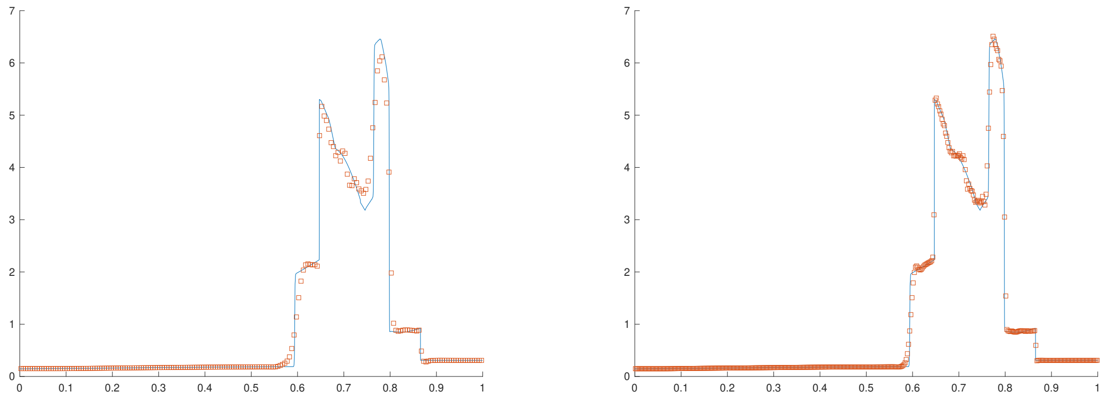
Table 5: Results of Example 5.5 at $T = 1$

Example 5.6. *We solve the one dimensional problem of blast waves in the domain* $\Omega = [0, 1]$ *with initial condition*

$$(\rho_0, u_0, p_0) = \begin{cases} (1, 0, 10^3) & 0 \le x < 0.1, \\ (1, 0, 10^{-2}) & 0.1 \le x < 0.9 \\ (1, 0, 10^2), & 0.9 \le x < 1 \end{cases}$$

and reflective boundary condition. The ratio of specific heat is $\gamma = 1.4$*.*

We plot the density of numerical solutions at $T = 0.38$ *for* $N = 200, N = 400$*, and compare them with the reference solution in Figure 5. Since the positivity-preserving limiter only works when the density or pressure is close to zero and no other limiters are used to stabilize shocks in this test, we can observe some oscillations in the figures.*



(a) $N = 200$

(b) $N = 400$

Figure 5: Results of Example 5.6 at $T = 0.038$. Solid line: reference solution; Squares: numerical solution (cell averages).

Example 5.7. *We solve two extreme Riemann problems in one space dimension. The first one is a double rarefaction problem in the domain* $\Omega = [-1, 1]$ *with initial condition*

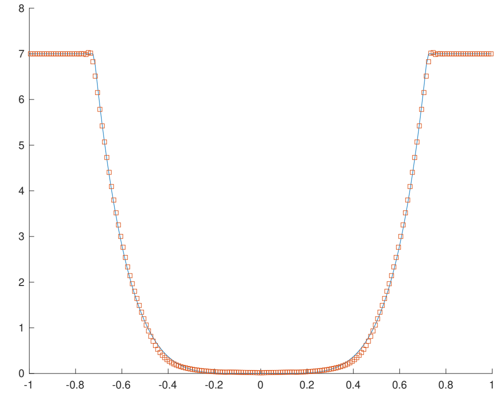
$$(\rho_0, u_0, p_0) = \begin{cases} (7, -1, 0.2), & x < 0 \\ (7, 1, 0.2), & x > 0. \end{cases}$$

The second one is the Leblanc shock tube problem in the domain $\Omega = [-10, 10]$ *with initial condition*

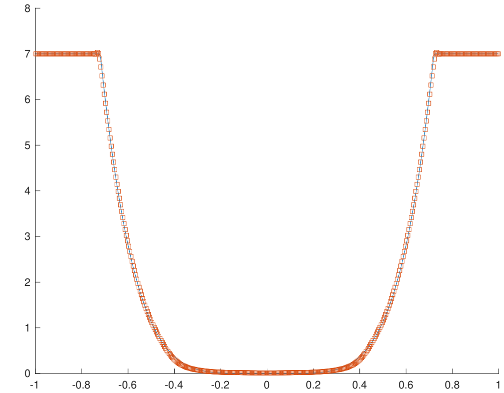
$$(\rho_0, u_0, p_0) = \begin{cases} (2, 0, 10^9), & x < 0 \\ (10^{-3}, 0, 1), & x > 0. \end{cases}$$

We take the ratio of specific heat $\gamma = 1.4$ *for both cases. In the first test example, vacuum (zero density) will be generated around the origin in the exact solution. For both problems, simulation will blow up without the positivity-preserving limiter in the tests.*

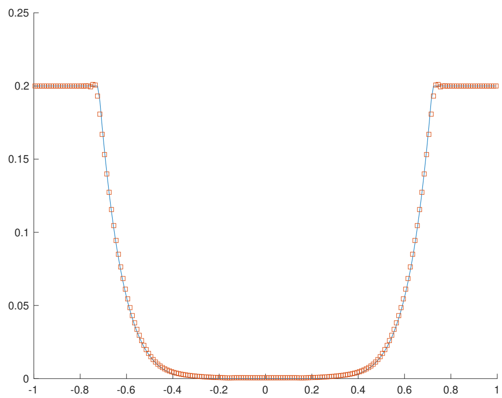
We plot the density of numerical solution of the double rarefaction problem at $T = 0.6$ *on* $N = 200$ *and* $N = 400$ *meshes, and compare them with the reference solution in Figure 6. The density of the numerical solution of the Leblanc shock tube problem at* $T = 0.0001$ *on* $N = 800$ *and* $N = 1600$ *meshes, together with the exact solution, are shown in Figure 7, where the y-axis uses* log *scales. From the figures, we can see that the positivity of density and pressure in both cases are preserved, and the numerical solutions agree with the exact solution well.*



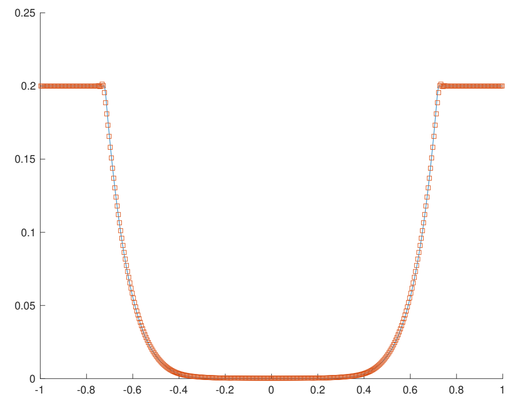
(a) Density



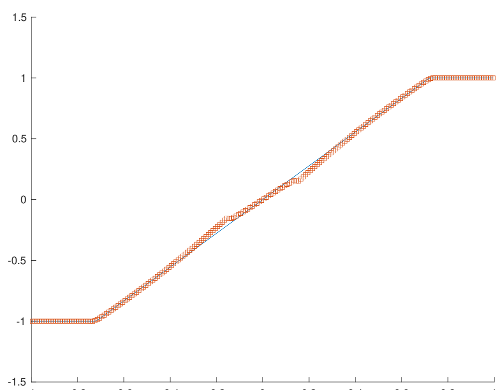
(b) Density



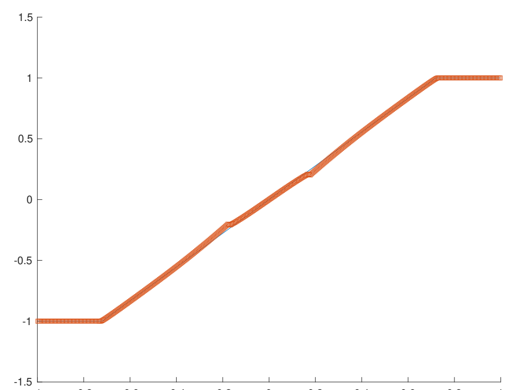
(c) Pressure



(d) Pressure

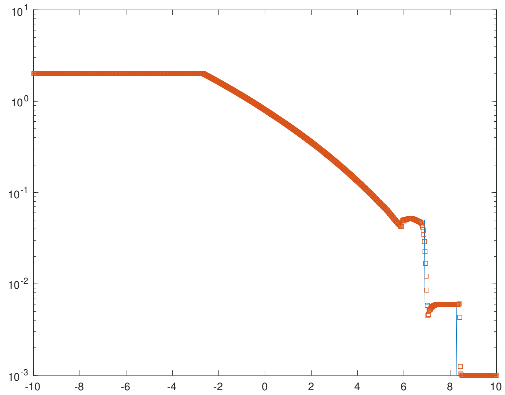


(e) Velocity

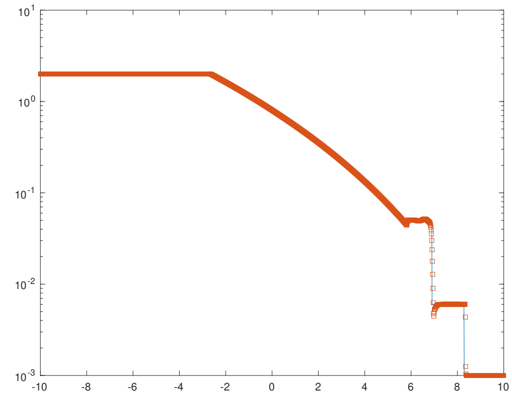


(f) Velocity

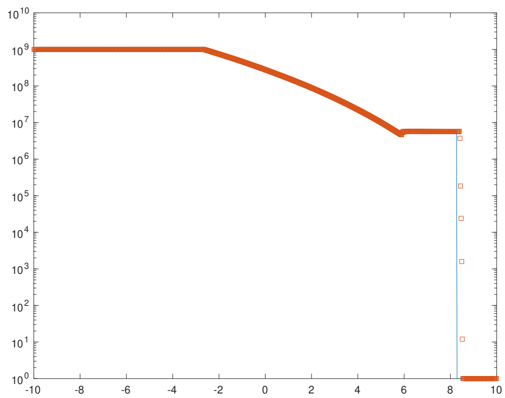
Figure 6: Results of Example 5.7, the double rarefaction problem, at $T = 0.6$. Solid line: reference solution; Squares: numerical solution (cell averages). Left: $N = 200$; Right:$N = 400$.



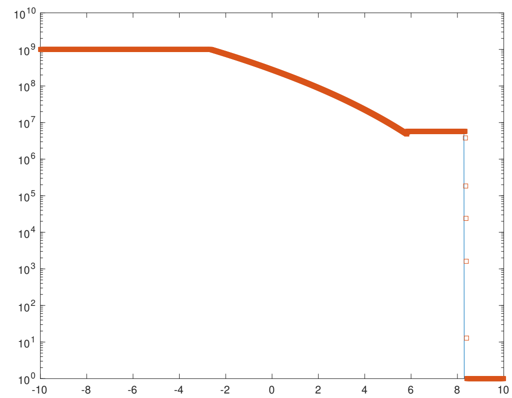
(a) Density, log scale



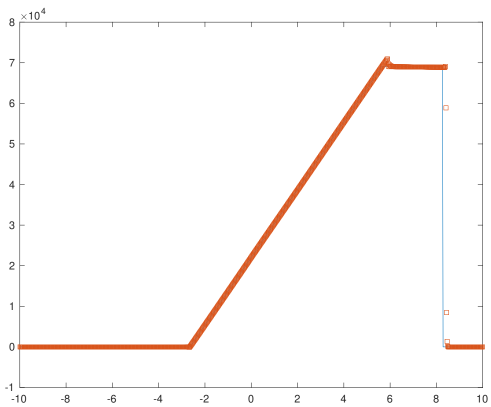
(b) Density, log scale



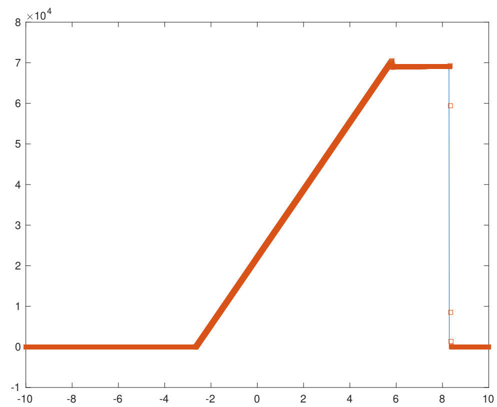
(c) Pressure, log scale



(d) Pressure, log scale



(e) Velocity



(f) Velocity

Figure 7: Results of Example 5.7, Leblanc shock tube problem, at $T = 0.0001$. Solid line: reference solution; Squares: numerical solution (cell averages). Left: $N = 800$; Right: $N = 1600$.

Example 5.8. *We solve the one dimensional Sedov point-blast wave problem in the domain* $\Omega = [-2, 2]$ *with the initial condition*

$$\rho_0 = 1, \quad u_0 = 0, \quad E_0 = \begin{cases} \frac{3200000}{\Delta x}, & |x| \le \frac{\Delta x}{2} \\ 10^{-12}, & otherwise. \end{cases}$$

The ratio of specific heat is $\gamma = 1.4$.

This example simulates the point-blast in air, which produces very low density after shock. The simulation will blow up without the positivity-preserving limiter due to the very low density in the exact solution. We plot the simulation results of density, pressure and velocity on $N = 201$ *and* $N = 401$ *meshes at* $T = 0.001$ *in Figure 8.*

Example 5.9. *We solve the two dimensional problem in the domain* $[0, 2\pi]^2$ *with the initial condition*

$$\rho_0(x, y) = 1 + 0.999 \sin(x + y), \quad u_0 = v_0 = p_0 = 1.$$

and periodic boundary conditions. The ratio of specific heat is $\gamma = 1.4$.

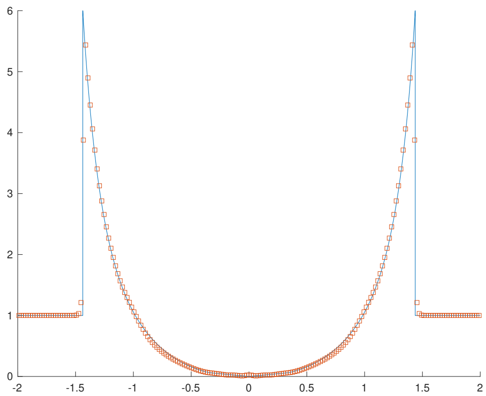
The exact solution of the problem is

$$\rho(x, y, t) = 1 + 0.999 \sin(x + y - 2t), \quad u(x, y, t) = v(x, y, t) = p(x, y, t) = 1.$$

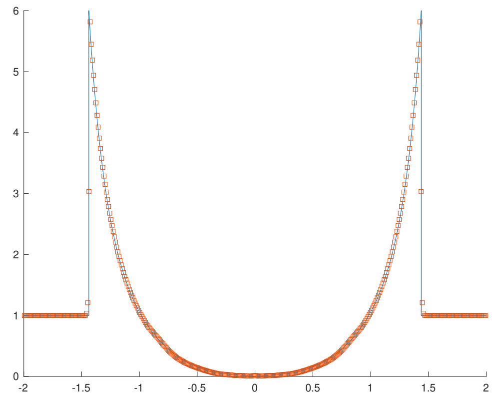
This is a low density problem with the minimum density 0.001. The positivity of density is preserved during simulation and the third order convergence of density at time $T = 0.1$ *is shown in Table 6.*

$N_x \times N_y$	L^1 error	order	L^∞ error	order
20×20	8.64E-03	–	1.23E-03	–
40×40	1.37E-03	2.65	2.12E-04	2.53
80×80	1.79E-04	2.94	2.71E-05	2.97
160×160	2.23E-05	3.00	3.33E-06	3.03
320×320	2.75E-06	3.02	4.12E-07	3.02

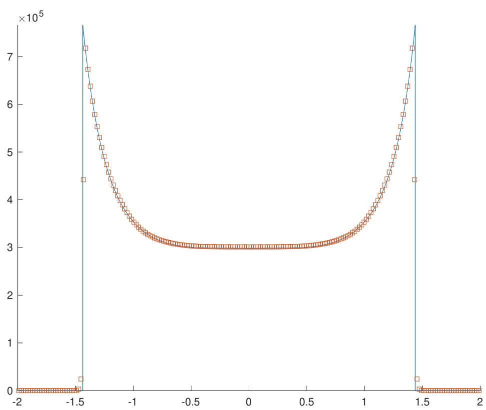
Table 6: Results of Example 5.9 at $T = 0.1$



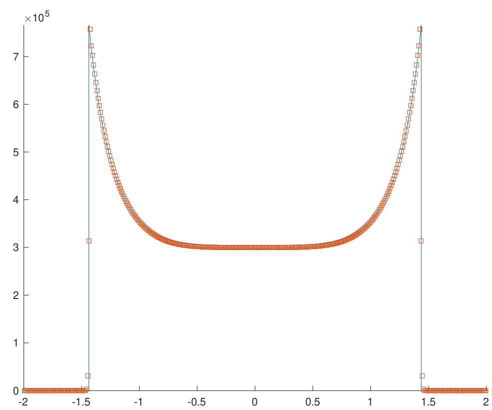
(a) Density



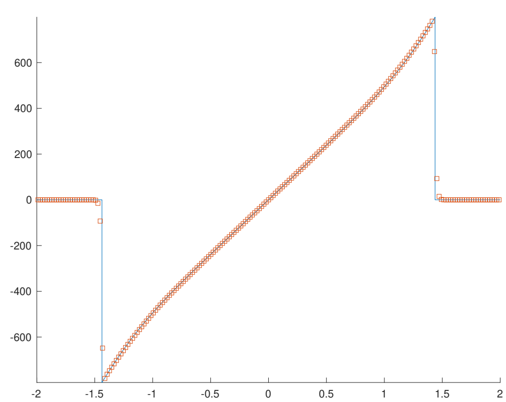
(b) Density



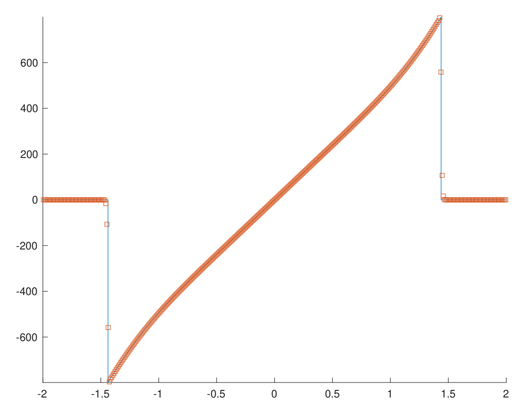
(c) Pressure



(d) Pressure



(e) Velocity



(f) Velocity

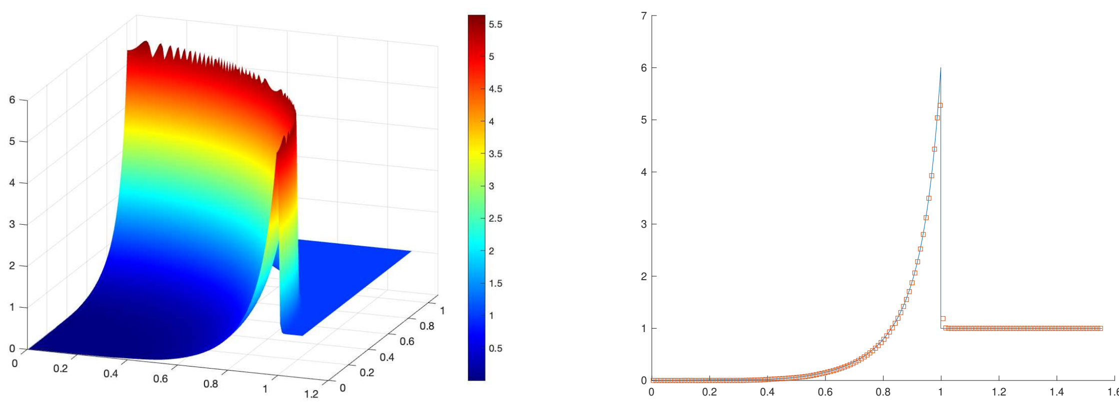
Figure 8: Results of Example 5.8 at $T = 0.001$. Solid line: reference solution; Squares: numerical solution (cell averages). Left: $N = 201$; Right: $N = 401$.

Example 5.10. *We solve the two dimensional Sedov point-blast wave problem in the domain* $\Omega = [0, 1.1] \times [0, 1.1]$ *with the initial condition*

$$\rho_0 = 1, \quad u_0 = v_0 = 0, \quad E_0 = \begin{cases} \frac{0.244816}{\Delta x \Delta y}, & (x, y) \in [0, \Delta x] \times [0, \Delta y] \\ 10^{-12}, & otherwise, \end{cases}$$

and the left and bottom boundary the reflective boundary, and other boundaries the outflow boundary. The ratio of specific heat is $\gamma = 1.4$.

We plot the density on Ω *and its profile cut along the diagonal of* Ω *at* $T = 1$ *on the* $N_x = 160, N_y = 160$ *mesh, see Figure 9. The simulation blows up if the positivity-preserving limiter is not used in the test.*



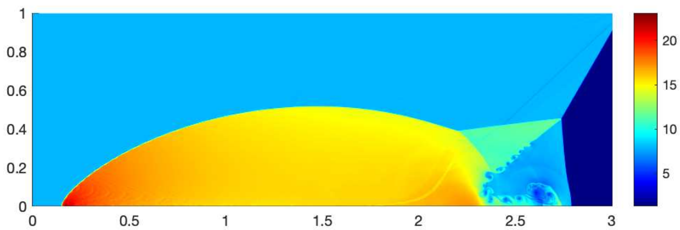
(a) Density in Ω (b) Cut of density along diagonal of Ω

Figure 9: Results of Example 5.10 at $T = 1$. Solid line: reference solution; Squares: numerical solution (cell averages).

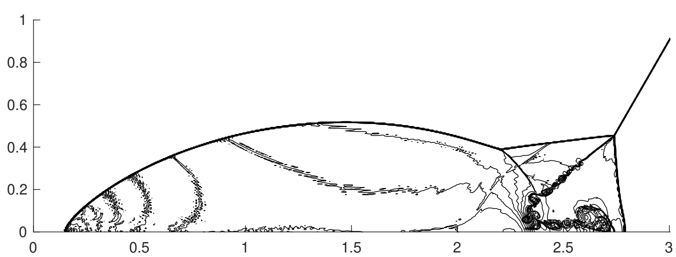
Example 5.11. *Consider the two-dimensional double Mach reflection problem with a Mach 10 shock in the domain* $\Omega = [0, 4] \times [0, 1]$, *with the initial condition*

$$(\rho_0, u_0, v_0, p_0) = \begin{cases} (8, \frac{33\sqrt{3}}{8}, -\frac{33}{8}, 116.5), & y > \sqrt{3}(x - \frac{1}{6}) \quad (post\text{-}shock) \\ (1.4, 0, 0, 1), & y < \sqrt{3}(x - \frac{1}{6}) \quad (pre\text{-}shock). \end{cases}$$

The left boundary is the inflow boundary, the right boundary is the outflow boundary, $\{0 \leq x < \frac{1}{6}, y = 0\}$ *on the bottom is the boundary with post-shock condition,* $\{\frac{1}{6} < x \leq 4, y = 0\}$ *on the bottom is the reflective* *boundary, and the condition on top boundary follows the motion of the shock. We show the results at* $T = 0.2$ *on the* $N_x = 960, N_y = 240$ *mesh in Figure 10. The results are comparable with the results in [36].*



(a) Density on $[0, 3] \times [0, 1]$



(b) 30 equally spaced contour lines from 1.394 to 23.083 for density

Figure 10: Results of Example 5.11 at $T = 0.2$ on $N_x = 960, N_y = 240$ mesh.

Example 5.12. *We solve the two dimensional problem of shock passing a backward facing corner in the domain* $\Omega = [1,13] \times [0,11] \cup [0,1] \times [6,11]$, *with the initial condition*

$$(\rho_0, u_0, v_0, p_0) = \begin{cases} (\rho_*, u_*, v_*, p_*), & x < 0.5 \quad \textit{(post-shock)} \\ (1.4, 0, 0, 1), & x > 0.5 \quad \textit{(pre-shock)} \end{cases},$$

where (ρ_*, u_*, v_*, p_*) *are taken such that the shock is right-moving with Mach number 5.09. The boundary* $\{x = 0, 6 \le y \le 11\}$ *is the inflow boundary,* $\{0 \le x \le 1, y = 6\}$ *and* $\{x = 1, 0 \le y \le 6\}$ *are reflexive boundaries,* $\{x = 13, 0 \le y \le 11\}$ *and* $\{1 \le x \le 13, y = 0\}$ *are outflow boundaries, and the boundary condition on* $\{0 \le x \le 13, y = 11\}$ *follows the motion of the shock.*

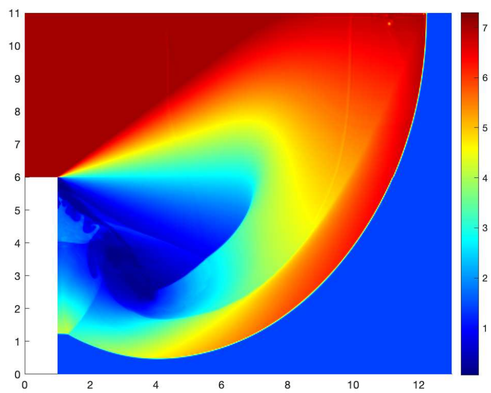
The density and pressure at $T = 2.3$ *with* $\Delta x = \Delta y = \frac{1}{32}$ *are presented in Figure 11. The results are comparable with the results in [36, 37]*

Example 5.13. *Consider the two-dimensional astrophysical jets problems with very high Mach number. We set the domain* $\Omega = [0, 0.5] \times [0, 0.25]$ *with initial condition* $\rho_0(x,y) = 0.5, u_0(x,y) = v_0(x,y) = 0, p_0(x,y) = 0.4127$. *The boundary conditions of the right and top are outflow; the bottom boundary is reflexive; the left boundary is inflow with* $(\rho, u, v, p) = (5, 800, 0, 0.4127)$ *if* $0 \le y \le 0.05$, *which corresponds to a jet flow of Mach number* 2000, *while* $(\rho, u, v, p) = (0.5, 0, 0, 0.4127)$ *otherwise. The ratio of specific heat is* $\gamma = 5/3$.

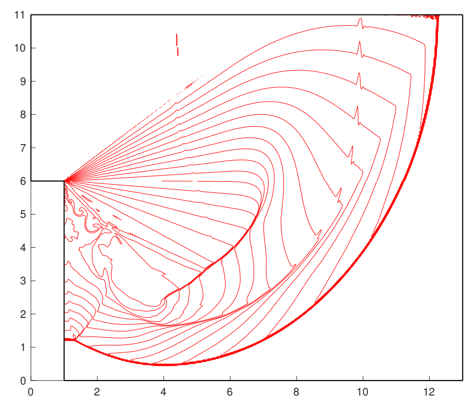
A combination of the total variation bounded limiter [8] and the flux limiter [41] are used before applying the positivity-preserving limiter in each time stage to reduce the spurious oscillations where the density and pressure are far above zero. We would like to note that, the positivity of density and pressure are preserved during simulation if only the positivity-preserving limiter is used, however, the simulation blows up very soon without the positivity-preserving limiter. We compute the solution on $N_x \times N_y = 320 \times 160$ *grid, and show the density and pressure at* $T = 5 \times 10^{-4}$ *in the Figure 5.13.*

6 Concluding remarks

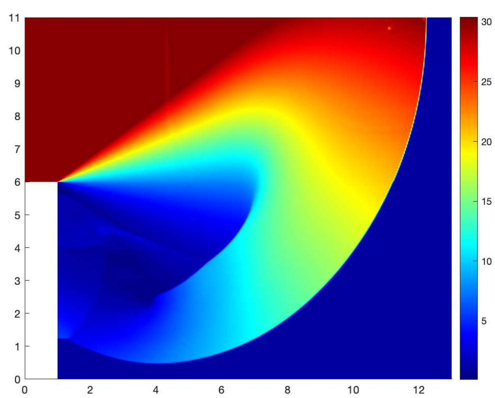
In this paper, we have proposed the third order maximum-principle-satisfying and positivity-preserving discontinuous Galerkin methods for scalar conservation laws and the Euler equations, respectively, based on the Lax-Wendroff time discretization. The main contribution of the paper is to prove rigorously that, under suitable CFL conditions, the cell average of the unmodulated LWDG scheme at the next time step is



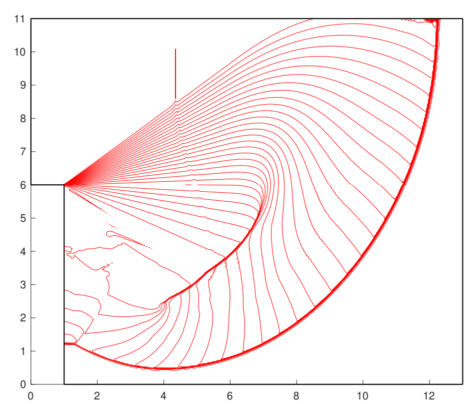
(a) Density in Ω



(b) 20 equally spaced contour lines from 0.066227 to 7.0668 for density

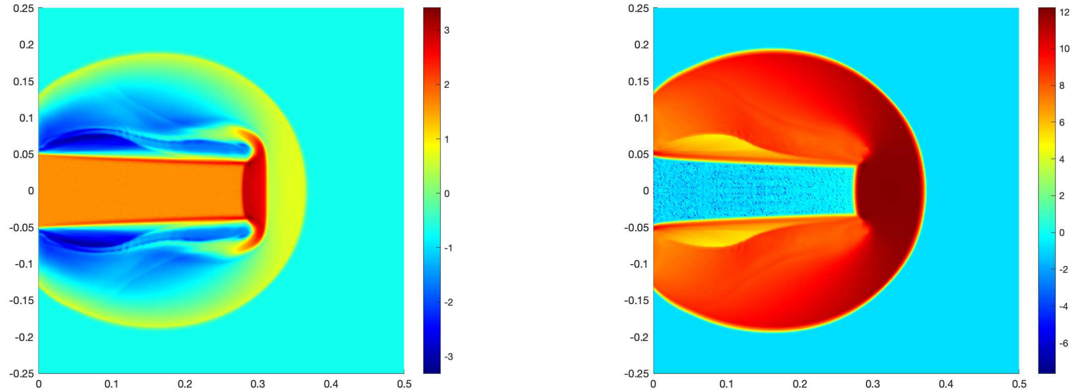


(c) Pressure in Ω



(d) 40 equally spaced contour lines from 0.091 to 37 for pressure

Figure 11: Results of Example 5.12 at $T = 2.3$.



(a) Density with log scale, lower part flipped from the upper part

(b) Pressure with log scale, lower part flipped from the upper part

Figure 12: Results of Example 5.13 at $T = 5 \times 10^{-4}$.

bounded, provided the solution stay in the desired bounds at the current time step. The scaling limiters, which were proved not to affect the high order accuracy and mass conservation, can then be used to enforce the bounds for the whole solution at the next time step, hence closing the loop of the bound-preserving LWDG algorithm.

In spite of the tedious CFL conditions derived for bound-preserving, in practice, one can use standard CFL conditions in computation, and rewind the computation back to the beginning of the step with halved time step-size when the cell averages exceeds their desired bounds at that step. The theoretical results in the paper guarantee that one only needs to halve the step-size finite number of times. We would also want to emphasize that, much of the heavy algebra in the paper is only for proof. It is not needed when implementing the algorithm.

Several possible extensions could be made in future works. For instance, it is of great importance to extend the algorithm to schemes with accuracy higher than third order. It is also meaningful to extend the algorithm from structured grids to unstructured meshes for geometry flexibility. The 3D case of the algorithm will also be studied in the future.

References

[1] S. Busto, S. Chiocchetti, M. Dumbser, E. Gaburro and I. Peshkov, High order ADER schemes for continuum mechanics, Frontiers in Physics, 8, 2020, 32.

[2] Z. Chen, H. Huang and J. Yan, Third order maximum-principle-satisfying direct discontinuous Galerkin methods for time dependent convection diffusion equations on unstructured triangular meshes, Journal of Computational Physics, 308, 2016, 198-217.

[3] B. Cockburn, S. Hou and C.-W. Shu, The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: The multidimensional case, Mathematics of Computation, 54, 1990, 545-581.

[4] B. Cockburn, S.-Y. Lin and C.-W. Shu, TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one-dimensional systems, Journal of Computational Physics, 84, 1989, 90-113.

[5] B. Cockburn, M. Luskin, C.-W. Shu and E. Süli, Enhanced accuracy by post-processing for finite element methods for hyperbolic equations, Mathematics of Computation, 72, 2003, 577-606.

[6] B. Cockburn and C.-W. Shu, TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II: General framework, Mathematics of computation, 52, 1989, 411-435.

[7] B. Cockburn and C.-W. Shu, The Runge-Kutta local projection-discontinuous-Galerkin finite element method for scalar conservation laws, ESAIM: Mathematical Modelling and Numerical Analysis, 25, 1991, 337-361.

[8] B. Cockburn and C.-W. Shu, The Runge–Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems, Journal of Computational Physics, 141, 1998, 199-224.

[9] M. Dumbser, C. Enaux and E.F. Toro, Finite volume schemes of very high order of accuracy for stiff hyperbolic balance laws, Journal of Computational Physics, 227, 2008, 3971-4001.

[10] M. Dumbser, F. Fambri, M. Tavelli, M. Bader and T. Weinzierl, Efficient implementation of ADER discontinuous Galerkin schemes for a scalable hyperbolic PDE engine, Axioms, 7, 2018, 63.

[11] M. Dumbser and M. Käser, Arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems, Journal of Computational Physics, 221, 2007, 693-723.

[12] M. Dumbser and C.D. Munz, Building blocks for arbitrary high order discontinuous Galerkin schemes, Journal of Scientific Computing, 27, 2006, 215-230.

[13] S. Gottlieb, D.I. Ketcheson and C.-W. Shu, High order strong stability preserving time discretizations, Journal of Scientific Computing, 38, 2009, 251-289.

[14] S. Gottlieb, C.-W. Shu and E. Tadmor, Strong stability-preserving high-order time discretization methods, SIAM review, 43, 2001, 89-112.

[15] W. Guo, J.-M. Qiu and J. Qiu, A new Lax-Wendroff discontinuous Galerkin method with superconvergence, Journal of Scientific Computing, 65, 2015, 299-326.

[16] D. Ling, J. Cheng and C.-W. Shu, Conservative high order positivity-preserving discontinuous Galerkin methods for linear hyperbolic and radiative transfer equations, Journal of Scientific Computing, 77, 2018, 1801-1831.

[17] H. Li, S. Xie and X. Zhang, A high order accurate bound-preserving compact finite difference scheme for scalar convection diffusion equations, SIAM Journal on Numerical Analysis, 56, 2018, 3308-3345.

[18] J. Li and Z. Du, A two-stage fourth order time-accurate discretization for Lax-Wendroff type flow solvers I. Hyperbolic conservation laws, SIAM Journal on Scientific Computing, 38, 2016, A3046-A3069.

[19] M. Li, P. Guyenne, F. Li and L. Xu, A positivity-preserving well-balanced central discontinuous Galerkin method for the nonlinear shallow water equations, Journal of Scientific Computing, 71, 2017, 994-1034.

[20] H. Liu and J. Yan, The direct discontinuous Galerkin (DDG) methods for diffusion problems, SIAM Journal on Numerical Analysis, 47, 2009, 675-698.

[21] S.A. Moe, J.A. Rossmanith and D.C. Seal, Positivity-preserving discontinuous Galerkin methods with Lax-Wendroff time discretizations, Journal of Scientific Computing, 71, 2017, 44-70.

[22] L. Pan, K. Xu, Q. Li and J. Li, An efficient and accurate two-stage fourth-order gas-kinetic scheme for the Euler and Navier–Stokes equations, Journal of Computational Physics, 326, 2016, 197-221.

[23] T. Qin and C.-W. Shu, Implicit positivity-preserving high-order discontinuous Galerkin methods for conservation laws, SIAM Journal on Scientific Computing, 40, 2018, A81-A107.

[24] J. Qiu, A numerical comparison of the Lax-Wendroff discontinuous Galerkin method based on different numerical fluxes, Journal of Scientific Computing, 30, 2007, 345-367.

[25] J. Qiu, Hermite WENO schemes with Lax-Wendroff type time discretizations for Hamilton-Jacobi equations, Journal of Computational Mathematics, 2007, 131-144.

[26] J. Qiu, M. Dumbser and C.-W. Shu, The discontinuous Galerkin method with Lax-Wendroff type time discretizations, Computer methods in applied mechanics and engineering, 194, 2005, 4528-4543.

[27] J. Qiu and C.-W. Shu, Finite Difference WENO Schemes with Lax-Wendroff-Type Time Discretizations, SIAM Journal on Scientific Computing, 24, 2003, 2185-2198.

[28] W.H. Reed and T.R. Hill, Triangular mesh methods for the neutron transport equation, Los Alamos Scientific Laboratory, 1973.

[29] D.C. Seal, Q. Tang, Z. Xu and A.J. Christlieb, An explicit high-order single-stage single-step positivity-preserving finite difference WENO method for the compressible Euler equations, Journal of Scientific Computing, 68, 2016, 171-190.

[30] C.-W. Shu, Total-variation-diminishing time discretizations, SIAM Journal on Scientific and Statistical Computing, 9, 1988, 1073-1084.

[31] V.A. Titarev and E.F. Toro, ADER: Arbitrary high order Godunov approach, Journal of Scientific Computing, 17, 2002, 609-618.

[32] C. Wang, X. Zhang, C.-W. Shu and J. Ning, Robust high order discontinuous Galerkin schemes for two-dimensional gaseous detonations, Journal of Computational Physics, 231, 2012, 653-665.

[33] Y. Xing and X. Zhang, Positivity-preserving well-balanced discontinuous Galerkin methods for the shallow water equations on unstructured triangular meshes, Journal of Scientific Computing, 57, 2013, 19-41.

[34] Y. Xing, X. Zhang and C.-W. Shu, Positivity-preserving high order well-balanced discontinuous Galerkin methods for the shallow water equations, Advances in Water Resources, 33, 2010, 1476-1493.

[35] Y. Yang, D. Wei and C.-W. Shu, Discontinuous Galerkin method for Krause's consensus models and pressureless Euler equations, Journal of Computational Physics, 252, 2013, 109-127.

[36] X. Zhang, On positivity-preserving high order discontinuous Galerkin schemes for compressible Navier–Stokes equations, Journal of Computational Physics, 328, 2017, 301-343.

[37] X. Zhang and C.-W. Shu, On positivity-preserving high order discontinuous Galerkin schemes for compressible Euler equations on rectangular meshes, Journal of Computational Physics, 229, 2010, 8918-8934.

[38] X. Zhang and C.-W. Shu, On maximum-principle-satisfying high order schemes for scalar conservation laws, Journal of Computational Physics, 229, 2010, 3091-3120.

[39] X. Zhang and C.-W. Shu, Positivity-preserving high order discontinuous Galerkin schemes for compressible Euler equations with source terms, Journal of Computational Physics, 230, 2011, 1238-1248.

[40] Y. Zhang, X. Zhang and C.-W. Shu, Maximum-principle-satisfying second order discontinuous Galerkin schemes for convection–diffusion equations on triangular meshes, Journal of Computational Physics, 234, 2013, 295-316.

[41] F. Zheng, C.-W. Shu and J. Qiu, A high order conservative finite difference scheme for compressible two-medium flows, Journal of Computational Physics, 445, 2021, 110597.

[42] X. Zhao, Y. Yang and C.E. Seyler, A positivity-preserving semi-implicit discontinuous Galerkin scheme for solving extended magnetohydrodynamics equations, Journal of Computational Physics, 278, 2014, 400-415.

Appendices

A Skipped details of CFL conditions and proofs of bound-preserving for the scalar conservation law and Euler equations

A.1 Constants in the CFL condition (2.25)

Denote

$M_1^f = \max_{m\le u\le M} |f'(u)|$,

$M_2^f = \max_{m\le u\le M} |f''(u)|$,

$M_1^g = \max_{m\le u\le M} |g'(u)|$,

$M_2^g = \max_{m\le u\le M} |g''(u)|$,

then the constants Q_1 and Q_2 in the CFL condition (2.25) are defined as:

$Q_1 = \min\{q_1^1, q_2^1, \dots, q_6^1\}$, where

$q_1^1 = \frac{1}{8M_1^f} \min_\gamma \hat{\omega}_\gamma$,

$q_2^1 = \frac{1}{4}\frac{4\beta_1-\frac{1}{2}}{5(M-m)M_2^f+\frac{4}{3}M_1^f}$,

$q_3^1 = \frac{1}{4}\frac{2-8\beta_1}{20(M-m)M_2^f+\frac{8}{3}M_1^f}$,

$q_4^1 = \frac{1}{4}\frac{\beta_0-\frac{3}{2}+4\beta_1}{15(M-m)M_2^f+\frac{4}{3}M_1^f}$,

$q_5^1 = \frac{1}{4}\frac{\hat{\omega}_1^{1/2}}{M_1^f(\beta_0-1+4\beta_1)^{1/2}}$,

$q_6^1 = \frac{1}{4}\frac{\hat{\omega}_{N_q}^{1/2}}{M_1^f(6-24\beta_1)^{1/2}}$,

$Q_2 = \min\{q_1^2, q_2^2, \dots, q_6^2\}$, where

$q_1^2 = \frac{1}{8M_1^g} \min_\gamma \hat{\omega}_\gamma$,

$q_2^2 = \frac{1}{4}\frac{4\beta_1-\frac{1}{2}}{5(M-m)M_2^g+\frac{4}{3}M_1^g}$,

$q_3^2 = \frac{1}{4}\frac{2-8\beta_1}{20(M-m)M_2^g+\frac{8}{3}M_1^g}$,

$q_4^2 = \frac{1}{4}\frac{\beta_0-\frac{3}{2}+4\beta_1}{15(M-m)M_2^g+\frac{4}{3}M_1^g}$,

$q_5^2 = \frac{1}{4}\frac{\hat{\omega}_1^{1/2}}{M_1^g(\beta_0-1+4\beta_1)^{1/2}}$,

$q_6^2 = \frac{1}{4}\frac{\hat{\omega}_{N_q}^{1/2}}{M_1^g(6-24\beta_1)^{1/2}}$,

Define

$c_1 = M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m){M_1^g}^2 M_2^f + 2M_1^f {M_1^g}^2)$,

$c_2 = M_1^f M_1^g + Q_1(10(M-m)M_1^f M_1^g M_2^f + 5(M-m){M_1^f}^2 M_2^g + 2M_1^g {M_1^f}^2)$, then Q_3 and Q_4 in (2.25) are defined as:

$Q_3 = \min\{q_1^3, q_2^3, q_3^3, q_4^3\}$, where

$q_1^3 = \frac{\hat{\omega}_1^2}{2\hat{\omega}_1\alpha_x^1 + 4Q_2c_1}$,

$q_2^3 = \frac{\hat{\omega}_1\hat{\omega}_{N_q}}{2\hat{\omega}_{N_q}\alpha_x^1 + 4Q_2c_1}$,

$q_3^3 = \frac{\hat{\omega}_1\alpha_y^1}{2c_2}$,

$q_4^3 = \frac{\hat{\omega}_{N_q}\alpha_y^1}{2c_2}$,

$Q_4 = \min\{q_1^4, q_2^4, q_3^4, q_4^4\}$, where

$q_1^4 = \frac{\hat{\omega}_1^2}{2\hat{\omega}_1\alpha_y^1 + 4Q_1c_2}$,

$q_2^4 = \frac{\hat{\omega}_1\hat{\omega}_{N_q}}{2\hat{\omega}_{N_q}\alpha_y^1 + 4Q_1c_2}$,

$q_3^4 = \frac{\hat{\omega}_1\alpha_x^1}{2c_1}$,

$q_4^4 = \frac{\hat{\omega}_{N_q}\alpha_x^1}{2c_1}$.

A.2 Coefficients in the expansion (2.26)

For convenience, we introduce the constants

$$d_1^\gamma = 2L'_{-1}(\hat{r}_\gamma), \quad d_2^\gamma = 2L'_0(\hat{r}_\gamma), \quad d_3^\gamma = 2L'_1(\hat{r}_\gamma), \quad \gamma = 1, 2, \ldots, 2N_q - 1,$$

where L_{-1}, L_0, L_1 are the Lagrange basis in (2.11) and $\{\hat{r}_\gamma, \gamma = 1, \ldots, 2N_q - 1\}$ are the Gauss-Lobatto points on $[-1, 1]$. It is clear that $|d_i^\gamma| \le 4$, for $i = 1, 2, 3, \gamma = 1, 2, \ldots, 2N_q - 1$.

The coefficients $z_1, \ldots, z_{14,\beta}$ in the expansion (2.26) are defined as follows.

$$z_1 = \lambda_x \left(\frac{1}{2}\hat{\omega}_1\alpha_x^1 + \lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(-\frac{1}{4}f'g'd_1^\beta + \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_1^\beta + \lambda_y f'g'^2 \right) (x_{i-\frac{1}{2}}^-, \hat{y}_\beta) \right)$$

$$z_2 = \lambda_x \left(\frac{1}{2}\hat{\omega}_{N_q}\alpha_x^1 + \lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(-\frac{1}{4}f'g'd_2^\beta + \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_2^\beta - 2\lambda_y f'g'^2 \right) (x_{i-\frac{1}{2}}^-, \hat{y}_\beta) \right)$$

$$z_3 = \lambda_x \left(\frac{1}{2}\hat{\omega}_{2N_q-1}\alpha_x^1 + \lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(-\frac{1}{4}f'g'd_3^\beta + \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_3^\beta + \lambda_y f'g'^2 \right) (x_{i-\frac{1}{2}}^-, \hat{y}_\beta) \right)$$

$$z_4 = \frac{1}{4}\hat{\omega}_1^2 - \frac{1}{2}\lambda_x\hat{\omega}_1\alpha_x^1 + \lambda_x\lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(-\frac{1}{4}f'g'd_1^\beta + \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_1^\beta + \lambda_y f'g'^2 \right) (x_{i-\frac{1}{2}}^+, \hat{y}_\beta)$$

$$z_5 = \frac{1}{4}\hat{\omega}_1\hat{\omega}_{N_q} - \frac{1}{2}\lambda_x\hat{\omega}_{N_q}\alpha_x^1 + \lambda_x\lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(-\frac{1}{4}f'g'd_2^\beta + \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_2^\beta - 2\lambda_y f'g'^2 \right) (x_{i-\frac{1}{2}}^+, \hat{y}_\beta)$$

$$z_6 = \frac{1}{4}\hat{\omega}_1\hat{\omega}_{2N_q-1} - \frac{1}{2}\lambda_x\hat{\omega}_{2N_q-1}\alpha_x^1 + \lambda_x\lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(-\frac{1}{4}f'g'd_3^\beta + \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_3^\beta + \lambda_y f'g'^2\right)(x_{i-\frac{1}{2}}^+, \hat{y}_\beta)$$

$$z_7 = \frac{1}{4}\hat{\omega}_1\hat{\omega}_{2N_q-1} - \frac{1}{2}\lambda_x\hat{\omega}_1\alpha_x^1 + \lambda_x\lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(\frac{1}{4}f'g'd_1^\beta - \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_1^\beta - \lambda_y f'g'^2\right)(x_{i+\frac{1}{2}}^-, \hat{y}_\beta)$$

$$z_8 = \frac{1}{4}\hat{\omega}_{N_q}\hat{\omega}_{2N_q-1} - \frac{1}{2}\lambda_x\hat{\omega}_{N_q}\alpha_x^1 + \lambda_x\lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(\frac{1}{4}f'g'd_2^\beta - \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_2^\beta + 2\lambda_y f'g'^2\right)(x_{i+\frac{1}{2}}^-, \hat{y}_\beta)$$

$$z_9 = \frac{1}{4}\hat{\omega}_{2N_q-1}\hat{\omega}_{2N_q-1} - \frac{1}{2}\lambda_x\hat{\omega}_{2N_q-1}\alpha_x^1 + \lambda_x\lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(\frac{1}{4}f'g'd_3^\beta - \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_3^\beta - \lambda_y f'g'^2\right)(x_{i+\frac{1}{2}}^-, \hat{y}_\beta)$$

$$z_{10} = \lambda_x \left(\frac{1}{2}\hat{\omega}_1\alpha_x^1 + \lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(\frac{1}{4}f'g'd_1^\beta - \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_1^\beta - \lambda_y f'g'^2\right)(x_{i+\frac{1}{2}}^+, \hat{y}_\beta)\right)$$

$$z_{11} = \lambda_x \left(\frac{1}{2}\hat{\omega}_{N_q}\alpha_x^1 + \lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(\frac{1}{4}f'g'd_2^\beta - \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_2^\beta + 2\lambda_y f'g'^2\right)(x_{i+\frac{1}{2}}^+, \hat{y}_\beta)\right)$$

$$z_{12} = \lambda_x \left(\frac{1}{2}\hat{\omega}_{2N_q-1}\alpha_x^1 + \lambda_y \sum_{\beta=1}^{2N_q-1} \hat{\omega}_\beta \left(\frac{1}{4}f'g'd_3^\beta - \frac{\Delta t}{12}(6f'g'g''u_y + 3g'^2f''u_y)d_3^\beta - \lambda_y f'g'^2\right)(x_{i+\frac{1}{2}}^+, \hat{y}_\beta)\right),$$

$$z_{13,\beta} = \frac{1}{4}\hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\alpha_x^1$$

$$z_{14,\beta} = \frac{1}{4}\hat{\omega}_1 - \frac{\lambda_x}{2}\alpha_x^1$$

Moreover, we have the following lower bound estimates for $z_1, \ldots, z_{14,\beta}$ under the CFL condition (2.25).

$$z_1 \geq \lambda_x \left(\frac{1}{2}\omega_1\alpha_x^1 - \lambda_y \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + Q_2 M_1^f M_1^{g2}\right)\right) \geq 0$$

$$z_2 \geq \lambda_x \left(\frac{1}{2}\omega_{N_q}\alpha_x^1 - \lambda_y \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + 2Q_2 M_1^f M_1^{g2}\right)\right) \geq 0$$

$$z_3 \geq \lambda_x \left(\frac{1}{2}\omega_{2N_q-1}\alpha_x^1 - \lambda_y \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + Q_2 M_1^f M_1^{g2}\right)\right) \geq 0$$

$$z_4 \geq \frac{1}{4}\omega_1^2 - \frac{1}{2}\lambda_x\omega_1\alpha_x^1 - \lambda_x Q_2 \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + Q_2 M_1^f M_1^{g2}\right) \geq 0$$

$$z_5 \geq \frac{1}{4}\omega_1\omega_{N_q} - \frac{1}{2}\lambda_x\omega_{N_q}\alpha_x^1 - \lambda_x Q_2 \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + 2Q_2 M_1^f M_1^{g2}\right) \geq 0$$

$$z_6 \geq \frac{1}{4}\omega_1\omega_{2N_q-1} - \frac{1}{2}\lambda_x\omega_{2N_q-1}\alpha_x^1 - \lambda_x Q_2 \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + Q_2 M_1^f M_1^{g2}\right) \geq 0$$

$$z_7 \geq \frac{1}{4}\omega_1\omega_{2N_q-1} - \frac{1}{2}\lambda_x\omega_1\alpha_x^1 - \lambda_x Q_2 \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + Q_2 M_1^f M_1^{g2}\right) \geq 0$$

$$z_8 \geq \frac{1}{4}\omega_{N_q}\omega_{2N_q-1} - \frac{1}{2}\lambda_x\omega_{N_q}\alpha_x^1 - \lambda_x Q_2 \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + 2Q_2 M_1^f M_1^{g2}\right) \geq 0$$

$$z_9 \geq \frac{1}{4}\omega_{2N_q-1}^2 - \frac{1}{2}\lambda_x\omega_{2N_q-1}\alpha_x^1 - \lambda_x Q_2 \left(M_1^f M_1^g + Q_2(10(M-m)M_1^f M_1^g M_2^g + 5(M-m)M_1^{g2}M_2^f) + Q_2 M_1^f M_1^{g2}\right) \geq 0$$

$$z_{10} \geq \lambda_x \left(\frac{1}{2}\omega_1 \alpha_x^1 - \lambda_y \left(M_1^f M_1^g + Q_2(10(M-m) M_1^f M_1^g M_2^g + 5(M-m) {M_1^g}^2 M_2^f) + Q_2 M_1^f {M_1^g}^2 \right) \right) \geq 0$$

$$z_{11} \geq \lambda_x \left(\frac{1}{2}\omega_{N_q} \alpha_x^1 - \lambda_y \left(M_1^f M_1^g + Q_2(10(M-m) M_1^f M_1^g M_2^g + 5(M-m) {M_1^g}^2 M_2^f) + 2Q_2 M_1^f {M_1^g}^2 \right) \right) \geq 0$$

$$z_{12} \geq \lambda_x \left(\frac{1}{2}\omega_{2N_q-1} \alpha_x^1 - \lambda_y \left(M_1^f M_1^g + Q_2(10(M-m) M_1^f M_1^g M_2^g + 5(M-m) {M_1^g}^2 M_2^f) + Q_2 M_1^f {M_1^g}^2 \right) \right) \geq 0$$

and $z_{13,\beta}, z_{14,\beta} \geq 0, \forall \beta$.

A.3 Coefficients in the expansion (3.11)

The coefficients of the expansion (3.11) are

$$z_{10} = \lambda^2 \left(\frac{\hat{\gamma}}{2}(4\beta_1 - \frac{1}{2}) + \frac{\Delta t}{12}\hat{\gamma}(3+\gamma)\,(u_x)^-_{j-\frac{1}{2}} + \lambda\hat{\gamma}u^-_{j-\frac{1}{2}} \right) e^+_{j-\frac{3}{2}}$$

$$z_{11} = \lambda^2 \left(\frac{\hat{\gamma}}{2}(2 - 8\beta_1) - \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)\,(u_x)^-_{j-\frac{1}{2}} - 2\lambda\hat{\gamma}u^-_{j-\frac{1}{2}} \right) e_{j-1}$$

$$z_{12} = \lambda^2 \left(\frac{\hat{\gamma}}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) + \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma\,(u_{xx})^-_{j-\frac{1}{2}} + \frac{\Delta t}{4}\hat{\gamma}(3+\gamma)\,(u_x)^-_{j-\frac{1}{2}} + \lambda\hat{\gamma}u^-_{j-\frac{1}{2}} \right) e^-_{j-\frac{1}{2}}$$

$$\begin{aligned} z_{13} =& \frac{1}{4}\omega_1 - \lambda^2 \left(\frac{\hat{\gamma}}{2}(4\beta_1 - \frac{1}{2}) + \frac{\hat{\gamma}}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) + \frac{\Delta t}{12}\hat{\gamma}(3+\gamma)\,(u_x)^-_{j+\frac{1}{2}} \right. \\ & \left. - \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma\,(u_{xx})^+_{j-\frac{1}{2}} + \frac{\Delta t}{4}\hat{\gamma}(3+\gamma)\,(u_x)^+_{j-\frac{1}{2}} + \lambda\hat{\gamma}u^-_{j+\frac{1}{2}} - \lambda\hat{\gamma}u^+_{j-\frac{1}{2}} \right) e^+_{j-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} z_{14} =& \frac{1}{4}\omega_{N_q} - \lambda^2 \left(\frac{\hat{\gamma}}{2}(2 - 8\beta_1) + \frac{\hat{\gamma}}{2}(2 - 8\beta_1) - \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)\,(u_x)^+_{j-\frac{1}{2}} \right. \\ & \left. - \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)\,(u_x)^-_{j+\frac{1}{2}} - 2\lambda\hat{\gamma}u^-_{j+\frac{1}{2}} + 2\lambda\hat{\gamma}u^+_{j-\frac{1}{2}} \right) e_j \end{aligned}$$

$$\begin{aligned} z_{15} =& \frac{1}{4}\omega_{2N_q-1} - \lambda^2 \left(\frac{\hat{\gamma}}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) + \frac{\hat{\gamma}}{2}(4\beta_1 - \frac{1}{2}) + \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma\,(u_{xx})^-_{j+\frac{1}{2}} \right. \\ & \left. + \frac{\Delta t}{12}\hat{\gamma}(3+\gamma)\,(u_x)^+_{j-\frac{1}{2}} + \frac{\Delta t}{4}\hat{\gamma}(3+\gamma)\,(u_x)^-_{j+\frac{1}{2}} - \lambda\hat{\gamma}u^+_{j-\frac{1}{2}} + \lambda\hat{\gamma}u^-_{j+\frac{1}{2}} \right) e^-_{j+\frac{1}{2}} \end{aligned}$$

$$z_{16} = \lambda^2 \left(\frac{\hat{\gamma}}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma\,(u_{xx})^+_{j+\frac{1}{2}} + \frac{\Delta t}{4}\hat{\gamma}(3+\gamma)\,(u_x)^+_{j+\frac{1}{2}} - \lambda\hat{\gamma}u^+_{j+\frac{1}{2}} \right) e^+_{j+\frac{1}{2}}$$

$$z_{17} = \lambda^2 \left(\frac{\hat{\gamma}}{2}(2 - 8\beta_1) - \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)\,(u_x)^+_{j+\frac{1}{2}} + 2\lambda\hat{\gamma}u^+_{j+\frac{1}{2}} \right) e_{j+1}$$

$$z_{18} = \lambda^2 \left(\frac{\hat{\gamma}}{2}(4\beta_1 - \frac{1}{2}) + \frac{\Delta t}{12}\hat{\gamma}(3+\gamma)\,(u_x)^+_{j+\frac{1}{2}} - \lambda\hat{\gamma}u^+_{j+\frac{1}{2}} \right) e^-_{j+\frac{3}{2}}$$

Under the condition $\lambda \leq \min\{q_7, q_8, q_9, q_{10}, q_{11}\}$, we have the estimates as follows

$$z_{10} \geq \lambda^2 \left(\frac{\hat{\gamma}}{2}(4\beta_1 - \frac{1}{2}) - \frac{\Delta t}{12}\hat{\gamma}(3+\gamma)||u_x||_\infty - \lambda\hat{\gamma}||u||_\infty \right) e^+_{j-\frac{3}{2}} \geq 0,$$

$$z_{11} \geq \lambda^2 \left(\frac{\hat{\gamma}}{2}(2 - 8\beta_1) - \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)||u_x||_\infty - 2\lambda\hat{\gamma}||u||_\infty \right) e_{j-1} \geq 0,$$

$$z_{12} \geq \lambda^2 \left(\frac{\hat{\gamma}}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma||u_{xx}||_\infty - \frac{\Delta t}{4}\hat{\gamma}(3+\gamma)||u_x||_\infty - \lambda\hat{\gamma}||u||_\infty \right) e^-_{j-\frac{1}{2}} \geq 0,$$

$$z_{13} \geq \frac{1}{4}\omega_1 - \lambda^2 \left(\frac{\hat{\gamma}}{2}(\beta_0 - 2 + 8\beta_1) + \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma||u_{xx}||_\infty + \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)||u_x||_\infty + 2\lambda\hat{\gamma}||u||_\infty \right) ||e||_\infty \geq 0,$$

$$z_{14} \geq \frac{1}{4}\omega_{N_q} - \lambda^2 \left(\hat{\gamma}(2 - 8\beta_1) + \frac{2\Delta t}{3}\hat{\gamma}(3+\gamma)||u_x||_\infty + 4\lambda\hat{\gamma}||u||_\infty \right) ||e||_\infty \geq 0,$$

$$z_{15} \geq \frac{1}{4}\omega_{2N_q-1} - \lambda^2 \left(\frac{\hat{\gamma}}{2}(\beta_0 - 2 + 8\beta_1) + \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma||u_{xx}||_\infty + \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)||u_x||_\infty + 2\lambda\hat{\gamma}||u||_\infty \right) ||e||_\infty \geq 0,$$

$$z_{16} \geq \lambda^2 \left(\frac{\hat{\gamma}}{2}(\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\Delta t^2}{12\lambda}\hat{\gamma}\gamma||u_{xx}||_\infty - \frac{\Delta t}{4}\hat{\gamma}(3+\gamma)||u_x||_\infty - \lambda\hat{\gamma}||u||_\infty \right) e^+_{j+\frac{1}{2}} \geq 0,$$

$$z_{17} \geq \lambda^2 \left(\frac{\hat{\gamma}}{2}(2 - 8\beta_1) - \frac{\Delta t}{3}\hat{\gamma}(3+\gamma)||u_x||_\infty - 2\lambda\hat{\gamma}||u||_\infty \right) e_{j+1} \geq 0,$$

$$z_{18} \geq \lambda^2 \left(\frac{\hat{\gamma}}{2}(4\beta_1 - \frac{1}{2}) - \frac{\Delta t}{12}\hat{\gamma}(3+\gamma)||u_x||_\infty - \lambda\hat{\gamma}||u||_\infty \right) e^-_{j+\frac{3}{2}} \geq 0,$$

A.4 Constants in the CFL condition (3.25)

$Q_1 = \min\{q^1_1, q^1_2, \ldots, q^1_{11}\}$, where

$q^1_1 = \frac{\hat{\omega}_1}{8||(|u|+c)||_\infty}$,

$q^1_2 = \frac{1}{4}\frac{6(\beta_0 - \frac{3}{2} + 4\beta_1)}{\Delta x^2||u_{xx}||_\infty + 6\Delta x||u_x||_\infty + 4||u||_\infty}$,

$q^1_3 = \frac{1}{4}\frac{3(2-8\beta_1)}{4(\Delta x||u_x||_\infty + ||u||_\infty)}$,

$q^1_4 = \frac{1}{4}\frac{3(4\beta_1 - \frac{1}{2})}{\Delta x||u_x||_\infty + 2||u||_\infty}$,

$q^1_5 = \frac{1}{8||u||_\infty}\left(\frac{\omega_1}{\beta_0 - 2 + 8\beta_1}\right)^{\frac{1}{2}}$,

$q^1_6 = \frac{1}{8||u||_\infty}\left(\frac{\omega_{N_q}}{2(2-8\beta_1)}\right)^{\frac{1}{2}}$,

$q^1_7 = \frac{1}{4}\frac{6(4\beta_1 - \frac{1}{2})}{(3+\gamma)\Delta x||u_x||_\infty + \hat{\gamma}\Delta x||v_y||_\infty + 12||u||_\infty}$,

$q^1_8 = \frac{1}{4}\frac{3(2-8\beta_1)}{2(3+\gamma)\Delta x||u_x||_\infty + 2\hat{\gamma}\Delta x||v_y||_\infty + 12||u||_\infty}$,

$q^1_9 = \frac{1}{4}\frac{6(\beta_0 - \frac{3}{2} + 4\beta_1)}{\gamma\Delta x^2||u_{xx}||_\infty + 3(3+\gamma)\Delta x||u_x||_\infty + 3\hat{\gamma}\Delta x||v_y||_\infty + 12||u||_\infty}$,

$q^1_{10} = \frac{1}{4}\left(\frac{\omega_1}{4\hat{\gamma}(\beta_0 - 2 + 8\beta_1)||e||_\infty}\right)^{\frac{1}{2}}$,

$q^1_{11} = \frac{1}{4}\left(\frac{\omega_{N_q}}{8\hat{\gamma}(2-8\beta_1)||e||_\infty}\right)^{\frac{1}{2}}$,

$Q_2 = \min\{q^2_1, q^2_2, \ldots, q^2_{11}\}$, where

$q^2_1 = \frac{\hat{\omega}_1}{8||(|v|+c)||_\infty}$,

$q^2_2 = \frac{1}{4}\frac{6(\beta_0 - \frac{3}{2} + 4\beta_1)}{\Delta y^2||v_{yy}||_\infty + 6\Delta y||v_y||_\infty + 4||v||_\infty}$,

$q^2_3 = \frac{1}{4}\frac{3(2-8\beta_1)}{4(\Delta y||v_y||_\infty + ||v||_\infty)}$,

$q_4^2 = \frac{1}{4}\frac{3(4\beta_1 - \frac{1}{2})}{\Delta y||v_y||_\infty + 2||v||_\infty}$,

$q_5^2 = \frac{1}{8||v||_\infty}\left(\frac{\omega_1}{\beta_0 - 2 + 8\beta_1}\right)^{\frac{1}{2}}$,

$q_6^2 = \frac{1}{8||v||_\infty}\left(\frac{\omega_{N_q}}{2(2 - 8\beta_1)}\right)^{\frac{1}{2}}$,

$q_7^2 = \frac{1}{4}\frac{6(4\beta_1 - \frac{1}{2})}{(3+\gamma)\Delta y||v_y||_\infty + \hat{\gamma}\Delta y||u_x||_\infty + 12||v||_\infty}$,

$q_8^2 = \frac{1}{4}\frac{3(2 - 8\beta_1)}{2(3+\gamma)\Delta y||v_y||_\infty + 2\hat{\gamma}\Delta y||u_x||_\infty + 12||v||_\infty}$,

$q_9^2 = \frac{1}{4}\frac{6(\beta_0 - \frac{3}{2} + 4\beta_1)}{\gamma\Delta y^2||v_{yy}||_\infty + 3(3+\gamma)\Delta y||v_y||_\infty + 3\hat{\gamma}\Delta y||u_x||_\infty + 12||v||_\infty}$,

$q_{10}^2 = \frac{1}{4}\left(\frac{\omega_1}{4\hat{\gamma}(\beta_0 - 2 + 8\beta_1)||e||_\infty}\right)^{\frac{1}{2}}$,

$q_{11}^2 = \frac{1}{4}\left(\frac{\omega_{N_q}}{8\hat{\gamma}(2 - 8\beta_1)||e||_\infty}\right)^{\frac{1}{2}}$,

Let

$$c_1 = 3\hat{\omega}_1\Delta x||(v_x u + v u_x)||_\infty + \hat{\omega}_1 Q_1\Delta x^2||A_4||_\infty + 12\left(||uv||_\infty + \frac{Q_1}{3}\Delta x||A_5||_\infty + \frac{Q_1}{3}||A_6||_\infty\right)$$

$$c_1' = 3\hat{\omega}_1\Delta y||(u_y v + u v_y)||_\infty + \hat{\omega}_1 Q_2\Delta y^2||A_1||_\infty + 12\left(||uv||_\infty + \frac{Q_2}{3}\Delta y||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty\right)$$

$$c_2 = 3\hat{\omega}_{N_q}\Delta x||(v_x u + v u_x)||_\infty + \hat{\omega}_{N_q} Q_1\Delta x^2||A_4||_\infty + 12\left(||uv||_\infty + \frac{Q_1}{3}\Delta x||A_5||_\infty + \frac{2Q_1}{3}||A_6||_\infty\right)$$

$$c_2' = 3\hat{\omega}_{N_q}\Delta y||(u_y v + u v_y)||_\infty + \hat{\omega}_{N_q} Q_2\Delta y^2||A_1||_\infty + 12\left(||uv||_\infty + \frac{Q_2}{3}\Delta y||A_2||_\infty + \frac{2Q_2}{3}||A_3||_\infty\right)$$

$$c_3 = 6\hat{\omega}_1\alpha_x^1 + 3\hat{\omega}_1 Q_2\Delta y||(u_y v + u v_y)||_\infty + \hat{\omega}_1 Q_2^2\Delta y^2||A_1||_\infty + 12Q_2\left(||uv||_\infty + \frac{Q_2}{3}\Delta y||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty\right)$$

$$c_3' = 6\hat{\omega}_1\alpha_y^1 + 3\hat{\omega}_1 Q_1\Delta x||(v_x u + v u_x)||_\infty + \hat{\omega}_1 Q_1^2\Delta x^2||A_4||_\infty + 12Q_1\left(||uv||_\infty + \frac{Q_1}{3}\Delta y||A_5||_\infty + \frac{Q_1}{3}||A_6||_\infty\right)$$

$$c_4 = 6\hat{\omega}_{N_q}\alpha_x^1 + 3\hat{\omega}_{N_q} Q_2\Delta y||(u_y v + u v_y)||_\infty + \hat{\omega}_{N_q} Q_2^2\Delta y^2||A_1||_\infty + 12Q_2\left(||uv||_\infty + \frac{Q_2}{3}\Delta y||A_2||_\infty + \frac{2Q_2}{3}||A_3||_\infty\right)$$

$$c_4' = 6\hat{\omega}_{N_q}\alpha_y^1 + 3\hat{\omega}_{N_q} Q_1\Delta x||(v_x u + v u_x)||_\infty + \hat{\omega}_{N_q} Q_1^2\Delta x^2||A_4||_\infty + 12Q_1\left(||uv||_\infty + \frac{Q_1}{3}\Delta x||A_5||_\infty + \frac{2Q_1}{3}||A_6||_\infty\right)$$

then

$Q_3 = \min\{q_1^3, q_2^3, q_3^3, q_4^3\}$, where

$q_1^3 = \frac{6\hat{\omega}_1\alpha_y^1}{c_1}$,

$q_2^3 = \frac{6\hat{\omega}_{N_q}\alpha_y^1}{c_2}$,

$q_3^3 = \frac{3\hat{\omega}_1^2}{c_3}$,

$q_4^3 = \frac{3\hat{\omega}_1\hat{\omega}_{N_q}}{c_4}$,

$Q_4 = \min\{q_1^4, q_2^4, q_3^4, q_4^4\}$, where

$q_1^4 = \frac{6\hat{\omega}_1\alpha_x^1}{c_1'}$,

$q_2^4 = \frac{6\hat{\omega}_{N_q}\alpha_x^1}{c_2'}$,

$q_3^4 = \frac{3\hat{\omega}_1^2}{c_3'}$,

$q_4^4 = \frac{3\hat{\omega}_1\hat{\omega}_{N_q}}{c_4'}$.

A.5 Coefficients in the expansion (3.27)

The coefficients $z_1, \dots, z_{16,\beta}$ in the expansion (3.27) are defined as follows.

$$z_1 = \lambda_x \left(\frac{1}{2}\hat{\omega}_1 \alpha_x^1 - \hat{\omega}_1 \frac{\lambda_y}{4}\Delta y(u_y v + u v_y)(x^-_{i-\frac{1}{2}}, y^+_{j-\frac{1}{2}}) + \hat{\omega}_1 \frac{\lambda_y^2}{12}\Delta y^2 A_1(x^-_{i-\frac{1}{2}}, y^+_{j-\frac{1}{2}}) \right.$$
$$\left. + \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(-\frac{1}{4} d_1^\gamma uv + \frac{\lambda_y}{12} d_1^\gamma \Delta y A_2 + \frac{\lambda_y}{3} A_3 \right)(x^-_{i-\frac{1}{2}}, \hat{y}_\gamma) \right)$$

$$z_2 = \lambda_x \left(\frac{1}{2}\hat{\omega}_{N_q} \alpha_x^1 - \hat{\omega}_{N_q} \frac{\lambda_y}{4}\Delta y(u_y v + u v_y)(x^-_{i-\frac{1}{2}}, y_j) + \hat{\omega}_{N_q} \frac{\lambda_y^2}{12}\Delta y^2 A_1(x^-_{i-\frac{1}{2}}, y_j) \right.$$
$$\left. + \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(-\frac{1}{4} d_2^\gamma uv + \frac{\lambda_y}{12} d_2^\gamma \Delta y A_2 - \frac{2\lambda_y}{3} A_3 \right)(x^-_{i-\frac{1}{2}}, \hat{y}_\gamma) \right)$$

$$z_3 = \lambda_x \left(\frac{1}{2}\hat{\omega}_{2N_q-1} \alpha_x^1 - \hat{\omega}_{2N_q-1} \frac{\lambda_y}{4}\Delta y(u_y v + u v_y)(x^-_{i-\frac{1}{2}}, y^-_{j+\frac{1}{2}}) + \hat{\omega}_{2N_q-1} \frac{\lambda_y^2}{12}\Delta y^2 A_1(x^-_{i-\frac{1}{2}}, y^-_{j+\frac{1}{2}}) \right.$$
$$\left. + \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(-\frac{1}{4} d_3^\gamma uv + \frac{\lambda_y}{12} d_3^\gamma \Delta y A_2 + \frac{\lambda_y}{3} A_3 \right)(x^-_{i-\frac{1}{2}}, \hat{y}_\gamma) \right)$$

$$z_4 = \frac{1}{4}\hat{\omega}_1^2 - \frac{\lambda_x}{2}\hat{\omega}_1 \alpha_x^1 - \hat{\omega}_1 \frac{\lambda_x \lambda_y}{4}\Delta y(u_y v + u v_y)(x^+_{i-\frac{1}{2}}, y^+_{j-\frac{1}{2}}) + \hat{\omega}_1 \frac{\lambda_x \lambda_y^2}{12}\Delta y^2 A_1(x^+_{i-\frac{1}{2}}, y^+_{j-\frac{1}{2}})$$
$$+ \lambda_x \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(-\frac{1}{4} d_1^\gamma uv + \frac{\lambda_y}{12} d_1^\gamma \Delta y A_2 + \frac{\lambda_y}{3} A_3 \right)(x^+_{i-\frac{1}{2}}, \hat{y}_\gamma)$$

$$z_5 = \frac{1}{4}\hat{\omega}_1 \hat{\omega}_{N_q} - \frac{\lambda_x}{2}\hat{\omega}_{N_q} \alpha_x^1 - \hat{\omega}_{N_q} \frac{\lambda_x \lambda_y}{4}\Delta y(u_y v + u v_y)(x^+_{i-\frac{1}{2}}, y_j) + \hat{\omega}_{N_q} \frac{\lambda_x \lambda_y^2}{12}\Delta y^2 A_1(x^+_{i-\frac{1}{2}}, y_j)$$
$$+ \lambda_x \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(-\frac{1}{4} d_2^\gamma uv + \frac{\lambda_y}{12} d_2^\gamma \Delta y A_2 - \frac{2\lambda_y}{3} A_3 \right)(x^+_{i-\frac{1}{2}}, \hat{y}_\gamma)$$

$$z_6 = \frac{1}{4}\hat{\omega}_1 \hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_{2N_q-1} \alpha_x^1 - \hat{\omega}_{2N_q-1} \frac{\lambda_x \lambda_y}{4}\Delta y(u_y v + u v_y)(x^+_{i-\frac{1}{2}}, y^-_{j+\frac{1}{2}}) + \hat{\omega}_{2N_q-1} \frac{\lambda_x \lambda_y^2}{12}\Delta y^2 A_1(x^+_{i-\frac{1}{2}}, y^-_{j+\frac{1}{2}})$$
$$+ \lambda_x \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(-\frac{1}{4} d_3^\gamma uv + \frac{\lambda_y}{12} d_3^\gamma \Delta y A_2 + \frac{\lambda_y}{3} A_3 \right)(x^+_{i-\frac{1}{2}}, \hat{y}_\gamma)$$

$$z_7 = \frac{1}{4}\hat{\omega}_1 \hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_1 \alpha_x^1 + \hat{\omega}_1 \frac{\lambda_x \lambda_y}{4}\Delta y(u_y v + u v_y)(x^-_{i+\frac{1}{2}}, y^+_{j-\frac{1}{2}}) - \hat{\omega}_1 \frac{\lambda_x \lambda_y^2}{12}\Delta y^2 A_1(x^-_{i+\frac{1}{2}}, y^+_{j-\frac{1}{2}})$$
$$+ \lambda_x \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(\frac{1}{4} d_1^\gamma uv - \frac{\lambda_y}{12} d_1^\gamma \Delta y A_2 - \frac{\lambda_y}{3} A_3 \right)(x^-_{i+\frac{1}{2}}, \hat{y}_\gamma)$$

$$z_8 = \frac{1}{4}\hat{\omega}_{N_q} \hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_{N_q} \alpha_x^1 + \hat{\omega}_{N_q} \frac{\lambda_x \lambda_y}{4}\Delta y(u_y v + u v_y)(x^-_{i+\frac{1}{2}}, y_j) - \hat{\omega}_{N_q} \frac{\lambda_x \lambda_y^2}{12}\Delta y^2 A_1(x^-_{i+\frac{1}{2}}, y_j)$$
$$+ \lambda_x \lambda_y \sum_{\gamma=1}^{2N_q-1} \hat{\omega}_\gamma \left(\frac{1}{4} d_2^\gamma uv - \frac{\lambda_y}{12} d_2^\gamma \Delta y A_2 + \frac{2\lambda_y}{3} A_3 \right)(x^-_{i+\frac{1}{2}}, \hat{y}_\gamma)$$

$$z_9 = \frac{1}{4}\hat{\omega}_{2N_q-1}\hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_{2N_q-1}\alpha_x^1 + \hat{\omega}_{2N_q-1}\frac{\lambda_x\lambda_y}{4}\Delta y(u_y v + uv_y)(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) - \hat{\omega}_{2N_q-1}\frac{\lambda_x\lambda_y^2}{12}\Delta y^2 A_1(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}})$$
$$+ \lambda_x\lambda_y\sum_{\gamma=1}^{2N_q-1}\hat{\omega}_\gamma\left(\frac{1}{4}d_3^\gamma uv - \frac{\lambda_y}{12}d_3^\gamma\Delta y A_2 - \frac{\lambda_y}{3}A_3\right)(x^-_{i+\frac{1}{2}}, \hat{y}_\gamma)$$

$$z_{10} = \lambda_x\left(\frac{1}{2}\hat{\omega}_1\alpha_x^1 + \hat{\omega}_1\frac{\lambda_y}{4}\Delta y(u_y v + uv_y)(x^+_{i+\frac{1}{2}}, y^+_{j-\frac{1}{2}}) - \hat{\omega}_1\frac{\lambda_y^2}{12}\Delta y^2 A_1(x^+_{i+\frac{1}{2}}, y^+_{j-\frac{1}{2}})\right.$$
$$\left. + \lambda_y\sum_{\gamma=1}^{2N_q-1}\hat{\omega}_\gamma\left(\frac{1}{4}d_1^\gamma uv - \frac{\lambda_y}{12}d_1^\gamma\Delta y A_2 - \frac{\lambda_y}{3}A_3\right)(x^+_{i+\frac{1}{2}}, \hat{y}_\gamma)\right)$$

$$z_{11} = \lambda_x\left(\frac{1}{2}\hat{\omega}_{N_q}\alpha_x^1 + \hat{\omega}_{N_q}\frac{\lambda_y}{4}\Delta y(u_y v + uv_y)(x^+_{i+\frac{1}{2}}, y_j) - \hat{\omega}_{N_q}\frac{\lambda_y^2}{12}\Delta y^2 A_1(x^+_{i+\frac{1}{2}}, y_j)\right.$$
$$\left. + \lambda_y\sum_{\gamma=1}^{2N_q-1}\hat{\omega}_\gamma\left(\frac{1}{4}d_2^\gamma uv - \frac{\lambda_y}{12}d_2^\gamma\Delta y A_2 + \frac{2\lambda_y}{3}A_3\right)(x^+_{i+\frac{1}{2}}, \hat{y}_\gamma)\right)$$

$$z_{12} = \lambda_x\left(\frac{1}{2}\hat{\omega}_{2N_q-1}\alpha_x^1 + \hat{\omega}_{2N_q-1}\frac{\lambda_y}{4}\Delta y(u_y v + uv_y)(x^+_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) - \hat{\omega}_{2N_q-1}\frac{\lambda_y^2}{12}\Delta y^2 A_1(x^+_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}})\right.$$
$$\left. + \lambda_y\sum_{\gamma=1}^{2N_q-1}\hat{\omega}_\gamma\left(\frac{1}{4}d_3^\gamma uv - \frac{\lambda_y}{12}d_3^\gamma\Delta y A_2 - \frac{\lambda_y}{3}A_3\right)(x^+_{i+\frac{1}{2}}, \hat{y}_\gamma)\right)$$

$$z_{13,\beta} = \lambda_x\left(\frac{1}{2}\alpha_x^1 - \frac{\lambda_y}{4}\Delta y(u_y v + uv_y)(x^-_{i-\frac{1}{2}}, \hat{y}_\beta) + \frac{\lambda_y^2}{12}\Delta y^2 A_1(x^-_{i-\frac{1}{2}}, \hat{y}_\beta)\right)$$

$$z_{14,\beta} = \lambda_x\left(\frac{1}{2}\alpha_x^1 + \frac{\lambda_y}{4}\Delta y(u_y v + uv_y)(x^+_{i+\frac{1}{2}}, \hat{y}_\beta) - \frac{\lambda_y^2}{12}\Delta y^2 A_1(x^+_{i+\frac{1}{2}}, \hat{y}_\beta)\right)$$

$$z_{15,\beta} = \frac{1}{4}\hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\alpha_x^1 + \frac{\lambda_x\lambda_y}{4}\Delta y(u_y v + uv_y)(x^-_{i+\frac{1}{2}}, \hat{y}_\beta) - \frac{\lambda_x\lambda_y^2}{12}\Delta y^2 A_1(x^-_{i+\frac{1}{2}}, \hat{y}_\beta)$$

$$z_{16,\beta} = \frac{1}{4}\hat{\omega}_1 - \frac{\lambda_x}{2}\alpha_x^1 - \frac{\lambda_x\lambda_y}{4}\Delta y(u_y v + uv_y)(x^+_{i-\frac{1}{2}}, \hat{y}_\beta) + \frac{\lambda_x\lambda_y^2}{12}\Delta y^2 A_1(x^+_{i-\frac{1}{2}}, \hat{y}_\beta)$$

Under the CFL condition (3.25), we have the following estimates.

$$z_1 \geq \lambda_x\left(\frac{1}{2}\hat{\omega}_1\alpha_x^1 - \hat{\omega}_1\frac{\lambda_y}{4}\Delta y||(u_y v + uv_y)||_\infty - \hat{\omega}_1\frac{\lambda_y}{12}Q_2\Delta y^2||A_1||_\infty\right.$$
$$\left. - \lambda_y\left(||uv||_\infty + \frac{Q_2}{3}\Delta y||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty\right)\right) \geq 0$$

$$z_2 \geq \lambda_x\left(\frac{1}{2}\hat{\omega}_{N_q}\alpha_x^1 - \hat{\omega}_{N_q}\frac{\lambda_y}{4}\Delta y||(u_y v + uv_y)||_\infty - \hat{\omega}_{N_q}\frac{\lambda_y}{12}Q_2\Delta y^2||A_1||_\infty\right.$$
$$\left. - \lambda_y\left(||uv||_\infty + \frac{Q_2}{3}\Delta y||A_2||_\infty + \frac{2Q_2}{3}||A_3||_\infty\right)\right) \geq 0$$

$$z_3 \geq \lambda_x\left(\frac{1}{2}\hat{\omega}_{2N_q-1}\alpha_x^1 - \hat{\omega}_{2N_q-1}\frac{\lambda_y}{4}\Delta y||(u_y v + uv_y)||_\infty - \hat{\omega}_{2N_q-1}\frac{\lambda_y}{12}Q_2\Delta y^2||A_1||_\infty\right.$$
$$\left. - \lambda_y\left(||uv||_\infty + \frac{Q_2}{3}\Delta y||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty\right)\right) \geq 0$$

$$z_4 \geq \frac{1}{4}\hat{\omega}_1^2 - \frac{\lambda_x}{2}\hat{\omega}_1 \alpha_x^1 - \hat{\omega}_1 \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_1 \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty$$
$$- \lambda_x Q_2 \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty \right) \geq 0$$

$$z_5 \geq \frac{1}{4}\hat{\omega}_1 \hat{\omega}_{N_q} - \frac{\lambda_x}{2}\hat{\omega}_{N_q} \alpha_x^1 - \hat{\omega}_{N_q} \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_{N_q} \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty$$
$$- \lambda_x Q_2 \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{2Q_2}{3}||A_3||_\infty \right) \geq 0$$

$$z_6 \geq \frac{1}{4}\hat{\omega}_1 \hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_{2N_q-1} \alpha_x^1 - \hat{\omega}_{2N_q-1} \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_{2N_q-1} \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty$$
$$- \lambda_x Q_2 \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty \right) \geq 0$$

$$z_7 \geq \frac{1}{4}\hat{\omega}_1 \hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_1 \alpha_x^1 - \hat{\omega}_1 \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_1 \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty$$
$$- \lambda_x Q_2 \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty \right) \geq 0$$

$$z_8 \geq \frac{1}{4}\hat{\omega}_{N_q} \hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_{N_q} \alpha_x^1 - \hat{\omega}_{N_q} \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_{N_q} \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty$$
$$- \lambda_x Q_2 \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{2Q_2}{3}||A_3||_\infty \right) \geq 0$$

$$z_9 \geq \frac{1}{4}\hat{\omega}_{2N_q-1} \hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\hat{\omega}_{2N_q-1} \alpha_x^1 - \hat{\omega}_{2N_q-1} \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_{2N_q-1} \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty$$
$$- \lambda_x Q_2 \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty \right) \geq 0$$

$$z_{10} \geq \lambda_x \left(\frac{1}{2}\hat{\omega}_1 \alpha_x^1 - \hat{\omega}_1 \frac{\lambda_y}{4} \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_1 \frac{\lambda_y}{12} Q_2 \Delta y^2 ||A_1||_\infty \right.$$
$$\left. - \lambda_y \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty \right) \right) \geq 0$$

$$z_{11} \geq \lambda_x \left(\frac{1}{2}\hat{\omega}_{N_q} \alpha_x^1 - \hat{\omega}_{N_q} \frac{\lambda_y}{4} \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_{N_q} \frac{\lambda_y}{12} Q_2 \Delta y^2 ||A_1||_\infty \right.$$
$$\left. - \lambda_y \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{2Q_2}{3}||A_3||_\infty \right) \right) \geq 0$$

$$z_{12} \geq \lambda_x \left(\frac{1}{2}\hat{\omega}_{2N_q-1} \alpha_x^1 - \hat{\omega}_{2N_q-1} \frac{\lambda_y}{4} \Delta y ||(u_y v + u v_y)||_\infty - \hat{\omega}_{2N_q-1} \frac{\lambda_y}{12} Q_2 \Delta y^2 ||A_1||_\infty \right.$$
$$\left. - \lambda_y \left(||uv||_\infty + \frac{Q_2}{3}\Delta y ||A_2||_\infty + \frac{Q_2}{3}||A_3||_\infty \right) \right) \geq 0$$

$$z_{13,\beta} \geq \lambda_x \left(\frac{1}{2}\alpha_x^1 - \frac{\lambda_y}{4}\Delta y ||(u_y v + u v_y)||_\infty - \frac{\lambda_y}{12} Q_2 \Delta y^2 ||A_1||_\infty \right) \geq 0, \quad \forall \beta$$

$$z_{14,\beta} \geq \lambda_x \left(\frac{1}{2}\alpha_x^1 - \frac{\lambda_y}{4}\Delta y ||(u_y v + u v_y)||_\infty - \frac{\lambda_y}{12} Q_2 \Delta y^2 ||A_1||_\infty \right) \geq 0, \quad \forall \beta$$

$$z_{15,\beta} \geq \frac{1}{4}\hat{\omega}_{2N_q-1} - \frac{\lambda_x}{2}\alpha_x^1 - \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty \geq 0, \quad \forall \beta$$

$$z_{16,\beta} \geq \frac{1}{4}\hat{\omega}_1 - \frac{\lambda_x}{2}\alpha_x^1 - \frac{\lambda_x}{4} Q_2 \Delta y ||(u_y v + u v_y)||_\infty - \frac{\lambda_x}{12} Q_2^2 \Delta y^2 ||A_1||_\infty \geq 0, \quad \forall \beta$$

A.6 Constants in the CFL condition (3.28)

The constants appearing in the CFL condition (3.28) are defined as follows.

$$
\begin{aligned}
C(x^-_{i+\frac{1}{2}},\hat{y}_\beta) =& \frac{\Delta x}{\alpha_x}\Big((2E(x^-_{i+\frac{1}{2}},\hat{y}_\beta)+p(x^-_{i+\frac{1}{2}},\hat{y}_\beta))\Big(|\tilde{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+2\rho(x^-_{i+\frac{1}{2}},\hat{y}_\beta)\Big(|\tilde{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+Q_1\frac{\Delta x}{\alpha_x}\Big(|\tilde{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\Big(|\tilde{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+\frac{1}{2}Q_1\frac{\Delta x}{\alpha_x}\Big(|\tilde{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)^2+\frac{1}{2}Q_1\frac{\Delta x}{\alpha_x}\Big(|\tilde{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)^2\\
&+(2|m(x^-_{i+\frac{1}{2}},\hat{y}_\beta)|+\frac{p(x^-_{i+\frac{1}{2}},\hat{y}_\beta)}{\alpha_x})\Big(|\tilde{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+2|n(x^-_{i+\frac{1}{2}},\hat{y}_\beta)|\Big(|\tilde{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
C(x^+_{i+\frac{1}{2}},\hat{y}_\beta) =& \frac{\Delta x}{\alpha_x}\Big((2E(x^+_{i+\frac{1}{2}},\hat{y}_\beta)+p(x^+_{i+\frac{1}{2}},\hat{y}_\beta))\Big(|\tilde{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+2\rho(x^+_{i+\frac{1}{2}},\hat{y}_\beta)\Big(|\tilde{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+Q_1\frac{\Delta x}{\alpha_x}\Big(|\tilde{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^1(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\Big(|\tilde{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^4(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+\frac{1}{2}Q_1\frac{\Delta x}{\alpha_x}\Big(|\tilde{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)^2+\frac{1}{2}Q_1\frac{\Delta x}{\alpha_x}\Big(|\tilde{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)^2\\
&+(2|m(x^+_{i+\frac{1}{2}},\hat{y}_\beta)|+\frac{p(x^+_{i+\frac{1}{2}},\hat{y}_\beta)}{\alpha_x})\Big(|\tilde{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^2(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\\
&+2|n(x^+_{i+\frac{1}{2}},\hat{y}_\beta)|\Big(|\tilde{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|+Q_1\Delta x|\check{f}^3(x_{i+\frac{1}{2}},\hat{y}_\beta)|\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
D(\hat{x}_\alpha,y^-_{j+\frac{1}{2}}) =& \frac{\Delta x}{\alpha_y}\Big((2E(\hat{x}_\alpha,y^-_{j+\frac{1}{2}})+p(\hat{x}_\alpha,y^-_{j+\frac{1}{2}}))\Big(|\tilde{f}^1(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^1(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)\\
&+2\rho(\hat{x}_\alpha,y^-_{j+\frac{1}{2}})\Big(|\tilde{f}^4(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^4(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)\\
&+Q_2\frac{\Delta y}{\alpha_y}\Big(|\tilde{f}^1(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^1(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)\Big(|\tilde{f}^4(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^4(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)\\
&+\frac{1}{2}Q_2\frac{\Delta y}{\alpha_y}\Big(|\tilde{f}^2(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^2(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)^2+\frac{1}{2}Q_2\frac{\Delta y}{\alpha_y}\Big(|\tilde{f}^3(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^3(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)^2\\
&+2|m(\hat{x}_\alpha,y^-_{j+\frac{1}{2}})|\Big(|\tilde{f}^2(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^2(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)\\
&+(2|n(\hat{x}_\alpha,y^-_{j+\frac{1}{2}})|+\frac{p(\hat{x}_\alpha,y^-_{j+\frac{1}{2}})}{\alpha_y})\Big(|\tilde{f}^3(\hat{x}_\alpha,y_{j+\frac{1}{2}})|+Q_2\Delta y|\check{f}^3(\hat{x}_\alpha,y_{j+\frac{1}{2}})|\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
D(\hat{x}_\alpha, y^+_{j+\frac{1}{2}}) = & \frac{\Delta x}{\alpha_y}\Big((2E(\hat{x}_\alpha, y^+_{j+\frac{1}{2}}) + p(\hat{x}_\alpha, y^+_{j+\frac{1}{2}}))\left(|\tilde{f}^1(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^1(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right) \\
& + 2\rho(\hat{x}_\alpha, y^+_{j+\frac{1}{2}})\left(|\tilde{f}^4(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^4(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right) \\
& + Q_2\frac{\Delta y}{\alpha_y}\left(|\tilde{f}^1(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^1(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right)\left(|\tilde{f}^4(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^4(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right) \\
& + \frac{1}{2}Q_2\frac{\Delta y}{\alpha_y}\left(|\tilde{f}^2(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^2(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right)^2 + \frac{1}{2}Q_2\frac{\Delta y}{\alpha_y}\left(|\tilde{f}^3(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^3(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right)^2 \\
& + 2|m(\hat{x}_\alpha, y^+_{j+\frac{1}{2}})|\left(|\tilde{f}^2(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^2(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right) \\
& + 2(|n(\hat{x}_\alpha, y^+_{j+\frac{1}{2}})| + \frac{p(\hat{x}_\alpha, y^+_{j+\frac{1}{2}})}{\alpha_y})\left(|\tilde{f}^3(\hat{x}_\alpha, y_{j+\frac{1}{2}})| + Q_2\Delta y|\check{f}^3(\hat{x}_\alpha, y_{j+\frac{1}{2}})|\right)\Big)
\end{aligned}
$$

B Derivatives in the Euler equations

To simplify the derivation and coding, we need to compute a lot of intermediate variables before finally obtaining m_t, m_{tt}, m_{ttt}, (and n_t, n_{tt}, n_{ttt} in $2D$), and E_t, E_{tt}, E_{ttt} to be used in the Lax-Wendroff procedure. The expressions of the intermediate and target variables are given as follows.

B.1 One dimensional space

$u = \frac{m}{\rho}$,

$u_x = \frac{m_x}{\rho} - \frac{u\rho_x}{\rho}$,

$u_{xx} = -\frac{2m_x\rho_x}{\rho^2} + \frac{m_{xx}}{\rho} + m(\frac{2\rho_x^2}{\rho^3} - \frac{\rho_{xx}}{\rho^2})$,

$\rho_t = -m_x$,

$m_t = -\left(\hat{\gamma}E_x + \frac{3-\gamma}{2}m_x u + \frac{3-\gamma}{2}m u_x\right)$,

$E_t = -\Big(\gamma E_x u + \gamma E u_x - \frac{\hat{\gamma}}{2}m_x u^2 - \hat{\gamma} m u u_x\Big)$,

$u_t = \frac{m_t}{\rho} - \frac{u\rho_t}{\rho}$,

$\rho_{tx} = -m_{xx}$,

$m_{tx} = -\left(\hat{\gamma}E_{xx} + \frac{3-\gamma}{2}m_{xx}u + (3-\gamma)m_x u_x + \frac{3-\gamma}{2}m u_{xx}\right)$,

$E_{tx} = -\Big(\gamma E_{xx}u + 2\gamma E_x u_x + \gamma E u_{xx} - \frac{\hat{\gamma}}{2}m_{xx}u^2 - 2\hat{\gamma}m_x u u_x - \hat{\gamma} m u_x^2 - \hat{\gamma} m u u_{xx}\Big)$,

$u_{tx} = \frac{m_{tx}}{\rho} - \frac{m_x\rho_t}{\rho^2} - \frac{m_t\rho_x + m\rho_{tx}}{\rho^2} + \frac{2u\rho_x\rho_t}{\rho^2}$,

$\rho_{tt} = -m_{tx}$,

$m_{tt} = -\left(\hat{\gamma}E_{tx} + \frac{3-\gamma}{2}m_{tx}u + \frac{3-\gamma}{2}m_x u_t + \frac{3-\gamma}{2}m_t u_x + \frac{3-\gamma}{2}m u_{tx}\right)$,

$E_{tt} = -\Big(\gamma E_{tx}u + \gamma E_x u_t + \gamma E_t u_x + \gamma E u_{tx} - \frac{\hat{\gamma}}{2}m_{tx}u^2 - \hat{\gamma}m_x u u_t - \hat{\gamma}m_t u u_x - \hat{\gamma}m u_t u_x - \hat{\gamma} m u u_{tx}\Big)$,

$$u_{tt} = -\frac{2m_t\rho_t}{\rho^2} + \frac{m_{tt}}{\rho} + u(\frac{2\rho_t^2}{\rho^2} - \frac{\rho_{tt}}{\rho}),$$

$$\rho_{ttt} = -(m_{tt})_x,$$

$$m_{ttt} = -\left(\hat{\gamma}E_{tt} + \frac{3-\gamma}{2}m_{tt}u + \frac{3-\gamma}{2}mu_{tt} + (3-\gamma)m_tu_t\right)_x,$$

$$E_{ttt} = -\left(\gamma E_{tt}u + \gamma Eu_{tt} + 2\gamma E_tu_t - \frac{\hat{\gamma}}{2}m_{tt}u^2 - \hat{\gamma}m(u_t^2 + uu_{tt}) - 2\hat{\gamma}m_tuu_t\right)_x.$$

B.2 Two dimensional space

$$u = \frac{m}{\rho},$$

$$v = \frac{n}{\rho},$$

$$u_x = \frac{m_x}{\rho} - \frac{u\rho_x}{\rho},$$

$$u_y = \frac{m_y}{\rho} - \frac{u\rho_y}{\rho},$$

$$v_x = \frac{n_x}{\rho} - \frac{v\rho_x}{\rho},$$

$$v_y = \frac{n_y}{\rho} - \frac{v\rho_y}{\rho},$$

$$u_{xx} = -\frac{2m_x\rho_x}{\rho^2} + \frac{m_{xx}}{\rho} + m(\frac{2\rho_x^2}{\rho^3} - \frac{\rho_{xx}}{\rho^2}),$$

$$u_{yy} = -\frac{2m_y\rho_y}{\rho^2} + \frac{m_{yy}}{\rho} + m(\frac{2\rho_y^2}{\rho^3} - \frac{\rho_{yy}}{\rho^2}),$$

$$u_{xy} = -\frac{\rho_ym_x}{\rho^2} - \frac{m_y\rho_x}{\rho^2} + \frac{2m\rho_y\rho_x}{\rho^3} + \frac{m_{xy}}{\rho} - \frac{m\rho_{xy}}{\rho^2},$$

$$v_{xx} = -\frac{2n_x\rho_x}{\rho^2} + \frac{n_{xx}}{\rho} + n(\frac{2\rho_x^2}{\rho^3} - \frac{\rho_{xx}}{\rho^2}),$$

$$v_{yy} = -\frac{2n_y\rho_y}{\rho^2} + \frac{n_{yy}}{\rho} + n(\frac{2\rho_y^2}{\rho^3} - \frac{\rho_{yy}}{\rho^2}),$$

$$v_{xy} = -\frac{\rho_yn_x}{\rho^2} - \frac{n_y\rho_x}{\rho^2} + \frac{2n\rho_y\rho_x}{\rho^3} + \frac{n_{xy}}{\rho} - \frac{n\rho_{xy}}{\rho^2},$$

$$\rho_t = -m_x - n_y,$$

$$m_t = -\left(\hat{\gamma}E_x + \frac{3-\gamma}{2}m_xu + \frac{3-\gamma}{2}mu_x - \frac{\hat{\gamma}}{2}n_xv - \frac{\hat{\gamma}}{2}nv_x + m_yv + mv_y\right),$$

$$n_t = -\left(n_xu + nu_x + \hat{\gamma}E_y - \frac{\hat{\gamma}}{2}m_yu - \frac{\hat{\gamma}}{2}mu_y + \frac{3-\gamma}{2}n_yv + \frac{3-\gamma}{2}nv_y\right),$$

$$E_t = -\left(\gamma E_xu + \gamma Eu_x - \frac{\hat{\gamma}}{2}m_xu^2 - \hat{\gamma}muu_x - \frac{\hat{\gamma}}{2}m_xv^2 - \hat{\gamma}mvv_x\right)$$
$$-\left(\gamma E_yv + \gamma Ev_y - \frac{\hat{\gamma}}{2}n_yu^2 - \hat{\gamma}nuu_y - \frac{\hat{\gamma}}{2}n_yv^2 - \hat{\gamma}nvv_y\right)$$

$$u_t = \frac{m_t}{\rho} - \frac{u\rho_t}{\rho},$$

$$v_t = \frac{n_t}{\rho} - \frac{v\rho_t}{\rho},$$

$$\rho_{tx} = -m_{xx} - n_{xy},$$

$$\rho_{ty} = -m_{xy} - n_{yy},$$

$$m_{tx} = -\left(\hat{\gamma}E_{xx} + \frac{3-\gamma}{2}m_{xx}u + (3-\gamma)m_xu_x + \frac{3-\gamma}{2}mu_{xx}\right.$$
$$\left.-\frac{\hat{\gamma}}{2}n_{xx}v - \hat{\gamma}n_xv_x - \frac{\hat{\gamma}}{2}nv_{xx} + m_{xy}v + m_yv_x + m_xv_y + mv_{xy}\right)$$

$$m_{ty} = -\big(\hat\gamma E_{xy} + \tfrac{3-\gamma}{2} m_{xy} u + \tfrac{3-\gamma}{2} m_x u_y + \tfrac{3-\gamma}{2} m_y u_x + \tfrac{3-\gamma}{2} m u_{xy}$$
$$-\tfrac{\hat\gamma}{2} n_{xy} v - \tfrac{\hat\gamma}{2} n_x v_y - \tfrac{\hat\gamma}{2} n_y v_x - \tfrac{\hat\gamma}{2} n v_{xy} + m_{yy} v + 2 m_y v_y + m v_{yy}\Big)$$
$$n_{tx} = -\Big(n_{xx} u + 2 n_x u_x + n u_{xx} + \hat\gamma E_{xy} - \tfrac{\hat\gamma}{2} m_{xy} u - \tfrac{\hat\gamma}{2} m_y u_x$$
$$-\tfrac{\hat\gamma}{2} m_x u_y - \tfrac{\hat\gamma}{2} m u_{xy} + \tfrac{3-\gamma}{2} n_{xy} v + \tfrac{3-\gamma}{2} n_y v_x + \tfrac{3-\gamma}{2} n_x v_y + \tfrac{3-\gamma}{2} n v_{xy}\Big)$$
$$n_{ty} = -\Big(n_{xy} u + n_x u_y + n_y u_x + n u_{xy} + \hat\gamma E_{yy} - \tfrac{\hat\gamma}{2} m_{yy} u - \hat\gamma m_y u_y - \tfrac{\hat\gamma}{2} m u_{yy}$$
$$+\tfrac{3-\gamma}{2} n_{yy} v + (3-\gamma) n_y v_y + \tfrac{3-\gamma}{2} n v_{yy}\big)$$
$$E_{tx} = -\Big(\gamma E_{xx} u + 2\gamma E_x u_x + \gamma E u_{xx} - \tfrac{\hat\gamma}{2} m_{xx} u^2 - 2\hat\gamma m_x u u_x - \hat\gamma m u_x^2 - \hat\gamma m u u_{xx} - \tfrac{\hat\gamma}{2} m_{xx} v^2 - 2\hat\gamma m_x v v_x$$
$$-\hat\gamma m v_x^2 - \hat\gamma m v v_{xx} + \gamma E_{xy} v + \gamma E_y v_x + \gamma E_x v_y + \gamma E v_{xy} - \tfrac{\hat\gamma}{2} n_{xy} u^2 - \hat\gamma n_y u u_x - \hat\gamma n_x u u_y$$
$$-\hat\gamma n u_x u_y - \hat\gamma n u u_{xy} - \tfrac{\hat\gamma}{2} n_{xy} v^2 - \hat\gamma n_y v v_x - \hat\gamma n_x v v_y - \hat\gamma n v_x v_y - \hat\gamma n v v_{xy}\Big)$$
$$E_{ty} = -\Big(\gamma E_{yy} v + 2\gamma E_y v_y + \gamma E v_{yy} - \tfrac{\hat\gamma}{2} n_{yy} v^2 - 2\hat\gamma n_y v v_y - \hat\gamma n v_y^2 - \hat\gamma n v v_{yy} - \tfrac{\hat\gamma}{2} n_{yy} u^2 - 2\hat\gamma n_y u u_y$$
$$-\hat\gamma n u_y^2 - \hat\gamma n u u_{yy} + \gamma E_{xy} u + \gamma E_x u_y + \gamma E_y u_x + \gamma E u_{xy} - \tfrac{\hat\gamma}{2} m_{xy} v^2 - \hat\gamma m_x v v_y - \hat\gamma m_y v v_x$$
$$-\hat\gamma m v_y v_x - \hat\gamma m v v_{xy} - \tfrac{\hat\gamma}{2} m_{xy} u^2 - \hat\gamma m_x u u_y - \hat\gamma m_y u u_x - \hat\gamma m u_y u_x - \hat\gamma m u u_{xy}\Big)$$
$$u_{tx} = \tfrac{m_{tx}}{\rho} - \tfrac{m_x \rho_t}{\rho^2} - \tfrac{m_t \rho_x + m \rho_{tx}}{\rho^2} + \tfrac{2u\rho_x \rho_t}{\rho^2},$$
$$u_{ty} = \tfrac{m_{ty}}{\rho} - \tfrac{m_y \rho_t}{\rho^2} - \tfrac{m_t \rho_y + m \rho_{ty}}{\rho^2} + \tfrac{2u\rho_y \rho_t}{\rho^2},$$
$$v_{tx} = \tfrac{n_{tx}}{\rho} - \tfrac{n_x \rho_t}{\rho^2} - \tfrac{n_t \rho_x + n \rho_{tx}}{\rho^2} + \tfrac{2v\rho_x \rho_t}{\rho^2},$$
$$v_{ty} = \tfrac{n_{ty}}{\rho} - \tfrac{n_y \rho_t}{\rho^2} - \tfrac{n_t \rho_y + n \rho_{ty}}{\rho^2} + \tfrac{2v\rho_y \rho_t}{\rho^2},$$
$$\rho_{tt} = -m_{tx} - n_{ty},$$
$$m_{tt} = -\big(\hat\gamma E_{tx} + \tfrac{3-\gamma}{2} m_{tx} u + \tfrac{3-\gamma}{2} m_x u_t + \tfrac{3-\gamma}{2} m_t u_x + \tfrac{3-\gamma}{2} m u_{tx}$$
$$-\tfrac{\hat\gamma}{2} n_{tx} v - \tfrac{\hat\gamma}{2} n_x v_t - \tfrac{\hat\gamma}{2} n_t v_x - \tfrac{\hat\gamma}{2} n v_{tx} + m_{ty} v + m_y v_t + m_t v_y + m v_{ty}\Big)$$
$$n_{tt} = -\Big(n_{tx} u + n_x u_t + n_t u_x + n u_{tx} + \hat\gamma E_{ty} - \tfrac{\hat\gamma}{2} m_{ty} u - \tfrac{\hat\gamma}{2} m_y u_t - \tfrac{\hat\gamma}{2} m_t u_y - \tfrac{\hat\gamma}{2} m u_{ty}$$
$$+\tfrac{3-\gamma}{2} n_{ty} v + \tfrac{3-\gamma}{2} n_y v_t + \tfrac{3-\gamma}{2} n_t v_y + \tfrac{3-\gamma}{2} n v_{ty}\big)$$
$$E_{tt} = -\Big(\gamma E_{tx} u + \gamma E_x u_t + \gamma E_t u_x + \gamma E u_{tx} - \tfrac{\hat\gamma}{2} m_{tx} u^2 - \hat\gamma m_x u u_t - \hat\gamma m_t u u_x - \hat\gamma m u_t u_x$$
$$-\hat\gamma m u u_{tx} - \tfrac{\hat\gamma}{2} m_{tx} v^2 - \hat\gamma m_x v v_t - \hat\gamma m_t v v_x - \hat\gamma m v_t v_x - \hat\gamma m v v_{tx} + \gamma E_{ty} v + \gamma E_y v_t$$
$$+\gamma E_t v_y + \gamma E v_{ty} - \tfrac{\hat\gamma}{2} n_{ty} u^2 - \hat\gamma n_y u u_t - \hat\gamma n_t u u_y - \hat\gamma n u_t u_y - \hat\gamma n u u_{ty} - \tfrac{\hat\gamma}{2} n_{ty} v^2$$
$$-\hat\gamma n_y v v_t - \hat\gamma n_t v v_y - \hat\gamma n v_t v_y - \hat\gamma n v v_{ty}\big)$$
$$u_{tt} = -\tfrac{2 m_t \rho_t}{\rho^2} + \tfrac{m_{tt}}{\rho} + u\big(\tfrac{2\rho_t^2}{\rho^2} - \tfrac{\rho_{tt}}{\rho}\big),$$
$$v_{tt} = -\tfrac{2 n_t \rho_t}{\rho^2} + \tfrac{n_{tt}}{\rho} + v\big(\tfrac{2\rho_t^2}{\rho^2} - \tfrac{\rho_{tt}}{\rho}\big),$$
$$\rho_{ttt} = -(m_{tt})_x - (n_{tt})_y,$$
$$m_{ttt} = -\Big(\hat\gamma E_{tt} + \tfrac{3-\gamma}{2} m_{tt} u + \tfrac{3-\gamma}{2} m u_{tt} + (3-\gamma) m_t u_t - \tfrac{\hat\gamma}{2} n_{tt} v - \tfrac{\hat\gamma}{2} n v_{tt} - \hat\gamma n_t v_t\Big)_x$$
$$-\left(m_{tt} v + m v_{tt} + 2 m_t v_t\right)_y$$

$$
\begin{aligned}
n_{ttt} =& -\left(nu_{tt} + n_{tt}u + 2n_tu_t\right)_x \\
& - \Big(\hat{\gamma}E_{tt} - \tfrac{\hat{\gamma}}{2}m_{tt}u - \tfrac{\hat{\gamma}}{2}mu_{tt} - \hat{\gamma}m_tu_t + \tfrac{3-\gamma}{2}n_{tt}v + \tfrac{3-\gamma}{2}nv_{tt} + (3-\gamma)n_tv_t\Big)_y \\
E_{ttt} =& -\Big(\gamma E_{tt}u + \gamma Eu_{tt} + 2\gamma E_tu_t - \tfrac{\hat{\gamma}}{2}m_{tt}u^2 - \hat{\gamma}m(u_t^2 + uu_{tt}) - 2\hat{\gamma}m_tuu_t \\
& - \tfrac{\hat{\gamma}}{2}m_{tt}v^2 - \hat{\gamma}m(v_t^2 + vv_{tt}) - 2\hat{\gamma}m_tvv_t\Big)_x \\
& - \Big(\gamma E_{tt}v + \gamma Ev_{tt} + 2\gamma E_tv_t - \tfrac{\hat{\gamma}}{2}n_{tt}u^2 - \hat{\gamma}n(u_t^2 + uu_{tt}) - 2\hat{\gamma}n_tuu_t \\
& - \tfrac{\hat{\gamma}}{2}n_{tt}v^2 - \hat{\gamma}n(v_t^2 + vv_{tt}) - 2\hat{\gamma}n_tvv_t\Big)_y
\end{aligned}
$$

C Maximum-principle-satisfying LWDG schemes for scalar conservation laws in one dimension on nonuniform meshes

We have discussed the bound-preserving LWDG schemes on uniform meshes in Sections 2 and 3. In this appendix, we show how to extend the technique to nonuniform meshes. For simplicity, we only consider the scalar conservation law in one space dimension, but the same methodology can be adopted to construct bound-preserving schemes for the Euler equations and multi-dimensional spaces.

We first introduce a direct extension of the maximum-principle-satisfying LWDG from uniform meshes, which is simple and efficient but has constraints on mesh sizes, i.e. $\frac{1}{2} < \frac{\Delta x_{j+1}}{\Delta x_j} < 2, \forall j$. Another way of extension is based on the composite Gauss-Lobatto rule, as used in [2], which removes the constraints on meshes but is less efficient. In practice, we recommend to combine both in the way that the composite Gauss-Lobatto rule is only used on the cells where it is necessary, i.e. the cells that violate $\frac{1}{2} < \frac{\Delta x_{j+1}}{\Delta x_j} < 2$.

C.1 A direct extension of the maximum-principle-satisfying LWDG scheme from uniform meshes

We define the DDG flux on nonuniform meshes as

$$
\widehat{u_x}^{\mathrm{DDG}}_{j+\frac{1}{2}} = \beta_{0,j+\frac{1}{2}} \frac{[u]_{j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} + \{u_x\}_{j+\frac{1}{2}} + \beta_{1,j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}} [u_{xx}]_{j+\frac{1}{2}} \tag{C.1}
$$

where $\Delta x_{j+\frac{1}{2}} = \min\{\Delta x_j, \Delta x_{j+1}\}$ and $\beta_{0,j+\frac{1}{2}}, \beta_{1,j+\frac{1}{2}}, j = 1, 2, \dots, N$ are penalty parameters satisfying (C.2) for the purpose of maximum-principle-preserving.

$$\begin{aligned} &\frac{1}{8}\max\{\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}}, \frac{\Delta x_{j+1}}{\Delta x_{j+\frac{1}{2}}}\} < \beta_{1,j+\frac{1}{2}} < \frac{1}{4}\min\{\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}}, \frac{\Delta x_{j+1}}{\Delta x_{j+\frac{1}{2}}}\}, \quad \forall j, \\ &\beta_{0,j+\frac{1}{2}} > \max\{\frac{3}{2}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} - 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x^2_{j+\frac{1}{2}}}{\Delta x_j^2}, \frac{3}{2}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}} - 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x^2_{j+\frac{1}{2}}}{\Delta x^2_{j+1}}\}, \quad \forall j \end{aligned} \tag{C.2}$$

Note that to make sense of (C.2), the nonuniform meshes must have a mild change in mesh size, i.e. $\frac{1}{2} < \frac{\Delta x_{j+1}}{\Delta x_j} < 2, \forall j$.

Similar to (2.9), we have the expansion of the DDG flux on nonuniform meshes.

Lemma C.1. *For $u \in V$, the DDG flux $\widehat{u_x}^{DDG}_{j+\frac{1}{2}}$ defined in* (C.1) *can be expanded on nonuniform meshes as*

$$\begin{aligned} \widehat{u_x}^{DDG}_{j+\frac{1}{2}} =& (\frac{1}{2\Delta x_j} - \frac{4\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{\Delta x_j^2})u^+_{j-\frac{1}{2}} + (-\frac{2}{\Delta x_j} + \frac{8\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{\Delta x_j^2})u_j \\ &+ (-\frac{\beta_{0,j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} + \frac{3}{2\Delta x_j} - \frac{4\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{\Delta x_j^2})u^-_{j+\frac{1}{2}} + (\frac{\beta_{0,j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2\Delta x_{j+1}} + \frac{4\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{\Delta x^2_{j+1}})u^+_{j+\frac{1}{2}} \\ &+ (\frac{2}{\Delta x_{j+1}} - \frac{8\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{\Delta x^2_{j+1}})u_{j+1} + (-\frac{1}{2\Delta x_{j+1}} + \frac{4\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{\Delta x^2_{j+1}})u^-_{j+\frac{3}{2}} \end{aligned} \tag{C.3}$$

The proof follows from direct computation and the fact that u is piecewise quadratic.

We now state the main result.

Theorem C.2. *Given $m \le u^n \le M$ and the DDG flux* (C.1) *with parameters* (C.2)*, the cell averages $\bar{u}_j^{n+1}, j = 1, 2, \dots, N$ of the solution of scheme* (2.4) *are bounded between m and M under the CFL condition* (C.4)*:*

$$\Delta t \le \min\{q_1, q_2, \dots, q_{10}\}, \tag{C.4}$$

where $q_1 = \frac{\hat{\omega}_1}{2M_1}\min_j \Delta x_j$, $q_2 = \min_j\{\frac{4\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} - \frac{1}{2}\Delta x_j}{5(M-m)M_2 + \frac{4}{3}M_1}\}$, $q_3 = \min_j\{\frac{4\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} - \frac{1}{2}\Delta x_{j+1}}{5(M-m)M_2 + \frac{4}{3}M_1}\}$, $q_4 = \min_j\{\frac{2\Delta x_j - 8\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{20(M-m)M_2 + \frac{8}{3}M_1}\}$,
$q_5 = \min_j\{\frac{2\Delta x_{j+1} - 8\beta_{1,j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{20(M-m)M_2 + \frac{8}{3}M_1}\}$, $q_6 = \min_j\{\frac{\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2} + 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}}{15(M-m)M_2 + \frac{4}{3}M_1}\Delta x_j\}$,
$q_7 = \min_j\{\frac{\beta_{0,j+\frac{1}{2}}\frac{\Delta x_{j+1}}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2} + 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}}}{15(M-m)M_2 + \frac{4}{3}M_1}\Delta x_{j+1}\}$,
$q_8 = \frac{1}{M_1}\min_j\left(\frac{2\omega_1\Delta x_j^2}{3(\beta_{0,j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}} - \frac{3}{2} + 4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}) + 3(4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} - \frac{1}{2})}\right)^{\frac{1}{2}}$,

$$q_9 = \frac{1}{M_1}\min_j \left(\frac{2\omega_{N_q}\Delta x_j^2}{3(2-8\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j})+3(2-8\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j})} \right)^{\frac{1}{2}},$$

$$q_{10} = \frac{1}{M_1}\min_j \left(\frac{2\omega_{2N_q-1}\Delta x_j^2}{3(\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}}-\frac{3}{2}+4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j})+3(4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}-\frac{1}{2})} \right)^{\frac{1}{2}}.$$

Proof. We have exactly the same results as in (2.13), (2.14), and (2.15), except that the coefficients in (2.15) are now

$$\begin{aligned}
z_1 =& \frac{\lambda_j^2}{4}{f'}^{2-}_{j-\frac{1}{2}}\left((4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}\Delta x_j}{\Delta x_{j-1}^2}-\frac{1}{2}\frac{\Delta x_j}{\Delta x_{j-1}})+\Delta t {f''}^{-}_{j-\frac{1}{2}} {u_x}^{-}_{j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-1}}+\frac{4\lambda_j}{3}{f'}^{-}_{j-\frac{1}{2}}\frac{\Delta x_j^2}{\Delta x_{j-1}^2}\right)\\
&+\frac{\lambda_j^2}{4}{f'}^{2+}_{j-\frac{1}{2}}(4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}\Delta x_j}{\Delta x_{j-1}^2}-\frac{1}{2}\frac{\Delta x_j}{\Delta x_{j-1}}),
\end{aligned}$$

$$\begin{aligned}
z_2 =& \frac{\lambda_j^2}{4}{f'}^{2-}_{j-\frac{1}{2}}\left((2\frac{\Delta x_j}{\Delta x_{j-1}}-8\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}\Delta x_j}{\Delta x_{j-1}^2})-4\Delta t {f''}^{-}_{j-\frac{1}{2}} {u_x}^{-}_{j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-1}}-\frac{8\lambda_j}{3}{f'}^{-}_{j-\frac{1}{2}}\frac{\Delta x_j^2}{\Delta x_{j-1}^2}\right)\\
&+\frac{\lambda_j^2}{4}{f'}^{2+}_{j-\frac{1}{2}}(2\frac{\Delta x_j}{\Delta x_{j-1}}-8\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}\Delta x_j}{\Delta x_{j-1}^2})
\end{aligned}$$

$$\begin{aligned}
z_3 =& \frac{\lambda_j^2}{4}{f'}^{2-}_{j-\frac{1}{2}}\left((\beta_{0,j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}}-\frac{3}{2}\frac{\Delta x_j}{\Delta x_{j-1}}+4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}\Delta x_j}{\Delta x_{j-1}^2})+3\Delta t {f''}^{-}_{j-\frac{1}{2}} {u_x}^{-}_{j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-1}}+\frac{4\lambda_j}{3}{f'}^{-}_{j-\frac{1}{2}}\frac{\Delta x_j^2}{\Delta x_{j-1}^2}\right)\\
&+\frac{\lambda_j^2}{4}{f'}^{2+}_{j-\frac{1}{2}}(\beta_{0,j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}}-\frac{3}{2}\frac{\Delta x_j}{\Delta x_{j-1}}+4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}\Delta x_j}{\Delta x_{j-1}^2})\\
z_4 =& \frac{1}{2}\omega_1-\frac{\lambda_j^2}{4}{f'}^{2-}_{j-\frac{1}{2}}(\beta_{0,j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}}-\frac{3}{2}+4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j})\\
&-\frac{\lambda_j^2}{4}{f'}^{2+}_{j-\frac{1}{2}}\left((\beta_{0,j-\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}}-\frac{3}{2}+4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j})+3\Delta t {f''}^{+}_{j-\frac{1}{2}} {u_x}^{+}_{j-\frac{1}{2}}-\frac{4\lambda_j}{3}{f'}^{+}_{j-\frac{1}{2}}\right)\\
&-\frac{\lambda_j^2}{4}{f'}^{2-}_{j+\frac{1}{2}}\left((4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}-\frac{1}{2})+\Delta t {f''}^{-}_{j+\frac{1}{2}} {u_x}^{-}_{j+\frac{1}{2}}+\frac{4\lambda_j}{3}{f'}^{-}_{j+\frac{1}{2}}\right)-\frac{\lambda_j^2}{4}{f'}^{2+}_{j+\frac{1}{2}}(4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}-\frac{1}{2})\\
z_5 =& \frac{1}{2}\omega_N-\frac{\lambda_j^2}{4}{f'}^{2-}_{j-\frac{1}{2}}(2-8\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j})-\frac{\lambda_j^2}{4}{f'}^{2+}_{j-\frac{1}{2}}\left((2-8\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j})-4\Delta t {f''}^{+}_{j-\frac{1}{2}} {u_x}^{+}_{j-\frac{1}{2}}+\frac{8\lambda_j}{3}{f'}^{+}_{j-\frac{1}{2}}\right)\\
&-\frac{\lambda_j^2}{4}{f'}^{2-}_{j+\frac{1}{2}}\left((2-8\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j})-4\Delta t {f''}^{-}_{j+\frac{1}{2}} {u_x}^{-}_{j+\frac{1}{2}}-\frac{8\lambda_j}{3}{f'}^{-}_{j+\frac{1}{2}}\right)-\frac{\lambda_j^2}{4}{f'}^{2+}_{j+\frac{1}{2}}(2-8\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j})\\
z_6 =& \frac{1}{2}\omega_{2N_q-1}-\frac{\lambda_j^2}{4}{f'}^{2-}_{j-\frac{1}{2}}(4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}-\frac{1}{2})-\frac{\lambda_j^2}{4}{f'}^{2+}_{j-\frac{1}{2}}\left((4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}-\frac{1}{2})+\Delta t {f''}^{+}_{j-\frac{1}{2}} {u_x}^{+}_{j-\frac{1}{2}}-\frac{4\lambda_j}{3}{f'}^{+}_{j-\frac{1}{2}}\right)\\
&-\frac{\lambda_j^2}{4}{f'}^{2-}_{j+\frac{1}{2}}\left((\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}}-\frac{3}{2}+4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j})+3\Delta t {f''}^{-}_{j+\frac{1}{2}} {u_x}^{-}_{j+\frac{1}{2}}+\frac{4\lambda_j}{3}{f'}^{-}_{j+\frac{1}{2}}\right)\\
&-\frac{\lambda_j^2}{4}{f'}^{2+}_{j+\frac{1}{2}}(\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}}-\frac{3}{2}+4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j})
\end{aligned}$$

$$z_7 = \frac{\lambda_j^2}{4} {f'^2}^-_{j+\frac{1}{2}} (\beta_{0,j+\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2} \frac{\Delta x_j}{\Delta x_{j+1}} + 4\beta_{1,j+\frac{1}{2}} \frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2})$$
$$+ \frac{\lambda_j^2}{4} {f'^2}^+_{j+\frac{1}{2}} \left((\beta_{0,j+\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2} \frac{\Delta x_j}{\Delta x_{j+1}} + 4\beta_{1,j+\frac{1}{2}} \frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2}) + 3\Delta t {f''}^+_{j+\frac{1}{2}} {u_x}^+_{j+\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j+1}} - \frac{4\lambda_j}{3} {f'}^+_{j+\frac{1}{2}} \frac{\Delta x_j^2}{\Delta x_{j+1}^2} \right)$$
$$z_8 = \frac{\lambda_j^2}{4} {f'^2}^-_{j+\frac{1}{2}} (2\frac{\Delta x_j}{\Delta x_{j+1}} - 8\beta_{1,j+\frac{1}{2}} \frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2})$$
$$+ \frac{\lambda_j^2}{4} {f'^2}^+_{j+\frac{1}{2}} \left((2\frac{\Delta x_j}{\Delta x_{j+1}} - 8\beta_{1,j+\frac{1}{2}} \frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2}) - 4\Delta t {f''}^+_{j+\frac{1}{2}} {u_x}^+_{j+\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j+1}} + \frac{8\lambda_j}{3} {f'}^+_{j+\frac{1}{2}} \frac{\Delta x_j^2}{\Delta x_{j+1}^2} \right)$$
$$z_9 = \frac{\lambda_j^2}{4} {f'^2}^-_{j+\frac{1}{2}} (4\beta_{1,j+\frac{1}{2}} \frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2} - \frac{1}{2} \frac{\Delta x_j}{\Delta x_{j+1}})$$
$$+ \frac{\lambda_j^2}{4} {f'^2}^+_{j+\frac{1}{2}} \left((4\beta_{1,j+\frac{1}{2}} \frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2} - \frac{1}{2} \frac{\Delta x_j}{\Delta x_{j+1}}) + \Delta t {f''}^+_{j+\frac{1}{2}} {u_x}^+_{j+\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j+1}} - \frac{4\lambda_j}{3} {f'}^+_{j+\frac{1}{2}} \frac{\Delta x_j^2}{\Delta x_{j+1}^2} \right)$$

It can be verified that

$$\frac{1}{2} \sum_{\gamma=2}^{N_q-1} \hat{\omega}_\gamma + \frac{1}{2} \sum_{\gamma=N_q+1}^{2N_q-2} \hat{\omega}_\gamma + z_1 + z_2 + \ldots + z_9 = \frac{1}{2}$$

and

$$z_1 \geq \frac{\lambda_j^2}{4} {f'^2}^-_{j-\frac{1}{2}} \left((4\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}} \Delta x_j}{\Delta x_{j-1}^2} - \frac{1}{2} \frac{\Delta x_j}{\Delta x_{j-1}}) - 5(M-m)M_2 \frac{\Delta t}{\Delta x_{j-1}} \frac{\Delta x_j}{\Delta x_{j-1}} - \frac{4\lambda_j}{3} M_1 \frac{\Delta x_j^2}{\Delta x_{j-1}^2} \right)$$
$$+ \frac{\lambda_j^2}{4} {f'^2}^+_{j-\frac{1}{2}} (4\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}} \Delta x_j}{\Delta x_{j-1}^2} - \frac{1}{2} \frac{\Delta x_j}{\Delta x_{j-1}}) \geq 0,$$

$$z_2 \geq \frac{\lambda_j^2}{4} {f'^2}^-_{j-\frac{1}{2}} \left((2\frac{\Delta x_j}{\Delta x_{j-1}} - 8\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}} \Delta x_j}{\Delta x_{j-1}^2}) - 20(M-m)M_2 \frac{\Delta t}{\Delta x_{j-1}} \frac{\Delta x_j}{\Delta x_{j-1}} - \frac{8\lambda_j}{3} M_1 \frac{\Delta x_j^2}{\Delta x_{j-1}^2} \right)$$
$$+ \frac{\lambda_j^2}{4} {f'^2}^+_{j-\frac{1}{2}} (2\frac{\Delta x_j}{\Delta x_{j-1}} - 8\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}} \Delta x_j}{\Delta x_{j-1}^2}) \geq 0,$$

$$z_3 \geq \frac{\lambda_j^2}{4} {f'^2}^-_{j-\frac{1}{2}} \left((\beta_{0,j-\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}} - \frac{3}{2} \frac{\Delta x_j}{\Delta x_{j-1}} + 4\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}} \Delta x_j}{\Delta x_{j-1}^2}) - 15(M-m)M_2 \frac{\Delta t}{\Delta x_{j-1}} \frac{\Delta x_j}{\Delta x_{j-1}} - \frac{4\lambda_j}{3} M_1 \frac{\Delta x_j^2}{\Delta x_{j-1}^2} \right)$$
$$+ \frac{\lambda_j^2}{4} {f'^2}^+_{j-\frac{1}{2}} (\beta_{0,j-\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}} - \frac{3}{2} \frac{\Delta x_j}{\Delta x_{j-1}} + 4\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}} \Delta x_j}{\Delta x_{j-1}^2}) \geq 0,$$

$$z_4 \geq \frac{1}{2}\omega_1 - \frac{\lambda_j^2}{4} M_1^2 (\beta_{0,j-\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}} - \frac{3}{2} + 4\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j})$$
$$- \frac{\lambda_j^2}{4} M_1^2 \left((\beta_{0,j-\frac{1}{2}} \frac{\Delta x_j}{\Delta x_{j-\frac{1}{2}}} - \frac{3}{2} + 4\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}) + 15(M-m)M_2\lambda_j + \frac{4\lambda_j}{3} M_1 \right)$$
$$- \frac{\lambda_j^2}{4} M_1^2 \left((4\beta_{1,j+\frac{1}{2}} \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} - \frac{1}{2}) + 5(M-m)M_2\lambda_j + \frac{4\lambda_j}{3} M_1 \right) - \frac{\lambda_j^2}{4} M_1^2 (4\beta_{1,j+\frac{1}{2}} \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} - \frac{1}{2}) \geq 0,$$

$$z_5 \geq \frac{1}{2}\omega_N - \frac{\lambda_j^2}{4} M_1^2 (2 - 8\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}) - \frac{\lambda_j^2}{4} M_1^2 \left((2 - 8\beta_{1,j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}) + 20(M-m)M_2\lambda_j + \frac{8\lambda_j}{3} M_1 \right)$$
$$- \frac{\lambda_j^2}{4} M_1^2 \left((2 - 8\beta_{1,j+\frac{1}{2}} \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}) + 20(M-m)M_2\lambda_j + \frac{8\lambda_j}{3} M_1 \right) - \frac{\lambda_j^2}{4} M_1^2 (2 - 8\beta_{1,j+\frac{1}{2}} \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}) \geq 0,$$

$$z_6 \geq \frac{1}{2}\omega_{2N_q-1} - \frac{\lambda_j^2}{4}M_1^2(4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} - \frac{1}{2}) - \frac{\lambda_j^2}{4}M_1^2\left((4\beta_{1,j-\frac{1}{2}}\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} - \frac{1}{2}) + 5(M-m)M_2\lambda_j + \frac{4\lambda_j}{3}M_1\right)$$
$$- \frac{\lambda_j^2}{4}M_1^2\left((\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2} + 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}) + 15(M-m)M_2\lambda_j + \frac{4\lambda_j}{3}M_1\right)$$
$$- \frac{\lambda_j^2}{4}M_1^2(\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2} + 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}) \geq 0,$$

$$z_7 \geq \frac{\lambda_j^2}{4}{f'}^{2-}_{j+\frac{1}{2}}(\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2}\frac{\Delta x_j}{\Delta x_{j+1}} + 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2})$$
$$+ \frac{\lambda_j^2}{4}{f'}^{2+}_{j+\frac{1}{2}}\left((\beta_{0,j+\frac{1}{2}}\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} - \frac{3}{2}\frac{\Delta x_j}{\Delta x_{j+1}} + 4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2}) - 15(M-m)M_2\frac{\Delta t}{\Delta x_{j+1}}\frac{\Delta x_j}{\Delta x_{j+1}} - \frac{4\lambda_j}{3}M_1\frac{\Delta x_j^2}{\Delta x_{j+1}^2}\right) \geq 0,$$

$$z_8 \geq \frac{\lambda_j^2}{4}{f'}^{2-}_{j+\frac{1}{2}}(2\frac{\Delta x_j}{\Delta x_{j+1}} - 8\beta_{1,j+\frac{1}{2}}\frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2})$$
$$+ \frac{\lambda_j^2}{4}{f'}^{2+}_{j+\frac{1}{2}}\left((2\frac{\Delta x_j}{\Delta x_{j+1}} - 8\beta_{1,j+\frac{1}{2}}\frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2}) - 20(M-m)M_2\frac{\Delta t}{\Delta x_{j+1}}\frac{\Delta x_j}{\Delta x_{j+1}} - \frac{8\lambda_j}{3}M_1\frac{\Delta x_j^2}{\Delta x_{j+1}^2}\right) \geq 0,$$

$$z_9 \geq \frac{\lambda_j^2}{4}{f'}^{2-}_{j+\frac{1}{2}}(4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2} - \frac{1}{2}\frac{\Delta x_j}{\Delta x_{j+1}})$$
$$+ \frac{\lambda_j^2}{4}{f'}^{2+}_{j+\frac{1}{2}}\left((4\beta_{1,j+\frac{1}{2}}\frac{\Delta x_j \Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}^2} - \frac{1}{2}\frac{\Delta x_j}{\Delta x_{j+1}}) - 5(M-m)M_2\frac{\Delta t}{\Delta x_{j+1}}\frac{\Delta x_j}{\Delta x_{j+1}} - \frac{4\lambda_j}{3}M_1\frac{\Delta x_j^2}{\Delta x_{j+1}^2}\right) \geq 0,$$

under the CFL condition (C.4).

Since II can be written as a half of a convex combination of point values of u^n, we still have $\frac{1}{2}m \leq \mathrm{II} \leq \frac{1}{2}M$ as before, which implies $m \leq \bar{u}_j^{n+1} \leq M, j = 1, 2, \ldots, N$ □

C.2 A maximum-principle-satisfying scheme on arbitrary nonuniform meshes

To construct the maximum-principle-satisfying scheme on arbitrary nonuniform meshes, we shall first introduce the composite quadrature rule to be used. Define $\Delta x_{j+\frac{1}{2}} = \frac{1}{3}\min\{\Delta x_j, \Delta x_{j+1}\}$ and denote by $\tilde{u}_j^1 = u(x_{j-\frac{1}{2}} - \Delta x_{j-\frac{1}{2}}), \tilde{u}_j^2 = u(x_{j-\frac{1}{2}} - \frac{1}{2}\Delta x_{j-\frac{1}{2}}), \tilde{u}_j^3 = u(x_{j-\frac{1}{2}} + \frac{1}{2}\Delta x_{j-\frac{1}{2}}), \tilde{u}_j^4 = u(x_{j-\frac{1}{2}} + \Delta x_{j-\frac{1}{2}}), \tilde{u}_j^5 = u(x_{j+\frac{1}{2}} - \Delta x_{j+\frac{1}{2}}), \tilde{u}_j^6 = u(x_{j+\frac{1}{2}} - \frac{1}{2}\Delta x_{j+\frac{1}{2}}), \tilde{u}_j^7 = u(x_{j+\frac{1}{2}} + \frac{1}{2}\Delta x_{j+\frac{1}{2}}), \tilde{u}_j^8 = u(x_{j+\frac{1}{2}} + \Delta x_{j+\frac{1}{2}})$, for the cell I_j.

We adopt the composite Gauss-Lobatto rule as follows: The interval I_j is divided into three subintervals, i.e. $I_j = [x_{j-\frac{1}{2}}, x_{j-\frac{1}{2}} + \Delta x_{j-\frac{1}{2}}] \cup [x_{j-\frac{1}{2}} + \Delta x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}} - \Delta x_{j+\frac{1}{2}}] \cup [x_{j+\frac{1}{2}} - \Delta x_{j+\frac{1}{2}}, x_{j+\frac{1}{2}}]$, and each subinterval is assigned with the $2N_q - 1$ Gauss-Lobatto quadrature rule, which results in the quadrature points

$\{\tilde{x}_1^j, \tilde{x}_2^j, \ldots, \tilde{x}_{6N_q-5}^j\}$ and quadrature weights $\{\tilde{\omega}_1^j, \omega_2^j, \ldots, \tilde{\omega}_{6N_q-5}^j\}$ on the interval I_j as follows,

$$\tilde{x}_\alpha^j = \begin{cases} x_{j-\frac{1}{2}} + \left(\frac{\hat{x}_\alpha+1}{2}\right)\Delta x_{j-\frac{1}{2}} & \alpha = 1,2,\ldots,2N_q-1, \\ x_{j-\frac{1}{2}} + \Delta x_{j-\frac{1}{2}} + \left(\frac{\hat{x}_{\alpha-2N_q+2}+1}{2}\right)\left(\Delta x_j - \Delta x_{j-\frac{1}{2}} - \Delta x_{j+\frac{1}{2}}\right) & \alpha = 2N_q,\ldots,4N_q-3, \\ x_{j+\frac{1}{2}} - \Delta x_{j+\frac{1}{2}} + \left(\frac{\hat{x}_{\alpha-4N_q+4}+1}{2}\right)\Delta x_{j+\frac{1}{2}} & \alpha = 4N_q-2,\ldots,6N_q-5, \end{cases}$$

and

$$\tilde{\omega}_\alpha^j = \begin{cases} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}\hat{\omega}_\alpha & \alpha = 1,2,\ldots,2N_q-2 \\ \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}\hat{\omega}_{2N_q-1} + \left(1 - \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} - \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}\right)\hat{\omega}_1 & \alpha = 2N_q-1, \\ \left(1 - \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} - \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}\right)\hat{\omega}_{\alpha-2N_q+2} & \alpha = 2N_q,\ldots,4N_q-4, \\ \left(1 - \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} - \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}\right)\hat{\omega}_{2N_q-1} + \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}\hat{\omega}_1 & \alpha = 4N_q-3, \\ \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}\hat{\omega}_{\alpha-4N_q+4} & \alpha = 4N_q-2,\ldots,6N_q-5, \end{cases}$$

respectively, where $\{\hat{x}_\alpha, \alpha = 1,2,\ldots,2N_q-1\}$ and $\{\hat{\omega}_\alpha, \alpha = 1,2,\ldots,2N_q-1\}$ are the Gauss-Lobatto points on $[-1,1]$ and weights satisfying $\sum_{\alpha=1}^{2N_q-1}\hat{\omega}_\alpha = 1$.

We redefine the DDG flux on nonuniform meshes:

$$\widehat{u_x}_{j+\frac{1}{2}}^{\mathrm{DDG}} = \beta_0 \frac{[u]_{j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} + \{u_x\}_{j+\frac{1}{2}} + \beta_1 \Delta x_{j+\frac{1}{2}} [u_{xx}]_{j+\frac{1}{2}}, \tag{C.5}$$

where β_0, β_1 are penalty parameters satisfying $\frac{1}{8} < \beta_1 < \frac{1}{4}, \beta_0 > \frac{3}{2} - 4\beta_1, j = 1,2,\ldots,N$ as in the uniform meshes.

Similarly, we have the expansion of DDG fluxes for $u \in V$.

$$\begin{aligned}\widehat{u_x}_{j+\frac{1}{2}}^{\mathrm{DDG}} =& \frac{1}{\Delta x_{j+\frac{1}{2}}}\Big((\frac{1}{2} - 4\beta_1)\tilde{u}^5 + (8\beta_1 - 2)\tilde{u}^6 + (-\beta_0 + \frac{3}{2} - 4\beta_1)u_{j+\frac{1}{2}}^- \\ &+ (\beta_0 - \frac{3}{2} + 4\beta_1)u_{j+\frac{1}{2}}^+ + (2 - 8\beta_1)\tilde{u}^7 + (4\beta_1 - \frac{1}{2})\tilde{u}^8\Big)\end{aligned}$$

and

$$\begin{aligned}\widehat{u_x}_{j-\frac{1}{2}}^{\mathrm{DDG}} =& \frac{1}{\Delta x_{j-\frac{1}{2}}}\Big((\frac{1}{2} - 4\beta_1)\tilde{u}^1 + (8\beta_1 - 2)\tilde{u}^2 + (-\beta_0 + \frac{3}{2} - 4\beta_1)u_{j-\frac{1}{2}}^- \\ &+ (\beta_0 - \frac{3}{2} + 4\beta_1)u_{j-\frac{1}{2}}^+ + (2 - 8\beta_1)\tilde{u}^3 + (4\beta_1 - \frac{1}{2})\tilde{u}^4\Big)\end{aligned}$$

The main result is as follows,

Theorem C.3. *Given $m \le u^n \le M$ and the DDG flux* (C.5)*, the cell averages $\bar{u}_j^{n+1}, j = 1,2,\ldots,N$ of the solution of scheme* (2.4) *are bounded between m and M under the CFL condition* (C.6).

$$\Delta t \le \min\{q_1, q_2, \dots, q_7\}, \tag{C.6}$$

where $q_1 = \frac{\hat{\omega}_1}{2M_1}\min_j \Delta x_j$, $q_2 = \frac{4\beta_1 - \frac{1}{2}}{5(M-m)M_2 + \frac{4}{3}M_1}\min_j \Delta x_{j+\frac{1}{2}}$, $q_3 = \frac{2-8\beta_1}{20(M-m)M_2+\frac{8}{3}M_1}\min_j \Delta x_{j+\frac{1}{2}}$, $q_4 = \frac{\beta_0 - \frac{3}{2} + 4\beta_1}{15(M-m)M_2 + \frac{4}{3}M_1}\min_j \Delta x_{j+\frac{1}{2}}$, $q_5 = \left(\frac{2\hat{\omega}_1}{3M_1^2(\beta_0 - \frac{3}{2}+4\beta_1)}\right)^{\frac{1}{2}}\min_j \Delta x_{j+\frac{1}{2}}$, $q_6 = \left(\frac{2\hat{\omega}_{N_q}}{3M_1^2(2-8\beta_1)}\right)^{\frac{1}{2}}\min_j \Delta x_{j+\frac{1}{2}}$, $q_7 = \left(\frac{4\hat{\omega}_1}{3M_1^2(4\beta_1-\frac{1}{2})}\right)^{\frac{1}{2}}\min_j \Delta x_{j+\frac{1}{2}}$

Proof. We have exactly the same results as in (2.13) and (2.14), but now II is expanded differently:

$$\begin{aligned}
\mathrm{II} =& \frac{1}{2}\sum_{\gamma=2}^{N_q-1}\omega_\gamma^j u^\gamma + \frac{1}{2}\sum_{\gamma=N_q+1}^{2N_q-2}\omega_\gamma^j u^\gamma + \frac{1}{2}\sum_{\gamma=2N_q}^{4N_q-4}\omega_\gamma^j u^\gamma + \frac{1}{2}\sum_{\gamma=4N_q-2}^{5N_q-5}\omega_\gamma^j u^\gamma + \frac{1}{2}\sum_{\gamma=5N_q-3}^{6N_q-5}\omega_\gamma^j u^\gamma \\
&+ z_1\tilde{u}^1 + z_2\tilde{u}^2 + z_3 u^-_{j-\frac{1}{2}} + z_4 u^+_{j-\frac{1}{2}} + z_5\tilde{u}^3 + z_6\tilde{u}^4 \\
&+ z_7\tilde{u}^5 + z_8\tilde{u}^6 + z_9 u^-_{j+\frac{1}{2}} + z_{10} u^+_{j+\frac{1}{2}} + z_{11}\tilde{u}^7 + z_{12}\tilde{u}^8,
\end{aligned}$$

where

$$z_1 = \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2-}_{j-\frac{1}{2}}\left((4\beta_1 - \frac{1}{2}) + \Delta t {f''}^-_{j-\frac{1}{2}}{u_x}^-_{j-\frac{1}{2}} + \frac{4}{3}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^-_{j-\frac{1}{2}}\right) + \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2+}_{j-\frac{1}{2}}(4\beta_1 - \frac{1}{2}),$$

$$z_2 = \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2-}_{j-\frac{1}{2}}\left((2-8\beta_1) - 4\Delta t {f''}^-_{j-\frac{1}{2}}{u_x}^-_{j-\frac{1}{2}} - \frac{8}{3}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^-_{j-\frac{1}{2}}\right) + \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2+}_{j-\frac{1}{2}}(2-8\beta_1)$$

$$z_3 = \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2-}_{j-\frac{1}{2}}\left((\beta_0 - \frac{3}{2} + 4\beta_1) + 3\Delta t {f''}^-_{j-\frac{1}{2}}{u_x}^-_{j-\frac{1}{2}} + \frac{4}{3}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^-_{j-\frac{1}{2}}\right) + \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2+}_{j-\frac{1}{2}}(\beta_0 - \frac{3}{2} + 4\beta_1)$$

$$z_4 = \frac{1}{2}\omega_1^j - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2-}_{j-\frac{1}{2}}(\beta_0 - \frac{3}{2} + 4\beta_1) - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2+}_{j-\frac{1}{2}}\left((\beta_0 - \frac{3}{2} + 4\beta_1) + 3\Delta t {f''}^+_{j-\frac{1}{2}}{u_x}^+_{j-\frac{1}{2}} - \frac{4}{3}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^+_{j-\frac{1}{2}}\right)$$

$$z_5 = \frac{1}{2}\omega_{N_q}^j - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2-}_{j-\frac{1}{2}}(2-8\beta_1) - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2+}_{j-\frac{1}{2}}\left((2-8\beta_1) - 4\Delta t {f''}^+_{j-\frac{1}{2}}{u_x}^+_{j-\frac{1}{2}} + \frac{8}{3}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^+_{j-\frac{1}{2}}\right)$$

$$z_6 = \frac{1}{2}\omega_{2N_q-1}^j - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2-}_{j-\frac{1}{2}}(4\beta_1 - \frac{1}{2}) - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^{2+}_{j-\frac{1}{2}}\left((4\beta_1 - \frac{1}{2}) + \Delta t {f''}^+_{j-\frac{1}{2}}{u_x}^+_{j-\frac{1}{2}} - \frac{4}{3}\frac{\Delta t}{\Delta x_{j-\frac{1}{2}}}{f'}^+_{j-\frac{1}{2}}\right)$$

$$z_7 = \frac{1}{2}\omega_{4N_q-3}^j - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^{2-}_{j+\frac{1}{2}}\left((4\beta_1 - \frac{1}{2}) + \Delta t {f''}^-_{j+\frac{1}{2}}{u_x}^-_{j+\frac{1}{2}} + \frac{4}{3}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^-_{j+\frac{1}{2}}\right) - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^{2+}_{j+\frac{1}{2}}(4\beta_1 - \frac{1}{2})$$

$$z_8 = \frac{1}{2}\omega_{5N_q-4}^j - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^{2-}_{j+\frac{1}{2}}\left((2-8\beta_1) - 4\Delta t {f''}^-_{j+\frac{1}{2}}{u_x}^-_{j+\frac{1}{2}} - \frac{8}{3}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^-_{j+\frac{1}{2}}\right) - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^{2+}_{j+\frac{1}{2}}(2-8\beta_1)$$

$$z_9 = \frac{1}{2}\omega_{6N_q-5}^j - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^{2-}_{j+\frac{1}{2}}\left((\beta_0 - \frac{3}{2} + 4\beta_1) + 3\Delta t {f''}^-_{j+\frac{1}{2}}{u_x}^-_{j+\frac{1}{2}} + \frac{4}{3}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^-_{j+\frac{1}{2}}\right) - \frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac{1}{2}}}{f'}^{2+}_{j+\frac{1}{2}}(\beta_0 - \frac{3}{2} + 4\beta_1)$$

$$z_{10}=\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^-_{j+\frac12}(\beta_0-\frac32+4\beta_1)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^+_{j+\frac12}\left((\beta_0-\frac32+4\beta_1)+3\Delta t {f''}^+_{j+\frac12}{u_x}^+_{j+\frac12}-\frac43\frac{\Delta t}{\Delta x_{j+\frac12}}{f'}^+_{j+\frac12}\right)$$

$$z_{11}=\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^-_{j+\frac12}(2-8\beta_1)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^+_{j+\frac12}\left((2-8\beta_1)-4\Delta t {f''}^+_{j+\frac12}{u_x}^+_{j+\frac12}+\frac83\frac{\Delta t}{\Delta x_{j+\frac12}}{f'}^+_{j+\frac12}\right)$$

$$z_{12}=\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^-_{j+\frac12}(4\beta_1-\frac12)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^+_{j+\frac12}\left((4\beta_1-\frac12)+\Delta t {f''}^+_{j+\frac12}{u_x}^+_{j+\frac12}-\frac43\frac{\Delta t}{\Delta x_{j+\frac12}}{f'}^+_{j+\frac12}\right)$$

One can verify that

$$\frac12\sum_{\gamma=2}^{N_q-1}\omega^j_\gamma+\frac12\sum_{\gamma=N_q+1}^{2N_q-2}\omega^j_\gamma+\frac12\sum_{\gamma=2N_q}^{4N_q-4}\omega^j_\gamma+\frac12\sum_{\gamma=4N_q-2}^{5N_q-5}\omega^j_\gamma+\frac12\sum_{\gamma=5N_q-3}^{6N_q-5}\omega^j_\gamma$$
$$+z_1+z_2+z_3+z_4+z_5+z_6+z_7+z_8+z_9+z_{10}+z_{11}+z_{12}=\frac12,$$

and

$$z_1\ge\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}{f'^2}^-_{j-\frac12}\left((4\beta_1-\frac12)-5(M-m)M_2\frac{\Delta t}{\Delta x_{j-\frac12}}-\frac43\frac{\Delta t}{\Delta x_{j-\frac12}}M_1\right)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}{f'^2}^+_{j-\frac12}(4\beta_1-\frac12)\ge0,$$

$$z_2\ge\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}{f'^2}^-_{j-\frac12}\left((2-8\beta_1)-20(M-m)M_2\frac{\Delta t}{\Delta x_{j-\frac12}}-\frac83\frac{\Delta t}{\Delta x_{j-\frac12}}M_1\right)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}{f'^2}^+_{j-\frac12}(2-8\beta_1)\ge0,$$

$$z_3\ge\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}{f'^2}^-_{j-\frac12}\left((\beta_0-\frac32+4\beta_1)-15(M-m)M_2\frac{\Delta t}{\Delta x_{j-\frac12}}-\frac43\frac{\Delta t}{\Delta x_{j-\frac12}}M_1\right)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}{f'^2}^+_{j-\frac12}(\beta_0-\frac32+4\beta_1)\ge0,$$

$$z_4\ge\frac12\omega^j_1-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}M_1^2(\beta_0-\frac32+4\beta_1)-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}M_1^2\left((\beta_0-\frac32+4\beta_1)+15(M-m)M_2\frac{\Delta t}{\Delta x_{j-\frac12}}+\frac43\frac{\Delta t}{\Delta x_{j-\frac12}}M_1\right)\ge0,$$

$$z_5\ge\frac12\omega^j_{N_q}-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}M_1^2(2-8\beta_1)-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}M_1^2\left((2-8\beta_1)+20(M-m)M_2\frac{\Delta t}{\Delta x_{j-\frac12}}+\frac83\frac{\Delta t}{\Delta x_{j-\frac12}}M_1\right)\ge0,$$

$$z_6\ge\frac12\omega^j_{2N_q-1}-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}M_1^2(4\beta_1-\frac12)-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j-\frac12}}M_1^2\left((4\beta_1-\frac12)+5(M-m)M_2\frac{\Delta t}{\Delta x_{j-\frac12}}+\frac43\frac{\Delta t}{\Delta x_{j-\frac12}}M_1\right)\ge0,$$

$$z_7\ge\frac12\omega^j_{4N_q-3}-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}M_1^2\left((4\beta_1-\frac12)+5(M-m)M_2\frac{\Delta t}{\Delta x_{j+\frac12}}+\frac43\frac{\Delta t}{\Delta x_{j+\frac12}}M_1\right)-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}M_1^2(4\beta_1-\frac12)\ge0,$$

$$z_8\ge\frac12\omega^j_{5N_q-4}-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}M_1^2\left((2-8\beta_1)+20(M-m)M_2\frac{\Delta t}{\Delta x_{j+\frac12}}+\frac83\frac{\Delta t}{\Delta x_{j+\frac12}}M_1\right)-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}M_1^2(2-8\beta_1)\ge0,$$

$$z_9\ge\frac12\omega^j_{6N_q-5}-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}M_1^2\left((\beta_0-\frac32+4\beta_1)+15(M-m)M_2\frac{\Delta t}{\Delta x_{j+\frac12}}+\frac43\frac{\Delta t}{\Delta x_{j+\frac12}}M_1\right)-\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}M_1^2(\beta_0-\frac32+4\beta_1)\ge$$

$$z_{10}\ge\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^-_{j+\frac12}(\beta_0-\frac32+4\beta_1)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^+_{j+\frac12}\left((\beta_0-\frac32+4\beta_1)-15(M-m)M_2\frac{\Delta t}{\Delta x_{j+\frac12}}-\frac43\frac{\Delta t}{\Delta x_{j+\frac12}}M_1\right)\ge0,$$

$$z_{11}\ge\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^-_{j+\frac12}(2-8\beta_1)+\frac{\lambda_j}{4}\frac{\Delta t}{\Delta x_{j+\frac12}}{f'^2}^+_{j+\frac12}\left((2-8\beta_1)-20(M-m)M_2\frac{\Delta t}{\Delta x_{j+\frac12}}-\frac83\frac{\Delta t}{\Delta x_{j+\frac12}}M_1\right)\ge0,$$

$$z_{12} \geq \frac{\lambda_j}{4} \frac{\Delta t}{\Delta x_{j+\frac{1}{2}}} f'^{2-}_{j+\frac{1}{2}} (4\beta_1 - \frac{1}{2}) + \frac{\lambda_j}{4} \frac{\Delta t}{\Delta x_{j+\frac{1}{2}}} f'^{2+}_{j+\frac{1}{2}} \left((4\beta_1 - \frac{1}{2}) - 5(M-m) M_2 \frac{\Delta t}{\Delta x_{j+\frac{1}{2}}} - \frac{4}{3} \frac{\Delta t}{\Delta x_{j+\frac{1}{2}}} M_1 \right) \geq 0,$$

under the CFL condition (C.6).

Therefore, we have $m \leq \bar{u}_j^{n+1} \leq M, j = 1, 2, \ldots, N$ following the same arguments as before. □

C.3 Numerical tests on nonuniform meshes

We demonstrate the accuracy and effectiveness of the maximum-principle-satisfying algorithm established in Section C.1 and Section C.2 on nonuniform meshes.

Example C.1. *We solve the linear equation $u_t + u_x = 0$ in the domain $\Omega = [-1, 1]$ with periodic boundary conditions and discontinuous initial condition*

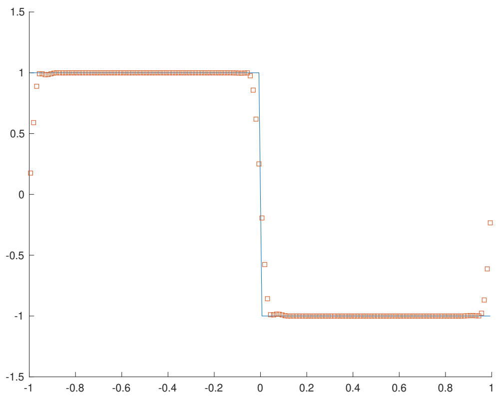
$$u_0(x) = \begin{cases} 1, & -1 \leq x \leq 0, \\ -1, & 0 \leq x \leq 1. \end{cases}$$

and take the terminal time $T = 100$ to show the effect of the maximum-principle-preserving.

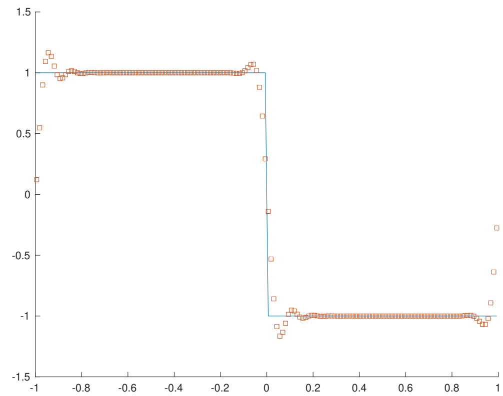
We solve the Burgers' equation $u_t + \left(\frac{u^2}{2}\right)_x = 0$ in the domain $\Omega = [0, 2\pi]$ with initial condition $u_0(x) = \frac{1}{2} + \sin(x)$ and periodic boundary conditions, and take the terminal time $T = 0.3$ to show the accuracy.

For the algorithm established in Section C.1, we generate the nonuniform meshes by adding uniformly distributed perturbation within $[-0.1\Delta x, 0.1\Delta x]$ on the inner nodes of the uniform mesh. For the algorithm established in Section C.2, we generate the nonuniform meshes by adding uniformly distributed perturbation within $[-0.3\Delta x, 0.3\Delta x]$ on the inner nodes of the uniform mesh.

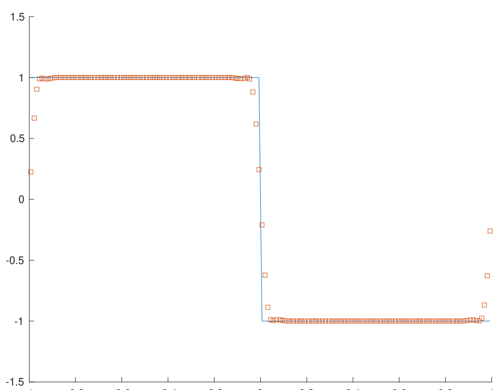
The results are given in Table 7 and Figure 13, from which we can observe the third order accuracy and maximum-principle-preserving effect.



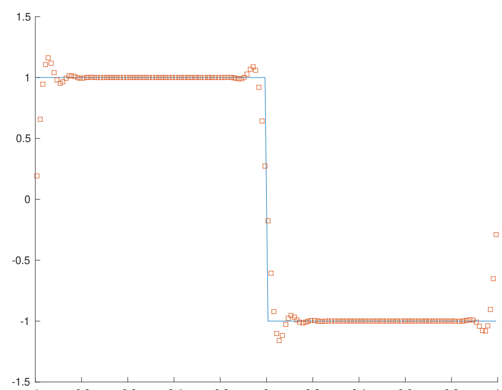
(a) Algorithm C.1 with limiter



(b) Algorithm C.1 without limiter



(c) Algorithm C.2 with limiter



(d) Algorithm C.2 without limiter

Figure 13: Results of Example C.1 with discontinuous initial condition at $T = 100$. $N = 160$. Solid line: exact solution; Squares: numerical solution (cell averages).

	Algorithm C.1				Algorithm C.2			
N	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
20	9.33E-04	–	1.50E-03	–	1.19E-03	–	2.37E-03	–
40	1.15E-04	3.02	2.38E-04	2.65	1.44E-04	3.05	3.61E-04	2.72
80	1.41E-05	3.03	4.09E-05	2.54	1.90E-05	2.92	8.56E-05	2.08
160	1.73E-06	3.03	5.56E-06	2.88	2.01E-06	3.24	9.33E-06	3.20
320	2.11E-07	3.03	8.14E-07	2.77	2.72E-07	2.89	1.69E-06	2.46
640	2.59E-08	3.03	1.04E-07	2.96	3.23E-08	3.07	1.93E-07	3.13

Table 7: Results of Example C.1, Burgers' equation at $T = 0.3$