ANALYSIS OF LOCAL DISCONTINUOUS GALERKIN METHODS WITH IMPLICIT-EXPLICIT TIME MARCHING FOR LINEARIZED KDV EQUATIONS *

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Abstract. In this paper, we present the stability and error analysis of two fully discrete IMEX-LDG schemes, combining local discontinuous Galerkin (LDG) spatial discretization with implicit-explicit (IMEX) Runge-Kutta temporal discretization, for the linearized one-dimensional KdV equations. The energy stability analysis begins with a series of temporal differences about stage solutions. Then by exploring the stability mechanism from the temporal differences, and by constructing the semi-negative definite *symmetric* form related to the discretization of the dispersion term, and by adopting the important relationships between the auxiliary variables with the prime variable to control the anti-dissipation terms, we derive the unconditional stability for a *discrete energy* involving the prime variable and all the auxiliary variables, in the sense that the time step is bounded by a constant that is independent of the spatial mesh size. We also propose a new projection technique and adopt the technique of summation by parts in the time direction to achieve the optimal order of accuracy. The new projection technique can serve as an analytical tool to be applied to general odd order wave equations. Finally, numerical experiments are shown to test the stability and accuracy of the considered schemes.

Key words. local discontinuous Galerkin, implicit-explicit, linearized KdV equation, stability analysis, error estimates.

MSC codes. 65M12, 65M15, 65M60

1. Introduction. The Korteweg–de Vries (KdV) equations are important mathematical models that describe the propagation of nonlinear dispersive waves in many engineering applications, such as aerology, geology, oceanography, plasma physic, etc. In this paper, we consider the stability and error analysis of a kind of fully discrete numerical schemes for solving the linearized one-dimensional KdV equation

$$U_t + aU_x = cU_{xxx}, \quad x \in \Omega = (x_l, x_r), \quad t \in (0, T], \tag{1.1}$$

coupled with the periodic boundary condition and the initial condition $U(x, 0) = U_0(x)$. Here aU_x and cU_{xxx} are called the convection term and the dispersion term, respectively. Without loss of generality, we assume that both a and c are positive constants. The fully discrete schemes are defined by following the method-of-lines framework, where the local discontinuous Galerkin (LDG) method [8] is applied in space and the Runge-Kutta (RK) type implicit-explicit (IMEX) method [2] is adopted in time. We call these fully discrete schemes as the IMEX-LDG schemes for short.

To motivate our effort, we first review some work in the literature. In [26], Yan and Shu proposed a first LDG scheme for the KdV equations, by introducing the first and the second order spatial derivatives of the prime variable (exact solution)

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as auxiliary variables, and they proved L^2 stability and sub-optimal error estimates of the semidiscrete LDG scheme for the linearized KdV equation. The optimal error estimates were later obtained by Xu and Shu in [24]. Superconvergent properties of the LDG scheme for linearized and nonlinear KdV equations were also studied, see for example [12, 3]. In the above works, upwind numerical flux is used for the convection term and alternating/upwind numerical flux is used for the dispersion term. Recently, Li et al. [13] presented the stability and optimal error analysis of LDG schemes with generalized numerical fluxes for the linearized KdV equation. Hybridizable discontinuous Galerkin (HDG) method for KdV equations were also analyzed [6, 9]. In the above mentioned semidiscrete works, purely explicit or purely implicit time discretization methods were adopted in numerical experiments.

As is well known, explicit time discretization methods often suffer from severe time step restriction to ensure numerical stability, when they are used for solving time-dependent partial differential equations (PDEs) with high order spatial derivatives. Even though implicit time discretization methods can overcome the small time step restriction, they always require solving nonlinear systems when there are nonlinear terms in the PDEs. IMEX time discretization methods, treating the linear (or stiff) part implicitly and the nonlinear (or non-stiff) part explicitly, can provide a good balance of the computational efficiency and numerical stability. It has been shown that such type of time discretization methods are efficient for dissipative equations, such as convection-diffusion equations [20] and time-dependent fourth order PDEs [22], where the time steps are allowed to be independent of the spatial mesh size. In fact, IMEX time discretization methods are also efficient for KdV type equations, see for example the numerical experiments and formal Fourier analysis in [23, 11, 29, 14, 27, 18, 17]. However, there are rarely rigourous theoretical analysis of the fully discrete IMEX-LDG schemes for solving KdV equations. The objective of this work is to attempt energy analysis on the stability and error estimates of fully discrete IMEX-LDG schemes for the linearized KdV equation. For equation (1.1), we discretize the convection term explicitly and the dispersion term implicitly.

The energy analysis of IMEX-LDG schemes for dispersive equations is a challenging work. On one hand, it inherits the difficulty of energy analysis for semidiscrete LDG methods. Namely, the dispersive equations lack coercivity, so the "stability" coming from the implicit discretization for the dispersion term is too weak to control the anti-dissipation coming from the explicit discretization for the convection term. On the other hand, it is not trivial to establish suitable energy equations for the fully discrete schemes. Totally different from the dissipative problems, the LDG operator for the dispersive equations is not symmetric so that the useful energy equations are not easily derived. We need to construct some "symmetric" forms about the LDG discretization for the dispersion term, with the purpose of getting a non-positive definite quadratic form. However, this task is generally hard to fulfill, especially for high order in time IMEX-LDG schemes.

To overcome the aforementioned difficulties, we first follow the idea of [25] and [19] to introduce a series of temporal differences about stage solutions, then we follow the practice in the semidiscrete analysis [24, 13] to establish energy equations for the prime variable as well as the auxiliary variables. During the process of energy analysis, we find out the stability mechanism from the temporal differences, meanwhile we successfully construct semi-negative symmetric from about the dispersion term. In addition, we adopt the important relationships between auxiliary variables and the prime variable (see Lemma 2.4) to estimate the anti-dissipation terms. Our main conclusion is that, for the first and second order in time IMEX-LDG schemes we consider in this paper, the "discrete energy" that involves the prime variable and all the auxiliary variables is unconditionally stable under the time step $\tau \le \tau_0$, where τ_0 is a fixed constant depending on the coefficients of convection and dispersion but independent of the mesh size h. The process of stability analysis is tedious, especially for the second order in time IMEX-LDG methods. Till now we have not obtained the similar result for higher order schemes, due to intricate relationships between each intermediate stages.

Regarding the error estimates, we would like to comment that it is a bit cumbersome to achieve the optimal error estimates by directly adopting the commonly-used Gauss-Radau (GR) projection [5]. Hence, we introduce a new projection technique, by which we solve the interactive influences of errors at different intermediate stages, and thus we achieve the optimal error estimates in a much simpler way. It is worth pointing out that, the new projection can also simplify the semidiscrete error analysis greatly, compared with that obtained in [24] by the GR projection. As far as we know, this is the first time that this kind of projection is used in the error analysis for KdV type problems. Besides, we emphasize the extra difficulty for the second order scheme, as there are some tricky terms that will prevent us from getting the optimal error estimates if they are dealt with directly at each time level. We will adopt the technique of summation by parts in the time direction to solve this problem.

This paper is organized as follows. In section 2 we present the semidiscrete LDG scheme and some related properties. In section 3 we give the stability analysis for the first order and the second order in time IMEX-LDG schemes. Section 4 is devoted to the error analysis for the IMEX-LDG schemes. In section 5 we present numerical results to test the stability and accuracy of the considered schemes. The conclusion is given in section 6 and the proof of a technical lemma is put in the Appendix.

2. Semidiscrete LDG scheme and related properties. In this section, we first define the semidiscrete LDG scheme for solving the linearized KdV equation (1.1), and then present some related properties that will be used in this paper.

2.1. The semidiscrete LDG scheme. Let $\mathcal{T}_h = \{I_j = (x_{j-1/2}, x_{j+1/2})\}_{j=1}^N$ be a partition of Ω, where $x_{1/2} = x_l$ and $x_{N+1/2} = x_r$. Denote by $h_j = x_{j+1/2} - x_{j-1/2}$ the length of cell I_j, for $j = 1, \ldots, N$. Define $h = \max_j h_j$ as the mesh size. In this paper, $\mathcal{T}_h$ is assumed to be quasi-uniform, namely, there exists a constant $\rho > 0$, such that all $h_j/h \ge \rho$ as h goes to zero. Here we call ρ the mesh regularity parameter.

Associated with $\mathcal{T}_h$, we introduce the discontinuous finite element space

$$V_h = V_h^k = \left\{ \phi \in L^2(\Omega) : \phi|_{I_j} \in \mathcal{P}_k(I_j),\ \forall j = 1, \ldots, N \right\}, \tag{2.1}$$

where $\mathcal{P}_k(I_j)$ denotes the space of polynomials in I_j of degree less than or equal to k. A main feature is that $\phi \in V_h$ may be discontinuous across the interfaces of elements. As usual, two traces along the left- and right-hand of the interface are denoted by ϕ^- and ϕ^+, respectively, and the jump is denoted by $[\![\phi]\!] = \phi^+ - \phi^-$.

Following the framework of LDG methods, we introduce two auxiliary variables $Q = U_x$ and $P = Q_x$ and then rewrite (1.1) into the following equivalent form

$$U_t + aU_x - cP_x = 0, \tag{2.2a}$$

$$P - Q_x = 0, \tag{2.2b}$$

$$Q - U_x = 0. \tag{2.2c}$$

The semidiscrete LDG scheme is then defined as follows: find the map

$$\boldsymbol{w}(\cdot, t) = (u(\cdot, t), p(\cdot, t), q(\cdot, t)) \colon [0, T] \to [V_h]^3,$$

such that the following variational forms [26, 24]

$$
\begin{aligned}
&(u_t, v)_j - a(u, v_x)_j + a\tilde{u}_{j+1/2}v^-_{j+1/2} - a\tilde{u}_{j-1/2}v^+_{j-1/2} \\
&\quad + c(p, v_x)_j - c\hat{p}_{j+1/2}v^-_{j+1/2} + c\hat{p}_{j-1/2}v^+_{j-1/2} = 0, \quad \forall v \in V_h, \qquad (2.3a)
\end{aligned}
$$

$$(p, \phi)_j + (q, \phi_x)_j - \hat{q}_{j+1/2}\phi^-_{j+1/2} + \hat{q}_{j-1/2}\phi^+_{j-1/2} = 0, \quad \forall \phi \in V_h, \tag{2.3b}$$

$$(q, \psi)_j + (u, \psi_x)_j - \hat{u}_{j+1/2}\psi^-_{j+1/2} + \hat{u}_{j-1/2}\psi^+_{j-1/2} = 0, \quad \forall \psi \in V_h, \tag{2.3c}$$

hold for any $t \in (0, T]$ and $j = 1, 2, \dots, N$. The definition of the numerical initial solution $\boldsymbol{w}(\cdot, 0)$ will be given in subsection 4.2.

In the above formulas, $(w, v)_j = \int_{I_j} w(x)v(x)dx$ and $\tilde{u}, \hat{p}, \hat{q}, \hat{u}$ are numerical fluxes. In this paper we take

$$\tilde{u} = u^- \tag{2.4}$$

for the convection term and

$$\hat{p} = p^+, \quad \hat{q} = q^-, \quad \hat{u} = u^- \tag{2.5}$$

for the dispersion term. Note that $w^-_{1/2} = w^-_{N+1/2}$ and $w^+_{N+1/2} = w^+_{1/2}$, due to periodic boundary condition. Based on the above choice of numerical fluxes, with the following notation

$$\mathcal{H}^\pm_j(w, v) = (w, v_x)_j - w^\pm_{j+1/2}v^-_{j+1/2} + w^\pm_{j-1/2}v^+_{j-1/2}, \tag{2.6}$$

we can rewrite the semidiscrete LDG scheme as

$$(u_t, v)_j = a\mathcal{H}^-_j(u, v) - c\mathcal{H}^+_j(p, v), \quad \forall v \in V_h, \tag{2.7a}$$

$$(p, \phi)_j = -\mathcal{H}^-_j(q, \phi), \quad \forall \phi \in V_h, \tag{2.7b}$$

$$(q, \psi)_j = -\mathcal{H}^-_j(u, \psi), \quad \forall \psi \in V_h. \tag{2.7c}$$

Furthermore, we denote $(\cdot, \cdot) = \sum_{j=1}^N (\cdot, \cdot)_j$ and $\mathcal{H}^\pm = \sum_{j=1}^N \mathcal{H}^\pm_j$. By summing the variational formulations (2.7) over all cells, we get the semidiscrete LDG scheme in the global form: find $\boldsymbol{w}(\cdot, t) \in [V_h]^3$, such that

$$(u_t, v) = a\mathcal{H}^-(u, v) - c\mathcal{H}^+(p, v), \quad \forall v \in V_h, \tag{2.8a}$$

$$(p, \phi) = -\mathcal{H}^-(q, \phi), \quad \forall \phi \in V_h, \tag{2.8b}$$

$$(q, \psi) = -\mathcal{H}^-(u, \psi), \quad \forall \psi \in V_h. \tag{2.8c}$$

Remark 2.1. We would like to comment that the choices of numerical fluxes are not unique. The basic principle is using upwind numerical flux for the convection term and alternating numerical flux between $\hat{p}$ and $\hat{u}$ related to the dispersion term, and the choice of $\hat{q}$ depends on the sign of c (upwind for the dispersive wave). If $c < 0$ we should take $\hat{q} = q^+$ for the purpose of stability.

2.2. Inverse inequality. Throughout this paper, the notation $\|\cdot\|_j$ is used to represent the standard L^2 norm in cell I_j and $\|\cdot\|$ means the L^2 norm in the whole domain Ω. As well known, there holds the inverse inequality for any function $v \in V_h$:

$$\max\left\{|v^+_{j-1/2}|, |v^-_{j+1/2}|\right\} \le \sqrt{\nu h_j^{-1}}\|v\|_j \le \sqrt{\nu(\rho h)^{-1}}\|v\|_j. \tag{2.9}$$

Here ν is called the inverse constant, which solely depends on the degree of polynomial space, one can refer to [1, 16] for more details.

2.3. Some properties. In the following we recall some properties with respect to the bilinear forms $\mathcal{H}^{\pm}$. For convenience of notation, we denote

$$\langle [\![w]\!], [\![v]\!] \rangle = \sum_{j=1}^{N} [\![w]\!]_{j-1/2} [\![v]\!]_{j-1/2}, \qquad [\![v]\!]^2 = \langle [\![v]\!], [\![v]\!] \rangle.$$

Some simple applications of integration by parts yield the following elemental conclusions, which have been given in [28, 20].

LEMMA 2.2. *For any $w, v \in V_h$, there hold the equalities*

$$\mathcal{H}^{\pm}(w, v) + \mathcal{H}^{\pm}(v, w) = \pm \langle [\![w]\!], [\![v]\!] \rangle, \tag{2.10}$$

$$\mathcal{H}^{\pm}(v, v) = \pm \frac{1}{2} [\![v]\!]^2, \tag{2.11}$$

$$\mathcal{H}^{-}(w, v) + \mathcal{H}^{+}(v, w) = 0. \tag{2.12}$$

By Cauchy-Schwarz inequality and the inverse inequality (2.9), we easily get the following boundedness properties [28, 20] of $\mathcal{H}^{\pm}$.

LEMMA 2.3. *For any $w, v \in V_h$, there hold the following inequalities*

$$|\mathcal{H}^{\pm}(w, v)| \leq \left(\|w_x\| + \sqrt{\nu(\rho h)^{-1}} [\![w]\!] \right) \|v\|, \tag{2.13}$$

$$|\mathcal{H}^{\pm}(w, v)| \leq \left(\|v_x\| + \sqrt{\nu(\rho h)^{-1}} [\![v]\!] \right) \|w\|. \tag{2.14}$$

The following compositions play important role in this paper. Let $m \geq 1$ be an integer, we consider a group of triple-functions pairs $(u_i, p_i, q_i) \in [V_h]^3$, for $i = 1, 2, \ldots, m$, satisfying the kernel relationship

$$(p_i, \phi) = -\mathcal{H}^{-}(q_i, \phi), \quad \forall \phi \in V_h, \tag{2.15a}$$

$$(q_i, \psi) = -\mathcal{H}^{-}(u_i, \psi), \quad \forall \psi \in V_h. \tag{2.15b}$$

If $m = 1$ the subscripts will be dropped for simplicity. From Lemma 2.2 we can easily obtain the following corollaries.

COROLLARY 2.1. *Let $m = 2$. For any two triple-functions pairs satisfying* (2.15), *we have*

$$\mathcal{H}^{+}(p_1, u_2) + \mathcal{H}^{+}(p_2, u_1) = \langle [\![q_1]\!], [\![q_2]\!] \rangle, \tag{2.16}$$

$$\mathcal{H}^{+}(p_1, u_1) = \frac{1}{2} [\![q_1]\!]^2. \tag{2.17}$$

COROLLARY 2.2. *Let $\mathbb{A} = \{a_{ij}\}_{i,j=1}^{m}$ be a symmetric positive semi-definite matrix, then for any triple-functions pairs satisfying* (2.15) *we have*

$$\underline{\mathcal{H}}^{+}(\boldsymbol{p}, \mathbb{A}\boldsymbol{u}) \doteq \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \mathcal{H}^{+}(p_i, u_j) \geq 0, \tag{2.18}$$

where $\boldsymbol{p} = (p_1, \cdots, p_m)^{\top}$ and $\boldsymbol{u} = (u_1, \cdots, u_m)^{\top}$.

Proof. From (2.16) and (2.17) in Corollary 2.1, we get

$$\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \mathcal{H}^{+}(p_i, u_j) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \langle [\![q_i]\!], [\![q_j]\!] \rangle.$$

This is a quadratic form, and (2.18) can be directly derived since $\mathbb{A}$ is positive semi-definite. ∎

The next lemma shows some important relationships between the numerical solutions of auxiliary variables and the prime variable, and it plays a key role in obtaining the stability results and error estimates in this paper.

LEMMA 2.4. *For any triple-functions pair (u,p,q) satisfying (2.15), there exists a positive constant $C_{\nu,\rho}$, solely dependent of ν and ρ, such that*

$$\|u_x\| + \sqrt{\nu(\rho h)^{-1}}[\![u]\!] \le C_{\nu,\rho}\|q\|, \tag{2.19}$$

$$\|q_x\| + \sqrt{\nu(\rho h)^{-1}}[\![q]\!] \le C_{\nu,\rho}\|p\|, \tag{2.20}$$

$$\|q\|^2 \le C_{\nu,\rho}\|u\|\|p\|. \tag{2.21}$$

Proof. One can refer to [20] for the proof of the first inequality, and the second inequality can be obtained similarly. The third inequality is directly obtained by taking $\psi = q$ in (2.15b) and using (2.14) and (2.20). ∎

3. IMEX-LDG schemes and stability analysis. For simplicity, let $\{t^n = n\tau\}_{n=0}^{M}$ be a uniform partition of the time interval $[0,T]$, with τ being the time step. We would like to compute the numerical solutions $\boldsymbol{w}^n = (u^n, p^n, q^n)$, at every time level t^n via the IMEXs-LDGk method, which adopts the s-th order RK type IMEX time discretization and the LDG spatial discretization with piecewise polynomials of degree at most k. In this paper, we mainly consider two schemes.

- The IMEX1-LDGk scheme is defined as follows: for any $n \ge 0$ there holds

$$(u^{n+1}, v) = (u^n, v) + a\tau\mathcal{H}^-(u^n, v) - c\tau\mathcal{H}^+(p^{n+1}, v), \quad \forall v \in V_h, \tag{3.1a}$$

  and the auxiliary variables satisfy

$$(p^n, \phi) = -\mathcal{H}^-(q^n, \phi), \quad \forall \phi \in V_h, \tag{3.1b}$$

$$(q^n, \psi) = -\mathcal{H}^-(u^n, \psi), \quad \forall \psi \in V_h. \tag{3.1c}$$

- The IMEX2-LDGk scheme [2] is defined as follows: for any $n \ge 0$ there holds

$$(u^{n,1}, v) = (u^n, v) + \gamma a\tau\mathcal{H}^-(u^n, v) - \gamma c\tau\mathcal{H}^+(p^{n,1}, v), \tag{3.2a}$$

$$\begin{aligned}(u^{n+1}, v) = {} & (u^n, v) + \delta a\tau\mathcal{H}^-(u^n, v) + (1-\delta)a\tau\mathcal{H}^-(u^{n,1}, v)\\ & - (1-\gamma)c\tau\mathcal{H}^+(p^{n,1}, v) - \gamma c\tau\mathcal{H}^+(p^{n+1}, v),\end{aligned} \tag{3.2b}$$

  where $\gamma = 1 - \frac{\sqrt{2}}{2}$ and $\delta = 1 - \frac{1}{2\gamma}$, and the auxiliary variables satisfy

$$(p^{n,\ell}, \phi) = -\mathcal{H}^-(q^{n,\ell}, \phi), \quad \ell = 0,1, \quad \forall \phi \in V_h, \tag{3.2c}$$

$$(q^{n,\ell}, \psi) = -\mathcal{H}^-(u^{n,\ell}, \psi), \quad \ell = 0,1, \quad \forall \psi \in V_h. \tag{3.2d}$$

  Here $\boldsymbol{w}^{n,0} = (u^{n,0}, p^{n,0}, q^{n,0}) = \boldsymbol{w}^n$, and $\boldsymbol{w}^{n,1} = (u^{n,1}, p^{n,1}, q^{n,1})$ mean the numerical solutions at the intermediate stage $t^{n,1} = t^n + \gamma\tau$.

In the next two subsections, we are going to carry out the stability analysis for the above two schemes. To that end, we define a "discrete energy" norm

$$\mathsf{E}^n = \|u^n\|^2 + c\tau\|p^n\|^2 + a\tau\|q^n\|^2, \tag{3.3}$$

and present the conclusion in the next theorem.

THEOREM 3.1. *There exist two constants $\tau_0 > 0$ and $C_\star > 1$, independent of n, h, τ, such that for $\tau \le \tau_0$ the above two schemes satisfy*

$$\mathsf{E}^n \le C_\star \mathsf{E}^0, \qquad \forall\ 1 \le n \le M. \tag{3.4}$$

Note that $C_\star$ depends on the final time T in general.

Remark 3.2. The stability result given in the above theorem is "sharp" to some extent, as to be seen in the numerical experiments. We observe that the L^2 norm of the numerical solution slightly increases with time under suitable time steps, which implies that the bounding constant $C_\star$ in the theorem can not be improved to be 1 in general. We also would like to mention that the constant $C_\star = 1$ seems to hold for lower order polynomials if the time step is small enough (perhaps being proportional to the mesh size), but this is not the direction that we pursue in this work. The aim of the present work is to get the *unconditional* stability, namely, the time step is independent of the mesh size.

The lines of proof for the two schemes are similar. Specially, we have to build up suitable energy equations and then carry out the energy analysis for the prime variable and auxiliary variables. During this process, the temporal difference technique plays an important role. In the following, the same notations with different meanings will be used in different subsections. The notations are independent in each subsection unless otherwise specified.

3.1. Proof for the IMEX1-LDGk scheme. Following the notations in [25, 19], we denote $\mathbb{D}_0 w^n = w^n$ and introduce the first order temporal difference $\mathbb{D}_1 w^n = w^{n+1} - w^n$ for $w = u, p, q$. It is obvious that

$$w^{n+1} = \mathbb{D}_1 w^n + \mathbb{D}_0 w^n. \tag{3.5}$$

Taking L^2 norm on both sides we get

$$\|w^{n+1}\|^2 - \|w^n\|^2 = \|\mathbb{D}_1 w^n\|^2 + 2(\mathbb{D}_1 w^n, \mathbb{D}_0 w^n) \doteq \mathcal{RHS}(w^n). \tag{3.6}$$

Note that this form of $\mathcal{RHS}(w^n)$ is not of very much use in deriving our desired stability results, since the first term $\|\mathbb{D}_1 w^n\|^2$ is an anti-dissipation term. If we estimate it directly then the mesh-size-dependent time step condition will be required. To obtain the unconditional stability, we consider the equivalent form

$$\mathcal{RHS}(w^n) = -\|\mathbb{D}_1 w^n\|^2 + \underbrace{2\|\mathbb{D}_1 w^n\|^2 + 2(\mathbb{D}_1 w^n, \mathbb{D}_0 w^n)}_{\mathfrak{R}(w^n)}. \tag{3.7}$$

The benefits of expressing $\|\mathbb{D}_1 w^n\|^2 = -\|\mathbb{D}_1 w^n\|^2 + 2\|\mathbb{D}_1 w^n\|^2$ are twofold. On the one hand, it provides an explicit stability description to control the anti-dissipation hidden in other terms. On the other hand, the combination term $\mathfrak{R}(w^n)$ will lead to a *symmetric* form about the spatial discretization of dispersion (see (3.11) below) that can be further converted to a semi-negative form.

With the above temporal differences, we would like to rewrite the fully discrete scheme as follows. From (3.1a) we get

$$(\mathbb{D}_1 u^n, v) = a\tau\mathcal{H}^-(\mathbb{D}_0 u^n, v) - c\tau\mathcal{H}^+(\mathbb{D}_0 p^n + \mathbb{D}_1 p^n, v), \tag{3.8a}$$

and by taking differences of (3.1b) and (3.1c) at two successive time levels, we have

$$(\mathbb{D}_\ell p^n, \phi) = -\mathcal{H}^-(\mathbb{D}_\ell q^n, \phi), \quad \text{for} \quad \ell = 0, 1, \tag{3.8b}$$

$$(\mathbb{D}_\ell q^n, \psi) = -\mathcal{H}^-(\mathbb{D}_\ell u^n, \psi), \quad \text{for} \quad \ell = 0, 1. \tag{3.8c}$$

These formulations will be used many times below.

In what follows we are going to carry out the energy analysis by three energy equations based on (3.6) and (3.7).

The first energy equation. By (3.8a), we can transfer $\mathfrak{R}(u^n)$ into the information of spatial discretization. Namely, it reads

$$\begin{aligned}\mathfrak{R}(u^n) &= 2(\mathbb{D}_1 u^n, \mathbb{D}_0 u^n + \mathbb{D}_1 u^n)\\ &= 2a\tau\mathcal{H}^-(\mathbb{D}_0 u^n, \mathbb{D}_0 u^n + \mathbb{D}_1 u^n) - 2c\tau\mathcal{H}^+(\mathbb{D}_0 p^n + \mathbb{D}_1 p^n, \mathbb{D}_0 u^n + \mathbb{D}_1 u^n)\\ &\doteq \mathsf{R}_a + \mathsf{R}_c,\end{aligned} \tag{3.9}$$

where R_a and R_c contain the informations related to the convection term and the dispersion term, respectively. According to (2.11), we have

$$\mathsf{R}_a = -a\tau[\![\mathbb{D}_0 u^n]\!]^2 + 2a\tau\mathcal{H}^-(\mathbb{D}_0 u^n, \mathbb{D}_1 u^n) \doteq \mathsf{R}_{a1} + \mathsf{R}_{a2}. \tag{3.10}$$

Owing to (3.5) and (2.17), we have

$$\mathsf{R}_c = -2c\tau\mathcal{H}^+(p^{n+1}, u^{n+1}) = -c\tau[\![q^{n+1}]\!]^2. \tag{3.11}$$

Therefore, we obtain the first energy equation from the above discussions that

$$\|u^{n+1}\|^2 - \|u^n\|^2 = -\|\mathbb{D}_1 u^n\|^2 - a\tau[\![\mathbb{D}_0 u^n]\!]^2 - c\tau[\![q^{n+1}]\!]^2 + \mathsf{R}_{a2}. \tag{3.12}$$

The second energy equation. To establish a proper energy equation for $\|p^n\|$, we start from two equivalent expressions of the term $2(\mathbb{D}_1 u^n, \mathbb{D}_1 q^n)$. Taking $v = 2\mathbb{D}_1 q^n$ in (3.8a), we have

$$\begin{aligned}2(\mathbb{D}_1 u^n, \mathbb{D}_1 q^n) &= 2a\tau\mathcal{H}^-(\mathbb{D}_0 u^n, \mathbb{D}_1 q^n) - 2c\tau\mathcal{H}^+(\mathbb{D}_0 p^n + \mathbb{D}_1 p^n, \mathbb{D}_1 q^n)\\ &\doteq \mathsf{V}_a + \mathsf{V}_c.\end{aligned} \tag{3.13}$$

By using formula (3.8c) and property (2.11), this term can also be expressed as

$$2(\mathbb{D}_1 q^n, \mathbb{D}_1 u^n) = -2\mathcal{H}^-(\mathbb{D}_1 u^n, \mathbb{D}_1 u^n) = [\![\mathbb{D}_1 u^n]\!]^2. \tag{3.14}$$

Combining the above two equalities, we have the identity

$$[\![\mathbb{D}_1 u^n]\!]^2 = \mathsf{V}_a + \mathsf{V}_c. \tag{3.15}$$

Then from (3.8b), we can use property (2.12) and the identity (3.15) to get

$$\begin{aligned}c\tau\mathfrak{R}(p^n) &= -2c\tau\mathcal{H}^-(\mathbb{D}_1 q^n, \mathbb{D}_0 p^n + \mathbb{D}_1 p^n)\\ &= 2c\tau\mathcal{H}^+(\mathbb{D}_0 p^n + \mathbb{D}_1 p^n, \mathbb{D}_1 q^n)\\ &= -\mathsf{V}_c = -[\![\mathbb{D}_1 u^n]\!]^2 + \mathsf{V}_a.\end{aligned} \tag{3.16}$$

This implies the second energy equation

$$c\tau(\|p^{n+1}\|^2 - \|p^n\|^2) = -c\tau\|\mathbb{D}_1 p^n\|^2 - [\![\mathbb{D}_1 u^n]\!]^2 + \mathsf{V}_a. \tag{3.17}$$

The third energy equation. From (3.6) we get the third energy equation

$$\begin{aligned}a\tau(\|q^{n+1}\|^2 - \|q^n\|^2) &= a\tau\|\mathbb{D}_1 q^n\|^2 + 2a\tau(\mathbb{D}_1 q^n, \mathbb{D}_0 q^n)\\ &= a\tau\|\mathbb{D}_1 q^n\|^2 - \mathsf{V}_a,\end{aligned} \tag{3.18}$$

where (3.8c) was used in the last step.

Now we are ready to get the final estimate. Adding up the above three energy equations, (3.12), (3.17) and (3.18), we can obtain

$$\mathsf{E}^{n+1} - \mathsf{E}^n + \mathsf{S} = \mathsf{R}_{a2} + a\tau\|\mathbb{D}_1 q^n\|^2 \doteq \mathsf{R}, \tag{3.19}$$

where the stability mechanism is explicitly expressed in the following terms

$$\mathsf{S} = \|\mathbb{D}_1 u^n\|^2 + c\tau\|\mathbb{D}_1 p^n\|^2 + c\tau[\![q^{n+1}]\!]^2 + a\tau[\![\mathbb{D}_0 u^n]\!]^2 + [\![\mathbb{D}_1 u^n]\!]^2. \tag{3.20}$$

Exploiting Lemma 2.3 and Lemma 2.4, we get

$$\begin{aligned}\mathsf{R} &\le 2a\tau(\|(\mathbb{D}_0 u^n)_x\| + \sqrt{\nu(\rho h)^{-1}}[\![\mathbb{D}_0 u^n]\!])\|\mathbb{D}_1 u^n\| + C_{\nu,\rho}a\tau\|\mathbb{D}_1 p^n\|\|\mathbb{D}_1 u^n\| \\ &\le C_{\nu,\rho}a\tau(2\|\mathbb{D}_0 q^n\| + \|\mathbb{D}_1 p^n\|)\|\mathbb{D}_1 u^n\|.\end{aligned} \tag{3.21}$$

Then using the Young's inequality yields

$$\mathsf{R} \le \|\mathbb{D}_1 u^n\|^2 + \frac{C_{\nu,\rho}^2 a^2}{2}\tau^2(4\|\mathbb{D}_0 q^n\|^2 + \|\mathbb{D}_1 p^n\|^2) \le \mathsf{S} + 2C_{\nu,\rho}^2 a^2\tau^2\|q^n\|^2, \tag{3.22}$$

as long as the time step is small enough such that

$$\tau \le \tau_0 \doteq \frac{2c}{C_{\nu,\rho}^2 a^2}. \tag{3.23}$$

Consequently, it follows from (3.19) that

$$\mathsf{E}^{n+1} - \mathsf{E}^n \le 2C_{\nu,\rho}^2 a^2\tau^2\|q^n\|^2 \le 2C_{\nu,\rho}^2 a\tau\mathsf{E}^n, \tag{3.24}$$

which implies (3.4) with the bounding constant $C_\star = \exp(2C_{\nu,\rho}^2 aT)$ under the condition (3.23), by an application of the discrete Gronwall's inequality. This completes the proof of Theorem 3.1 for the IMEX1-LDGk scheme.

3.2. Stability for the IMEX2-LDGk scheme. Similarly as the previous subsection, we denote $\mathbb{D}_0 w^n = w^n$ and define a series of temporal difference about the stage solutions

$$\mathbb{D}_1 w^n = \frac{1}{\gamma}(w^{n,1} - w^n), \quad \mathbb{D}_2 w^n = 2w^{n+1} - \frac{2}{\gamma}w^{n,1} + (\frac{2}{\gamma} - 2)w^n, \tag{3.25}$$

for $w = u, p, q$. It is easy to check that

$$2w^{n+1} = \mathbb{D}_2 w^n + 2\mathbb{D}_1 w^n + 2\mathbb{D}_0 w^n, \tag{3.26}$$

and to get the trivial energy equation

$$\begin{aligned}4(\|w^{n+1}\|^2 - \|w^n\|^2) &= \|\mathbb{D}_2 w^n\|^2 + 4\|\mathbb{D}_1 w^n\|^2 + 4(\mathbb{D}_2 w^n, \mathbb{D}_1 w^n) \\ &\quad + 4(\mathbb{D}_2 w^n, \mathbb{D}_0 w^n) + 8(\mathbb{D}_1 w^n, \mathbb{D}_0 w^n) \\ &\doteq \mathcal{RHS}(w^n).\end{aligned} \tag{3.27}$$

To find out some stability terms provided by the temporal differences and a semi-negative definite symmetric form about the spatial discretization of the dispersion term, we would like to rewrite $\mathcal{RHS}(w^n)$ in the form

$$\mathcal{RHS}(w^n) = \alpha\|\mathbb{D}_2 w^n\|^2 + \beta\|\mathbb{D}_1 w^n\|^2 + \mathfrak{R}(w^n), \tag{3.28}$$

hoping that the undetermined parameters α and β are not larger than zero. Here

$$\mathfrak{R}(w^n) = (\mathbb{D}_1 w^n, \mathbb{E}_1 w^n) + (\mathbb{D}_2 w^n, \mathbb{E}_2 w^n), \tag{3.29}$$

with

$$\mathbb{E}_1 w^n = 8\mathbb{D}_0 w^n + (4-\beta)\mathbb{D}_1 w^n + (4-\theta)\mathbb{D}_2 w^n, \tag{3.30a}$$

$$\mathbb{E}_2 w^n = 4\mathbb{D}_0 w^n + \theta\mathbb{D}_1 w^n + (1-\alpha)\mathbb{D}_2 w^n. \tag{3.30b}$$

We remark that an additional parameter θ is introduced here in order to accomplish the purpose of constructing symmetric forms.

With the temporal differences, we have the following formulas for the considered scheme. From (3.2a)-(3.2b), we have

$$(\mathbb{D}_\ell u^n, v) = a\tau\mathcal{H}^-(\mathbb{D}_{\ell-1} u^n, v) - c\tau\mathcal{H}^+(\mathbb{F}_\ell p^n, v), \quad \ell = 1, 2, \tag{3.31a}$$

and from (3.2c)-(3.2d) we get

$$(\mathbb{D}_\ell p^n, \phi) = -\mathcal{H}^-(\mathbb{D}_\ell q^n, \phi), \quad \ell = 0, 1, 2, \tag{3.31b}$$

$$(\mathbb{D}_\ell q^n, \psi) = -\mathcal{H}^-(\mathbb{D}_\ell u^n, \psi), \quad \ell = 0, 1, 2. \tag{3.31c}$$

In the above formulas, we adopted the short notations

$$\mathbb{F}_1 w^n = \gamma\mathbb{D}_1 w^n + \mathbb{D}_0 w^n, \quad \mathbb{F}_2 w^n = \gamma\mathbb{D}_2 w^n + 2\gamma(1-\gamma)\mathbb{D}_1 w^n. \tag{3.32}$$

Determine parameters. By (3.29) and (3.31a), we can express $\mathfrak{R}(u^n)$ by the spatial discretization for convection and dispersion, namely,

$$\mathfrak{R}(u^n) = \mathsf{R}_a + \mathsf{R}_c, \tag{3.33}$$

where

$$\mathsf{R}_a = a\tau\big[\mathcal{H}^-(\mathbb{D}_0 u^n, \mathbb{E}_1 u^n) + \mathcal{H}^-(\mathbb{D}_1 u^n, \mathbb{E}_2 u^n)\big], \tag{3.34a}$$

$$\mathsf{R}_c = -c\tau\big[\mathcal{H}^+(\mathbb{F}_1 p^n, \mathbb{E}_1 u^n) + \mathcal{H}^+(\mathbb{F}_2 p^n, \mathbb{E}_2 u^n)\big]. \tag{3.34b}$$

Let us skip the term R_a and focus on how to ensure the term R_c to be a semi-negative definite symmetric form. Due to the linear structure of the bilinear form $\mathcal{H}^+(\cdot,\cdot)$, a simple expansion yields

$$\mathsf{R}_c = -c\tau\underline{\mathcal{H}}^+(\boldsymbol{p}^n, \mathbb{A}\boldsymbol{u}^n), \tag{3.35}$$

where the definition of $\underline{\mathcal{H}}^+(\cdot,\cdot)$ can be found in (2.18). Here

$$\boldsymbol{p}^n = (\mathbb{D}_0 p^n, \mathbb{D}_1 p^n, \mathbb{D}_2 p^n)^\top, \quad \boldsymbol{u}^n = (\mathbb{D}_0 u^n, \mathbb{D}_1 u^n, \mathbb{D}_2 u^n)^\top,$$

and

$$\mathbb{A} = \begin{pmatrix} 8 & 4-\beta & 4-\theta \\ 8\gamma + 8\gamma(1-\gamma) & \gamma(4-\beta) + 2\gamma(1-\gamma)\theta & \gamma(4-\theta) + 2\gamma(1-\gamma)(1-\alpha) \\ 4\gamma & \gamma\theta & \gamma(1-\alpha) \end{pmatrix}.$$

Due to Corollary 2.2, in order to ensure $\mathsf{R}_c \le 0$ we demand the matrix $\mathbb{A}$ to be symmetric positive semi-definite. To be symmetric, we require

$$\begin{cases} 4-\beta = 8\gamma + 8\gamma(1-\gamma), \\ 4-\theta = 4\gamma, \\ \gamma(4-\theta) + 2\gamma(1-\gamma)(1-\alpha) = \gamma\theta. \end{cases} \tag{3.36}$$

This system has the unique solution

$$\alpha = 2\sqrt{2} - 3, \quad \beta = 0, \quad \theta = 2\sqrt{2}, \tag{3.37}$$

which fortunately satisfies our purpose that $\alpha, \beta \le 0$ and $\mathbb{A}$ is positive semi-definite, since the eigenvalues of $\mathbb{A}$ are all non-negative. Moreover, there holds the equality

$$4 - \theta = 1 - \alpha = 4\gamma, \tag{3.38}$$

which will be used later.

In what follows, we are going to build up three energy (in)equalities based on (3.27) with the above parameters α, β and θ; see (3.37).

The first energy inequality. Since $\mathsf{R}_c \le 0$, we only need to give a good estimate to R_a. To do that, we use relationship (3.38) and split R_a into two parts, namely

$$\mathsf{R}_a = \mathsf{R}_{a1} + \mathsf{R}_{a2}, \tag{3.39}$$

with

$$\begin{aligned} \mathsf{R}_{a1} =\,& a\tau[8\mathcal{H}^-(\mathbb{D}_0 u^n, \mathbb{D}_0 u^n) + 4\mathcal{H}^-(\mathbb{D}_0 u^n, \mathbb{D}_1 u^n) + 4\mathcal{H}^-(\mathbb{D}_1 u^n, \mathbb{D}_0 u^n) \\ & + \theta\mathcal{H}^-(\mathbb{D}_1 u^n, \mathbb{D}_1 u^n)], \end{aligned} \tag{3.40a}$$

$$\mathsf{R}_{a2} = 4\gamma a\tau\mathcal{H}^-(\mathbb{D}_0 u^n + \mathbb{D}_1 u^n, \mathbb{D}_2 u^n). \tag{3.40b}$$

By using (2.10) and (2.11), we get

$$\begin{aligned} \mathsf{R}_{a1} =& -a\tau \left(4[\![\mathbb{D}_0 u^n]\!]^2 + 4\langle [\![\mathbb{D}_0 u^n]\!], [\![\mathbb{D}_1 u^n]\!] \rangle + \frac{\theta}{2}[\![\mathbb{D}_1 u^n]\!]^2 \right) \\ =& \begin{pmatrix} [\![\mathbb{D}_0 u^n]\!] \\ [\![\mathbb{D}_1 u^n]\!] \end{pmatrix}^\top \begin{pmatrix} 4 & 2 \\ 2 & \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} [\![\mathbb{D}_0 u^n]\!] \\ [\![\mathbb{D}_1 u^n]\!] \end{pmatrix}. \end{aligned} \tag{3.41}$$

The involved matrix is denoted by $\mathbb{B}$. It is easy to see that this matrix is symmetric positive definite and has the smallest eigenvalue

$$\lambda = 2 + \tfrac{\sqrt{2}}{4} - \tfrac{1}{2\sqrt{34 - 8\sqrt{2}}} \approx 0.3256.$$

Since $\mathbb{B} - \lambda\mathbb{I}$ is positive semi-definite, where $\mathbb{I}$ is the identity matrix, we have

$$\mathsf{R}_{a1} \le -\lambda a\tau([\![\mathbb{D}_0 u^n]\!]^2 + [\![\mathbb{D}_1 u^n]\!]^2). \tag{3.42}$$

Skipping the estimate to R_{a2} and summing up the above discussions, we arrive at the first energy inequality

$$4(\|u^{n+1}\|^2 - \|u^n\|^2) \le \alpha\|\mathbb{D}_2 u^n\|^2 - \lambda a\tau([\![\mathbb{D}_0 u^n]\!]^2 + [\![\mathbb{D}_1 u^n]\!]^2) + \mathsf{R}_{a2}. \tag{3.43}$$

The second energy inequality. Similarly as the treatment in the first order scheme, we start from two equivalent expressions of the term

$$\mathsf{V}(u^n; q^n) \doteq (\mathbb{D}_1 u^n, \mathbb{E}_3 q^n) + (\mathbb{D}_2 u^n, \mathbb{E}_4 q^n), \tag{3.44}$$

with two short notations for any $w = u, p, q$,

$$\mathbb{E}_3 w^n = 8\mathbb{D}_1 w^n + 4\mathbb{D}_2 w^n, \qquad \mathbb{E}_4 w^n = 4\mathbb{D}_1 w^n + 4\mathbb{D}_2 w^n. \tag{3.45}$$

On one hand, by using (3.31a) we can transfer this term into the information of spatial discretization, i.e.

$$\mathsf{V}(u^n;q^n)=\mathsf{V}_a+\mathsf{V}_c, \tag{3.46}$$

where

$$\begin{aligned}\mathsf{V}_a&=a\tau\big[\mathcal{H}^-(\mathbb{D}_0u^n,\mathbb{E}_3q^n)+\mathcal{H}^-(\mathbb{D}_1u^n,\mathbb{E}_4q^n)\big],\\ \mathsf{V}_c&=-c\tau\big[\mathcal{H}^+(\mathbb{F}_1p^n,\mathbb{E}_3q^n)+\mathcal{H}^+(\mathbb{F}_2p^n,\mathbb{E}_4q^n)\big].\end{aligned}$$

On the other hand, by making a new combination mode for (3.44) we get

$$\mathsf{V}(u^n;q^n)=(\mathbb{D}_1q^n,\mathbb{E}_3u^n)+(\mathbb{D}_2q^n,\mathbb{E}_4u^n).$$

Then using the formula (3.31c) and applying the properties (2.10) and (2.11), we have

$$\begin{aligned}\mathsf{V}(u^n;q^n)&=-\mathcal{H}^-(\mathbb{D}_1u^n,\mathbb{E}_3u^n)-\mathcal{H}^-(\mathbb{D}_2u^n,\mathbb{E}_4u^n)\\ &=4[\![\mathbb{D}_1u^n]\!]^2+4\langle[\![\mathbb{D}_1u^n]\!],[\![\mathbb{D}_2u^n]\!]\rangle+2[\![\mathbb{D}_2u^n]\!]^2\ge[\![\mathbb{D}_2u^n]\!]^2.\end{aligned}\tag{3.47}$$

As a consequence, we have

$$\mathsf{V}_a+\mathsf{V}_c\ge[\![\mathbb{D}_2u^n]\!]^2. \tag{3.48}$$

Now we turn to set up the second energy inequality. By (3.29) and (3.31b) we get

$$\mathfrak{R}(p^n)=-\mathcal{H}^-(\mathbb{D}_1q^n,\mathbb{E}_1p^n)-\mathcal{H}^-(\mathbb{D}_2q^n,\mathbb{E}_2p^n). \tag{3.49}$$

Plugging the equivalent expressions (due to (3.38))

$$\mathbb{E}_1p^n=8\mathbb{D}_0p^n+4\mathbb{D}_1p^n+4\gamma\mathbb{D}_2p^n,\quad \mathbb{E}_2p^n=4\mathbb{D}_0p^n+4(1-\gamma)\mathbb{D}_1p^n+4\gamma\mathbb{D}_2p^n$$

into the above formula, and making some expansion and combination manipulations, we get a new formulation

$$\mathfrak{R}(p^n)=-\mathcal{H}^-(\mathbb{E}_3q^n,\mathbb{F}_1p^n)-\mathcal{H}^-(\mathbb{E}_4q^n,\mathbb{F}_2p^n), \tag{3.50}$$

where $\mathbb{E}_3w^n,\mathbb{E}_4w^n$ are defined in (3.45), and $\mathbb{F}_1w^n,\mathbb{F}_2w^n$ are defined in (3.32). In the above procedure, we have split some coefficients by the following simple identities (due to the value of γ)

$$4=8\gamma+8\gamma(1-\gamma),\quad 4(1-\gamma)=4\gamma+8\gamma(1-\gamma).$$

Then using (2.12) in Lemma 2.2, and noticing the definition of V_c, we get

$$c\tau\mathfrak{R}(p^n)=-\mathsf{V}_c. \tag{3.51}$$

At last, by combining (3.48) and (3.51), we can get from (3.27) and (3.28) the second energy inequality

$$4c\tau(\|p^{n+1}\|^2-\|p^n\|^2)\le\alpha c\tau\|\mathbb{D}_2p^n\|^2-[\![\mathbb{D}_2u^n]\!]^2+\mathsf{V}_a. \tag{3.52}$$

The third energy equality. It follows from (3.27) that

$$\begin{aligned}4a\tau(\|q^{n+1}\|^2-\|q^n\|^2)&=a\tau\|\mathbb{D}_2q^n\|^2+a\tau\left[(\mathbb{D}_0q^n,\mathbb{E}_3q^n)+(\mathbb{D}_1q^n,\mathbb{E}_4q^n)\right]\\ &=a\tau\|\mathbb{D}_2q^n\|^2-\mathsf{V}_a,\end{aligned}\tag{3.53}$$

where (3.31c) has been used in the last step.

Now we can prove the stability theorem for the second order scheme. Adding up the above three conclusions, (3.43), (3.52) and (3.53), we then obtain

$$4(\mathsf{E}^{n+1}-\mathsf{E}^{n})+\mathsf{S}\leq \mathsf{R}_{a2}+a\tau\|\mathbb{D}_2 q^n\|^2\doteq \mathsf{R}, \tag{3.54}$$

where the stability mechanism is explicitly shown by

$$\mathsf{S}=-\alpha\|\mathbb{D}_2 u^n\|^2-\alpha c\tau\|\mathbb{D}_2 p^n\|^2+[\![\mathbb{D}_2 u^n]\!]^2+\lambda a\tau([\![\mathbb{D}_0 u^n]\!]^2+[\![\mathbb{D}_1 u^n]\!]^2). \tag{3.55}$$

Noting that $\mathbb{D}_0 u^n+\mathbb{D}_1 u^n=u^{n+1}-\frac{1}{2}\mathbb{D}_2 u^n$, we can derive

$$\begin{aligned}
\mathsf{R}=&\,4\gamma a\tau\mathcal{H}^-(u^{n+1},\mathbb{D}_2 u^n)-2\gamma a\tau\mathcal{H}^-(\mathbb{D}_2 u^n,\mathbb{D}_2 u^n)+a\tau\|\mathbb{D}_2 q^n\|^2\\
\leq&\,4\gamma a\tau(\|u_x^{n+1}\|+\sqrt{\nu(\rho h)^{-1}}[\![u^{n+1}]\!])\|\mathbb{D}_2 u^n\|+\gamma a\tau[\![\mathbb{D}_2 u^n]\!]^2\\
&+C_{\nu,\rho}a\tau\|\mathbb{D}_2 p^n\|\|\mathbb{D}_2 u^n\|\\
\leq&\,4\gamma C_{\nu,\rho}a\tau\|q^{n+1}\|\|\mathbb{D}_2 u^n\|+\gamma a\tau[\![\mathbb{D}_2 u^n]\!]^2+C_{\nu,\rho}a\tau\|\mathbb{D}_2 p^n\|\|\mathbb{D}_2 u^n\|,
\end{aligned} \tag{3.56}$$

where Lemma 2.3, (2.11) and Lemma 2.4 have been used. Then by the Young's inequality, we obtain

$$\begin{aligned}
\mathsf{R}\leq&\,-\alpha\|\mathbb{D}_2 u^n\|^2+\gamma a\tau[\![\mathbb{D}_2 u^n]\!]^2+\frac{8\gamma^2C_{\nu,\rho}^2a^2}{-\alpha}\tau^2\|q^{n+1}\|^2+\frac{C_{\nu,\rho}^2a^2}{-2\alpha}\tau^2\|\mathbb{D}_2 p^n\|^2\\
\leq&\,-\alpha\|\mathbb{D}_2 u^n\|^2+[\![\mathbb{D}_2 u^n]\!]^2-\alpha c\tau\|\mathbb{D}_2 p^n\|^2+\frac{8\gamma^2C_{\nu,\rho}^2a}{-\alpha}a\tau^2\|q^{n+1}\|^2\\
\leq&\,\mathsf{S}+\frac{8\gamma^2C_{\nu,\rho}^2a}{-\alpha}a\tau^2\|q^{n+1}\|^2,
\end{aligned} \tag{3.57}$$

provided that $\tau\leq\tau_0'\doteq\min\left\{\frac{1}{\gamma a},\frac{2\alpha^2c}{C_{\nu,\rho}^2a^2}\right\}$. Hence, from (3.54) and (3.57) we can get

$$\mathsf{E}^{n+1}-\mathsf{E}^{n}\leq\frac{2\gamma^2C_{\nu,\rho}^2a}{-\alpha}a\tau^2\|q^{n+1}\|^2\leq\frac{2\gamma^2C_{\nu,\rho}^2a}{-\alpha}\tau\mathsf{E}^{n+1}. \tag{3.58}$$

As a result, if

$$\tau\leq\tau_0\doteq\min\left\{\tau_0',\frac{-\alpha}{2.5\gamma^2C_{\nu,\rho}^2a}\right\}, \tag{3.59}$$

we can get the conclusion (3.4) by using the discrete Gronwall's inequality. This completes the proof for the IMEX2-LDGk scheme.

4. Error estimates. In this section, we will obtain the optimal error estimates of the two IMEX-LDG schemes considered in the previous section for the linearized KdV equation. In the standard analysis for finite element methods, we usually conduct error estimates with the help of some suitable projections. The GR projection [5] or the generalized GR (GGR) projection [15, 7] are commonly used in this topic. The advantage of them is that they can simultaneously eliminate the projection errors in the element interior and at the element interface, which is greatly helpful for obtaining the optimal error order. In [24] and [13], GR/GGR projections were adopted to achieve the optimal error estimates of semidiscrete LDG methods for linearized KdV equations. However, in this work it is difficult to get the optimal error estimates by the GR projection, due to the troubles caused by interactive influences of errors at different intermediate stages. This motivates us to find a new projection technique.

4.1. Projection. Following the idea of proving optimal error estimates for the fourth order PDEs [10, 22], we introduce the following projection for the KdV equation.

Let $U(x)$ be a given function satisfying the periodic boundary condition, and denote $\boldsymbol{W} = (U, P, Q)$ with $Q = U_x$ and $P = Q_x$. The projection $\boldsymbol{W}_h = (U_h, P_h, Q_h) \in [V_h]^3$ is defined by the following conditions:

$$\mathcal{H}^+(P, v) = \mathcal{H}^+(P_h, v), \quad \forall v \in V_h, \tag{4.1a}$$

$$(P_h, \phi) = -\mathcal{H}^-(Q_h, \phi), \quad \forall \phi \in V_h, \tag{4.1b}$$

$$(Q_h, \psi) = -\mathcal{H}^-(U_h, \psi), \quad \forall \psi \in V_h, \tag{4.1c}$$

where the bilinear forms $\mathcal{H}^\pm(\cdot, \cdot)$ are the same as before. Moreover, to make $\boldsymbol{W}_h$ well-defined, we demand

$$(P_h, 1) = (Q_h, 1) = (U - U_h, 1) = 0. \tag{4.1d}$$

LEMMA 4.1. *The projection $\boldsymbol{W}_h$ exists uniquely.*

Proof. We first point out that (4.1) forms a linear system with the same number of unknowns and restriction conditions, because

$$\mathcal{H}^\pm(w, 1) = 0 \tag{4.2}$$

holds automatically for any $w \in V_h$. Hence, to prove the existence and uniqueness of the projection, we only need to show $\boldsymbol{W}_h = 0$ for $\boldsymbol{W} = 0$.

Taking $\phi = P_h$ in (4.1b) and using the property (2.12), we get

$$\|P_h\|^2 = -\mathcal{H}^-(Q_h, P_h) = \mathcal{H}^+(P_h, Q_h) = \mathcal{H}^+(P, Q_h) = 0, \tag{4.3}$$

which implies $P_h = 0$. Taking $\phi = Q_h$ in (4.1b) and using the property (2.11) yield

$$[\![Q_h]\!]^2 = 0, \tag{4.4}$$

and hence Q_h is continuous across each element interface. Then taking $\phi = U_h$ in (4.1b) and $\psi = Q_h$ in (4.1c), adding them together we have

$$\|Q_h\|^2 = -\mathcal{H}^-(Q_h, U_h) - \mathcal{H}^-(U_h, Q_h) = \langle [\![Q_h]\!], [\![U_h]\!] \rangle = 0, \tag{4.5}$$

where the property (2.10) is used. Thus $Q_h = 0$. Lastly, from Lemma 2.4 we get

$$\|(U_h)_x\| + \sqrt{\nu(\rho h)^{-1}} [\![U_h]\!] \le C_{\nu,\rho} \|Q_h\| = 0. \tag{4.6}$$

This implies that U_h is a constant. Due to (4.1d), we have $(U_h, 1) = 0$ and get $U_h = 0$. Now we complete the proof of this lemma. ∎

Remark 4.2. The projection defined in (4.1) is similar to the elliptic projection which is usually used in the error analysis for harmonic and bi-harmonic problems, such as convection-diffusion equation [21], the time-dependent fourth order PDEs [10, 22], the Swift-Hohenberg equation [30], and so on. However, such type of projection has rarely been analyzed and adopted for the KdV type and other type of odd order wave equations.

The main advantage of this projection lies in that, all the projection terms vanish in the error equations with respect to the auxiliary variables, and this greatly simplify the error analysis. Similar to the elliptic projection, the optimal approximation property of the projection defined in (4.1) can be obtained by the aid of the adjoint problem. We present the optimal approximation property of this projection in the following lemma, whose proof is put aside in the Appendix.

LEMMA 4.3. *If $U \in H^{k+3}(\Omega)$ and $\boldsymbol{W} = (U, P, Q)$ satisfy the periodic boundary condition, then the projection $\boldsymbol{W}_h$ defined in (4.1) satisfies*

$$\|U_h - U\| + \|P_h - P\| + \|Q_h - Q\| \leq Ch^{k+1}, \tag{4.7}$$

if h is small enough. Here the bounding constant C depends on $\|U\|_{H^{k+3}(\Omega)}$.

4.2. Main conclusion. Before presenting the main result about the error estimates, we would like to give the definition of the initial values, which are taken as the projection of (U_0, P_0, Q_0) defined in (4.1), namely

$$(u^0, p^0, q^0) = (U_h^0, P_h^0, Q_h^0), \tag{4.8}$$

where U_0 is the given initial condition, $Q_0 = U_0'$ and $P_0 = U_0''$.

Remark 4.4. Such choice for the initial condition is only for the purpose of error analysis, in numerical experiments $u^0 \in V_h$ can be taken as other projection of U_0.

The next theorem states the error estimates for the two IMEX-LDG schemes proposed in subsection 3.1.

THEOREM 4.5. *Let u^n be the numerical solution of the IMEXs-LDGk schemes (3.1) and (3.2), respectively, with $s = 1$ and $s = 2$, where the numerical initial solutions are given as (4.8). Let $U(x,t)$ be the exact solution of (1.1) and denote $U^n = U(x, t^n)$. Assume $U(x,t)$ is sufficiently smooth, then we have the optimal error estimate*

$$\|U^n - u^n\| \leq C(h^{k+1} + \tau^s), \quad \forall n \geq 0, \tag{4.9}$$

if the mesh size h is small enough and the time step $\tau \leq \tau_1$. Here τ_1 is independent of h, and $C > 0$ is the bounding constant independent of h and τ.

Next we will take the IMEX2-LDGk scheme as an example to prove this theorem. The proof for the IMEX1-LDGk scheme is much simpler, so we omit it to save space. Even though the proof line is similar to that of the stability analysis, we have to overcome some troubles resulting from the projection errors. In what follows, the notations follow from those in subsection 3.2, and we will carry out the proof of Theorem 4.5 for the IMEX2-LDGk scheme in three steps.

Step 1: Error decomposition. For $\ell = 0, 1$, let $\boldsymbol{W}^{n,\ell} = (U^{n,\ell}, P^{n,\ell}, Q^{n,\ell}) = (U(x, t^{n,\ell}), P(x, t^{n,\ell}), Q(x, t^{n,\ell}))$ be the reference solution of (2.2) at time level $t^{n,\ell}$, the corresponding errors are denoted by

$$\boldsymbol{e}^{n,\ell} = (e_u^{n,\ell}, e_p^{n,\ell}, e_q^{n,\ell}) = (U^{n,\ell} - u^{n,\ell}, P^{n,\ell} - p^{n,\ell}, Q^{n,\ell} - q^{n,\ell}) = \boldsymbol{W}^{n,\ell} - \boldsymbol{w}^{n,\ell}. \tag{4.10}$$

We will omit the superscript ℓ if $\ell = 0$. Based on the projection (4.1), we divide the errors in the form $\boldsymbol{e}^{n,\ell} = \boldsymbol{\xi}^{n,\ell} - \boldsymbol{\eta}^{n,\ell}$, with

$$\boldsymbol{\xi}^{n,\ell} = \boldsymbol{W}_h^{n,\ell} - \boldsymbol{w}^{n,\ell}, \quad \boldsymbol{\eta}^{n,\ell} = \boldsymbol{W}_h^{n,\ell} - \boldsymbol{W}^{n,\ell}.$$

From Lemma 4.3 and the linearity of the projection $\boldsymbol{W}_h$, it can be derived that

$$\|\eta_u^n\| + \|\eta_q^n\| + \|\eta_p^n\| \le Ch^{k+1}, \tag{4.11a}$$

$$\|\mathbb{D}_\ell \eta_u^n\| + \|\mathbb{D}_\ell \eta_q^n\| \le Ch^{k+1}\tau, \quad \ell = 1, 2, \tag{4.11b}$$

$$\|\eta_u^{n+1} - 2\eta_u^n + \eta_u^{n-1}\| \le Ch^{k+1}\tau^2, \tag{4.11c}$$

where the constant $C > 0$ depends on $\|U\|_{L^\infty(H^{k+3})}$, $\|U_t\|_{L^\infty(H^{k+3})}$ and $\|U_{tt}\|_{L^\infty(H^{k+3})}$, respectively. In addition, since $w^{n+1} - w^n = \mathbb{D}_1 w^n + \frac{1}{2}\mathbb{D}_2 w^n$, we also have

$$\|\eta_u^{n+1} - \eta_u^n\| + \|\eta_q^{n+1} - \eta_q^n\| \le Ch^{k+1}\tau. \tag{4.11d}$$

The remaining work is to estimate $\boldsymbol{\xi}$, which lies in the finite element space. To do that, we need to set up the corresponding error equations.

Step 2: Error equations. Since the exact solution is assumed to be smooth enough, it is easy to see that $\boldsymbol{W}^{n,\ell}$ satisfies the following variational forms:

$$(\mathbb{D}_\ell U^n, v) = a\tau\mathcal{H}^-(\mathbb{D}_{\ell-1}U^n, v) - c\tau\mathcal{H}^+(\mathbb{F}_\ell P^n, v) + (\kappa_{2\ell}\varsigma^n, v), \quad \ell = 1, 2, \tag{4.12a}$$

$$(\mathbb{D}_\ell P^n, \phi) = -\mathcal{H}^-(\mathbb{D}_\ell Q^n, \phi), \quad \ell = 0, 1, 2, \tag{4.12b}$$

$$(\mathbb{D}_\ell Q^n, \psi) = -\mathcal{H}^-(\mathbb{D}_\ell U^n, \psi), \quad \ell = 0, 1, 2, \tag{4.12c}$$

for any $v, \phi, \psi \in V_h$. Here $\kappa_{21} = 0$, $\kappa_{22} = 1$, and ς^n is the local truncation error satisfying

$$\|\varsigma^n\| \le C\tau^3, \tag{4.13a}$$

$$\|\varsigma^{n+1} - \varsigma^n\| \le C\tau^4, \tag{4.13b}$$

where the constant $C > 0$ depends on $\|U_{ttt}\|_{L^\infty(L^2)}$ and $\|U_{tttt}\|_{L^\infty(L^2)}$, respectively.

From the definition of the projection $\boldsymbol{W}_h$ and the variational forms (4.12b), (4.12c), we can easily get

$$0 = \mathcal{H}^+(\mathbb{D}_\ell \eta_p^n, v),\ (\mathbb{D}_\ell \eta_p^n, \phi) = -\mathcal{H}^-(\mathbb{D}_\ell \eta_q^n, \phi),\ (\mathbb{D}_\ell \eta_q^n, \psi) = -\mathcal{H}^-(\mathbb{D}_\ell \eta_u^n, \psi). \tag{4.14}$$

So subtracting (3.2) from (4.12), we get the following error equations

$$\begin{aligned}(\mathbb{D}_\ell \xi_u^n, v) = {}& a\tau\mathcal{H}^-(\mathbb{D}_{\ell-1}\xi_u^n, v) - c\tau\mathcal{H}^+(\mathbb{F}_\ell \xi_p^n, v)\\ &+ (\mathbb{D}_\ell \eta_u^n + \kappa_{2\ell}\varsigma^n, v) - a\tau\mathcal{H}^-(\mathbb{D}_{\ell-1}\eta_u^n, v), \quad \ell = 1, 2,\end{aligned} \tag{4.15a}$$

$$(\mathbb{D}_\ell \xi_p^n, \phi) = -\mathcal{H}^-(\mathbb{D}_\ell \xi_q^n, \phi), \quad \ell = 0, 1, 2, \tag{4.15b}$$

$$(\mathbb{D}_\ell \xi_q^n, \psi) = -\mathcal{H}^-(\mathbb{D}_\ell \xi_u^n, \psi), \quad \ell = 0, 1, 2. \tag{4.15c}$$

Step 3: Energy estimates for $\boldsymbol{\xi}$. Along the same procedure as the stability analysis in subsection 3.2, we establish three energy (in)equalities for ξ_u^n, ξ_p^n and ξ_q^n, which are similar to the energy (in)equalities for u^n, p^n and q^n, with some extra projection related terms. Then we can obtain the energy estimate

$$4(\mathsf{E}_\xi^{n+1} - \mathsf{E}_\xi^n) + \mathsf{S}_\xi \le \mathsf{R}_\xi + \mathsf{R}_{\eta,t}^u + \mathsf{R}_{\eta,s}^u + \mathsf{R}_{\eta,t}^q + \mathsf{R}_{\eta,s}^q, \tag{4.16}$$

where $\mathsf{E}_\xi^n, \mathsf{S}_\xi, \mathsf{R}_\xi$ are defined similarly as $\mathsf{E}^n, \mathsf{S}, \mathsf{R}$ that appeared in subsection 3.2. The only change is replacing (u, p, q) with (ξ_u, ξ_p, ξ_q). In detail,

$$\begin{aligned}\mathsf{E}_\xi^n &= \|\xi_u^n\|^2 + c\tau\|\xi_p^n\|^2 + a\tau\|\xi_q^n\|^2,\\ \mathsf{S}_\xi &= -\alpha\|\mathbb{D}_2\xi_u^n\|^2 - \alpha c\tau\|\mathbb{D}_2\xi_p^n\|^2 + [\![\mathbb{D}_2\xi_u^n]\!]^2 + \lambda a\tau([\![\mathbb{D}_0\xi_u^n]\!]^2 + [\![\mathbb{D}_1\xi_u^n]\!]^2),\\ \mathsf{R}_\xi &= 4\gamma a\tau\mathcal{H}^-(\mathbb{D}_0\xi_u^n + \mathbb{D}_1\xi_u^n, \mathbb{D}_2\xi_u^n) + a\tau\|\mathbb{D}_2\xi_q^n\|^2,\end{aligned}$$

and the projection related terms

$$\mathsf{R}^u_{\eta,t} = (\mathbb{D}_1\eta^n_u, \mathbb{E}_1\xi^n_u) + (\mathbb{D}_2\eta^n_u + \varsigma^n, \mathbb{E}_2\xi^n_u), \tag{4.17a}$$

$$\begin{aligned}\mathsf{R}^u_{\eta,s} &= -a\tau[\mathcal{H}^-(\mathbb{D}_0\eta^n_u, \mathbb{E}_1\xi^n_u) + \mathcal{H}^-(\mathbb{D}_1\eta^n_u, \mathbb{E}_2\xi^n_u)]\\ &= a\tau[(\mathbb{D}_0\eta^n_q, \mathbb{E}_1\xi^n_u) + (\mathbb{D}_1\eta^n_q, \mathbb{E}_2\xi^n_u)],\end{aligned} \tag{4.17b}$$

$$\mathsf{R}^q_{\eta,t} = (\mathbb{D}_1\eta^n_u, \mathbb{E}_3\xi^n_q) + (\mathbb{D}_2\eta^n_u + \varsigma^n, \mathbb{E}_4\xi^n_q), \tag{4.17c}$$

$$\begin{aligned}\mathsf{R}^q_{\eta,s} &= -a\tau[\mathcal{H}^-(\mathbb{D}_0\eta^n_u, \mathbb{E}_3\xi^n_q) + \mathcal{H}^-(\mathbb{D}_1\eta^n_u, \mathbb{E}_4\xi^n_q)]\\ &= a\tau[(\mathbb{D}_0\eta^n_q, \mathbb{E}_3\xi^n_q) + (\mathbb{D}_1\eta^n_q, \mathbb{E}_4\xi^n_q)],\end{aligned} \tag{4.17d}$$

where $\mathbb{E}_i w^n$, for $i = 1, \dots, 4$, have been defined in (3.30) and (3.45). Note that the third equation in (4.14) is also used to deal with $\mathsf{R}^u_{\eta,s}$ and $\mathsf{R}^q_{\eta,s}$.

First we estimate $\mathsf{R}^u_{\eta,t} + \mathsf{R}^u_{\eta,s}$. Since the stability term S_ξ provides a good control only for the terms relating to $\mathbb{D}_2\xi^n_u$, we have to carefully deal with those terms involving $\mathbb{D}_1\xi^n_u$. To do that, we use the relationship $\mathbb{D}_1 w^n = w^{n+1} - w^n - \frac{1}{2}\mathbb{D}_2 w^n$ to rewrite $\mathbb{E}_1\xi^n_u$ and $\mathbb{E}_2\xi^n_u$ in the form

$$\begin{aligned}\mathbb{E}_1\xi^n_u &= 4\xi^n_u + 4\xi^{n+1}_u + (2-\theta)\mathbb{D}_2\xi^n_u,\\ \mathbb{E}_2\xi^n_u &= (4-\theta)\xi^n_u + \theta\xi^{n+1}_u + (1 - \alpha - \tfrac{\theta}{2})\mathbb{D}_2\xi^n_u.\end{aligned}$$

Then by using the Cauchy-Schwarz inequality, the properties (4.11a), (4.11b), (4.13a) and the Young's inequality, we can obtain (note that $\alpha < 0$)

$$\begin{aligned}\mathsf{R}^u_{\eta,t} + \mathsf{R}^u_{\eta,s} &\le C(h^{k+1}\tau + \tau^3)(\|\xi^n_u\| + \|\xi^{n+1}_u\| + \|\mathbb{D}_2\xi^n_u\|)\\ &\le \varepsilon_0\tau(\|\xi^n_u\|^2 + \|\xi^{n+1}_u\|^2) - \frac{\alpha}{4}\|\mathbb{D}_2\xi^n_u\|^2 + C(h^{2k+2}\tau + \tau^5),\end{aligned} \tag{4.18}$$

here and below ε_0 is an arbitrary positive constant.

Next we estimate $\mathsf{R}^q_{\eta,t} + \mathsf{R}^q_{\eta,s}$. Similarly as the above argument, we must carefully deal with those terms involving $\mathbb{D}_1\xi^n_q$. By the identity $2\mathbb{D}_1 w^n + \mathbb{D}_2 w^n = 2(w^{n+1} - w^n)$, we can rewrite $\mathbb{E}_3\xi^n_q$ and $\mathbb{E}_4\xi^n_q$ in the form

$$\mathbb{E}_3\xi^n_q = 8(\xi^{n+1}_q - \xi^n_q), \qquad \mathbb{E}_4\xi^n_q = 4(\xi^{n+1}_q - \xi^n_q) + 2\mathbb{D}_2\xi^n_q.$$

After some expansion manipulations, we can get the following formulas

$$\begin{aligned}\mathsf{R}^q_{\eta,t} &= (2\mathbb{D}_1\eta^n_u + \mathbb{D}_2\eta^n_u + \varsigma^n, 4(\xi^{n+1}_q - \xi^n_q)) + (\mathbb{D}_2\eta^n_u + \varsigma^n, 2\mathbb{D}_2\xi^n_q)\\ &= 8(\eta^{n+1}_u - \eta^n_u, \xi^{n+1}_q - \xi^n_q) + 4(\varsigma^n, \xi^{n+1}_q - \xi^n_q) + 2(\mathbb{D}_2\eta^n_u + \varsigma^n, \mathbb{D}_2\xi^n_q),\\ \mathsf{R}^q_{\eta,s} &= 8a\tau(\mathbb{D}_0\eta^n_q, \xi^{n+1}_q - \xi^n_q) + 4a\tau(\mathbb{D}_1\eta^n_q, \xi^{n+1}_q - \xi^n_q) + 2a\tau(\mathbb{D}_1\eta^n_q, \mathbb{D}_2\xi^n_q).\end{aligned}$$

With a new combination we have

$$\mathsf{R}^q_{\eta,t} + \mathsf{R}^q_{\eta,s} = \mathsf{V}^n + \Lambda^n, \tag{4.19}$$

where

$$\begin{aligned}\mathsf{V}^n &= 2(\mathbb{D}_2\eta^n_u + \varsigma^n, \mathbb{D}_2\xi^n_q) + 4a\tau(\mathbb{D}_1\eta^n_q, \xi^{n+1}_q - \xi^n_q) + 2a\tau(\mathbb{D}_1\eta^n_q, \mathbb{D}_2\xi^n_q),\\ \Lambda^n &= 8(\eta^{n+1}_u - \eta^n_u, \xi^{n+1}_q - \xi^n_q) + 4(\varsigma^n, \xi^{n+1}_q - \xi^n_q) + 8a\tau(\mathbb{D}_0\eta^n_q, \xi^{n+1}_q - \xi^n_q).\end{aligned}$$

By using the Cauchy-Schwarz inequality, properties (4.11b), (4.13a) and the Young's inequality, we can easily get the estimate for the first term

$$\begin{aligned}\mathsf{V}^n &\le C(h^{k+1}\tau + \tau^3)\|\mathbb{D}_2\xi^n_q\| + Ch^{k+1}\tau^2(\|\xi^n_q\| + \|\xi^{n+1}_q\|)\\ &\le \varepsilon_0 a\tau^2(\|\xi^n_q\|^2 + \|\xi^{n+1}_q\|^2) + a\tau\|\mathbb{D}_2\xi^n_q\|^2 + C(h^{2k+2}\tau + \tau^5).\end{aligned}$$

Then applying (2.21) and the Young's inequality to estimate $a\tau\|\mathbb{D}_2\xi_q^n\|^2$, we get

$$\begin{aligned}\mathsf{V}^n \leq &\varepsilon_0 a\tau^2(\|\xi_q^n\|^2 + \|\xi_q^{n+1}\|^2) - \frac{\alpha}{4}\|\mathbb{D}_2\xi_u^n\|^2 - \frac{C_{\nu,\rho}^2 a^2}{\alpha}\tau^2\|\mathbb{D}_2\xi_p^n\|^2\\ &+ C(h^{2k+2}\tau + \tau^5).\end{aligned} \tag{4.20}$$

However, it is not easy to achieve a good estimate for the second term Λ^n. With the direct application of Cauchy-Schwarz inequality and Young's inequality, the best result that we can get is $\Lambda^n \leq \tau^2(\|\xi_q^n\|^2 + \|\xi_q^{n+1}\|^2) + C(h^{2k+2} + \tau^4)$, and the expected factor τ is missing in the last term. This rough estimate prevents us from getting the optimal error order. Hence we suspend the estimate to Λ^n at this moment and will control their sum as a whole item before adopting the discrete Gronwall's inequality.

The estimate for R_ξ is similar to that for R in subsection 3.2 (see (3.56)), namely

$$\begin{aligned}\mathsf{R}_\xi \leq &4\gamma C_{\nu,\rho} a\tau\|\xi_q^{n+1}\|\|\mathbb{D}_2\xi_u^n\| + \gamma a\tau[\![\mathbb{D}_2\xi_u^n]\!]^2 + C_{\nu,\rho}a\tau\|\mathbb{D}_2\xi_p^n\|\|\mathbb{D}_2\xi_u^n\|\\ \leq &-\frac{\alpha}{2}\|\mathbb{D}_2\xi_u^n\|^2 - \frac{16\gamma^2 C_{\nu,\rho}^2 a^2}{\alpha}\tau^2\|\xi_q^{n+1}\|^2 - \frac{C_{\nu,\rho}^2 a^2}{\alpha}\tau^2\|\mathbb{D}_2\xi_p^n\|^2 + \gamma a\tau[\![\mathbb{D}_2\xi_u^n]\!]^2.\end{aligned} \tag{4.21}$$

Combining (4.16), (4.18), (4.19), (4.20) and (4.21) we obtain

$$\begin{aligned}4(\mathsf{E}_\xi^{n+1} - \mathsf{E}_\xi^n) + \mathsf{S}_\xi \leq &\varepsilon_0\tau\mathsf{E}_\xi^n + \tilde{C}\tau\mathsf{E}_\xi^{n+1} + \Lambda^n + C(h^{2k+2}\tau + \tau^5)\\ &- \alpha\|\mathbb{D}_2\xi_u^n\|^2 - \frac{2C_{\nu,\rho}^2 a^2}{\alpha}\tau^2\|\mathbb{D}_2\xi_p^n\|^2 + \gamma a\tau[\![\mathbb{D}_2\xi_u^n]\!]^2,\end{aligned} \tag{4.22}$$

where $\tilde{C} = \varepsilon_0 - \frac{16\gamma^2 C_{\nu,\rho}^2 a}{\alpha}$. Thus,

$$4(\mathsf{E}_\xi^{i+1} - \mathsf{E}_\xi^i) \leq \varepsilon_0\tau\mathsf{E}_\xi^i + \tilde{C}\tau\mathsf{E}_\xi^{i+1} + \Lambda^i + C(h^{2k+2}\tau + \tau^5), \quad \forall\, i, \tag{4.23}$$

as long as $\tau \leq \tau_1' \doteq \min\left\{\frac{1}{4\gamma a}, \frac{\alpha^2 c}{2C_{\nu,\rho}^2 a^2}\right\}$.

Summing (4.23) from $i=0$ to $i=n$ we get

$$4(\mathsf{E}_\xi^{n+1} - \mathsf{E}_\xi^0) \leq \varepsilon_0\tau\sum_{i=0}^n \mathsf{E}_\xi^i + \tilde{C}\tau\sum_{i=0}^n \mathsf{E}_\xi^{i+1} + \sum_{i=0}^n \Lambda^i + C(h^{2k+2} + \tau^4). \tag{4.24}$$

Next we will adopt the technique of summation by parts in time direction to estimate the term $\sum_{i=0}^n \Lambda^i$. Specifically, $\sum_{i=0}^n \Lambda^i = \Lambda_1 + \Lambda_2 + \Lambda_3$, where

$$\begin{aligned}\Lambda_1 =&8(\eta_u^{n+1} - \eta_u^n, \xi_q^{n+1}) - 8\sum_{i=1}^n(\eta_u^{i+1} - 2\eta_u^i + \eta_u^{i-1}, \xi_q^i) - 8(\eta_u^1 - \eta_u^0, \xi_q^0),\\ \Lambda_2 =&4(\varsigma^n, \xi_q^{n+1}) - 4\sum_{i=1}^n(\varsigma^i - \varsigma^{i-1}, \xi_q^i) - 4(\varsigma^0, \xi_q^0),\\ \Lambda_3 =&8a\tau(\eta_q^n, \xi_q^{n+1}) - 8a\tau\sum_{i=1}^n(\eta_q^i - \eta_q^{i-1}, \xi_q^i) - 8a\tau(\eta_q^0, \xi_q^0).\end{aligned}$$

The benefit of such treatment lies in that, we can transfer the temporal difference of $\xi_q^{i+1} - \xi_q^i$ to the corresponding temporal differences of $\eta_u^{i+1} - 2\eta_u^i + \eta_u^{i-1}$, $\eta_q^i - \eta_q^{i-1}$ and $\varsigma^i - \varsigma^{i-1}$, which will provide an extra factor τ compared with the direct treatment as illustrated before, so it helps us to get the optimal error estimates. Owing to the choice of the initial values, we have $\xi_q^0 = 0$. So by the Cauchy-Schwarz and Young's inequalities, and applying the properties (4.11c)-(4.11d) to the term Λ_1, applying (4.13) to Λ_2, applying (4.11a) and (4.11d) to Λ_3, we arrive at

$$\begin{aligned}\sum_{i=0}^{n}\Lambda^i &\le C(h^{k+1}\tau+\tau^3)\|\xi_q^{n+1}\| + \sum_{i=1}^{n} C(h^{k+1}\tau^2+\tau^4)\|\xi_q^i\| \\ &\le \varepsilon_0 a\tau^2\|\xi_q^{n+1}\|^2 + \varepsilon_0 a\tau^2\sum_{i=1}^{n}\|\xi_q^i\|^2 + C(h^{2k+2}+\tau^4).\end{aligned} \tag{4.25}$$

As a result, plugging the above estimate into (4.24) yields

$$4(\mathsf{E}_\xi^{n+1} - \mathsf{E}_\xi^0) \le (\tilde{C} + 2\varepsilon_0)\tau\sum_{i=0}^{n+1}\mathsf{E}_\xi^i + C(h^{2k+2}+\tau^4). \tag{4.26}$$

Taking $\varepsilon_0 = \frac{\gamma^2 C_{\nu,\rho}^2 a}{-\alpha}$, we obtain

$$\mathsf{E}_\xi^{n+1} - \mathsf{E}_\xi^0 \le \frac{5\gamma^2 C_{\nu,\rho}^2 a}{-\alpha}\tau\sum_{i=0}^{n+1}\mathsf{E}_\xi^i + C(h^{2k+2}+\tau^4). \tag{4.27}$$

As a consequence, if

$$\tau \le \tau_1 \doteq \min\left\{\tau_1', \frac{-\alpha}{10\gamma^2 C_{\nu,\rho}^2 a}\right\} = \frac{1}{4}\tau_0, \tag{4.28}$$

where τ_0 is given in (3.59), then by the Gronwall inequality and $\mathsf{E}_\xi^0 = 0$, we get

$$\mathsf{E}_\xi^n \le C(h^{2k+2}+\tau^4). \tag{4.29}$$

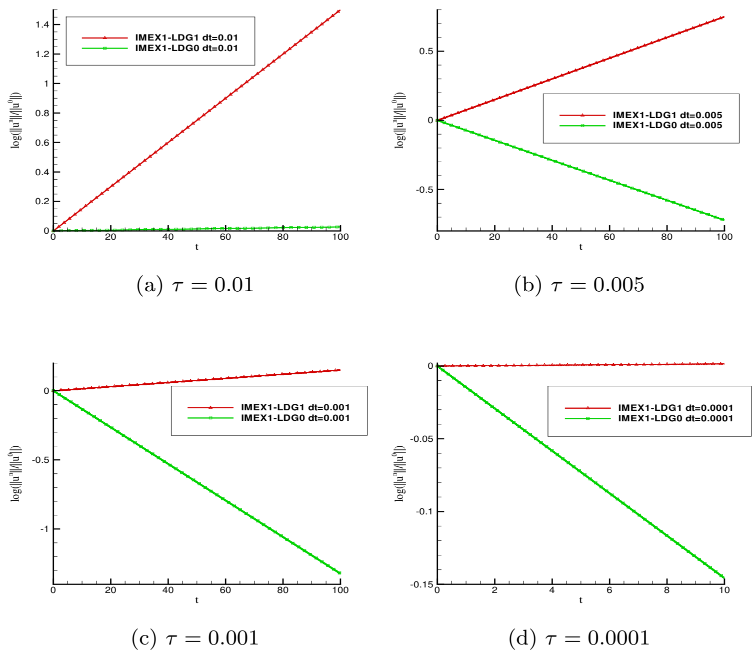
Therefore, $\|\xi_u^n\| \le C(h^{k+1}+\tau^2)$, combining $\|\eta_u^n\| \le Ch^{k+1}$ leads to (4.9). Thus we complete the proof of Theorem 4.5.

5. Numerical experiments. In this section, we present some numerical experiments to verify the stability results and error estimates of the fully discrete IMEX-LDG schemes considered in this paper.

EXAMPLE 5.1. *Consider the model equation (1.1) in the interval* $[0, 2\pi]$. *The initial condition is* $U_0(x) = \sin(x)$ *and the exact solution is* $U(x,t) = \sin(x-(a+c)t)$.

First we observe the stability performances. We take $a = 2, c = 1$ and uniform mesh with mesh number $N = 640$. Since the behavior of the energy norm E^n is the similar to that of the L^2 norm of u^n, we only present the results of the latter. The behavior of $\log(\|u^n\|/\|u^0\|)$ versus time for different schemes are displayed in Figure 1 and Figure 2, respectively. From these figures, we see that, for the IMEX1-LDG1 and IMEX2-LDG2 schemes, the L^2 norm of u^n increases with time, even under very small time step (here $\tau = 0.0001$ is about h^2). These show that the result given in Theorem 3.1 is sharp in the sense that the bounding constant can not be improved to $C_\star = 1$.

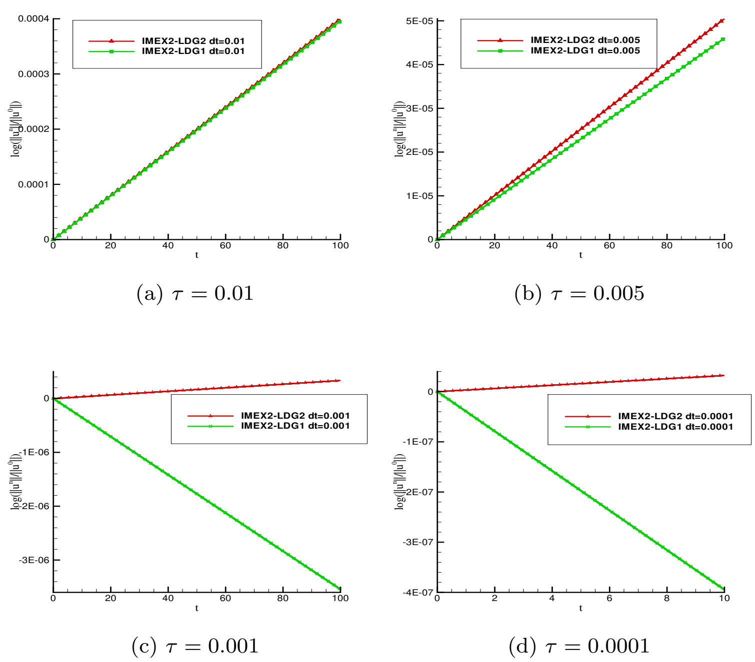
For the IMEX1-LDG0 and IMEX2-LDG1 scheme, the L^2 norm of u^n also increases with time for large time steps, see Figure 1-(a) and Figure 2-(a),(b). However, it decreases with time if the time step is smaller, see Figure 1-(b),(c),(d) and Figure 2-(c),(d). It seems that there holds the strong stability $\|u^n\| \le \|u^0\|$ when the time



(a) $\tau = 0.01$ (b) $\tau = 0.005$

(c) $\tau = 0.001$ (d) $\tau = 0.0001$

FIG. 1. $\log \frac{\|u^n\|}{\|u^0\|}$ *vs. time for schemes IMEX1-LDG1 and IMEX1-LDG0 with different time steps.*



(a) $\tau = 0.01$ (b) $\tau = 0.005$

(c) $\tau = 0.001$ (d) $\tau = 0.0001$

FIG. 2. $\log \frac{\|u^n\|}{\|u^0\|}$ *vs. time for schemes IMEX2-LDG2 and IMEX2-LDG1 with different time steps.*

step is small in the order of $\tau = \mathcal{O}(h)$. The rigorous proof will be done in the further work.

Next we test the accuracy order of the considered two fully-discrete schemes. Besides, we also test two higher order IMEX-LDG schemes, where the 3rd and 4th order IMEX-RK schemes [2, 4] are used.

We take $c = 1$ and $a = 0, 1, 3, 5$, respectively. The computing time is $T = 2$ and the time step is taken as $\tau = 0.5h$. The L^2 errors and orders of accuracy for the four schemes are listed in Table 1, from which we clearly observe the optimal accuracy for all the experiments. We also find that the convection term affects the magnitude of the error for the schemes, it implies that the magnitude of the error is larger if the coefficient of convection is larger.

TABLE 1

Example 5.1: The L^2 errors and orders of accuracy for IMEXs-LDGk schemes, for equation (1.1) with $c = 1$ and different values of a. $T = 2$, $\tau = 0.5h$, uniform mesh.

scheme	N	$a=0$		$a=1$		$a=3$		$a=5$	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
IMEX1 -LDG0	40	5.34E-01	-	6.82E-01	-	1.73E-01	-	3.64E+00	-
	80	2.84E-01	0.91	3.68E-01	0.89	6.59E-02	1.40	1.49E+00	1.29
	160	1.46E-01	0.96	1.91E-01	0.94	2.96E-02	1.15	6.64E-01	1.17
	320	7.41E-02	0.98	9.76E-02	0.97	1.44E-02	1.04	3.13E-01	1.08
	640	3.73E-02	0.99	4.93E-02	0.99	7.12E-03	1.01	1.52E-01	1.04
IMEX2 -LDG1	40	3.96E-03	-	1.08E-02	-	1.94E-01	-	8.24E-01	-
	80	9.91E-04	2.00	2.74E-03	1.98	4.88E-02	1.99	2.01E-01	2.03
	160	2.48E-04	2.00	6.84E-04	2.00	1.22E-02	2.00	5.01E-02	2.01
	320	6.20E-05	2.00	1.71E-04	2.00	3.05E-03	2.00	1.25E-02	2.00
	640	1.55E-05	2.00	4.28E-05	2.00	7.63E-04	2.00	3.13E-03	2.00
IMEX3 -LDG2	40	6.88E-05	-	4.66E-04	-	1.54E-02	-	1.07E-01	-
	80	8.65E-06	2.99	5.91E-05	2.98	1.96E-03	2.97	1.38E-02	2.95
	160	1.08E-06	3.00	7.39E-06	3.00	2.46E-04	3.00	1.73E-03	3.00
	320	1.35E-07	3.00	9.27E-07	3.00	3.08E-05	3.00	2.17E-04	3.00
	640	1.69E-08	3.00	1.23E-07	2.91	3.86E-06	3.00	2.71E-05	3.00
IMEX4 -LDG3	40	4.84E-07	-	1.08E-05	-	1.10E-03	-	1.12E-02	-
	80	3.03E-08	4.00	6.83E-07	3.98	6.97E-05	3.98	7.05E-04	3.99
	160	1.90E-09	4.00	4.28E-08	4.00	4.36E-06	4.00	4.41E-05	4.00
	320	1.18E-10	4.00	2.68E-09	4.00	2.73E-07	4.00	2.76E-06	4.00
	640	7.41E-12	4.00	1.67E-10	4.00	1.71E-08	4.00	1.73E-07	4.00

EXAMPLE 5.2. *We also consider the nonlinear KdV equation*

$$U_t + (3U^2)_x + U_{xxx} = 0, \tag{5.1}$$

with the exact solution $U(x,t) = \frac{1}{2}\text{sech}^2\left(\frac{1}{2}(x-t)\right)$.

The computational domain is taken as $[-25, 25]$. The final computing time is $T = 2$ and the time step is $\tau = 0.1h$. In this example, we take central numerical flux for the convection term. The L^2 errors and orders of accuracy for the four schemes are listed in Table 2, and we can also observe the optimal accuracy for all these schemes.

6. Conclusion. We studied the stability analysis and error estimates of first and second order in time fully discrete IMEX-LDG methods for the linearized one-dimensional KdV equations. Utilizing the temporal difference technique, we establish proper energy equations and explore the stability mechanism of the schemes. By constructing semi-negative symmetric forms about the spatial discretization of the dispersion, and by the aid of the stability provided by the temporal differences, we

Table 2
Example 5.2: the L^2 errors and orders of accuracy for IMEXs-LDGk schemes. $T = 2$, $\tau = 0.1h$, uniform mesh.

scheme	IMEX1-LDG0		IMEX2-LDG1		IMEX3-LDG2		IMEX4-LDG3	
N	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
40	2.55E-01	-	4.91E-02	-	5.91E-03	-	8.94E-04	-
80	1.48E-01	0.79	1.13E-02	2.12	7.37E-04	3.00	4.16E-05	4.43
160	8.34E-02	0.83	2.69E-03	2.07	9.42E-05	2.97	2.55E-06	4.03
320	4.48E-02	0.90	6.63E-04	2.02	1.19E-05	2.98	1.57E-07	4.02
640	2.33E-02	0.94	1.65E-04	2.01	1.50E-06	2.99	9.79E-09	4.00

obtain the energy stability analysis for a "discrete energy", which is unconditionally stable under the time step restriction $\tau \le \tau_0$, with τ_0 independent of the spatial mesh size h. Besides, a new projection technique and a technique of summation by parts in the time direction are successfully utilized to obtain the optimal order accuracy for the considered schemes. Our next goal is to investigate the stability and optimal error analysis of higher order in time IMEX-LDG schemes for the KdV equations and other odd order wave equations.

Appendix: Proof of Lemma 4.3. In this appendix we prove Lemma 4.3 by the aid of two GR projections [5], which are denoted by π_h^+ and π_h^-. For simplicity of statement, we would like to only present the proof for $k \ge 1$. The proof for $k = 0$ can be obtained by a minor modification and is omitted to save space.

For any function $w \in H^m(\Omega)$ with $m \ge 1$, the GR projections $\pi_h^\pm w \in V_h$ are respectively defined as follows: for $j = 1, \cdots, N$, there hold

(A.1)
$$\begin{cases} (\pi_h^- w - w, v)_j = 0, \ \forall v \in \mathcal{P}_{k-1}(I_j), \\ (\pi_h^- w)^-_{j+1/2} = w^-_{j+1/2}. \end{cases} \qquad \begin{cases} (\pi_h^+ w - w, v)_j = 0, \ \forall v \in \mathcal{P}_{k-1}(I_j), \\ (\pi_h^+ w)^+_{j-1/2} = w^+_{j-1/2}. \end{cases}$$

The projections exist uniquely and satisfy the optimal approximation property [5]

$$\|w - \pi_h^\pm w\|_{H^\ell(I_j)} \le C h^{\min(k+1,m)-\ell} |w|_{H^m(I_j)}, \quad \forall j, \qquad 0 \le \ell \le m, \tag{A.2}$$

where the bounding constant $C > 0$ is independent of h and j.

Denote $\eta_u = U_h - U, \eta_p = P_h - P$ and $\eta_q = Q_h - Q$. Rewrite them by the aid of the GR projections $(\pi_h^- U, \pi_h^+ P, \pi_h^- Q)$, namely

$$\eta_u = \pi_h^- U - U + \pi_h^- \eta_u, \quad \eta_p = \pi_h^+ P - P + \pi_h^+ \eta_p, \quad \eta_q = \pi_h^- Q - Q + \pi_h^- \eta_q. \tag{A.3}$$

From the definition of the GR projections, we can easily get

$$\mathcal{H}^-(\pi_h^- w - w, v) = \mathcal{H}^+(\pi_h^+ w - w, v) = 0, \quad \forall v \in V_h. \tag{A.4}$$

Thus, we can derive from (4.14) that

$$0 = \mathcal{H}^+(\pi_h^+ \eta_p, v), \tag{A.5a}$$

$$(\pi_h^+ \eta_p, \phi) = (P - \pi_h^+ P, \phi) - \mathcal{H}^-(\pi_h^- \eta_q, \phi), \tag{A.5b}$$

$$(\pi_h^- \eta_q, \psi) = (Q - \pi_h^- Q, \psi) - \mathcal{H}^-(\pi_h^- \eta_u, \psi). \tag{A.5c}$$

Estimate $\|\pi_h^+ \eta_p\|$. Taking $v = -\pi_h^- \eta_q$ and $\phi = \pi_h^+ \eta_p$ in Eq. (A.5), using the properties (2.10) and (A.2) we get

$$\|\pi_h^+ \eta_p\|^2 = (P - \pi_h^+ P, \pi_h^+ \eta_p) \le C h^{k+1} \|\pi_h^+ \eta_p\|,$$

and hence

$$\|\pi_h^+ \eta_p\| \le Ch^{k+1}. \tag{A.6}$$

Estimate $\|\pi_h^- \eta_q\|$. Taking $\phi = \pi_h^- \eta_u - \pi_h^- \eta_q$ and $\psi = \pi_h^- \eta_q - \pi_h^- \eta_u$ in Eq. (A.5), and using the properties (2.10) and (2.11) we get

$$\begin{aligned}\|\pi_h^- \eta_q\|^2 =& (\pi_h^- \eta_q, \pi_h^- \eta_u) - (\pi_h^+ \eta_p, \pi_h^- \eta_u - \pi_h^- \eta_q)\\ &+ (P - \pi_h^+ P, \pi_h^- \eta_u - \pi_h^- \eta_q) + (Q - \pi_h^- Q, \pi_h^- \eta_q - \pi_h^- \eta_u)\\ &- \left(\frac{1}{2}[\![\pi_h^- \eta_q]\!]^2 - \langle \pi_h^- \eta_q, \pi_h^- \eta_u\rangle + \frac{1}{2}[\![\pi_h^- \eta_u]\!]^2\right).\end{aligned} \tag{A.7}$$

Then using the Cauchy-Schwarz inequality, (A.6), (A.2) and the Young's inequality we obtain

$$\begin{aligned}\|\pi_h^- \eta_q\|^2 &\le \|\pi_h^- \eta_u\|\|\pi_h^- \eta_q\| + Ch^{k+1}(\|\pi_h^- \eta_u\| + \|\pi_h^- \eta_q\|) - \frac{1}{2}[\![\pi_h^- \eta_q - \pi_h^- \eta_u]\!]^2\\ &\le \frac{1}{2}\|\pi_h^- \eta_q\|^2 + \frac{3}{2}\|\pi_h^- \eta_u\|^2 + Ch^{2k+2}.\end{aligned} \tag{A.8}$$

Consequently,

$$\|\pi_h^- \eta_q\| \le C(\|\pi_h^- \eta_u\| + h^{k+1}). \tag{A.9}$$

Estimate $\|\pi_h^- \eta_u\|$. Motivated by the idea of [10, 22], we will estimate $\|\pi_h^- \eta_u\|$ by the aid of the adjoint problem. For any given (periodic) function $z \in V_h$ satisfying $\int_\Omega z dx = 0$, we introduce the following adjoint problem

$$\sigma_x = \omega, \quad \omega_x = \zeta, \quad \zeta_x = z, \tag{A.10}$$

where σ, ω and ζ are periodic and continuous functions satisfying the uniqueness condition

$$\int_\Omega \sigma dx = \int_\Omega \omega dx = \int_\Omega \zeta dx = 0. \tag{A.11}$$

The solution of this adjoint problem exists uniquely, due to the following discussion. Specifically, letting $z_j(x) = z(x)|_{I_j}$, we get $\zeta(x)|_{I_j} = \sum_{i=1}^{j-1} \int_{I_i} z_i(x)dx + \int_{x_{j-1/2}}^x z_j(y)dy + \zeta_0$, where the constant ζ_0 can be uniquely determined by the condition $\int_\Omega \zeta dx = 0$. After that, ω and σ exist uniquely by the existence theorem for primitive functions and the condition (A.11).

According to the aforementioned argument, we also easily have the following regularity

$$|\zeta|_{H^1(\Omega_h)}, |\omega|_{H^2(\Omega_h)}, |\sigma|_{H^3(\Omega_h)} \le \|z\|_{L^2(\Omega)}, \tag{A.12}$$

where $|\cdot|_{H^m(\Omega_h)} = \sqrt{\sum_{j=1}^N |\cdot|^2_{H^m(I_j)}}$.

Along the similar argument to Lemma A.1 in [22], we can obtain the following conclusion: for any $z \in V_h$ satisfying $\int_\Omega z dx = 0$, we have

$$\begin{aligned}(\pi_h^- \eta_u, z) =& -(Q - \pi_h^- Q, \zeta - \pi_h^+ \zeta) + (\pi_h^- \eta_q, \zeta - \pi_h^+ \zeta) + (Q - \pi_h^- Q, (\omega - \pi_h^+ \omega)_x)\\ &+ (P - \pi_h^+ P, \omega - \pi_h^+ \omega) - (\pi_h^+ \eta_p, \omega - \pi_h^+ \omega) - (P - \pi_h^+ P, (\sigma - \pi_h^- \sigma)_x).\end{aligned} \tag{A.13}$$

Using Cauchy-Schwarz inequality and the property (A.2), we get

$$|(\pi_h^- \eta_u, z)| \le Ch^{k+1}|Q|_{H^{k+1}} h^{\min\{k,1\}}(|\zeta|_{H^1} + |\omega|_{H^2}) + C\|\pi_h^- \eta_q\| h^{\min\{k+1,1\}}|\zeta|_{H^1}$$
$$+ Ch^{k+1}|P|_{H^{k+1}} h^{\min\{k,2\}}(|\omega|_{H^2} + |\sigma|_{H^3}) + C\|\pi_h^+ \eta_p\| h^{\min\{k+1,2\}}|\omega|_{H^2}.$$

Then by (A.6), (A.9) and (A.12), we obtain

$$\begin{aligned} |(\pi_h^- \eta_u, z)| &\le C(h^{k+1} h^{\min\{k,1\}} + h\|\pi_h^- \eta_q\|)\|z\| \\ &\le C(h^{k+1} h^{\min\{k,1\}} + h\|\pi_h^- \eta_u\|)\|z\|. \end{aligned} \tag{A.14}$$

Now we take $z = \pi_h^- \eta_u \in V_h$. Remark that $\int_\Omega \pi_h^- \eta_u dx = \int_\Omega \eta_u dx = \int_\Omega (U_h - U) dx = 0$, so this action is reasonable. Then we can get from the previous inequality that $\|\pi_h^- \eta_u\| \le C(h^{k+2} + h\|\pi_h^- \eta_u\|)$. Thus, if h is small enough we can obtain $\|\pi_h^- \eta_u\| \le Ch^{k+2}$, and the estimate for $\|\eta_u\|$ can be obtained by a simple use of triangle inequality as well as the approximation property of the GR projection. Finally, noticing (A.6) and (A.9) we get the estimates for $\|\eta_p\|$, $\|\eta_q\|$ and complete the proof of Lemma 4.3.

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