A LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE NOVIKOV EQUATION*

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Abstract. In this paper, we propose a local discontinuous Galerkin (LDG) method for the Novikov equation that contains cubic nonlinear high-order derivatives. Flux correction techniques are used to ensure the stability of the numerical scheme. The H^1-norm stability of the general solution and the error estimate for smooth solutions without using any priori assumptions are presented. Numerical examples demonstrate the accuracy and capability of the LDG method for solving the Novikov equation.

Key words. local discontinuous Galerkin, Novikov equation, stability, error estimates.

MSC codes. 65M12, 65M15, 65M60

1. Introduction.

In this paper, we consider numerical approximations to the Novikov equation

$$M_t + 4U^2U_x - 3UU_xU_{xx} - U^2U_{xxx} = 0, \tag{1.1a}$$

$$M - U + U_{xx} = 0, \tag{1.1b}$$

which is an integrable analogue of the Camassa-Holm (CH) equation with cubic (rather than quadratic) nonlinearities. Similar to the CH equation, the Novikov equation also admits peakon solutions and possesses infinitely many conserved quantities, one of which is the following H^1-norm:

$$E(U, U_x) = \int_\Omega U^2 + U_x^2 \, dx.$$

However, unlike the CH equation, the cubic nonlinearity and nonconservation with respect to M in the Novikov equation bring great difficulties to the numerical calculation and numerical analysis. It is very challenging to design an accurate and stable finite element method (FEM) to solve it.

We develop a class of local discontinuous Galerkin (LDG) methods with some flux correction terms for the Novikov equation. Our proposed scheme is consistent, high-order accurate, nonlinearly stable (discrete H^1-norm can be conserved or dissipated), and flexible for arbitrary h and p adaptivity. The proof of the H^1-norm stability of the

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*Submitted to the editors DATE.

Funding: Q. Tao's Research is partially supported by NSFC (Grant No. 12301464). X.K. Chang's research is partially supported in part by the NSFC (Grant Nos. 12222119, 12288201, 12171461) and the Youth Innovation Promotion Association CAS. Y. Liu's Research is partially supported by NSFC (Grant Nos. 12201621, 12288201) and the Youth Innovation Promotion Association CAS. C.-W. Shu's research is partially supported by NSF grant DMS-2309249.

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scheme is given for the general solution. Error estimates for smooth solutions without using any priori assumptions are presented. To our best knowledge, this is the first provably high-order accurate and stable LDG method for the Novikov equation.

In 1993, the CH equation [2] was derived as an integrable model of unidirectional wave propagation for shallow water by executing an asymptotic expansion of the Hamiltonian for Euler's equations of hydrodynamics. It has attracted a great deal of attention due to its intriguing properties, such as allowing the existence of peakon solutions [1, 2] and describing the breakdown of regularity [1, 13, 31]. Later on, in addition to the CH equation, there subsequently appear many integrable equations with the presence of peakon solutions. These integrable PDEs are sometimes called CH-type equations, among which the Degasperis-Procesi (DP) equation was discovered in 1998 as the second one. While both the CH and DP equations involve quadratic nonlinearities, the Novikov equation was later discovered by Novikov [19, 33] in 2009 as a new member of CH-type equations with cubic nonlinearities. In recent years, there has been a growing interest in the Novikov equation. For example, the peakon solution was investigated in [3, 4, 18, 17] etc. Stability of peakons was considered in [25, 7] etc. See also e.g. [6, 16, 22, 34, 32, 20, 38, 44] for the Cauchy problem for the Novikov equation including global solutions and blow-up phenomenon.

The main motivation for studying the LDG method for the Novikov equation is twofold. On the one hand, as pointed out in the review article [29] in 2022, the literature surrounding the Novikov equation has not been as extensive as the CH and DP equations, but there is no shortage of articles on the analysis aspects of PDE well-posedness and peakon problems for the Novikov equation as mentioned above. However, the numerical analysis community has not yet jumped on the bandwagon. There are only a few numerical works in the literature about the Novikov equation. In [5, 8], Chen et. al. developed a second-order conservative finite difference scheme for the Novikov equation. Wang and Yan [37] applied the multi-layer physics-informed neural networks (PINNs) deep learning to study the Novikov equation. On the other hand, it has been verified that the LDG method is a good tool for solving nonlinear higher-order equations [42, 43, 23, 39, 41, 46].

Recently, there are many works about LDG methods for the CH and DP equations as well as their generalizations [39, 41, 46, 24, 27, 28]. However, as an intriguing analogue of the CH equation, there is little work about LDG schemes for the Novikov equation in the decades since it was discovered. The main difficulties to develop the LDG method for the Novikov equation (1.1) are as follows:

(i) Upon rewriting the Novikov equation (1.1a) in the following form

$$M_t + f(U)_x - (U^2U_x)_{xx} + (U_x^2U)_x + U_x(U_xU)_x = 0,$$

with $f(U) = \frac{4}{3}U^3$, it is evident that the Novikov equation is nonconserved with respect to M. Therefore, it is difficult to design a stable numerical scheme to solve the Novikov equation;

(ii) The nonlinear term in the Novikov equation (1.1) is cubic which is higher than the quadratic nonlinearities in the CH and DP equations.

In order to design a stable numerical scheme, we notice that the numerical flux is the key point to ensure the stability of DG schemes. Therefore, we introduce some flux correction terms ($\mathcal{B}$ in LDG scheme (2.4)) in the LDG scheme to overcome the first difficulty. These flux correction terms are introduced artificially and their design needs to follow two principles. To be precise, we require them to balance the flux terms that may cause instability, and not to destroy the consistency of the numerical scheme. The introduction of flux correction terms is a highlight of the numerical scheme. The energy boundedness of semi-discrete schemes is obtained thanks to the flux correction terms. It is noted that the rigorous energy boundedness of a fully discrete scheme for such nonlinear equations is out of the scope of this paper, and this aspect will be left for future work. Although we do not discuss the energy boundedness of fully discretization schemes in this paper, one can adopt the relaxation Runge-Kutta (RRK) method [21] to achieve the conservation or dissipation numerically for the fully discrete energy. Here we refer to [39, 41, 46] for the LDG method for the CH and DP equations using the explicit RK method to ensure the efficiency of computing.

The error estimation of the LDG method for nonlinear wave equations with high-order derivatives is challenging. In [39], Xu and Shu proved k-th order (k is the highest degree of polynomial in finite element space) error estimates in L^2-norm for the LDG solution of the CH equation with an a priori assumption,

$$\|u - u_h\| \leq h, \quad \text{for small enough } h.$$

This assumption is always used to deal with the error estimates of the DG method for the nonlinear equations [41, 46]. In our analysis, we first introduce some suitable projections and present the projection error estimates. Then the errors between the projection and the numerical solution are the main part we need to estimate. To this end, we take advantage of the fact that the nonlinear terms have a polynomial structure in CH-type equations. Therefore, the nonlinear error term can be decomposed into two parts: one includes the error term between the projection and the exact solution, and the other includes the error term between the projection and the numerical solution. Since the projection and the numerical solution are both in our finite element space, the nonlinear stability and the error energy equation are used to estimate them in our analysis. These techniques allow us to obtain an error estimate without using any priori assumptions even though the Novikov equation has cubic nonlinear derivative terms.

The DG method we discuss in this work is a class of finite element methods (FEMs) using completely discontinuous basis functions. It was first designed and has been successful in solving first-order PDEs such as nonlinear conservation laws [10, 12]. The DG method has many good properties such as extremely local data structure, high parallel efficiency, and the allowance of arbitrary triangulation with hanging nodes. The LDG method is an extension of the DG method for solving the higher-order PDEs. The main idea of the LDG method is to rewrite the high-order equations into a first-order system and then use the DG method to solve the first-order system. The LDG method was first developed by Cockburn and Shu for convection-diffusion equations [11]. Later, the LDG method has been applied for KdV equations [43], Burgers-Poisson equations [26], Zakharov system [36], and so on. For more information about the LDG method, we refer to the review article [40].

The paper is organized as follows. In section 2, we present the LDG scheme for the Novikov equation. In section 3, we give a proof of the discrete H^1-stability. Error estimates of the LDG scheme are given in section 4. Section 5 contains numerical experiments to confirm the theoretical analysis and the good performance of the numerical scheme. Some concluding remarks are given in section 6. Technical proofs of several lemmas are collected in the appendix.

2. The LDG scheme for the Novikov equation.

In this section, we introduce the semi-discrete LDG scheme for solving the Novikov equation (1.1). First of all, we give some notations to define the LDG scheme.

2.1. Notations. Let $\Omega = (x_l, x_r)$ be our computation domain, $\Omega_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=1}^{N}$ be the partition of Ω, where $x_{\frac{1}{2}} = x_l$ and $x_{N+\frac{1}{2}} = x_r$. Denote the cell length as $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $j = 1, \ldots, N$, and $h = \max_j h_j$. In this paper, we assume Ω_h is quasi-uniform, i.e., there exists a positive constant ρ, such that for all j there holds $h_j/h \geq \rho$ as h tends to zero.

Associated with the partition Ω_h, we define the discontinuous finite element space

$$V_h^k = \left\{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j), \, \forall j = 1, \ldots, N \right\}, \tag{2.1}$$

where $\mathcal{P}_k(I_j)$ denotes the space of polynomials in I_j of degree at most $k \geq 0$. We define the broken Sobolev space, for $m \geq 1$,

$$H^m(\Omega_h) = \left\{ w \in L^2(\Omega) : w|_{I_j} \in H^m(I_j), \, \forall j = 1, \ldots, N \right\}.$$

It is not hard to see $V_h^k \subset H^m(\Omega_h)$. It is allowed to have discontinuities across element interfaces, so we define $v^{\pm}_{j+\frac{1}{2}} = \lim_{\epsilon \to 0^+} v(x_{j+\frac{1}{2}} \pm \epsilon)$ and denote its jump as $[\![v]\!]_{j+\frac{1}{2}} = v^+_{j+\frac{1}{2}} - v^-_{j+\frac{1}{2}}$. Furthermore, we denote

$$(w, v)_j = \int_{I_j} w(x)v(x)dx, \quad \|v\|_j = \|v\|_{L^2(I_j)}, \quad \|v\| = \|v\|_{L^2(\Omega)}, \quad [\![v]\!]^2 = \sum_{j=1}^{N} [\![v]\!]^2_{j-\frac{1}{2}},$$

$$\|v\|^2_{\partial I_j} = v(x^-_{j+\frac{1}{2}})^2 + v(x^+_{j-\frac{1}{2}})^2, \quad \|v\|^2_{\partial \Omega_h} = \sum_{j=1}^{N} \|v\|^2_{\partial I_j}, \quad \|v\|_\infty = \|v\|_{L^\infty(\Omega)}.$$

2.2. The LDG scheme. Following the framework of LDG methods, we introduce two auxiliary variables $R = U_x$, $P = (U^2R)_x$ and we rewrite (1.1) into the following equivalent form

$$M_t + f(U)_x - P_x + (R^2U)_x + R(RU)_x = 0, \tag{2.2a}$$
$$P - (U^2R)_x = 0, \tag{2.2b}$$
$$R - U_x = 0, \tag{2.2c}$$
$$M - U + R_x = 0, \tag{2.2d}$$

with the initial condition

$$U(x, 0) = U_0(x), \tag{2.3}$$

and periodic boundary conditions. The LDG scheme is defined as follows (we omit the subscript h in the numerical solution to simplify notations): Let $u(\cdot, 0) \in V_h^k$ be an approximation of the initial data $U_0(x)$, and for any $t \in (0, T]$ we find $u(\cdot, t)$, $p(\cdot, t)$, $r(\cdot, t)$ and $m(\cdot, t) \in V_h^k$, such that for each cell I_j and any test functions $v, q, \psi, \varphi \in V_h^k$ satisfying

$$(m_t, v)_j - (f(u), v_x)_j + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} + (p, v_x)_j - \hat{p}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{p}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$$
$$- (r^2u, v_x)_j + \widehat{(r^2u)}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \widehat{(r^2u)}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} + (r(ru)_x, v)_j + \mathcal{B}_j(u, r; v) = 0, \tag{2.4a}$$

$$(p, q)_j + (u^2r, q_x)_j - \widehat{(u^2r)}_{j+\frac{1}{2}} q^-_{j+\frac{1}{2}} + \widehat{(u^2r)}_{j-\frac{1}{2}} q^+_{j-\frac{1}{2}} = 0, \tag{2.4b}$$

$$(r, \psi)_j + (u, \psi_x)_j - \hat{u}_{j+\frac{1}{2}} \psi^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} \psi^+_{j-\frac{1}{2}} = 0, \tag{2.4c}$$

$$(m, \varphi)_j - (u, \varphi)_j - (r, \varphi_x)_j + \hat{r}_{j+\frac{1}{2}} \varphi^-_{j+\frac{1}{2}} - \hat{r}_{j-\frac{1}{2}} \varphi^+_{j-\frac{1}{2}} = 0, \tag{2.4d}$$

where

- $\hat{f}$, $\hat{p}$, $\widehat{(r^2u)}$, $\widehat{(u^2r)}$, $\hat{u}$ and $\hat{r}$ are numerical fluxes. We choose

$$\hat{p} = p^-, \quad \widehat{(r^2u)} = (r^+)^2u^+, \quad \widehat{(u^2r)} = (u^-)^2r^-, \quad \hat{u} = u^+, \quad \hat{r} = r^-. \tag{2.5a}$$

For $\hat{f}$, we can choose the following different fluxes such that the numerical scheme is conserved or dissipated:

(i) For a dissipative scheme, we choose the monotone flux. Since $f'(u) = 4u^2 \geq 0$, we take the upwind flux

$$\hat{f}(u^+, u^-) = f(u^-). \tag{2.5b}$$

(ii) For a conservative scheme, we take the central flux

$$\hat{f}(u^+, u^-) = \frac{1}{3}(u^+ + u^-)\big((u^-)^2 + (u^-)^2\big). \tag{2.5c}$$

- $\mathcal{B}_j(u, r; v)$ is a flux correction term and is defined as follows:

$$\mathcal{B}_j(u, r; v) = \big(r^-u^-[\![r]\!]\big)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \big(r^+r^+[\![u]\!]\big)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}. \tag{2.6}$$

For the exact solution, we have $\mathcal{B}_j(U, R; v) = 0$, which makes the numerical scheme consistent. In addition, $\mathcal{B}_j(u, r; v)$ is very crucial to ensure the stability of the scheme and is also the highlight of the whole scheme.

The definition of the algorithm is now complete. Based on the above choice of numerical fluxes, we introduce

$$\mathcal{H}^{\pm}_j(w, v) = (w, v_x)_j - w^{\pm}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + w^{\pm}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}, \tag{2.7}$$

$$\mathcal{G}_j(f(w), v) = (f(w), v_x)_j - \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}. \tag{2.8}$$

Furthermore, we omit the subscript j to denote the sum over j. After summing the variational formulations (2.4) over all cells, we get the LDG scheme in the global form:

$$(m_t, v) - \mathcal{G}(f(u), v) + \mathcal{H}^-(p, v) - \mathcal{H}^+(r^2u, v) + (r(ru)_x, v) + \mathcal{B}(u, r; v) = 0, \tag{2.9a}$$

$$(p, q) + \mathcal{H}^-(u^2r, q) = 0, \tag{2.9b}$$

$$(r, \psi) + \mathcal{H}^+(u, \psi) = 0, \tag{2.9c}$$

$$(m, \varphi) - (u, \varphi) - \mathcal{H}^-(r, \varphi) = 0. \tag{2.9d}$$

Remark 2.1. We should notice that we cannot rewrite (1.1) into a conservation form for M, which brings essential difficulties to the construction of the numerical scheme. The flux correction term $\mathcal{B}_j(u, r; v)$ is carefully designed to ensure nonlinear stability, see Section 3.

To conclude this section, we recall some standard inverse inequalities for the discrete space V_h^k and some properties of bilinear forms $\mathcal{H}^{\pm}$.

LEMMA 2.2. *(Inverse inequalities) There exists an inverse constant $\nu = \nu(k)$, such that for any $v \in V_h^k$*

$$\|v_x\|_j \leq \nu(\rho h)^{-1}\|v\|_j, \quad \|v\|_{L^\infty(I_j)} \leq \sqrt{\nu(\rho h)^{-1}}\|v\|_j. \tag{2.10}$$

We refer to [9] for these standard inverse inequalities. We give the discrete Sobolev inequality in the following lemma:

Lemma 2.3. *[36, Lemma 4.1] There exists a constant C independent of h, such that for any $v \in V_h^k$*

$$\|v\|_\infty^2 \le C\|v\|(\|v\| + \|v_x\| + h^{-\frac{1}{2}}[\![v]\!]). \tag{2.11}$$

In the next lemmas, we recall some properties of bilinear forms $\mathcal{H}^\pm$.

Lemma 2.4. *For any $w, v \in H^1(\Omega_h)$, there holds*

$$\mathcal{H}^-(w,v) + \mathcal{H}^+(v,w) = 0, \tag{2.12}$$

$$|\mathcal{H}^\pm(v,w)| \le \Big(\|v\| + \sqrt{\nu^{-1}(\rho h)}\|v\|_{\partial\Omega_h}\Big)\Big(\|w_x\| + \sqrt{\nu(\rho h)^{-1}}[\![w]\!]\Big). \tag{2.13}$$

Proof. The proof is the standard argument in the DG framework, thus we omit them and refer to [45] for more details. ∎

The next lemma establishes an important relationship between the auxiliary variables and the prime variables, which plays a key role in error estimates.

Lemma 2.5. *For $w \in V_h^k$ and $f \in L^2(\Omega)$, if $\mathcal{H}_j^\pm(w,v) = (f,v)_j$ $\forall v \in V_h^k$, $j = 1, \cdots, N$, then there exists a positive constant $C_{\nu,\rho}$ dependent of ν and ρ, such that*

$$\|w_x\| + \sqrt{\nu(\rho h)^{-1}}[\![w]\!] \le C_{\nu,\rho}\|f\|. \tag{2.14}$$

Proof. We refer to [35] for the details of the proof. ∎

3. The stability of the LDG scheme.

In this section, we study the stability of the LDG scheme (2.4) for solving the Novikov equation (1.1).

Theorem 3.1. *Let u and r be the solution of the scheme (2.4a)-(2.4d), then the discrete energy $E(u,r) = \|u\|^2 + \|r\|^2$ satisfies:*

- *For the dissipative scheme with the numerical flux (2.5a) and (2.5b):*

$$\frac{d}{dt}E(u,r) \le 0. \tag{3.1}$$

- *For the conservative scheme with the numerical flux (2.5a) and (2.5c):*

$$\frac{d}{dt}E(u,r) = 0. \tag{3.2}$$

Proof. **The first energy equation.** We choose $v = u$, $q = -r$ and $\psi = p$ in (2.9a)-(2.9c), respectively, to obtain

$$(m_t, u) - \mathcal{G}(f(u), u) + \mathcal{H}^-(p,u) - \mathcal{H}^+(r^2u, u) + (r(ru)_x, u) + \mathcal{B}(u, r; u) = 0, \tag{3.3a}$$

$$-(p, r) - \mathcal{H}^-(u^2r, r) = 0, \tag{3.3b}$$

$$(r, p) + \mathcal{H}^+(u, p) = 0. \tag{3.3c}$$

By summing up the above three equations in (3.3), it follows from Lemma 2.4 that

$$(m_t, u) - \mathcal{G}(f(u), u) - \mathcal{H}^+(r^2u, u) - \mathcal{H}^-(u^2r, r) + (r(ru)_x, u) + \mathcal{B}(u, r; u) = 0.$$

By the definition of $\mathcal{H}^\pm$, $\mathcal{G}$ and $\mathcal{B}$, it is not hard to get

$$-\mathcal{H}^+(r^2u, u) - \mathcal{H}^-(u^2r, r) + (r(ru)_x, u) + \mathcal{B}(u, r; u) = 0, \tag{3.4}$$

$$\mathcal{G}(f(u), u) = -\sum_{j=1}^{N} \Theta_{j+\frac{1}{2}}, \tag{3.5}$$

where

$$\Theta_{j+\frac{1}{2}} = \int_{u^-_{j+\frac{1}{2}}}^{u^+_{j+\frac{1}{2}}} f(s) - \hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}})\, ds.$$

Therefore, we have

$$(m_t, u) = -\sum_{j=1}^{N} \Theta_{j+\frac{1}{2}}. \tag{3.6}$$

In addition, it is easy to check $\sum_{j=1}^{N} \Theta_{j+\frac{1}{2}} \geq 0$ for the flux (2.5b) and $\sum_{j=1}^{N} \Theta_{j+\frac{1}{2}} = 0$ for the flux (2.5c).

The second energy equation. We firstly choose $\psi = r_t$ in (2.9c), then we take the time derivative in (2.9d) and choose $\varphi = -u$ in (2.9d), to get

$$(r, r_t) + \mathcal{H}^+(u, r_t) = 0, \tag{3.7a}$$

$$-(m_t, u) + (u_t, u) + \mathcal{H}^-(r_t, u) = 0. \tag{3.7b}$$

By summing up the above two equations in (3.7), we obtain from Lemma 2.4

$$(r, r_t) - (m_t, u) + (u_t, u) = 0. \tag{3.8}$$

Therefore, by (3.6) and (3.8), we have

$$(r, r_t) + (u, u_t) = -\sum_{j=1}^{N} \Theta_{j+\frac{1}{2}}, \tag{3.9}$$

which yields (3.1) and (3.2). ∎

4. Error estimates of the LDG method.

In this section, we consider the error estimate of the LDG scheme (2.4) for solving the Novikov equation (1.1). To save space, we only consider the dissipative scheme in the error estimates, and the results can be easily transferred to the conservative scheme. For the dissipative scheme, we have

$$\mathcal{G}(f(w), v) = \mathcal{H}^-(f(w), v) \quad \forall v \in V_h^k.$$

We assume that the exact solution $U(x,t)$ satisfies the following regularity assumption

$$U, U_t \in L^\infty(0, T; H^{k+3}(\Omega)). \tag{4.1}$$

4.1. Projections.

We introduce some projections which will be used in our error estimates.

- The L^2 projection P_h. For $\forall w \in L^2(\Omega)$, $P_h w \in V_h^k$ is defined as following: In each interval I_j, there holds

$$(P_h w - w, v)_j = 0 \quad \forall v \in \mathcal{P}_k(I_j).$$

• The Gauss-Radau projections $P_h^{\pm}$. For $\forall w \in H^m(\Omega_h)\,(m\geq 1)$, $P_h^{\pm}w \in V_h^k$ is defined as following: In each interval I_j, there holds

$$(P_h^{\pm}w - w, v)_j = 0 \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (P_h^{\pm}w)^{\pm}_{j\mp\frac{1}{2}} = w^{\pm}_{j\mp\frac{1}{2}}.$$

By a standard scaling argument [9], it is easy to obtain the following approximation property for the projection errors

$$h^l\|w-\pi_h w\|_{H^l(I_j)} + h^{\frac{1}{2}}\|w-\pi_h w\|_{L^\infty(I_j)} \leq Ch^{\min(k+1,m)}\|w\|_{H^m(I_j)}, \tag{4.2}$$

where $0 \leq l \leq m$, $j = 1,\cdots,N$, $\pi_h = P_h, P_h^{\pm}$ and $C > 0$ is a bounded constant independent of h and j. Furthermore, from the definition of the projections, we can easily get

$$(w - P_h w, v) = 0, \quad \mathcal{H}^{\pm}(w - P_h^{\pm}w, v) = 0 \quad \forall v \in V_h^k. \tag{4.3}$$

4.2. Error equations. We denote

$$(e_u, e_p, e_r, e_m) = (U - u, P - p, R - r, M - m).$$

By the aid of the above projections, we split the errors into two parts, namely

$$(e_u, e_p, e_r, e_m) = (\eta_u - \xi_u, \eta_p - \xi_p, \eta_r - \xi_r, \eta_m - \xi_m),$$

where

$$\begin{aligned} \eta_u &= U - P_h^+U, & \xi_u &= u - P_h^+U;\\ \eta_p &= P - P_h^-P, & \xi_p &= p - P_h^-P;\\ \eta_r &= R - P_hR, & \xi_r &= r - P_hR;\\ \eta_m &= M - P_hM, & \xi_m &= m - P_hM. \end{aligned}$$

Note that the exact solutions (U,P,R,M) also satisfy the LDG scheme (2.9a)-(2.9d), hence we have the following error equations: For any test functions $v, q, \psi, \varphi \in V_h^k$,

$$\begin{aligned} ((\xi_m)_t, v) = {} & ((\eta_m)_t, v) - \mathcal{H}^-(f(U) - f(u), v) + \mathcal{H}^-(\eta_p - \xi_p, v)\\ & -\mathcal{H}^+(R^2U - r^2u, v) + (R(RU)_x - r(ru)_x, v) - \mathcal{B}(u, r; v), \end{aligned} \tag{4.4a}$$

$$(\xi_p, q) = (\eta_p, q) + \mathcal{H}^-(U^2R - u^2r, q), \tag{4.4b}$$

$$(\xi_r, \psi) = (\eta_r, \psi) + \mathcal{H}^+(\eta_u - \xi_u, \psi), \tag{4.4c}$$

$$(\xi_m, \varphi) - (\xi_u, \varphi) = (\eta_m, \varphi) - (\eta_u, \varphi) - \mathcal{H}^-(\eta_r - \xi_r, \varphi). \tag{4.4d}$$

By Lemma 2.3, Lemma 2.5 and (4.3), we can get the following corollary, which states the important relationships between ξ_u and ξ_r.

Corollary 4.1. Suppose ξ_u and ξ_r satisfy (4.4c), then we have

$$\|(\xi_u)_x\| + \sqrt{\nu(\rho h)^{-1}}[\![\xi_u]\!] \leq C_{\nu,\rho}\|\xi_r\|, \tag{4.5}$$

$$\|(\xi_u)\|_\infty \leq C(\|\xi_u\| + \|\xi_r\|), \tag{4.6}$$

where C is a constant independent of h.

Before presenting the energy estimates, let us first discuss the setting of the numerical initial condition.

4.3. The numerical initial condition. The initial condition plays an important role in the proof of the error estimates. Firstly, we take

$$u(0) = P_h^+ U_0. \tag{4.7}$$

It is noted that $r(0)$ can be obtained by the scheme (2.4c). By taking $u = P_h^+ U_0$ in (2.4c), thanks to the definition of the projection P_h^+, we obtain

$$r(0) = P_h(R(x,0)), \tag{4.8}$$

where $R(x,0) = U_0'(x)$. Thus, we can easily get the following initial error estimates.

Lemma 4.2. *Assume that the initial condition $U_0(x) \in H^{k+1}(\Omega)$, and the numerical initial conditions $u(0), r(0)$ satisfy (4.7) and (4.8), respectively, then*

$$\|\xi_u\|(0) = 0, \quad \|\xi_r\|(0) = 0. \tag{4.9}$$

4.4. Error analysis. We first give the following lemma, which presents the energy equation for ξ_u and ξ_r.

Lemma 4.3. *The following equation holds:*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E(\xi_u,\xi_r) =& -(\eta_p,\xi_r)+((\eta_u)_t,\xi_u)+\mathcal{H}^-((\eta_r)_t,\xi_u)-\mathcal{H}^-(f(U)-f(u),\xi_u)\\ &-\mathcal{H}^+(R^2U-r^2u,\xi_u) - \mathcal{H}^-(U^2R-u^2r,\xi_r)\\ &+(R(RU)_x-r(ru)_x,\xi_u)-\mathcal{B}(u,r;\xi_u).\end{aligned} \tag{4.10}$$

Proof. Taking $v = \xi_u$, $q = -\xi_r$, and $\psi = \xi_p$ in (4.4a)-(4.4c), respectively, and owing to (4.3), we have

$$\begin{aligned}((\xi_m)_t,\xi_u) &= -\mathcal{H}^-(f(U)-f(u),\xi_u) - \mathcal{H}^-(\xi_p,\xi_u) - \mathcal{H}^+(R^2U-r^2u,\xi_u)\\ &\quad +(R(RU)_x-r(ru)_x,\xi_u)-\mathcal{B}(u,r;\xi_u),\end{aligned} \tag{4.11a}$$

$$-(\xi_p,\xi_r) = -(\eta_p,\xi_r) - \mathcal{H}^-(U^2R-u^2r,\xi_r), \tag{4.11b}$$

$$(\xi_r,\xi_p) = -\mathcal{H}^+(\xi_u,\xi_p). \tag{4.11c}$$

By summing up the above three equations in (4.11), and using Lemma 2.4, we get

$$\begin{aligned}((\xi_m)_t,\xi_u) &= -\mathcal{H}^-(f(U)-f(u),\xi_u)+(R(RU)_x-r(ru)_x,\xi_u)-\mathcal{B}(u,r;\xi_u)\\ &\quad -(\eta_p,\xi_r) - \mathcal{H}^-(U^2R-u^2r,\xi_r) - \mathcal{H}^+(R^2U-r^2u,\xi_u).\end{aligned} \tag{4.12}$$

Next, we choose $\psi = (\xi_r)_t$ in (4.4c) and take time derivative in (4.4d) and choose $\varphi = -\xi_u$. Owing to (4.3) we have

$$\begin{aligned}(\xi_r,(\xi_r)_t) &= -\mathcal{H}^+(\xi_u,(\xi_r)_t),\\ -((\xi_m)_t,\xi_u) + ((\xi_u)_t,\xi_u) &= ((\eta_u)_t,\xi_u) + \mathcal{H}^-((\eta_r)_t,\xi_u) - \mathcal{H}^-((\xi_r)_t,\xi_u).\end{aligned} \tag{4.13}$$

By summing up the above two equations in (4.13), and empolying Lemma 2.4, we get

$$(\xi_r,(\xi_r)_t) - ((\xi_m)_t,\xi_u) + ((\xi_u)_t,\xi_u) = ((\eta_u)_t,\xi_u) + \mathcal{H}^-((\eta_r)_t,\xi_u). \tag{4.14}$$

Combine (4.12) and (4.14), we obtain (4.10). ∎

Next, we need to estimate the terms on the right hand side of (4.10) to obtain the estimation for ξ_u and ξ_r. Thus, we denote

$$\begin{aligned}
&\Theta_1 := -(\eta_p, \xi_r) + ((\eta_u)_t, \xi_u) + \mathcal{H}^-((\eta_r)_t, \xi_u);\\
&\Theta_2 := -\mathcal{H}^-(f(U) - f(u), \xi_u);\\
&\Theta_3 := -\mathcal{H}^+(R^2U - r^2u, \xi_u) - \mathcal{H}^-(U^2R - u^2r, \xi_r);\\
&\Theta_4 := (R(RU)_x - r(ru)_x, \xi_u);\\
&\Theta_5 := -\mathcal{B}(u, r; \xi_u).
\end{aligned}$$

In the estimate of the Θ_1 - Θ_5 we assume $k \geq 1$ and $h < 1$. In addition, under the smoothness assumption (4.1), the constant "C" in Lemma 4.4 - Lemma 4.10 is dependent on the smoothness of the exact solution and is independent of h.

LEMMA 4.4. **(The estimate for Θ_1 and Θ_2)** *For $k \geq 1$, we have the following estimates for the terms Θ_1 and Θ_2*

$$\Theta_1 \leq C\|\xi_u\|^2 + C\|\xi_r\|^2 + Ch^{2k+2}, \tag{4.15}$$

$$\Theta_2 \leq C\|\xi_u\|^2 + C\|\xi_r\|^2 + Ch^{2k+2}, \tag{4.16}$$

where C is a constant independent of h.

Proof. According to the projection properties (4.2), Lemma 2.4 and Corollary 4.1, we get the estimates for Θ_1. To estimate Θ_2, we first rewrite the error $f(U) - f(u)$ in the following form:

$$f(U) - f(u) = \frac{4}{3}e_u(3U^2 - 3Ue_u + e_u^2),$$

from which we get

$$\Theta_2 = -\mathcal{H}^-(4U^2e_u, \xi_u) + \mathcal{H}^-(4Ue_u^2, \xi_u) - \mathcal{H}^-(\frac{4}{3}e_u^3, \xi_u). \tag{4.17}$$

For the first term in (4.17), by Lemma 2.4, Corollary 4.1 and inverse inequality for ξ_u, we have

$$\begin{aligned}
-\mathcal{H}^-(4U^2e_u, \xi_u) &\leq C(\|\eta_u\| + h^{1/2}\|\eta_u\|_{\partial\Omega_h} + \|\xi_u\|)\|\xi_r\|\\
&\leq C\|\xi_u\|^2 + C\|\xi_r\|^2 + Ch^{2k+2}.
\end{aligned}$$

The last inequality is derived by the error estimate of projections (4.2). For the second term in (4.17), by Lemma 2.3, Lemma 2.5 and (2.9c), we have

$$\|u\|_\infty^2 \leq C\|u\|(\|u\| + \|u_x\| + h^{-1/2}[\![u]\!]) \leq C\|u\|(\|u\| + \|r\|).$$

From the energy stability result in Theorem 3.1 and the boundedness of projections (4.7)-(4.8), we obtain

$$\|u\|_\infty^2 \leq C(\|u(0)\| + \|r(0)\|)^2 \leq C(\|U_0\|_\infty + \|U_0'\|)^2.$$

Thus we have the estimation for $\|e_u\|_\infty$:

$$\|e_u\|_\infty \leq \|U\|_\infty + \|u\|_\infty \leq \|U\|_\infty + C(\|U_0\|_\infty + \|U_0'\|) \leq C. \tag{4.18}$$

Furthermore, by Lemma 2.4 and Corollary 4.1, we have

$$\begin{aligned}\mathcal{H}^-(4Ue_u^2,\xi_u) &\le C(\|e_u^2\| + h^{1/2}\|e_u^2\|_{\partial\Omega_h})\|\xi_r\|\\ &\le C(\|e_u\| + h^{1/2}\|e_u\|_{\partial\Omega_h})\|\xi_r\| \quad \text{(by (4.18))}\\ &\le C\|\xi_u\|^2 + C\|\xi_r\|^2 + Ch^{2k+2}.\end{aligned}$$

For the third term in (4.17), we can do a similar analysis as that for the second term. This completes the proof. ∎

Remark 4.5. We would like to point out that we do not need to use a priori error assumption as in [39] to estimate Θ_2, since the nonlinear term is a polynomial and thanks to the relationship between u and r. The energy stability gives us the boundedness for u in L^∞-norm to deal with the high-order terms in Θ_2. In addition, we again use the relationship between ξ_u and ξ_r to deal with the derivative of ξ_u and boundary terms, which makes the estimates easier.

The estimates for the Θ_3, Θ_4, and Θ_5 are very technical since they include nonlinear differential terms and nonlinear boundary terms. The main idea in our analysis is to make use of the nonlinear stability as given in (3.4). However, since the stability results are only valid for functions in V_h^k, we need to decompose the error with the help of projections and use the following property

$$\mathcal{H}^+(\xi_r^2\xi_u,\xi_u) + \mathcal{H}^-(\xi_u^2\xi_r,\xi_r) - (\xi_r(\xi_r\xi_u)_x,\xi_u) - \mathcal{B}(\xi_u,\xi_r;\xi_u) = 0. \tag{4.19}$$

We use the following lemmas to estimate Θ_3, Θ_4 and Θ_5, for which the details of the proof will be placed in the Appendix for the convenience of the readers.

Lemma 4.6. **(The estimate for Θ_3)** *There exists a constant C independent of h, such that for $k \ge 1$*

$$\begin{aligned}\Theta_3 \le\ & Ch^{-1}(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + Ch^k\|\xi_u\| + Ch^k\|\xi_r\|\\ & + \mathcal{H}^+(\xi_r^2\xi_u,\xi_u) + \mathcal{H}^-(\xi_u^2\xi_r,\xi_r) - \frac{1}{2}\sum_{j=1}^{N}(U[\![\xi_r]\!])^2_{j+\frac12} + \Gamma_1,\end{aligned} \tag{4.20}$$

where Γ_1 is a boundary term and is defined as follows

$$\begin{aligned}\Gamma_1 =& \sum_{j=1}^{N}\Big(\big(U(\xi_r^+)^2 + 2R\xi_r^+\xi_u^+\big)[\![\xi_u]\!] + \big(R(\xi_u^-)^2 + 2U\xi_r^-\xi_u^-\big)[\![\xi_r]\!]\Big)_{j+\frac12}\\ &+ \sum_{j=1}^{N}\Big(U(\xi_r^-)^2\xi_u^- - U(\xi_r^+)^2\xi_u^+ + R(\xi_u^-)^2\xi_r^- - R(\xi_u^+)^2\xi_r^+\Big)_{j+\frac12}.\end{aligned}$$

Proof. The proof of this lemma is provided in Appendix A.1. ∎

Lemma 4.7. **(The estimate for Θ_4)** *There exists a constant C independent of h, such that for $k \ge 1$*

$$\begin{aligned}\Theta_4 \le\ & Ch^{-1}(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + Ch^k\|\xi_u\|\\ & - (\xi_r(\xi_r\xi_u)_x,\xi_u) + \Gamma_2,\end{aligned} \tag{4.21}$$

where Γ_2 is a boundary term and is defined as follows

$$\Gamma_2 = \sum_{j=1}^{N}\Big(-RU\xi_r^-\xi_u^- + RU\xi_r^+\xi_u^+ - R\xi_r^-(\xi_u^-)^2 + R\xi_r^+(\xi_u^+)^2 - \frac12 U\xi_u^-(\xi_r^-)^2 + \frac12 U\xi_u^+(\xi_r^+)^2\Big)_{j+\frac12}.$$

Proof. The proof of this lemma is provided in Appendix A.2. ∎

LEMMA 4.8. **(The estimate for Θ_5)** *There exists a constant C independent of h, such that for $k \geq 1$*

$$\Theta_5 \leq C(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + Ch^k\|\xi_u\| + Ch^k\|\xi_r\| - \mathcal{B}(\xi_u, \xi_r; \xi_u) + \Gamma_3, \tag{4.22}$$

where Γ_3 is a boundary term and is defined as follows

$$\Gamma_3 = \sum_{j=1}^{N} \Big((-RU - R\xi_u^- - U\xi_r^-)\xi_u^- [\![\xi_r]\!] - 2R\xi_r^+\xi_u^+[\![\xi_u]\!] \Big)_{j+\frac{1}{2}}.$$

Proof. The proof of this lemma is provided in Appendix A.3. ∎

LEMMA 4.9. *There exists a constant C independent of h, such that for $k \geq 1$*

$$\Gamma_1 + \Gamma_2 + \Gamma_3 - \frac{1}{2}\sum_{j=1}^{N}(U[\![\xi_r]\!])^2_{j+\frac{1}{2}} \leq Ch^{-1}\|\xi_u\|^4 + Ch^{-1}\|\xi_r\|^4 + C\|\xi_r\|^2. \tag{4.23}$$

Proof. The proof of this lemma is provided in Appendix A.4. ∎

LEMMA 4.10. *For $k \geq 1$, the ξ_u and ξ_r satisfy*

$$\frac{d}{dt}E(\xi_u, \xi_r) \leq Ch^{-1}(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + Ch^{2k}. \tag{4.24}$$

where C is a bounding constant independent of h.

Proof. By Lemma 4.3 and combining the estimate for $\Theta_1 - \Theta_5$ in Lemma 4.4, Lemma 4.6 – Lemma 4.9 , we have

$$\begin{aligned}\frac{d}{dt}E(\xi_u, \xi_r) \leq& Ch^{-1}(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + Ch^{2k} \\ &+ \mathcal{H}^+(\xi_r^2\xi_u, \xi_u) + \mathcal{H}^-(\xi_u^2\xi_r, \xi_r) - (\xi_r(\xi_u\xi_r)_x, \xi_u) - \mathcal{B}(\xi_u, \xi_r; \xi_u).\end{aligned}$$

Using the stability results (4.19), the last line on the right-hand side will vanish. Hence, we obtain (4.24). ∎

LEMMA 4.11. *If $k \geq 1$, $A(0) \leq Ch^{2k+2}$ and $A(t)$ satisfies the following inequality*

$$A'(t) \leq C(h^{-1}A^2 + A + h^{2k}), \quad 0 \leq t \leq T, \tag{4.25}$$

where C is a constant independent of h and t. Then when h is small enough, we have

$$A(t) \leq \tilde{C}h^{2k}, \quad 0 \leq t \leq T,$$

where $\tilde{C}$ is a constant independent of h and dependent on T.

Proof. The proof of this lemma is provided in Appendix A.5. ∎

Finally, we present our main result in this section by the following Theorem.

THEOREM 4.12. *Let (U, P, R, M) be the exact solution of the Novikov equation (1.1) satisfying the smoothness assumption (4.1), and let (u, p, r, m) be the numerical solution of the LDG scheme (2.4), then under the initial condition in Lemma 4.2 and for $k \geq 1$, we have*

$$\|U - u\|^2 + \|R - r\|^2 \leq Ch^{2k}, \tag{4.26}$$

where C is a bounded constant independent of h and dependent on the $\|U\|_{L^\infty([0,T],H^{k+3}(\Omega))}$.

Proof. By using Lemma 4.10, Lemma 4.11, and the estimate of the initial condition in Lemma 4.2, we have

$$E(\xi_u, \xi_r) \leq Ch^{2k}.$$

Combining the approximation property for the projection error and using the triangle inequality we get

$$\|U - u\| + \|R - r\| \leq \|\eta_u\| + \|\eta_r\| + \|\xi_u\| + \|\xi_r\| \leq Ch^k.$$

This completes the proof. ∎

5. Numerical experiments. In this section, we present some numerical examples to confirm our theoretical results. We adopt the classical fourth-order Runge-Kutta method as our time-stepping method for the numerical examples unless otherwise specified. The CFL condition is $\Delta t = O(h)$, where Δt and h are temporal step size and spatial step size, respectively. We measure the error in the energy norm, that is $\sqrt{E(U-u, R-r)}$. The computations are (partly) done on the high performance computers of State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences.

Example 5.1. We test the accuracy of our proposed LDG method. We take a source term $s(x,t)$ on the right side of the Novikov equation (1.1a) such that its exact solution is $U(x,t) = \cos(\pi(x-t))$. The computational domain is $\Omega = (0,2)$ with the periodic boundary condition.

We test this example on the uniform and non-uniform meshes for $k = 0, 1, 2, 3$. The non-uniform mesh is generated by perturbing randomly 10% on the uniform mesh. The numerical errors and convergence rates are shown in Table 1. We can observe the $(k+1)$-th optimal convergence rates of dissipative and conservative schemes on both uniform and non-uniform meshes.

Example 5.2. In this example, we consider a smooth soliton solution as shown in [30]. The exact solution can be written in the following form

$$U^2 = \frac{2\kappa^3(\cosh\xi + \frac{1+2\alpha^2}{1-\alpha^2})^2}{\cosh 2\xi + \frac{8(2+\alpha^2)}{4-\alpha^2}\cosh\xi + \frac{3(4-\alpha^2+3\alpha^4)}{(1-\alpha^2)(4-\alpha^2)}},$$

where

$$x - ct - x_0 = \frac{\xi}{\alpha} + \frac{1}{2}\ln\left(\frac{\tanh^2\frac{\xi}{2} - \frac{2}{\alpha}\tanh\frac{\xi}{2} + \frac{4-\alpha^2}{3\alpha^2}}{\tanh^2\frac{\xi}{2} + \frac{2}{\alpha}\tanh\frac{\xi}{2} + \frac{4-\alpha^2}{3\alpha^2}}\right),$$

with

$$c = \frac{\kappa^3(4-\alpha^2)}{1-\alpha^2},$$

is the velocity of the soliton.

We test this example on uniform meshes for $\kappa = 1$, $x_0 = 0$, and three distinct values of $\alpha = 0.7, 0.85, 0.95$. The computational domain is set as $\Omega = (-20, 20)$ with compact support boundary conditions and the terminal time $T = 0.1$. We show the exact solution for $\alpha = 0.7, 0.85, 0.95$ in Figure 1. The asymptotic behavior of U as

		uniform mesh				non-uniform mesh			
		Dissipative scheme		Conservative scheme		Dissipative scheme		Conservative scheme	
	N	Error	Order	Error	Order	Error	Order	Error	Order
$\mathcal{P}_0$	20	3.75E+00	–	4.36E+00	–	5.02E+00	–	4.42E+00	–
	40	1.43E+00	1.39	1.60E+00	1.44	3.94E+00	0.35	1.80E+00	1.29
	80	9.37E-01	0.61	8.98E-01	0.84	1.39E+00	1.50	9.07E-01	0.99
	160	5.59E-01	0.75	5.32E-01	0.75	7.42E-01	0.91	5.38E-01	0.75
	320	3.02E-01	0.89	2.88E-01	0.88	4.03E-01	0.88	3.18E-01	0.76
	640	1.59E-01	0.93	1.52E-01	0.93	2.14E-01	0.91	1.90E-01	0.74
$\mathcal{P}_1$	20	9.67E-02	–	9.85E-02	–	1.02E-01	–	2.62E-01	–
	40	2.41E-02	2.00	2.44E-02	2.02	2.47E-02	2.04	2.64E-02	3.31
	80	6.11E-03	1.98	6.14E-03	1.99	6.48E-03	1.93	1.03E-02	1.35
	160	1.53E-03	1.99	1.54E-03	2.00	1.55E-03	2.06	1.55E-03	2.74
	320	3.83E-04	2.00	3.84E-04	2.00	3.87E-04	2.00	3.92E-04	1.98
	640	9.58E-05	2.00	9.59E-05	2.00	9.78E-05	1.99	9.69E-05	2.02
$\mathcal{P}_2$	20	3.79E-03	–	3.80E-03	–	4.14E-03	–	4.04E-03	–
	40	5.64E-04	2.75	5.64E-04	2.75	5.71E-04	2.86	5.72E-04	2.82
	80	7.66E-05	2.88	7.66E-05	2.88	7.77E-05	2.88	7.73E-05	2.89
	160	1.03E-05	2.89	1.03E-05	2.89	1.04E-05	2.90	1.04E-05	2.89
	320	1.41E-06	2.87	1.41E-06	2.87	1.43E-06	2.86	1.43E-06	2.86
	640	1.95E-07	2.85	1.95E-07	2.85	1.99E-07	2.85	1.98E-07	2.85
$\mathcal{P}_3$	20	1.02E-04	–	1.02E-04	–	1.16E-04	–	1.11E-04	–
	40	4.80E-06	4.41	4.80E-06	4.41	5.06E-06	4.52	4.91E-06	4.50
	80	2.79E-07	4.11	2.79E-07	4.11	2.85E-07	4.15	2.87E-07	4.10
	160	1.45E-08	4.26	1.45E-08	4.26	1.46E-08	4.29	1.47E-08	4.29
	320	6.43E-10	4.50	6.43E-10	4.50	6.83E-10	4.42	6.69E-10	4.45

TABLE 1

Example 5.1: Energy errors and orders of numerical solutions at the terminal time $T = 1.0$.

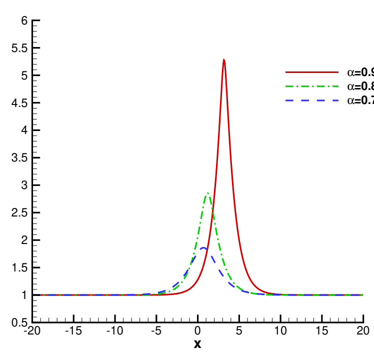

FIG. 1. *Example 5.2: The exact solution at time* $T = 0.1$ *for* $\alpha = 0.7, 0.85, 0.95$.

α tends to 1 was explored in [30], which shows that it forms a peak. The errors and orders of numerical solutions of dissipative schemes and conservative schemes are shown in Table 2 and Table 3 respectively. Again we observe the $(k+1)$-th optimal convergence rates for all numerical solutions.

Example 5.3. We consider the peakon solution [19] $U(x,t) = \sqrt{c}e^{-|x-ct|}$ with $c = 0.36$ of the Novikov equation (1.1), which is a right-going traveling wave solution, and bright peakon solution.

The computational domain is $\Omega = (-10, 10)$ with compact support boundary

		$\alpha = 0.7$		$\alpha = 0.85$		$\alpha = 0.95$	
	N	Error	Order	Error	Order	Error	Order
$\mathcal{P}_0$	160	9.79E-02	–	5.26E-01	–	3.22E+00	–
	320	5.01E-02	0.97	3.14E-01	0.74	2.40E+00	0.42
	640	2.54E-02	0.98	1.77E-01	0.83	1.69E+00	0.51
	1280	1.28E-02	0.99	9.53E-02	0.89	1.11E+00	0.61
	2560	6.39E-03	1.00	4.97E-02	0.94	6.83E-01	0.70
$\mathcal{P}_1$	20	1.40E-01	–	7.35E-01	–	3.58E+00	–
	40	4.54E-02	1.63	2.75E-01	1.42	1.97E+00	0.86
	80	1.13E-02	2.01	8.57E-02	1.68	1.13E+00	0.81
	160	2.86E-03	1.98	1.99E-02	2.10	4.68E-01	1.27
	320	7.26E-04	1.98	4.76E-03	2.07	1.38E-01	1.76
	640	1.83E-04	1.99	1.17E-03	2.02	2.68E-02	2.36
$\mathcal{P}_2$	20	3.94E-02	–	2.80E-01	–	2.18E+00	–
	40	7.80E-03	2.34	5.47E-02	2.36	8.68E-01	1.33
	80	1.28E-03	2.61	1.08E-02	2.34	3.53E-01	1.30
	160	1.29E-04	3.31	1.37E-03	2.98	7.43E-02	2.25
	320	1.13E-05	3.51	1.47E-04	3.22	7.32E-03	3.34
	640	1.38E-06	3.03	1.82E-05	3.02	6.63E-04	3.46
$\mathcal{P}_3$	20	1.01E-02	–	1.34E-01	–	1.52E+00	–
	40	8.37E-04	3.59	1.89E-02	2.82	4.52E-01	1.75
	80	9.46E-05	3.15	1.53E-03	3.63	1.34E-01	1.75
	160	7.09E-06	3.74	1.06E-04	3.85	1.33E-02	3.34
	320	5.93E-07	3.58	7.71E-06	3.78	5.99E-04	4.47
	640	3.07E-08	4.27	4.34E-07	4.15	4.10E-05	3.87

TABLE 2

Example 5.2: Errors and orders of the dissipative scheme at the terminal time $T = 0.1$.

conditions and the terminal time is $T = 10.0$. We plot the profile of numerical solutions at different times for $k = 3$ on the uniform mesh with $N = 320$, see Figure 2. We observe the LDG method can capture the peakon structure well. We also show the differences $|E(t) - E(0)|$ in Figure 2, it shows that the conservative scheme can preserve the error of the total energy near the machine error level and the dissipative scheme has monotone energy stability.

Example 5.4. In the last example, we consider the periodic peakon solution $U(x,t) = \sqrt{c}\,\mathrm{sech}\pi\cosh(x - ct - 2\pi\lfloor (x-ct)/(2\pi)\rfloor - \pi)$ in [15, 14]. The periodic domain is $\Omega = (-3\pi, 3\pi)$ and the terminal time is $T = 10.0$.

We solve this example by using the LDG method for $k = 3$ on uniform mesh with $N = 320$. The profile of numerical solutions u at different times are shown in Figure 3. The energy error $|E(t) - E(0)|$ against time is also plotted in Figure 3. We again observe the energy conservation and energy stability for the conservative and dissipative schemes respectively.

6. Conclusion. We have developed a LDG method to solve the Novikov equation. The energy stability is proven for general solutions, and an a priori error estimate is obtained for smooth solutions. The nonlinear stability helps us to deal with the nonlinear spatial discretization terms and obtain the error estimate without using any priori assumptions. Numerical examples demonstrate our proposed schemes have

		$\alpha = 0.7$		$\alpha = 0.85$		$\alpha = 0.95$	
	N	Error	Order	Error	Order	Error	Order
$\mathcal{P}_0$	160	8.26E-02	–	4.23E-01	–	2.50E+00	–
	320	4.14E-02	1.00	2.47E-01	0.78	1.84E+00	0.44
	640	2.07E-02	1.00	1.37E-01	0.84	1.30E+00	0.50
	1280	1.03E-02	1.00	7.36E-02	0.90	8.69E-01	0.58
	2560	5.16E-03	1.00	3.83E-02	0.94	5.45E-01	0.67
$\mathcal{P}_1$	20	1.39E-01	–	6.49E-01	–	3.18E+00	–
	40	4.58E-02	1.60	2.59E-01	1.33	1.86E+00	0.78
	80	1.12E-02	2.03	8.10E-02	1.68	1.09E+00	0.76
	160	2.86E-03	1.97	1.93E-02	2.07	4.59E-01	1.25
	320	7.26E-04	1.98	4.71E-03	2.04	1.36E-01	1.76
	640	1.83E-04	1.99	1.17E-03	2.01	2.66E-02	2.36
$\mathcal{P}_2$	20	4.04E-02	–	2.83E-01	–	2.17E+00	–
	40	8.19E-03	2.30	5.20E-02	2.44	8.43E-01	1.36
	80	1.28E-03	2.68	1.07E-02	2.28	3.47E-01	1.28
	160	1.30E-04	3.31	1.37E-03	2.97	7.35E-02	2.24
	320	1.12E-05	3.53	1.47E-04	3.22	7.27E-03	3.34
	640	1.38E-06	3.03	1.82E-05	3.02	6.62E-04	3.46
$\mathcal{P}_3$	20	9.84E-03	–	1.29E-01	–	1.52E+00	–
	40	8.06E-04	3.61	1.86E-02	2.79	4.46E-01	1.77
	80	9.27E-05	3.12	1.51E-03	3.62	1.33E-01	1.74
	160	7.06E-06	3.72	1.06E-04	3.84	1.32E-02	3.33
	320	5.94E-07	3.57	7.71E-06	3.78	5.97E-04	4.47
	640	3.07E-08	4.27	4.34E-07	4.15	4.10E-05	3.87

TABLE 3
Example 5.2: Errors and orders of the conservative scheme at the terminal time $T = 0.1$.

arbitrarily high-order accuracy and the capability of capturing the peakon solutions. Only the semi-discrete scheme is analyzed in this paper and the numerical analysis of the fully discrete scheme will be left for our future work.

Appendix A. Proof of a few technical lemmas.

A.1. The proof for Lemma 4.6.

Proof. We recall the definition of Θ_3,

$$\Theta_3 = -\mathcal{H}^+(R^2U - r^2u, \xi_u) - \mathcal{H}^-(U^2R - u^2r, \xi_r).$$

The main difficulty in estimating Θ_3 is caused by the nonlinear terms, especially $\mathcal{H}^+(\xi_r^2\xi_u, \xi_u)$ and $\mathcal{H}^-(\xi_u^2\xi_r, \xi_r)$ contained in Θ_3. If we estimate these two terms directly with inverse inequalities, we cannot obtain the error estimate results, not even for suboptimal results. Therefore, our main idea is to decompose the error to extract these two terms and treat them with nonlinear stability (4.19), then estimate the remaining part.

Step 1: Error decomposition.

Firstly, we have

$$R^2 - r^2 = e_r(2R - e_r) = \eta_r(2R - \eta_r) + 2\eta_r\xi_r - 2R\xi_r - \xi_r^2.$$

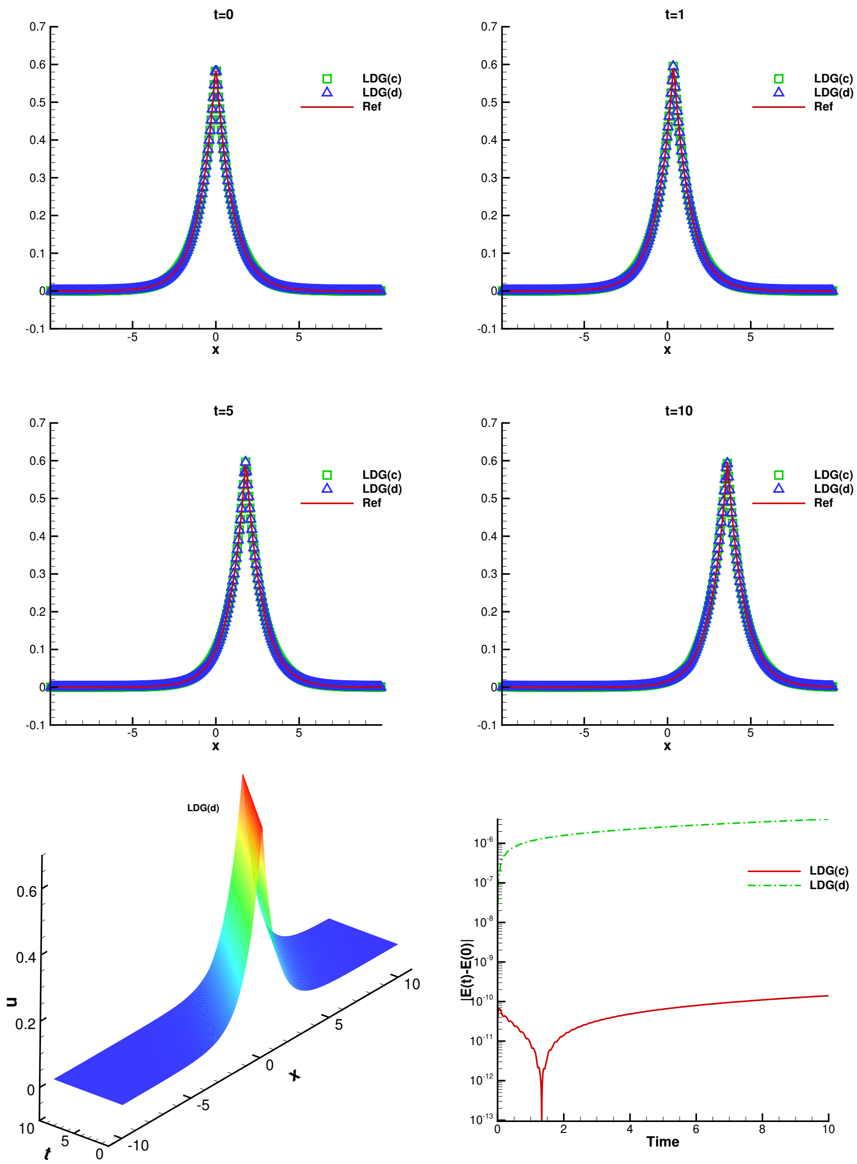

FIG. 2. *Example 5.3: The numerical solutions u for $k = 3$ on uniform mesh with $N = 320$.*

The projection error η_u and η_r are high order terms, since we have the projection error estimates results (4.2). Therefore, we put together the terms that have projection errors and denote

$$A_1 = \eta_r(2R - \eta_r) + 2\eta_r\xi_r.$$

Therefore,

$$\begin{aligned}&-(R^2U - r^2u)\\ =&-U(R^2 - r^2) - R^2(U - u) + (U - u)(R^2 - r^2)\end{aligned}$$

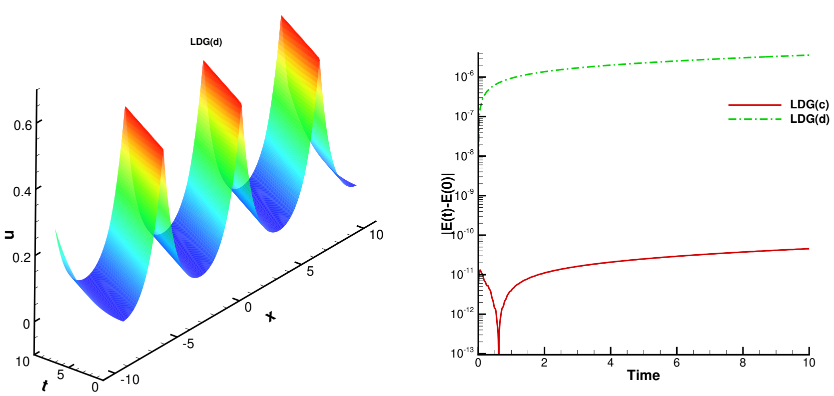

FIG. 3. *Example 5.4: The numerical solutions u for k = 3 on uniform mesh with N = 320.*

$$
\begin{aligned}
&= -U(A_1 - 2R\xi_r - \xi_r^2) - R^2\eta_u + R^2\xi_u + (\eta_u - \xi_u)(A_1 - 2R\xi_r - \xi_r^2) \\
&:= \Pi_1 + \Pi_2 + \xi_u\xi_r^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_1 &= -UA_1 - R^2\eta_u + \eta_u(A_1 - 2R\xi_r - \xi_r^2) - \xi_u A_1, \\
\Pi_2 &= 2UR\xi_r + U\xi_r^2 + R^2\xi_u + 2R\xi_r\xi_u.
\end{aligned}
$$

Similarly, we denote $A_2 = \eta_u(2U - \eta_u) + 2\eta_u\xi_u$, then

$$-(U^2R - u^2r) := \Pi_3 + \Pi_4 + \xi_r\xi_u^2,$$

where

$$\Pi_3 = -RA_2 - U^2\eta_r + \eta_r(A_2 - 2U\xi_u - \xi_u^2) - \xi_r A_2,$$
$$\Pi_4 = 2UR\xi_u + R\xi_u^2 + U^2\xi_r + 2U\xi_u\xi_r.$$

Hence, we have

$$\Theta_3 = \mathcal{H}^+(\Pi_1 + \Pi_2, \xi_u) + \mathcal{H}^-(\Pi_3 + \Pi_4, \xi_r) + \mathcal{H}^+(\xi_r^2\xi_u, \xi_u) + \mathcal{H}^-(\xi_u^2\xi_r, \xi_r).$$

After the error decomposition we extract $\mathcal{H}^+(\xi_r^2\xi_u, \xi_u)$ and $\mathcal{H}^-(\xi_u^2\xi_r, \xi_r)$, and make each term in Π_1 and Π_3 containing a projection error. Thus, it is easy to obtain the estimates for Π_1 and Π_3 by the projection properties and inverse inequalities. However, the terms in Π_2 and Π_4 should be treated carefully.

Step 2: Estimates.

The estimates for $\mathcal{H}^+(\Pi_1, \xi_u) + \mathcal{H}^-(\Pi_3, \xi_r)$:

By the projection property (4.2) and inverse inequalities (2.10), we have

$$\|\Pi_1\| \le Ch^{k+\frac{1}{2}}\|\xi_r\| + Ch^{k+\frac{1}{2}}\|\xi_u\| + Ch^k\|\xi_r\|\|\xi_u\| + Ch^k\|\xi_r\|^2 + Ch^{k+1},$$
$$\|\Pi_3\| \le Ch^{k+\frac{1}{2}}\|\xi_r\| + Ch^{k+\frac{1}{2}}\|\xi_u\| + Ch^k\|\xi_r\|\|\xi_u\| + Ch^k\|\xi_u\|^2 + Ch^{k+1}.$$

Therefore,

$$\begin{aligned}
&\mathcal{H}^+(\Pi_1, \xi_u) + \mathcal{H}^-(\Pi_3, \xi_r)\\
\le &Ch^{-1}\|\Pi_1\|\|\xi_u\| + Ch^{-1}\|\Pi_3\|\|\xi_r\|\\
\le &C\|\xi_r\|^2 + C\|\xi_u\|^2 + Ch^k\|\xi_u\| + Ch^k\|\xi_r\| + C\|\xi_r\|^4 + C\|\xi_u\|^4.
\end{aligned}$$

The estimates for $\mathcal{H}^+(\Pi_2, \xi_u) + \mathcal{H}^-(\Pi_4, \xi_r)$:

By integration by parts, we have

$$\mathcal{H}^+(2UR\xi_r, \xi_u) + \mathcal{H}^-(2UR\xi_u, \xi_r) = -2((UR)_x\xi_u, \xi_r),$$
$$\mathcal{H}^+(U\xi_r^2 + 2R\xi_r\xi_u, \xi_u) + \mathcal{H}^-(R\xi_u^2 + 2U\xi_r\xi_u, \xi_r) = -(U_x\xi_r^2, \xi_u) - (R_x\xi_u^2, \xi_r) + \Gamma_1.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathcal{H}^+(2UR\xi_r + U\xi_r^2 + 2R\xi_r\xi_u, \xi_u) + \mathcal{H}^-(2UR\xi_u + R\xi_u^2 + 2U\xi_r\xi_u, \xi_r)\\
= &-2((UR)_x\xi_u, \xi_r) - (U_x\xi_r^2, \xi_u) - (R_x\xi_u^2, \xi_r) + \Gamma_1\\
\le &C\|\xi_u\|\|\xi_r\| + \|\xi_r\xi_u\|(\|\xi_u\| + \|\xi_r\|) + \Gamma_1\\
\le &Ch^{-1/2}(\|\xi_r\|^2\|\xi_u\| + \|\xi_u\|^2\|\xi_r\|) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + \Gamma_1\\
\le &Ch^{-1}(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + \Gamma_1.
\end{aligned}$$

Here we used the inverse inequality for $\|\xi_u\xi_r\|$ as follows:

$$\|\xi_u\xi_r\| \le \|\xi_u\|_\infty\|\xi_r\| \le Ch^{-1/2}\|\xi_u\|\|\xi_r\|. \tag{A.1}$$

By Lemma 2.4 and Corollary 4.1, we have

$$\mathcal{H}^+(R^2\xi_u, \xi_u) \le C\|\xi_r\|^2 + C\|\xi_u\|^2.$$

By integration by parts, we have

$$\mathcal{H}^-(U^2\xi_r,\xi_r) = -(UU_x,\xi_r^2) - \frac{1}{2}\sum_{j=1}^{N}(U[\![\xi_r]\!])^2_{j+\frac{1}{2}} \le C\|\xi_r\|^2 - \frac{1}{2}\sum_{j=1}^{N}(U[\![\xi_r]\!])^2_{j+\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}&\mathcal{H}^+(\Pi_2,\xi_u) + \mathcal{H}^-(\Pi_4,\xi_r)\\ \le\, &Ch^{-1}(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) - \frac{1}{2}\sum_{j=1}^{N}(U[\![\xi_r]\!])^2_{j+\frac{1}{2}} + \Gamma_1.\end{aligned}$$

Thanks to the linearity of the operator $\mathcal{H}^\pm$ we obtain (4.20). ∎

A.2. The proof for Lemma 4.7.

Proof. Recall the definition of Θ_4,

$$\Theta_4 = \big(R(RU)_x - r(ru)_x, \xi_u\big).$$

Similar to the estimate of Θ_3, we also need to extract the terms in $R(RU)_x - r(ru)_x$ that cannot directly use the inverse inequalities to estimate. Therefore, we firstly do error decomposition.

Step 1: Error decomposition.

$$(RU - ru)_x = (Re_u + Ue_r - e_re_u)_x := B_1 + B_2,$$

where

$$\begin{aligned}B_1 &= (R\eta_u + U\eta_r - \eta_u\eta_r + \eta_r\xi_u + \eta_u\xi_r)_x - R_x\xi_u - U_x\xi_r,\\ B_2 &= -R(\xi_u)_x - U(\xi_r)_x - (\xi_u\xi_r)_x.\end{aligned}$$

Here we collect the terms containing the derivative of ξ_u or ξ_r but not the projection error in B_2. Then, we have

$$\begin{aligned}R(RU)_x - r(ru)_x &= R(RU - ru)_x + (RU)_xe_r - (RU - ru)_x(\eta_r - \xi_r)\\ &= RB_1 + RB_2 + (RU)_xe_r - \eta_r(B_1 + B_2) + \xi_r(B_1 + B_2)\\ &= \Lambda_1 + \Lambda_2 - \xi_r(\xi_u\xi_r)_x,\end{aligned}$$

where

$$\Lambda_1 = RB_1 + (RU)_xe_r - \eta_r(B_1 + B_2) + \xi_rB_1, \quad \Lambda_2 = RB_2 - \xi_r\big(R(\xi_u)_x + U(\xi_r)_x\big).$$

Therefore,

$$\Theta_4 = (\Lambda_1,\xi_u) + (\Lambda_2,\xi_u) - (\xi_r(\xi_u\xi_r)_x,\xi_u).$$

After the error decomposition, we extract $(\xi_r(\xi_u\xi_r)_x,\xi_u)$ and Λ_2, which cannot directly use the inverse inequalities to estimate.

Step 2: Estimates.

The estimates for Λ_1:

By the properties of projections in (4.2), inverse inequalities in Lemma 2.2 and (A.1), we have

$$\|B_1\| \le Ch^k + C\|\xi_u\| + C\|\xi_r\|, \tag{A.2}$$

$$\|B_2\| \le Ch^{-1}(\|\xi_u\| + \|\xi_r\|) + Ch^{-\frac{3}{2}}\|\xi_u\|\|\xi_r\|. \tag{A.3}$$

Then, by (A.1) – (A.3) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(\Lambda_1, \xi_u) &\le C\|B_1\|\|\xi_u\| + C\|e_r\|\|\xi_u\| + \|\eta_r\|_\infty(\|B_1\| + \|B_2\|)\|\xi_u\| + \|B_1\|\|\xi_r\xi_u\| \\ &\le Ch^{-1}(\|\xi_r\|^4 + \|\xi_u\|^4) + C(\|\xi_r\|^2 + \|\xi_u\|^2) + Ch^k\|\xi_u\|.\end{aligned}$$

The estimates for Λ_2:

$$(\Lambda_2, \xi_u) = (-R^2(\xi_u)_x - RU(\xi_r)_x - R(\xi_u\xi_r)_x - R\xi_r(\xi_u)_x - U\xi_r(\xi_r)_x,\ \xi_u).$$

By the relationship between ξ_u and ξ_r in (4.5), inverse inequalities in Lemma 2.2 and (A.1), we have

$$(-R^2(\xi_u)_x - R\xi_r(\xi_u)_x,\ \xi_u) \le Ch^{-1}\|\xi_r\|^4 + C(\|\xi_r\|^2 + \|\xi_u\|^2).$$

Apply integration by parts to obtain

$$\begin{aligned}&-(RU(\xi_r)_x, \xi_u) - (R(\xi_u\xi_r)_x, \xi_u) - (U\xi_r(\xi_r)_x, \xi_u) \\ =&((RU\xi_u)_x, \xi_r) + ((R\xi_u)_x, \xi_u\xi_r) + \frac{1}{2}((U\xi_u)_x, \xi_r^2) + \Gamma_2 \\ \le& Ch^{-1}\|\xi_r\|^4 + C(\|\xi_r\|^2 + \|\xi_u\|^2) + \Gamma_2.\end{aligned}$$

The last inequality is obtained by using the relationship between ξ_u and ξ_r in (4.5), inverse inequalities in Lemma 2.2 and (A.1). We put all boundary terms coming from integration by parts in Γ_2. Therefore,

$$(\Lambda_2, \xi_u) \le Ch^{-1}\|\xi_r\|^4 + C(\|\xi_r\|^2 + \|\xi_u\|^2) + \Gamma_2.$$

Combining the estimates for Λ_1 and Λ_2, we obtain (4.21). ∎

A.3. The proof for Lemma 4.8.

Proof. Since the exact solutions are continuous at the cell boundary, we get

$$\Theta_5 = -\mathcal{B}(u, r; \xi_u) = \sum_{j=1}^{N}\left(r^-u^-[\![e_r]\!]\xi_u^- + r^+r^+[\![e_u]\!]\xi_u^+\right)_{j+\frac{1}{2}}.$$

Similarly, we do the error decomposition first to extract the term $\mathcal{B}(\xi_u, \xi_r; \xi_u)$.
Step 1: Error decomposition.

$$\begin{aligned}ru &= RU - (RU - ru) = RU - R\eta_u - U\eta_r + R\xi_u + U\xi_r + \eta_r\eta_u - \xi_r\eta_u - \eta_r\xi_u + \xi_r\xi_u, \\ r^2 &= R^2 - (R^2 - r^2) = R^2 - 2R\eta_r + 2R\xi_r + \eta_r^2 - 2\xi_r\eta_r + \xi_r^2.\end{aligned}$$

Then, we rewrite $ru[\![e_r]\!]$ and $r^2[\![e_u]\!]$ in the following form:

$$\begin{aligned}ru[\![e_r]\!] &= ru[\![\eta_r - \xi_r]\!] = \Upsilon_1 + \Upsilon_2 - \xi_r\xi_u[\![\xi_r]\!], \\ r^2[\![e_u]\!] &= r^2[\![\eta_u - \xi_u]\!] = \Upsilon_3 + \Upsilon_4 - \xi_r^2[\![\xi_u]\!],\end{aligned}$$

where

$$\begin{aligned}\Upsilon_1 &= (RU - (RU - ru))[\![\eta_r]\!] + (R\eta_u + U\eta_r - \eta_r\eta_u + \xi_r\eta_u + \eta_r\xi_u)[\![\xi_r]\!], \\ \Upsilon_2 &= -(RU + R\xi_u + U\xi_r)[\![\xi_r]\!],\end{aligned}$$

$$\Upsilon_3 = (R^2 - (R^2 - r^2))[\![\eta_u]\!] + (-R^2 + 2R\eta_r - \eta_r^2 + 2\xi_r\eta_r)[\![\xi_u]\!],$$
$$\Upsilon_4 = -2R\xi_r[\![\xi_u]\!].$$

The above error decomposition ensures that each term in Υ_1 has a projection error which is a high-order term. Similar idea is used to decompose $r^2[\![e_u]\!]$ and obtain Υ_3. We also put $-R^2[\![\xi_u]\!]$ in Υ_3, since we have the estimate for $[\![\xi_u]\!]$ in Corollary 4.1.

Step 2: Estimates.

By the boundedness of the exact solutions, the properties of projections in (4.2), inverse inequalities (2.10) and Corollary 4.1, we have

$$\begin{aligned}&\sum_{j=1}^{N}\Big(\Upsilon_1(x^-_{j+\frac12})\xi_u^-|_{j+\frac12} + \Upsilon_3(x^+_{j+\frac12})\xi_u^+|_{j+\frac12}\Big)\\ &\le Ch^k\|\xi_u\| + Ch^k\|\xi_r\| + C\|\xi_u\|^2 + C\|\xi_r\|^2 + C\|\xi_u\|^4 + C\|\xi_r\|^4.\end{aligned}$$

By the definition of $\mathcal{B}(\cdot,\cdot;\cdot)$, we have

$$\sum_{j=1}^{N}\Big(-\xi_r^-\xi_u^-[\![\xi_r]\!]\xi_u^- - \xi_r^+\xi_r^+[\![\xi_u]\!]\xi_u^+\Big)_{j+\frac12} = -\mathcal{B}(\xi_u,\xi_r;\xi_u).$$

Since we cannot directly use inverse inverse inequalities to estimate Υ_2 and Υ_4, we leave them and estimate them together with boundary terms in Θ_3 and Θ_4. Upon denoting

$$\sum_{j=1}^{N}\Big(\Upsilon_2(x^-_{j+\frac12})\xi_u^- + \Upsilon_4(x^+_{j+\frac12})\xi_u^+\Big)_{j+\frac12} = \Gamma_3,$$

we obtain (4.22). ∎

A.4. The proof for Lemma 4.9.

Proof. By the definition of the Γ_1, Γ_2 and Γ_3, we have

$$\begin{aligned}&\Gamma_1+\Gamma_2+\Gamma_3-\frac12\sum_{j=1}^{N}(U[\![\xi_r]\!])^2_{j+\frac12}\\ =&\sum_{j=1}^{N}\Big\{\Big(\big(U(\xi_r^+)^2+2R\xi_r^+\xi_u^+\big)[\![\xi_u]\!]+\big(R(\xi_u^-)^2+2U\xi_r^-\xi_u^-\big)[\![\xi_r]\!]+U(\xi_r^-)^2\xi_u^- - U(\xi_r^+)^2\xi_u^+\\ &\quad + R(\xi_u^-)^2\xi_r^- - R(\xi_u^+)^2\xi_r^+\Big)+\Big(-RU\xi_r^-\xi_u^- + RU\xi_r^+\xi_u^+ - R\xi_r^-(\xi_u^-)^2 + R\xi_r^+(\xi_u^+)^2\\ &\quad -\frac12 U\xi_u^-(\xi_r^-)^2+\frac12 U\xi_u^+(\xi_r^+)^2\Big)+\Big(-(RU+R\xi_u^-+U\xi_r^-)\xi_u^-[\![\xi_r]\!]-2R\xi_r^+\xi_u^+[\![\xi_u]\!]\Big)\\ &\quad -\frac12(U[\![\xi_r]\!])^2\Big\}_{j+\frac12}\\ =&\sum_{j=1}^{N}\Big(U(\xi_r^+)^2[\![\xi_u]\!]+U\xi_r^-\xi_u^-[\![\xi_r]\!]+\frac12U(\xi_r^-)^2\xi_u^- - \frac12U(\xi_r^+)^2\xi_u^+ - RU\xi_r^-\xi_u^- + RU\xi_r^+\xi_u^+\\ &\quad - RU\xi_u^-[\![\xi_r]\!]-\frac12(U[\![\xi_r]\!])^2\Big)_{j+\frac12}.\end{aligned}$$

Using the following identities

$$a^-b^- - a^+b^+ = -a^-[\![b]\!] - b^+[\![a]\!],$$
$$a^-(b^-)^2 - a^+(b^+)^2 = -a^-b^+[\![b]\!] - a^-b^-[\![b]\!] - (b^+)^2[\![a]\!].$$

We obtain

$$\sum_{j=1}^{N}\Big(\frac{1}{2}U(\xi_r^-)^2\xi_u^- - \frac{1}{2}U(\xi_r^+)^2\xi_u^+ - RU\xi_r^-\xi_u^- + RU\xi_r^+\xi_u^+\Big)_{j+\frac{1}{2}}$$
$$=\sum_{j=1}^{N}\Big(-\frac{1}{2}U\xi_u^-\xi_r^+[\![\xi_r]\!] - \frac{1}{2}U\xi_u^-\xi_r^-[\![\xi_r]\!] - \frac{1}{2}U(\xi_r^+)^2[\![\xi_u]\!] + RU\xi_u^-[\![\xi_r]\!] + RU\xi_r^+[\![\xi_u]\!]\Big)_{j+\frac{1}{2}}.$$

Therefore,

$$\Gamma_1+\Gamma_2+\Gamma_3 - \sum_{j=1}^{N}\Big(\frac{1}{2}(U[\![\xi_r]\!])^2\Big)_{j+\frac{1}{2}}$$
$$=\sum_{j=1}^{N}\frac{1}{2}\Big(U\xi_r^-\xi_u^-[\![\xi_r]\!] - U\xi_u^-\xi_r^+[\![\xi_r]\!] + U\xi_r^+\xi_r^+[\![\xi_u]\!] + 2RU\xi_r^+[\![\xi_u]\!] - (U[\![\xi_r]\!])^2\Big)_{j+\frac{1}{2}}$$
$$\le\sum_{j=1}^{N}\frac{1}{2}\Big((\xi_r^-\xi_u^-)^2 + (\xi_u^-\xi_r^+)^2 + \frac{1}{2}(U[\![\xi_r]\!])^2 + U\xi_r^+\xi_r^+[\![\xi_u]\!] + 2RU\xi_r^+[\![\xi_u]\!] - (U[\![\xi_r]\!])^2\Big)_{j+\frac{1}{2}}$$
$$\le\sum_{j=1}^{N}\frac{1}{2}\Big((\xi_r^-\xi_u^-)^2 + (\xi_u^-\xi_r^+)^2 + U\xi_r^+\xi_r^+[\![\xi_u]\!] + 2RU\xi_r^+[\![\xi_u]\!]\Big)_{j+\frac{1}{2}}$$
$$\le Ch^{-1}\|\xi_u\|_\infty^2\|\xi_r\|^2 + C\|\xi_r\|_\infty\|\xi_r\|h^{-\frac{1}{2}}[\![\xi_u]\!] + C\|\xi_r\|h^{-\frac{1}{2}}[\![\xi_u]\!]$$
$$\le Ch^{-1}\|\xi_u\|^4 + Ch^{-1}\|\xi_r\|^4 + C\|\xi_r\|^2,$$

where (4.5) and (4.6) are used in the last inequality. Therefore, we obtain (4.23). ∎

A.5. The proof for Lemma 4.11.

Proof. We denote

$$L(t) = h^{2k} + \int_0^t h^{-1}A^2(\tau) + A(\tau)d\tau.$$

Since $A'(t) \le C(h^{-1}A^2 + A + h^{2k}), A(0) \le Ch^{2k+2}$, we have $A(t) \le (2C+CT)L(t)$ and

$$L'(t) = h^{-1}A^2(t) + A(t) \le C^\star\Big(h^{-1}L^2(t) + L(t)\Big),$$

where $C^\star = \max\{(2C+CT)^2, 2C+CT\}$. Therefore,

$$\int_0^t \frac{L'(\tau)}{h^{-1}L^2(\tau)+L(\tau)}d\tau \le C^\star T,$$

and

$$\mathcal{F}\Big(\frac{L(t)}{L(0)}\Big) := \int_1^{\frac{L(t)}{L(0)}} \frac{1}{y + h^{2k-1}y^2}dy$$

$$= \int_1^{\frac{L(t)}{L(0)}} \frac{1}{y + h^{-1} y^2 L(0)} dy = \int_0^t \frac{L'(\tau)}{h^{-1} L^2(\tau) + L(\tau)} d\tau.$$

It is easy to check $\mathcal{F}(\omega) = \ln\left(\frac{\omega}{1+h^{2k-1}\omega}\right) + \ln(1+h^{2k-1})$ and

$$\mathcal{F}\Big(\frac{2e^{C^\star T+1}}{1 - h^{2k-1} e^{C^\star T+1}}\Big) \geq C^\star T.$$

Upon denoting $C_1 = 4e^{C^\star T+1}$ and $C_1 \geq \frac{2e^{C^\star T+1}}{1-h^{2k-1}e^{C^\star T+1}}$ as h small enough. Since $\mathcal{F}(\omega)$ is increasing with ω and $\mathcal{F}(C_1) \geq \mathcal{F}\left(\frac{2e^{C^\star T+1}}{1-h^{2k-1}e^{C^\star T+1}}\right) \geq C^\star T \geq \mathcal{F}\left(\frac{L(t)}{L(0)}\right)$, then we have $\frac{L(t)}{L(0)} \leq C_1$ and $A(t) \leq C^\star L(t) \leq C^\star C_1 L(0) \leq C^\star C_1 h^{2k}$. ∎

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