On the convergence of the discontinuous Galerkin scheme for Einstein-scalar equations

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Abstract

We prove the stability and convergence of the high order discontinuous Galerkin scheme to Einstein-scalar equations for the large initial data problem.

Key Words: Einstein-scalar equations; Discontinuous Galerkin scheme.
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Contents

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1 Introduction

The numerical solutions of Einstein's equations have a wide range of important applications, including the numerical simulation of black holes, neutron stars, and gravitational waves [24, 6, 25, 22, 4, 44]. The numerical simulation of black hole spacetime plays a vital role in detecting and analyzing gravitational waves observed by gravitational wave detectors.

The history of numerical relativity can be traced back to the 1960s when, under the influence of John Wheeler, some of his students started working on numerical calculations of Einstein's equations [34]. The groundbreaking paper by Arnowitt, Deser, and Misner [2] marked the beginning of the computational efforts. Concurrently, Wheeler introduced the concept of geometrodynamics and coined the terms "lapse" and "shift" [48]. In 1964, Hahn and Lindquist [27] presented the initial numerical simulation of a binary black hole collision, which unfortunately resulted in program crashing after a few steps without any physical outcomes. In the 1970s, Eppley [23] and Smarr [45] delved into coordinate selection for numerical Einstein equation calculations but faced issues with computational stability. By the 1980s, Piran et al. [36, 46] explored numerical calculations of axisymmetric problems, revealing challenges in achieving stability. Factors influencing the stability of numerical Einstein equations include computational resolution [1, 8], boundary conditions [26], computational methods [31], different forms of Einstein's equations (e.g., BSSN form) [43, 5] and others. Progress on the stability of numerical relativity was stagnant until around the 2000s. In 2005, Pretorius [37, 38] utilized generalized harmonic coordinates to achieve numerical stability and successfully simulated the merger process of binary black holes. Subsequently, NASA [3] and the UTB numerical relativity group [10] independently resolved the stability issue. Since 2006, numerous research teams worldwide have addressed the computational stability problem, including [28, 32, 9, 49, 11, 47] and others.

From an application point of view, numerical relativity has been successful. However, the above work on numerical stability is based on experience, and little has been done to rigorously analyze the stability and convergence of the numerical scheme to Einstein equations mathematically. Since Einstein's equations are highly nonlinear hyperbolic systems, such an analysis is difficult. To reduce the difficulty, we may start with the Einstein equations under certain symmetries. In order to investigate the stability of numerical schemes for the Einstein equations, we first need to investigate how to analyze the Einstein equations at the PDE level. Historically, Christodoulo studied the spherically symmetric Einstein massless scalar fields equations [12, 13, 14, 15]. In [12], he demonstrated that a global classical solution exists if the initial data is sufficiently small. In [13], he examined the global initial value problem of spherically symmetric Einstein-scalar field equations on large scales, introduced the concept of generalized solutions, and established the existence of generalized solutions that are not constrained by the size of the initial data. He proved that when the final Bondi mass M is non-zero, a black hole forms with a mass M surrounded by vacuum as the retarded time approaches infinity [14].

We believe that the high-order discontinuous Galerkin (DG) scheme will play an important role in the numerical simulation of Einstein equations, so we will prove the stability and convergence of the DG scheme for Einstein equations. Studying problems involving simplifying assumptions, such as spherical or axial symmetry, is of utmost importance in the development of methods that have the potential to tackle more general problems. With this motivation in mind, we delve into the study the high order DG scheme of Einstein-scalar equations with spherical symmetry. The DG method belongs to the family of finite element methods. The distinguishing feature of the DG method is its utilization of discontinuous piecewise polynomial space for both the numerical solution and test functions in the spatial variables. By employing this approach, the DG method allows for accurate representation of complex geometries and sharp gradients within a computational domain. This makes it particularly suitable for problems involving complex geometry domains, such as the merger of binary black holes. To ensure stability and efficiency, the DG method is often combined with explicit and nonlinearly stable high order Runge-Kutta time discretization [42]. The initial application of the DG method can be traced back to 1973 when Reed and Hill [40] employed it to solve the neutron transport equation, a linear hyperbolic equation that is not time-dependent. Cockburn et al. have made a significant breakthrough in the DG method through their series of papers [17, 18, 19, 20, 21] . They have developed an effective framework for solving nonlinear time-dependent problems, such as the Euler equations of gas dynamics. This is achieved by using explicit, nonlinearly stable high-order Runge-Kutta time discretizations [42] and DG discretization in space with interface fluxes based on exact or approximate Riemann solvers. To ensure non-oscillatory behavior for strong shocks, they employ total variation bounded (TVB) nonlinear limiters [41]. DG methods are widely used in numerical simulations due to their numerous appealing characteristics. For instance, achieving arbitrary high accuracy order easily, efficient *hp* adaptivity, performing computations in complex geometric domains, and exhibiting excellent parallel efficiency. Optimal *a priori* error estimates $O(h^{k+1})$ for DG scheme with piecewise polynomials of degree k have been established for smooth solutions to linear conservation laws on one-dimensional and multi-dimensional tensor product meshes, as well as other structured mesh cases. Furthermore, error estimates $O(h^{k+\frac{1}{2}})$ have been derived for other cases, including both steady state solutions and space-time DG discretization [33, 39, 30]. The optimality in the general case has been demonstrated in [35]. Zhang and Shu presented *a priori* error estimates for fully discrete Runge-Kutta DG methods applied to scalar nonlinear conservation laws [50] and symmetrizable systems [51], assuming smooth solutions.

The rest of the paper is organized as follows. In section 2, we introduce the spherically symmetric Einstein-scalar equations. In section 3, we present the DG scheme for Einstein-scalar equations. In section 4, we prove the L^2 stability of the DG scheme. In Section 5, for a class of initial data, we establish the *a priori* estimates for the exact solution and demonstrate the global existence, that is, $u(t,r) \in C^1([0,\infty) \times [0,b])$. These *a priori* estimates are essential for error analysis. Furthermore, we will show that a black hole will form from this class of initial data. In section 6, we prove the the convergence theorem of DG scheme for $k \geq 1$. In section 7, we prove the the convergence theorem of DG scheme for the P^0 case. In section 8, we show the numerical tests. We write all the details of some of the more technical proofs in the Appendix. We summarize the main results as follows. In Theorem 4, we show that for high order DG scheme ($k \geq 1$), the optimal error estimate can be obtained $\|u(t,\cdot) - u_h(t,\cdot)\| \lesssim e^{ct}h^{k+1}$. In Theorem 5, we show the error estimate for P^0 DG scheme $\|u(t_n,\cdot) - u_h(t_n,\cdot)\|_\infty \lesssim e^{ct_n}h$.

2 The spherically symmetric Einstein-scalar equations

We introduce the Bondi coordinate system (u, r, θ, ϕ) [7], and assume the spacetime metric takes the Bondi-Sachs form [12],

$$ds^2 = -g(u,r)\tilde{g}(u,r)du^2 - 2g(u,r)dudr + r^2 d\Omega^2,$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

In order to make the Einstein-scalar equations more friendly to readers in the field of numerical computation, we change the notation of the coordinate system to the following,

$$t := u.$$

Then, the metric can be expressed as

$$ds^2 = -g(t,r)\tilde{g}(t,r)dt^2 - 2g(t,r)dtdr + r^2 d\Omega^2.$$

The Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu},$$

where $g_{\mu\nu}$ is the space-time metric given above, $R_{\mu\nu}$ is the Ricci curvature, R is the scalar curvature and $T_{\mu\nu}$ is the energy-momentum tensor of the massless scalar field φ.

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\sigma,$$

where $\sigma = g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi$. By simple calculation, we have

$$R = 8\pi\sigma,$$

then, the Einstein equation can be reduced to

$$R_{\mu\nu} = 8\pi\partial_\mu\varphi\partial_\nu\varphi.$$

Let $E_{\mu\nu}$ be the tensor,

$$E_{\mu\nu} = R_{\mu\nu} - 8\pi\partial_\mu\varphi\partial_\nu\varphi,$$

taking the two nonvanishing components of $E_{\mu\nu}$, we get the following equations

$$\begin{aligned} &\frac{2}{rg}g_r = 8\pi(\varphi_r)^2, && (E_{rr} = 0), \\ &(r\tilde{g})_r = g, && (E_{\theta\theta} = 0). \end{aligned}$$

The conservation of the energy-momentum tensor $\nabla^\mu T_{\mu\nu} = 0$, gives the wave equation for φ $\nabla^\mu\partial_\mu\varphi = 0$, then, the massless scalar field equation is given by (for more detail, one can see the Appendix or [12])

$$-2(\varphi_{tr} + \frac{1}{r}\varphi_t) + \frac{1}{r}\left(\tilde{g}(r\varphi)_r\right)_r - \frac{1}{r}\varphi\tilde{g}_r = 0.$$

The computational domain is given by

$$Q = \{(t, r) : [0, T] \times [0, b]\}.$$

In order to express the equations of motion in a simpler form, we introduce the following variables

$$\begin{aligned} u(t, r) &:= (r\varphi)_r, \\ \tilde{u}(t, r) &:= \varphi(t, r) = \frac{1}{r}\int_0^r u(t, s)ds. \end{aligned}$$

To simplify our expressions, we rescale and let $4\pi = 1$. The resulting equations reduce to [12]:

$$u_t - (\frac{1}{2}\tilde{g}u)_r = -\frac{1}{2}\tilde{g}_r\tilde{u}, \tag{1}$$

$$g_r = \frac{1}{r}g(u - \tilde{u})^2, \tag{2}$$

$$\tilde{g}_r = \frac{1}{r}(g - \tilde{g}). \tag{3}$$

We can solve equations (2) and (3) to get

$$\tilde{g}(r) = \frac{1}{r}\int_0^r g(s)ds, \qquad g(r) = \exp\left(-\int_r^b \frac{1}{s}(u - \tilde{u})^2 ds\right).$$

We collect the useful properties as follows [12, 13].

Lemma 1.

$$0 \leq \tilde{g} \leq 1, \tag{4}$$

$$0 \leq g \leq 1, \tag{5}$$

$$\tilde{g} \leq g. \tag{6}$$

2.1 Boundary condition

Observing the equations (1) and (4), we know that the information is transmitted from b to 0. The solution of (1)-(2) satisfies the asymptotic condition:

$$g(b) = 1.$$

So, in equation (2), we integrate ODEs from b to 0. The boundary condition for u is given by

$$u(t, b) = U_b.$$

3 DG scheme for Einstein-scalar equations

In this section, we will build a semi-discrete DG scheme for (1)-(2)-(3). We assume the following uniform mesh (for simplicity) to cover $[0, b]$, consisting of the cells $I_i = [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}]$, for $1 \leq i \leq N$,where

$$0 = r_{\frac{1}{2}} < r_{\frac{3}{2}} < ... < r_{N+\frac{1}{2}} = b.$$

The mesh size is

$$h = r_{i+\frac{1}{2}} - r_{i-\frac{1}{2}}.$$

We define a piecewise polynomial space

$$V_h^k = \{\phi : \phi|_{I_i} \in P^k(I_i); 1 \leq i \leq N\}, \tag{7}$$

where $P^k(I_i)$ denotes the set of polynomials of degree up to k defined on I_i.

In the first step, we describe the DG scheme for Einstein constrain equation (2)-(3). The numerical solutions to u and g are denoted by u_h and g_h respectively, and

$$\tilde{u}_h = \frac{1}{r}\int_0^r u_h \, ds, \tag{8}$$

$$g_h = \exp\left(-\int_r^b \frac{1}{s}(u_h - \tilde{u}_h)^2 \, ds\right), \tag{9}$$

$$\tilde{g}_h = \frac{1}{r}\int_0^r g_h \, ds. \tag{10}$$

Next, we define the DG scheme for equation (1): Find a $u_h \in V_h^k$ such that $\forall v \in V_h^k$:

$$\int_{I_i} (u_h)_t v + \frac{1}{2}\tilde{g}_h u_h v_r \, dr - \left(\frac{1}{2}\tilde{g}_{h,i+\frac{1}{2}}\hat{u}_{h,i+\frac{1}{2}}v^-_{i+\frac{1}{2}} - \frac{1}{2}\tilde{g}_{h,i-\frac{1}{2}}\hat{u}_{h,i-\frac{1}{2}}v^+_{i-\frac{1}{2}}\right) = -\frac{1}{2}\int_{I_i} \tilde{g}_{h,r}\tilde{u}_h v \, dr \tag{11}$$

where the numerical flux is given by the upwinding choice $\hat{u}_{h,i+\frac{1}{2}} = u^+_{h,i+\frac{1}{2}}$ and $\tilde{g}_{h,r} = \frac{1}{r}(g_h - \tilde{g}_h)$. We denote by $\|\cdot\|$ and $\|\cdot\|_\infty$ the usual L^2 norm and L^∞ norm, respectively. We denote $A \lesssim B$ if there exist a constant $c_0 > 0$ independent of h such that $A \leq c_0 B$.

4 L^2 stability

4.1 L^2 estimation for equation (1)

Before studying the stability of the numerical scheme, we need to study the L^2 estimation of equation (1).

Lemma 2.

$$\int_0^b u^2(t,r)dr \leq (\frac{1}{2}U_b^2 + 1)t + \int_0^b u_0^2(r)dr. \tag{12}$$

Proof. By multiplying both sides of equation (1) by u and integrating by parts, we get

$$\int_0^b (\frac{1}{2}u^2)_t + (\frac{1}{4}\tilde{g})(u^2)_r dr = \frac{1}{4}\tilde{g}u^2|_0^b - \int_0^b \frac{1}{2}\tilde{g}_r\tilde{u}u\ dr.$$

Using equations (3), (6), (2) and the following relation

$$\frac{1}{2}u^2 - \tilde{u}u = \frac{1}{2}(u-\tilde{u})^2 - \frac{1}{2}\tilde{u}^2, \tag{13}$$

we obtain

$$\int_0^b (u^2)_t dr - \frac{1}{2}\tilde{g}u^2|_0^b = \int_0^b \tilde{g}_r(\frac{1}{2}u^2 - \tilde{u}u)dr.$$

We use the boundary condition $u(t,b) = U_b$ and $\tilde{g} \leq 1$, then

$$\begin{aligned}
\int_0^b (u^2)_t dr &\leq \frac{1}{2}U_b^2 + \int_0^b \tilde{g}_r(\frac{1}{2}u^2 - \tilde{u}u)dr \\
&= \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2}\tilde{g}_r((u-\tilde{u})^2 - \tilde{u}^2)dr \quad (\text{use } (13)\ ), \\
&= \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2r}(g-\tilde{g})((u-\tilde{u})^2 - \tilde{u}^2)dr \quad (\text{use } (3)\text{: } \tilde{g}_r = \frac{1}{r}(g-\tilde{g})), \\
&\leq \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2r}g(u-\tilde{u})^2 dr \quad (\text{use } 0 \leq \tilde{g} \leq g), \\
&= \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2}g_r dr \quad (\text{use } (2)\text{: } g_r = \frac{g}{r}(u-\tilde{u})^2), \\
&\leq \frac{1}{2}U_b^2 + 1.
\end{aligned}$$

Then,

$$\int_0^b u^2(t,r)dr \leq (\frac{1}{2}U_b^2 + 1)t + \int_0^b u_0^2(r)dr.$$

□

4.2 L^2 stability of the DG scheme

In this section, we study the L^2 stability of the DG scheme (11). In the first step, we will prove a cell entropy inequality. Next, we mimic Lemma 2 to prove the L^2 stability of the scheme (11). Following the line in [29], we can prove a similar cell entropy inequality for the square entropy.

Define the entropy $\eta(u_h) = \dfrac{u_h^2}{2}$, we have

Lemma 3. *The following cell entropy inequality holds*

$$\int_{I_i} (\eta_t - \frac{1}{2}\tilde{g}_{h,r}\eta + \frac{1}{2}\tilde{g}_{h,r}u_h\tilde{u}_h)\ dr + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} \le 0, \tag{14}$$

where

$$\hat{F}_{i+\frac{1}{2}} = \frac{1}{4}\tilde{g}_{h,i+\frac{1}{2}}(u^-_{h,i+\frac{1}{2}})^2 - \frac{1}{2}\tilde{g}_{h,i+\frac{1}{2}}u^+_{h,i+\frac{1}{2}}u^-_{h,i+\frac{1}{2}}.$$

Proof. We take the test function $v = u_h$ in scheme (11) to obtain

$$\begin{aligned} 0 &= \int_{I_i} \left((\frac{1}{2}u_h^2)_t + \frac{1}{2}\tilde{g}_h \left(\frac{u_h^2}{2}\right)_r + \frac{1}{2}\tilde{g}_{h,r}\tilde{u}_h u_h \right)\ dr + B_i^1 \\ &= \int_{I_i} (\eta_t + \frac{1}{2}\tilde{g}_h\eta_r + \frac{1}{2}\tilde{g}_{h,r}\tilde{u}_h u_h)\ dr + B_i^1, \end{aligned} \tag{15}$$

where

$$B_i^1 = -\frac{1}{2}\left(\tilde{g}_{h,i+\frac{1}{2}}u^+_{h,i+\frac{1}{2}}u^-_{h,i+\frac{1}{2}} - \tilde{g}_{h,i-\frac{1}{2}}u^+_{h,i-\frac{1}{2}}u^+_{h,i-\frac{1}{2}}\right).$$

Integrating by parts in (15), we have

$$0 = \int_{I_i} (\eta_t - \frac{1}{2}\tilde{g}_{h,r}\eta + \frac{1}{2}\tilde{g}_{h,r}\tilde{u}_h u_h)\ dr + B_i^1 + B_i^2, \tag{16}$$

where

$$B_i^2 = \frac{1}{2}\tilde{g}_{h,i+\frac{1}{2}}\eta^-_{i+\frac{1}{2}} - \frac{1}{2}\tilde{g}_{h,i-\frac{1}{2}}\eta^+_{i-\frac{1}{2}}.$$

Next, we will decompose the term $B_i^1 + B_i^2$ into a flux difference plus a remainder

$$B_i^1 + B_i^2 = \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} + \theta_{i-\frac{1}{2}},$$

where

$$\begin{aligned} \hat{F}_{i+\frac{1}{2}} &= \frac{1}{4}\tilde{g}_{h,i+\frac{1}{2}}(u^-_{h,i+\frac{1}{2}})^2 - \frac{1}{2}\tilde{g}_{h,i+\frac{1}{2}}u^+_{h,i+\frac{1}{2}}u^-_{h,i+\frac{1}{2}}, \\ \theta_{i-\frac{1}{2}} &= \frac{1}{4}\tilde{g}_{h,i-\frac{1}{2}}(u^{-2}_{h,i-\frac{1}{2}} - 2u^+_{h,i-\frac{1}{2}}u^-_{h,i-\frac{1}{2}} + 2u^+_{h,i-\frac{1}{2}}u^+_{h,i-\frac{1}{2}} - u^{+2}_{h,i-\frac{1}{2}}). \end{aligned}$$

We can verify $\theta_{i-\frac{1}{2}} \geq 0$ as follows

$$\begin{aligned}\theta_{i-\frac{1}{2}} &= \frac{1}{4}\tilde{g}_{h,i-\frac{1}{2}}(u_{h,i-\frac{1}{2}}^{-2} - 2u_{h,i-\frac{1}{2}}^{+}u_{h,i-\frac{1}{2}}^{-} + 2u_{h,i-\frac{1}{2}}^{+}u_{h,i-\frac{1}{2}}^{+} - u_{h,i-\frac{1}{2}}^{+2})\\ &= \frac{1}{4}\tilde{g}_{h,i-\frac{1}{2}}(u_{h,i-\frac{1}{2}}^{+} - u_{h,i-\frac{1}{2}}^{-})^2\\ &\geq 0,\end{aligned}$$

then we have the cell entropy inequality

$$\int_{I_i}(\eta_t - \frac{1}{2}\tilde{g}_{h,r}\eta + \frac{1}{2}\tilde{g}_{h,r}u_h\tilde{u}_h)dr + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} \leq 0.$$

□

The cell entropy inequality (14) implies an L^2 stability of u_h.

Theorem 1. *The solution u_h to the DG scheme (11) satisfies the following L^2 stability*

$$\int_0^b u_h^2(t,r)dr \leq (\frac{1}{2}U_b^2 + 1)t + \int_0^b u_0^2(r)dr. \tag{17}$$

Proof. Summing up the cell entropy inequality (11) over i, we have

$$\int_0^b (\frac{1}{2}u_h^2)_t \leq \int_0^b \frac{1}{2}\tilde{g}_{h,r}(\frac{1}{2}u_h^2 - \tilde{u}_h u_h)\ dr + \hat{F}_{\frac{1}{2}} - \hat{F}_{N+\frac{1}{2}}.$$

We can clearly see that $\hat{F}_{\frac{1}{2}} = 0$. Moreover, since

$$\begin{aligned}\hat{F}_{N+\frac{1}{2}} &= \frac{1}{4}\tilde{g}_{h,N+\frac{1}{2}}\left((u_{h,N+\frac{1}{2}}^{-})^2 - 2U_b u_{h,N+\frac{1}{2}}^{-}\right)\\ &= \frac{1}{4}\tilde{g}_{h,N+\frac{1}{2}}\left((u_{h,N+\frac{1}{2}}^{-} - U_b)^2 - (U_b)^2\right),\end{aligned}$$

then

$$\begin{aligned}\int_0^b (\frac{1}{2}u_h^2)_t\ dr &\leq \frac{1}{2}\int_0^b \tilde{g}_{h,r}(\frac{1}{2}u_h^2 - \tilde{u}_h u_h)\ dr + \frac{1}{4}\tilde{g}_{N+\frac{1}{2}}(U_b)^2\\ &\leq \frac{1}{2}\int_0^b \tilde{g}_{h,r}(\frac{1}{2}u_h^2 - \tilde{u}_h u_h)\ dr + \frac{1}{4}U_b^2.\end{aligned}$$

So, we have

$$\begin{aligned}\int_0^b (u_h^2)_t dr &\le \frac{1}{2}U_b^2 + \int_0^b \tilde{g}_{h,r}(\frac{1}{2}u_h^2 - \tilde{u}_h u_h)dr \\ &= \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2}\tilde{g}_{h,r}((u_h - \tilde{u}_h)^2 - \tilde{u}_h^2)dr \\ &= \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2r}(g_h - \tilde{g}_h)((u_h - \tilde{u}_h)^2 - \tilde{u}_h^2)dr \\ &\le \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2r}g_h(u_h - \tilde{u}_h)^2 dr \\ &= \frac{1}{2}U_b^2 + \int_0^b \frac{1}{2}g_{h,r}dr \\ &\le \frac{1}{2}U_b^2 + 1.\end{aligned}$$

Then,

$$\int_0^b u_h^2(t,r)dr \le (\frac{1}{2}U_b^2 + 1)t + \int_0^b u_0^2(r)dr.$$

□

5 Global existence and black hole formation for a class of large data problems

In this section, we will address the issue of the large data problem. For arbitrarily large initial data, Christodoulo [13] demonstrated the existence of a unique solution defined on $(t,r) \in [0,\infty) \times (0,\infty)$ but not at $r = 0$. Our goal is to establish conditions for the formation of a black hole and the definition of the solution on $(t,r) \in [0,\infty) \times [0,b]$ (in numerical computation, we choose b to be finite) with large initial data. To achieve this, we introduce a set of large initial data: ensuring that $\int_0^b r u_0^2(r)\, dr$ is sufficiently large to yield a metric decay estimate of $g(t,r) \lesssim \exp(-ct)$ for $r \le 2M_1$ and a globally existing solution. We define some constants

$$\varepsilon = \frac{1}{2}\frac{B^2}{b},\ \ \varepsilon_0 = \frac{1}{100},\ \ \lambda = \frac{9}{10},\ \ g_0 = \frac{1}{100},\ \ \delta = \frac{1}{100},\ \ \theta = \frac{2}{100},$$

where B is a constant which will be defined in Lemma 11. We define the mass $m(t,r)$ by

$$m(t,r) := \frac{r}{2}(1 - \frac{\tilde{g}}{g}).$$

The Bondi mass $M(t)$ is given by

$$M(t) = m(t,b).$$

We denote the initial Bondi mass as M_0, and choose r_0 to satisfy

$$2M_0 = (1+\delta)r_0, \tag{18}$$

and define M_1 as

$$2M_1 = \lambda r_0. \tag{19}$$

We define

$$F(t,r) := \int_r^b s u^2(t,s)\, ds.$$

The initial data $u_0(r) \in C^1([0,b])$ satisfy the following conditions

$$F(0,r) > \left(\gamma - \log\left(2c_0(r_0 - \lambda c_0)\frac{1}{r_2^2}\right)\right) r_2^2, \tag{20}$$

$$g(0,r) \le g_0 \le g_0 \log(\frac{1}{g_0}) \le \frac{1}{100},\ \forall r < r_0, \tag{21}$$

$$r_0 \ge \frac{b}{2}, \tag{22}$$

$$\sup_{r\le 2M_1} |u_r(0,r)| \le \left(\frac{c_0/r_2^2}{12M_0}\right)^{\frac{1}{2}} e^{\frac{1}{2}\beta^2}, \tag{23}$$

where γ is a constant defined in (84), c_0 is a constant satisfying $0 < 2c_0 < r_0$, and r_2 is a constant satisfying $2M_0 < r_2 < b$. We can choose the initial data large enough such that $\frac{F(0,r_0)}{r_2^2} - \gamma > 0$, and define a constant β as $\beta := \left(-\gamma + \frac{F(0,r_0)}{r_2^2}\right)^{\frac{1}{2}}$.

Denote

$$r_* = r_0 - \frac{1}{2}\int_0^t \tilde{g}(s, r(s))\, ds.$$

Theorem 2. *For large initial data satisfying (20)-(21)-(22)-(23), then*

$$g(t, r_*) \le \exp\left(\gamma - \frac{F_0}{r_2^2} - \frac{c_0}{r_2^2}t\right),$$

where $F_0 := F(0, r_0)$. As $t \to \infty$, the event horizon will form.

Theorem 3. *For large initial data satisfying (20)-(21)-(22)-(23), we have the a-priori estimate*

$$\|u(t,r)\|_\infty \le C_1 \exp(\frac{t}{M_1}), \tag{24}$$

$$\|u_r(t,r)\|_\infty \le C_2 \exp\left(-1 + e^{\frac{2t}{M_1}}\right), \tag{25}$$

where C_1, C_2 depend only on the initial data. There is a unique global solution $u(t,r) \in C^1([0,T]\times[0,b])$.

Remark: By the same spirit, we can also show that, for any $k \ge 1$, if the initial data is of class C^k, then $u(t,r)$ belongs to $C^k([0,T]\times[0,b])$. The proof of Theorems 2 and 3 will be given in Appendix A.4.

6 Convergence analysis for P^k, $k \geq 1$.

In this section, we will give the error estimate of the DG scheme for $P^k, k \geq 1$. We need the assumption that the initial data is of class C^{k+1} such that $u(t,r) \in C^{k+1}([0,T] \times [0,b])$.

6.1 Projection and inverse properties

We list some important properties of the L^2 type projections. Assume $u(x)$ is sufficiently smooth, the L^2 projection of u into V_h is denoted by Πu,

$$\int_{I_i} (\Pi u(x) - u(x))v(x)\ dx = 0,\ \forall v \in P^k(I_i),$$

and the Gauss-Radau projections $\Pi^{\pm}$ into V_h satisfy

$$\int_{I_i} (\Pi^{\pm} u(x) - u(x))v(x)\ dx = 0,\ \forall v \in P^{k-1}(I_i)$$

and

$$\Pi^+ u(x^+_{i-\frac{1}{2}}) = u(x_{i-\frac{1}{2}}), \qquad \Pi^- u(x^-_{i+\frac{1}{2}}) = u(x_{i+\frac{1}{2}}).$$

Let $\eta = \Pi u(x) - u(x)$ or $\eta = \Pi^{\pm} u(x) - u(x)$, then, we have [16]

$$\|\eta\| + \|\eta\|_\infty + h^{\frac{1}{2}}\|\eta\|_{\Gamma_h} + h\|\eta_r\| \leq Ch^{k+1}, \tag{26}$$

where here and below C is a constant independent of h but depends on different norms of the exact solution u (assumed to be smooth), and Γ_h denotes the set of boundary points of all elements I_i. For any $u_h \in V_h$, there is a positive constant C independent of u_h and h, such that [16]

$$\|(u_h)_r\| \leq Ch^{-1}\|u_h\|, \tag{27}$$

$$\|u_h\|_\infty \leq h^{-\frac{1}{2}}\|u_h\|, \tag{28}$$

$$\|u_h\|_{\Gamma_h} \leq Ch^{-\frac{1}{2}}\|u_h\|. \tag{29}$$

For more details, we refer to [16].

6.2 Some Lemmas

Given two functions $u(r)$ and $u_h(r)$, we define their averages to be $\tilde{u}(r)$ and $\tilde{u}_h(r)$, respectively

$$\tilde{u}(r) = \frac{1}{r}\int_0^r u(s)\ ds, \qquad \tilde{u}_h(r) = \frac{1}{r}\int_0^r u_h(s)\ ds,$$

and we define $e, \tilde{e}$ as follows

$$e = u - u_h, \qquad \tilde{e} = \tilde{u} - \tilde{u}_h = \frac{1}{r}\int_0^r e\ ds.$$

Let u_h be the numerical solution to u, and define

$$\xi := u_h - \Pi u,$$

we can get

$$e = u - u_h = u - \Pi u - (u_h - \Pi u) = \eta - \xi.$$

166 By direct calculation we can get the following lemma:

Lemma 4.

$$(u_h - \tilde{u}_h)^2 = (u - \tilde{u})^2 + (e - \tilde{e})^2 + 2(u - \tilde{u})(-e + \tilde{e}). \tag{30}$$

167 We would like to make a bootstrap assumption

Assumption 1.

$$\|\xi\| \leq h^{\frac{3}{2}}, \tag{31}$$

168 and we will improve this assumption by the end of the proof of Theorem 4. Then, we have
169 the following corollary.

Corollary 1.

$$\|e\|_\infty \leq Ch, \tag{32}$$

$$\|e_r\| \leq Ch^{\frac{1}{2}}, \tag{33}$$

170 *where C is independent of h.*

We define

$$\Delta = g - g_h, \quad \Delta_r = g_r - g_{h,r}, \quad \tilde{\Delta} = \tilde{g} - \tilde{g}_h, \quad \tilde{\Delta}_r = \tilde{g}_r - \tilde{g}_{h,r} = \frac{1}{r}(\Delta - \tilde{\Delta}).$$

171 Under the assumption (31), then the following estimates hold:

Lemma 5.

$$\|\tilde{u}\|_\infty \leq \|u\|_\infty, \|\tilde{e}\|_\infty \leq \|e\|_\infty, \tag{34}$$

$$\|\tilde{u}\| \leq 2\|u\|, \|\tilde{e}\| \leq 2\|e\|, \tag{35}$$

$$\|\tilde{u}_r\|_\infty \leq \|u_r\|_\infty, \tag{36}$$

$$\|\tilde{u}_r\| \leq 2\|u_r\|, \tag{37}$$

$$\|\tilde{g}_r\|_\infty \leq \|g_r\|_\infty \leq Ce^{c_3 t}, \tag{38}$$

$$\|\tilde{e}\|_\infty \lesssim h^{k+1} + h^{-\frac{1}{2}}\|\xi\|, \tag{39}$$

$$\|\tilde{e}_r\|_\infty \lesssim h^k + h^{-\frac{3}{2}}\|\xi\|, \tag{40}$$

$$\|\tilde{e}_r\| \lesssim h^k + h^{-1}\|\xi\|, \tag{41}$$

and

$$\|\Delta\|_\infty \le c_1\|e\|, \tag{42}$$
$$\|\tilde{\Delta}\|_\infty \le c_1\|e\|, \tag{43}$$
$$\|\Delta_r\| \le c_2\|e\|, \tag{44}$$
$$\|\tilde{\Delta}_r\| \le c_2\|e\|, \tag{45}$$

where c_1, c_2 are constants independent of h and are positive. c_3 and C depend on the initial conditions and are positive.

The proof of this lemma will be given in Appendix A.1.

6.3 Error estimate

To simplify the expression, let us denote

$$v = -\frac{1}{2}\tilde{g}, \qquad \Omega = -\frac{1}{2}\tilde{g}_r, \qquad f = uv.$$

We define v_h as the numerical solutions of v, and define

$$\mathring{e} := v - v_h.$$

We will drive the error equation below. Since

$$\int_{I_i} u_{h,t}\phi - f_h\phi_r \; dr + \hat{f}_{h,i+\frac{1}{2}}\phi^-_{i+\frac{1}{2}} - \hat{f}_{h,i-\frac{1}{2}}\phi^+_{i-\frac{1}{2}} = \int_{I_i} \Omega_h\tilde{u}_h\phi \; dr$$
$$\int_{I_i} u_t\phi - f\phi_r \; dr + f_{i+\frac{1}{2}}\phi^-_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\phi^+_{i-\frac{1}{2}} = \int_{I_i} \Omega\tilde{u}\phi \; dr,$$

then, the error equation is given by

$$\begin{aligned}&\int_{I_i} (u-u_h)_t\phi - (f-f_h)\phi_r \; dr + (f_{i+\frac{1}{2}} - \hat{f}_{i+\frac{1}{2}})\phi^-_{i+\frac{1}{2}} - (f_{i-\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}})\phi^+_{i-\frac{1}{2}} \\ =& \int_{I_i} (\Omega\tilde{u} - \Omega_h\tilde{u}_h)\phi \; dr,\end{aligned}$$

where

$$\begin{aligned} f - f_h &= uv - u_hv_h \\ &= (u-u_h)v + (v-v_h)u_h \\ &= ev + u_h\mathring{e}, \\ f_{i+\frac{1}{2}} - \hat{f}_{i+\frac{1}{2}} &= u_{i+\frac{1}{2}}v_{i+\frac{1}{2}} - u^+_{h,i+\frac{1}{2}}v_{h,i+\frac{1}{2}} \\ &= (u_{i+\frac{1}{2}} - u^+_{h,i+\frac{1}{2}})v_{i+\frac{1}{2}} + (v_{i+\frac{1}{2}} - v_{h,i+\frac{1}{2}})u^+_{h,i+\frac{1}{2}} \\ &= e^+_{i+\frac{1}{2}}v_{i+\frac{1}{2}} + \mathring{e}_{i+\frac{1}{2}}u^+_{h,i+\frac{1}{2}}. \end{aligned}$$

Taking $\phi = \xi$, we get the error equation

$$\begin{aligned}
&\int_{I_i} e_t\xi - (ev + u_h\mathring{e})\xi_r \; dr \\
&+ e^+ v\xi^-|_{i+\frac{1}{2}} - e^+ v\xi^+|_{i-\frac{1}{2}} \\
&+ \mathring{e}u_h^+\xi^-|_{i+\frac{1}{2}} - \mathring{e}u_h^+\xi^+|_{i-\frac{1}{2}} \\
&= \int_{I_i} \Omega(\tilde{u} - \tilde{u}_h)\xi + \tilde{u}_h(\Omega - \Omega_h)\xi \; dr. \qquad (46)
\end{aligned}$$

Using $e = \eta - \xi$, we have

$$\int_{I_i} \eta_t\xi - \eta v\xi_r \; dr + \eta^+ v\xi^-|_{i+\frac{1}{2}} - \eta^+ v\xi^+|_{i-\frac{1}{2}} + \mathcal{Q}_{2,i} \qquad (47)$$

$$= \int_{I_i} \xi\xi_t - \xi v\xi_r \; dr + \xi^+ v\xi^-|_{i+\frac{1}{2}} - \xi^+ v\xi^+|_{i-\frac{1}{2}} + \mathcal{Q}_{3,i}, \qquad (48)$$

where

$$\begin{aligned}
\mathcal{Q}_{2,i} &= \int_{I_i} -u_h\mathring{e}\xi_r \; dr + \mathring{e}u_h^+\xi^-|_{i+\frac{1}{2}} - \mathring{e}u_h^+\xi^+|_{i-\frac{1}{2}} \\
&= \int_{I_i} -u_h\mathring{e}\xi_r \; dr + \mathring{e}u_h^+\xi^-|_{i+\frac{1}{2}} - \mathring{e}u_h^+\xi^+|_{i-\frac{1}{2}} + \left(\mathring{e}u_h^-\xi^-\right)_{i+\frac{1}{2}} - \left(\mathring{e}u_h^-\xi^-\right)_{i+\frac{1}{2}} \\
&= \int_{I_i} (u_h\mathring{e})_r\xi \; dr + \left(\mathring{e}(u_h^+ - u_h^-)\xi^-\right)_{i+\frac{1}{2}}, \\
\mathcal{Q}_{3,i} &= \int_{I_i} \Omega(\tilde{u} - \tilde{u}_h)\xi + \tilde{u}_h(\Omega - \Omega_h)\xi \; dr.
\end{aligned}$$

To prove the error estimate, we need the following lemma to control some of the terms $\mathcal{Q}_{2,i}, \mathcal{Q}_{3,i}$ that appear in the error equation.

Lemma 6. *We set* $\mathcal{Q}_2, \mathcal{Q}_3$ *as*

$$\begin{aligned}
\mathcal{Q}_2 &= \sum_{i=1}^{N} \mathcal{Q}_{2,i}, \\
\mathcal{Q}_3 &= \sum_{i=1}^{N} \mathcal{Q}_{3,i}.
\end{aligned}$$

Then, we obtain the estimate

$$\|\mathcal{Q}_2\|_\infty \lesssim \|\xi\|^2 + \|\eta\|^2, \qquad (49)$$

$$\|\mathcal{Q}_3\|_\infty \lesssim \|\xi\|^2 + \|\eta\|^2. \qquad (50)$$

Proof.

$$
\begin{aligned}
\sum_{i=1}^{N} \mathcal{Q}_{2,i} &= \sum_{i=1}^{N} \int_{I_i} (u_h \mathring{e})_r \xi \ dr + \sum_{i=1}^{N} \left(\mathring{e}(u_h^+ - u_h^-)\xi^-\right)_{i+\frac{1}{2}} \\
&= \sum_{i=1}^{N} \int_{I_i} u_{h,r} \mathring{e} \xi + u_h \mathring{e}_r \xi \ dr + \sum_{i=1}^{N} \left(\mathring{e}(u_h^+ - u_h^-)\xi^-\right)_{i+\frac{1}{2}}.
\end{aligned}
$$

The first part $\displaystyle\int_I u_{h,r} \mathring{e} \xi \ dr$ can be controlled as

$$
\begin{aligned}
\left| \int_I u_{h,r} \mathring{e} \xi \ dr \right| &= \left| \int_I u_{h,r} \frac{1}{2} \tilde{\Delta} \xi \ dr \right| \\
&= \frac{1}{2} \left| \int_I (u-e)_r \tilde{\Delta} \xi \ dr \right| \\
&\lesssim \|\tilde{\Delta}\|_\infty \|u_r - e_r\| \|\xi\| \\
&\lesssim \|\tilde{\Delta}\|_\infty \left(\|u_r\| + \|e_r\|\right) \|\xi\| \\
&\lesssim \|e\| \|\xi\|, \quad \text{(by (43) (33) and (25))}, \\
&\lesssim \|\xi\|^2 + \|\eta\|^2.
\end{aligned}
$$

The second part $\displaystyle\int_I u_h \mathring{e}_r \xi \ dr = \frac{1}{2} \int_I u_h \tilde{\Delta}_r \xi \ dr$ is controlled as

$$
\begin{aligned}
\frac{1}{2} \left| \int_I u_h \tilde{\Delta}_r \xi \ dr \right| &= \frac{1}{2} \left| \int_I (u-e)\, \tilde{\Delta}_r \xi \ dr \right| \\
&\leq \|u - e\|_\infty \|\tilde{\Delta}_r\| \|\xi\|, \\
&\leq \left(\|u\|_\infty + \|e\|_\infty\right) \|\tilde{\Delta}_r\| \|\xi\|, \\
&\lesssim \|e\| \|\xi\|, \quad \text{(by (45) (32) and the a-priori estimate (24))}, \\
&\lesssim \|\eta\|^2 + \|\xi\|^2.
\end{aligned}
$$

The boundary part $\sum_{i=1}^{N}\left(\mathring{e}(u_h^+ - u_h^-)\xi^-\right)_{i+\frac{1}{2}}$ can be controlled by

$$
\begin{aligned}
&\sum_{i=1}^{N}\left(\mathring{e}(u_h^+ - u_h^-)\xi^-\right)_{i+\frac{1}{2}} \\
&= \sum_{i=1}^{N}\left(\mathring{e}(u_h^+ - \Pi u + \Pi u - u_h^-)\xi^-\right)_{i+\frac{1}{2}} \\
&\leq \|\mathring{e}\|_\infty \sum_{i=1}^{N}\left(|(u_h^+ - \Pi u + \Pi u - u_h^-)\xi^-|\right)_{i+\frac{1}{2}} \\
&\lesssim \|\tilde{\Delta}\|_\infty \sum_{i=1}^{N}\left((\xi^+)^2 + (\xi^-)^2\right)_{i+\frac{1}{2}}, \quad \text{(using (43), (31) and (29))} \\
&\lesssim \|e\|\frac{1}{h}\|\xi\|^2, \\
&\lesssim \|\xi\|^2.
\end{aligned}
$$

178 So,

$$\|\mathcal{Q}_2\|_\infty \lesssim \|\eta\|^2 + \|\xi\|^2. \tag{51}$$

Next, we will consider $\mathcal{Q}_3$,

$$
\begin{aligned}
\mathcal{Q}_3 &= \int_I \Omega(\tilde{u} - \tilde{u}_h)\xi + \tilde{u}_h(\Omega - \Omega_h)\xi \; dr, \\
\|\mathcal{Q}_3\|_\infty &\leq \int_I |\Omega(\tilde{u} - \tilde{u}_h)\xi| \; dr + \int_I |\tilde{u}_h(\Omega - \Omega_h)\xi| \; dr.
\end{aligned}
$$

For the first part

$$
\begin{aligned}
&\int_I |\Omega(\tilde{u} - \tilde{u}_h)\xi| \; dr \\
&\leq \|\Omega\|_\infty \|\tilde{u} - \tilde{u}_h\| \|\xi\| \\
&\lesssim \|g_r\|_\infty \|e\| \|\xi\|, \quad \text{(use } \Omega = -\frac{1}{2}\tilde{g}_r \text{ and (38))} \\
&\lesssim \|g_r\|_\infty \|e\| \|\xi\| \\
&\lesssim \|\xi\|^2 + \|\eta\|^2.
\end{aligned}
$$

The second part is given by

$$\begin{aligned}
&\int_I |\tilde{u}_h(\Omega-\Omega_h)\xi|\ dr \\
&= \int_I |(\tilde{u}-\tilde{e})(\Omega-\Omega_h)\xi|\ dr \\
&\le \|\tilde{u}-\tilde{e}\|_\infty \int_I |(\Omega-\Omega_h)\xi|\ dr \\
&\le (\|\tilde{u}\|_\infty + \|\tilde{e}\|_\infty)\, \|\tilde{\Delta}_r\| \|\xi\|, \quad \text{(using (45)-(24) and } \|\tilde{e}\|_\infty \le \|e\|_\infty \le Ch.\text{)} \\
&\lesssim \|e\|\|\xi\| \\
&\lesssim \|\eta\|^2 + \|\xi\|^2.
\end{aligned}$$

Then, we obtain

$$\|\mathcal{Q}_3\|_\infty \lesssim \|\xi\|^2 + \|\eta\|^2. \tag{52}$$

179 □

Theorem 4. *For any given integer $k \ge 1$, we have*

$$\|u(t,\cdot) - u_h(t,\cdot)\| \lesssim e^{ct} h^{k+1}.$$

Proof. The boundary terms in (47) can be eliminated if we use the Gauss-Radau projection, namely

$$\eta^+_{i-\frac{1}{2}} = 0.$$

The first part of (48) can be rewritten as

$$\begin{aligned}
&\int_{I_i} \xi\xi_t - \xi v \xi_r\ dr + \xi^+ v \xi^-|_{i+\frac{1}{2}} - \xi^+ v \xi^+|_{i-\frac{1}{2}} \\
&= \int_{I_i} \frac{1}{2}(\xi^2)_t + \frac{1}{2}\xi^2 (v)_r\ dr + \mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} + \vartheta_{i+\frac{1}{2}},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_{i+\frac{1}{2}} &= \frac{1}{2} v (\xi^+)^2|_{i+\frac{1}{2}}, \\
\vartheta_{i+\frac{1}{2}} &= v\xi^-(\xi^+ - \frac{1}{2}\xi^-)|_{i+\frac{1}{2}} - \frac{1}{2} v (\xi^+)^2|_{i+\frac{1}{2}}, \\
&= -\frac{1}{2} v (\xi^+ - \xi^-)^2|_{i+\frac{1}{2}}.
\end{aligned}$$

180 Since $v = -\frac{1}{2}\tilde{g} \le 0$, it is easy to check that $\vartheta_{i+\frac{1}{2}} \ge 0$.

Then (47)-(48) becomes

$$\begin{aligned}
&\int_{I_i} \eta_t \xi - \eta v \xi_r\ dr + \mathcal{Q}_{2,i} \\
&\ge \int_{I_i} \frac{1}{2}(\xi^2)_t + \frac{1}{2}\xi^2 (v)_r\ dr + \mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} + \mathcal{Q}_{3,i}.
\end{aligned}$$

Summing up over i, we have

$$\begin{aligned}
&\int_I \eta_t \xi - \eta v \xi_r \ dr + \mathcal{Q}_2 \\
\geq &\int_I \frac{1}{2}(\xi^2)_t + \frac{1}{2}\xi^2 (v)_r \ dr + \mathcal{F}_{N+\frac{1}{2}} - \mathcal{F}_{\frac{1}{2}} + \mathcal{Q}_3. \qquad (53)
\end{aligned}$$

By Lemma 6, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ can be controlled.

As for the estimate to $\displaystyle\int_I \eta v \xi_r \ dr$, since

$$\begin{aligned}
\int_I \eta v \xi_r \ dr &= \sum_{i=1}^{N} \int_{I_i} (v - \bar{v}_i + \bar{v}_i)\eta \xi_r \ dr \\
&= \sum_{i=1}^{N} \int_{I_i} (v - \bar{v}_i)\eta \xi_r \ dr,
\end{aligned}$$

in which we have we used the orthogonal property of the projection Π, where $\bar{v}_i$ is the cell integral average of v in each cell I_i. Next, we use the inverse properties $\|\xi_r\| \leq C\frac{1}{h}\|\xi\|$ and $\|v - \bar{v}_i\|_\infty \lesssim h$, then

$$\begin{aligned}
|\int_I \eta v \xi_r \ dr| &\leq \sum_{i=1}^{N} \int_{I_i} |v - \bar{v}_i||\eta||\xi_r| \ dr \\
&\lesssim h\|\eta\|\|\xi_r\| \\
&\lesssim h\frac{1}{h}\|\eta\|\|\xi\| \\
&\lesssim h^{2k+2} + \|\xi\|^2, \qquad (54)
\end{aligned}$$

where we have used the optimal approximation properties of Π in (26). For the boundary terms $\mathcal{F}_{N+\frac{1}{2}}, \mathcal{F}_{\frac{1}{2}}$, since

$$0 = (u - u_h)^+_{N+\frac{1}{2}} = \eta^+_{N+\frac{1}{2}} - \xi^+_{N+\frac{1}{2}},$$

and $\eta^+_{N+\frac{1}{2}} = 0$, then

$$\xi^+_{N+\frac{1}{2}} = 0,$$

we have

$$\mathcal{F}_{N+\frac{1}{2}} = \left(\frac{1}{2}v(\xi^+)^2\right)_{N+\frac{1}{2}} = 0, \qquad (55)$$

and

$$\mathcal{F}_{\frac{1}{2}} = \left(\frac{1}{2}v(\xi^+)^2\right)_{\frac{1}{2}} \leq 0. \qquad (56)$$

Collecting (54), (55), (56), (49) and (50) into (53) and using $\|\eta_t\| \lesssim h^{k+1}\|u_t\|_{k+1}$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\xi\|^2 &\le \int_I |\eta_t \xi| \; dr + \int_I |\eta v \xi_r| \; dr + \frac{1}{2}\|v_r\|_\infty \int_I \xi^2 \; dr + \|\mathcal{Q}_2\|_\infty + \|\mathcal{Q}_3\|_\infty \\
&\lesssim h^{2k+2} + \|\xi\|^2 + \frac{\|v_r\|_\infty}{2}\|\xi\|^2 + (h^{2k+2} + \|\xi\|^2) \\
&\lesssim (\frac{1}{2}\|v_r\|_\infty + C)\|\xi\|^2 + Ch^{2k+2}, \qquad \text{(using (38), } \|v_r\|_\infty \text{ is bounded)} \\
&\lesssim \|\xi\|^2 + h^{2k+2}.
\end{aligned}$$

By Gronwall's inequality,

$$\|\xi\|^2 \le Ch^{2k+2}\exp(ct).$$

Taking h small enough such that $\sqrt{C}h^{k+1}\exp(\frac{c}{2}t) < h^{\frac{3}{2}}, k \ge 1$, then we improve the bootstrap assumption (31) and close the loop. Finally, we have

$$\|u(t,\cdot) - u_h(t,\cdot)\| \lesssim e^{ct}h^{k+1}.$$

□

7 Convergence analysis for the P^0 case

In this section, we only need the exact solution $u \in C^1$. We shall give a-priori estimate for the full discrete P^0 DG scheme

$$\frac{1}{\tau}(u^{n+1}_{h,i} - u^n_{h,i}) - \frac{1}{2h}\tilde{g}^n_{h,i+\frac{1}{2}}(u^n_{h,i+1} - u^n_{h,i}) = \frac{1}{2r_{i+\frac{1}{2}}}(g^n_{h,i+\frac{1}{2}} - \tilde{g}^n_{h,i+\frac{1}{2}})(u^n_{h,i} - \varphi^n_{h,i}), \tag{57}$$

where τ satisfies the CFL condition $\tau \le 2h$. To simplify the expression, we use φ to represent $\tilde{u}$ and define

$$\tilde{g}'_{h,i+\frac{1}{2}} := \frac{1}{r_{i+\frac{1}{2}}}(g_{h,i+\frac{1}{2}} - \tilde{g}_{h,i+\frac{1}{2}}), \quad D^+u_i := \frac{1}{h}(u_{i+1} - u_i), \quad v_{h,i} := D^+u_{h,i}.$$

7.1 The main ideas

This part is completely different from Section 6, where the L^∞ norm of the error is estimated instead of the L^2 norm. In order to get the error estimate, the most important component is the a-priori estimate of u_h and D^+u_h. The method of these a-priori estimates is almost the same as the proof of Theorem 3. The proofs are rather technical, so we proceed in several steps to clarify the main ideas.

Step 1: Following Theorem 2, we can demonstrate that the numerical solution for g satisfies the same exponential decay estimate: $g_h(t,r) \le C\exp(-ct), \forall r \le r_\diamond := \frac{99}{100}\lambda r_0$. This proof is provided in Lemma 7.

Step 2: Using a bootstrap technique, we can show that $\|u_h\|_\infty \le C\exp(ct)$. It is not difficult to prove that ,

$$\sup_{r > r_\diamond} |D^+u^n_h| \le C\exp\left(3c\|u_0\|^2_\infty(-1 + e^{\frac{4n\tau}{r_\diamond}})\right).$$

We show this in Lemma 8.

Step 3: In Lemma 9, using $g_h \le C\exp(-ct), \forall r \le r_\diamond$, we can prove a-priori estimate for D^+u_h in the region $r \le r_\diamond$:

$$\sup_{r \le r_\diamond} |D^+u_h| \le C_1.$$

Combining Lemma 8 and Lemma 9, we have a globally a-priori estimate for u_h and D^+u_h.

Step 4: In Theorem 5, we show the error estimate

$$\|e^n\|_\infty \lesssim e^{c\tau n}h.$$

7.2 The details

In the first step, we need a bootstrap assumption

Assumption 2.

$$\|u - u_h\|_\infty \le h^{\frac{1}{2}}. \tag{58}$$

Lemma 7. *For large data satisfying (20)-(21)-(22)-(23), we have*

$$g_h(t, r_*) \le \exp\left(\gamma - \frac{F_0}{c_2^2} - \frac{c_0}{c_2^2}t + O(h^{\frac{1}{2}})\right).$$

As $t \to \infty$, the event horizon will form, where $r_*, \gamma, F_0, c_2, c_0$ are defined in Theorem 5.

Proof. Due to the bootstrap assumption (58), we have

$$u = u_h + O(h^{\frac{1}{2}}), \qquad u_h^2 \ge u^2 + O(h^{\frac{1}{2}}).$$

Therefore, using equations (87) and (84), we have

$$\begin{aligned}
-\int_r^b su^2 + O(h^{\frac{1}{2}})\ ds + \gamma \ge -\int_r^b su_h^2\ ds + \gamma,& \\
g_h(t,r) \le \exp\left(\gamma - \frac{1}{c_2^2}\int_r^b su_h^2\ ds\right)& \\
\le \exp\left(\gamma - \frac{1}{c_2^2}\int_r^b su^2 + O(h^{\frac{1}{2}})\ ds\right)& \\
\le \exp\left(\gamma - \frac{F_0}{c_2^2} - \frac{c_0}{c_2^2}t + O(h^{\frac{1}{2}})\right)&.
\end{aligned}$$

Following the standard process used in the proof of Lemma 14, we can get

$$r_* \ge \lambda r_0 + O(h^{\frac{1}{2}}).$$

□

Taking h sufficient small such that $\lambda r_0 + O(h^{\frac{1}{2}}) > \frac{99}{100}\lambda r_0$. Define $r_\diamond := \lambda r_0 \frac{99}{100}$,

$$\chi_1(r) := \begin{cases} 1, & r \le r_\diamond, \\ 0, & r > r_\diamond, \end{cases}$$

and

$$\chi_2(r) := \begin{cases} 1, & r \ge r_\diamond, \\ 0, & r < r_\diamond. \end{cases}$$

Lemma 8.

$$\|u_h^n\|_\infty \le c\ \exp(\frac{4n\tau}{r_\diamond})\|u_0\|_\infty,$$

and for $r > r_\diamond$,

$$\|v_h^n \chi_2\|_\infty \le \|v_0\|_\infty \exp\left(3c\|u_0\|_\infty^2(-1 + e^{\frac{4n\tau}{r_\diamond}})\right).$$

To prove this Lemma, we need a Corollary of Lemma 19, which is a P^0 version of Lemma 5.

Corollary 2.

$$\begin{aligned} \|\tilde{g}_h^{'}\|_\infty &\le \|g_h^{'}\|_\infty, \\ |D^+\tilde{g}_h| &\le \|g_h^{'}\|_\infty, \\ |D^+\tilde{g}_{h,i}^{'}| &\le \frac{3}{r_{i+1}}\|g_h^{'}\|_\infty, \\ \frac{1}{r_i}|u_{h,i} - \varphi_{h,i}| &\le \|v_h\|_\infty. \end{aligned}$$

The proof of Lemma 19 is given in Appendix A.5. Next, we give the proof of Lemma 8 below.

Proof. The estimate of $\|u_h\|_\infty$ is given by the bootstrap assumption (58)

$$\|u - u_h\|_\infty \le h^{\frac{1}{2}}.$$

Using a-priori estimate for the exact solution (24), we have

$$\|u_h\|_\infty \le h^{\frac{1}{2}} + \|u\|_\infty \le c \exp(\frac{4n\tau}{r_\diamond})\|u_0\|_\infty.$$

The equation of v_h is given by

$$\begin{aligned} \frac{1}{\tau}(v_{h,i}^{n+1} - v_{h,i}^n) - \frac{1}{2}\tilde{g}_{h,i+\frac{1}{2}}^n D^+ v_{h,i}^n = \frac{1}{2}(\tilde{g}_{h,i+\frac{1}{2}}^{'})^n v_{h,i}^n + \frac{1}{2}D^+\tilde{g}_{h,i+\frac{1}{2}}^n v_{h,i+1}^n \\ + \frac{1}{2}D^+(\tilde{g}_{h,i+\frac{1}{2}}^{'})^n(u_{h,i+1}^n - \varphi_{h,i+1}^n) - \frac{1}{2}(\tilde{g}_{h,i+\frac{1}{2}}^{'})^n D^+\varphi_i^n, \quad (59) \end{aligned}$$

where

$$\tilde{g}^{'}_{h,i+\frac{1}{2}} = \frac{1}{r_{i+\frac{1}{2}}}(g_{h,i+\frac{1}{2}} - \tilde{g}_{h,i+\frac{1}{2}}).$$

Setting

$$RHS = \frac{1}{2}(\tilde{g}^{'}_{h,i+\frac{1}{2}})^n v^n_{h,i} + \frac{1}{2}D^+\tilde{g}^n_{h,i+\frac{1}{2}} v^n_{h,i+1} + \frac{1}{2}D^+(\tilde{g}^{'}_{h,i+\frac{1}{2}})^n(u^n_{h,i+1} - \varphi^n_{h,i+1}) - \frac{1}{2}(\tilde{g}^{'}_{h,i+\frac{1}{2}})^n D^+\varphi^n_i,$$

we shall show that the RHS can be controlled

$$|RHS| \le \frac{12}{r_\diamond}\|u^n_h\|^2_\infty\|v^n_h\|_\infty.$$

Since $\tilde{g}^{'}_h = \frac{1}{r}(g_h - \tilde{g}_h)$, by Corollary 2, we have

$$\|\tilde{g}^{'}_h\|_\infty \le \|g^{'}_h\|_\infty.$$

So, we only need to consider the bound of $g^{'}_h$. For $r \ge r_\diamond$,

$$\begin{aligned} g^{'}_h &= \frac{g_h}{r}(u_h - \varphi_h)^2, \\ |g^{'}_h| &\le \frac{1}{r_\diamond}(u_h - \varphi_h)^2 \\ &\le 2\|v_h\|_\infty\|u_h\|_\infty, \quad (\text{using } \frac{1}{r}|u_h - \varphi_h| \le \|v_h\|_\infty), \end{aligned}$$

or

$$g^{'}_h \le \frac{4}{r_\diamond}\|u_h\|^2_\infty, \quad (\text{using } |\varphi_h| \le \|u_h\|_\infty)\ .$$

209 Therefore

$$\|\tilde{g}^{'}_h\|_\infty \le \frac{4}{r_\diamond}\|u_h\|^2_\infty, \text{ or } \quad \|\tilde{g}^{'}_h\|_\infty \le 2\|u_h\|_\infty\|u_h\|_\infty. \tag{60}$$

By Corollary 2 and (60), we have

$$|D^+\tilde{g}^{'}_{h,i+\frac{1}{2}}| \le \frac{6}{r_\diamond}\|v_h\|_\infty\|u_h\|_\infty,$$

then, collecting the above, we have

$$|RHS| \le \frac{12}{r_\diamond}\|u_h\|^2_\infty\|v_h\|_\infty.$$

For $r \ge r_\diamond$,

$$\begin{aligned} v^{n+1}_{h,i} &\le (1 - \frac{\tau}{2h}\tilde{g}^n_{h,i+\frac{1}{2}})v^n_{h,i} + \frac{\tau}{2h}\tilde{g}^n_{h,i+\frac{1}{2}}v^n_{h,i+1} + \frac{12\tau}{r_\diamond}\|u^n_h\|^2_\infty\|v^n_h\|_\infty, \\ \|v^{n+1}_h\|_\infty &\le \|v^n_h\|_\infty + \frac{12\tau}{r_\diamond}\|u^n_h\|^2_\infty\|v^n_h\|_\infty. \end{aligned}$$

Using Gronwall's inequality, we obtain

$$\|v_h^n\|_\infty \le \|v_h^0\|_\infty \exp\left(\sum_{j=0}^{n} \tau \frac{12}{r_\diamond}\|u_h^j\|_\infty^2\right).$$

By Lemma 8, $\|u_h^j\|_\infty^2 \le c\|u_0\|_\infty^2 e^{\frac{4j\tau}{r_\diamond}}$, then

$$\sum_{j=0}^{n} \tau\|u_h^j\|_\infty^2 \le c\frac{1}{4}\|u_0\|_\infty^2 r_\diamond(-1+e^{\frac{4\tau n}{r_\diamond}}).$$

By Gronwall's inequality, we have

$$\begin{aligned}\|v_h^n\chi_2\|_\infty \le \|v_h^n\|_\infty &\le \|v_h^0\|_\infty \exp\left(c\frac{12}{r_\diamond}\frac{1}{4}\|u_0\|_\infty^2(-1+e^{\frac{4n\tau}{r_\diamond}})\right)\\ &\le \|v_h^0\|_\infty \exp\left(3c\|u_0\|_\infty^2(-1+e^{\frac{4n\tau}{r_\diamond}})\right).\end{aligned}$$

□

Lemma 9. *For $r \le r_\diamond$, we have*

$$\|D^+u_h\chi_1\|_\infty \le C_1,$$

where C_1 depend only on the initial data.

Proof. Since $g_h' := g_h\frac{1}{r}(u_h-\varphi_h)^2$, $|(u_h-\varphi_h)\chi_1| \le 2r\|D^+u_h\chi_1\|_\infty$ and $|\varphi_h\chi_1| \le \|u_h\chi_1\|_\infty$, then

$$|g_h'| \le 4g_h r_\diamond\|D^+u_h\chi_1\|_\infty^2.$$

Using Lemma 7, we have

$$|g_h'| \le 4r_\diamond \exp(-\beta^2-\alpha t)\|D^+u_h\chi_1\|_\infty^2, \tag{61}$$

where

$$-\beta^2 := \gamma - \frac{F_0}{c_2^2}, \qquad \alpha := \frac{c_0}{c_2^2},$$

in Theorem 5. The equation of $v_{h,i}$ is given by equation (59).

We define a useful quantity $\mathcal{G}$ as

$$\mathcal{G} := \sup_{0\le r\le b}|g_h'|.$$

Using (61), we have

$$\mathcal{G} \le 4r_\diamond \exp(-\beta^2-\alpha t)\|D^+u_h\chi_1\|_\infty^2. \tag{62}$$

Using Lemma 19, we have

$$\begin{aligned}|\tilde{g}_h'| &\le \mathcal{G},\\ |D^+\tilde{g}| &\le \mathcal{G},\\ |D^+\tilde{g}_h| &\le \frac{1}{r+h}\mathcal{G},\\ \frac{1}{r}|u_h-\varphi_h| &\le 2\|D^+u_h\|_\infty.\end{aligned}$$

Therefore, we establish the following bounds of the right hand side of (59):

$$\frac{1}{2}|D^+\tilde{g}_{h,i+\frac{1}{2}}v_{h,i+1}| \le \frac{1}{2}\mathcal{G}\|v_h\chi_1\|_\infty,$$
$$\frac{1}{2}|D^+\tilde{g}'_{i+\frac{1}{2}}(u_{h,i+1}-\varphi_{h,i+1})| \le \frac{\mathcal{G}}{2r_{i+1}}|u_{h,i+1}-\varphi_{h,i+1}| \le \mathcal{G}\|v_h\chi_1\|_\infty,$$
$$\frac{1}{2}\tilde{g}'_{i+\frac{1}{2}}|D^+(u_{h,i}-\varphi_{h,i})| \le \mathcal{G}\|v_h\chi_1\|_\infty.$$

By using (62) we have

$$\frac{1}{\tau}(v_{h,i}^{n+1}-v_{h,i}^n)-\frac{1}{2}\tilde{g}_{h,i+\frac{1}{2}}^n D^+v_i^n \le 3\mathcal{G}\|v_h^n\chi_1\|_\infty \le 12r_\diamond \exp(-\beta^2-\alpha t)\|v_h^n\chi_1\|_\infty^3 := c_3e^{-\alpha t}\|v_h^n\chi_1\|_\infty^3, \tag{63}$$

then, we have

$$|v_{h,i}^{n+1}| \le |v_{h,i}^n|(1-\frac{\tau}{2h}\tilde{g}_{i+\frac{1}{2}}^n) + |v_{h,i+1}^n|\frac{\tau}{2h}\tilde{g}_{i+\frac{1}{2}}^n + \tau c_3e^{-\alpha\tau n}\|v_h^n\chi\|_\infty^3,$$

therefore

$$\|v_h^{n+1}\chi_1\|_\infty \le \|v_h^n\chi_1\|_\infty + \tau c_3e^{-\alpha\tau n}\|v_h^n\chi_1\|_\infty^3. \tag{64}$$

Here we use the bound of $\|v_h^n(1-\chi_1)\|_\infty$. We define

$$\delta\|v_h^n\chi_1\|_\infty := \frac{1}{\tau}(\|v_h^{n+1}\chi_1\|_\infty - \|v_h^n\chi_1\|_\infty),$$

and using (64) we have

$$\frac{\delta\|v_h^n\chi_1\|_\infty}{\|v_h^n\chi_1\|_\infty^3} \le c_3e^{-\alpha\tau n}.$$

We mention that

$$\delta(-\frac{1}{2\|v_h^n\chi_1\|_\infty}) \le \frac{\delta\|v_h^n\chi_1\|_\infty}{\|v_h^n\chi_1\|_\infty^3},$$

which is because

$$\delta(-\frac{1}{2\|v_h^n\chi_1\|_\infty^2}) = \delta\|v_h^n\chi_1\|_\infty\frac{1}{\xi^3}, \quad \xi \text{ is between } \|v_h^n\chi_1\|_\infty \text{ and } \|v_h^{n+1}\chi_1\|_\infty\ .$$

If $\|v_h^n\chi_1\|_\infty \le \|v_h^{n+1}\chi_1\|_\infty$, then $\frac{\delta\|v_h^n\chi_1\|_\infty}{\xi^3} \le \frac{\delta\|v_h^n\chi_1\|_\infty}{\|v_h^n\chi_1\|_\infty^3}$.

If $\|v_h^n\chi_1\|_\infty \ge \|v_h^{n+1}\chi_1\|_\infty$, then $\frac{\delta\|v_h^n\chi_1\|_\infty}{\xi^3} \le \frac{\delta\|v_h^n\chi_1\|_\infty}{\|v_h^n\chi_1\|_\infty^3}$.

So,

$$\delta(-\frac{1}{2\|v_h^n\chi_1\|_\infty^2}) \le \frac{\delta\|v_h^n\chi_1\|_\infty}{\|v_h^n\chi_1\|_\infty^3}.$$

Then we have

$$\delta(-\frac{1}{2\|v_h^n\chi_1\|_\infty^2}) \le c_3e^{-\alpha\tau n}. \tag{65}$$

Multiplying both sides of (65) by τ and summing over n index gives us the following result.

$$\begin{aligned}\frac{1}{\|v_h^n\chi_1\|_\infty^2} - \frac{1}{\|v_h^0\chi_1\|_\infty^2} &\geq -2\tau\sum_{j=0}^{n} c_3 e^{-\alpha\tau j}\\ &\geq \sum_{j=0}^{n}\frac{2c_3}{\alpha}(e^{-\alpha\tau(j+1)} - e^{-\alpha\tau j})\\ &\geq -2\frac{c_3}{\alpha}.\end{aligned}$$

Finally, we have

$$\|v_h^n\chi_1\|_\infty \leq \left(\|v(0)\chi_1\|_\infty^{-2} - \frac{2c_3}{\alpha}\right)^{-\frac{1}{2}} := C_1. \tag{66}$$

If the initial data satisfy $\|v(0)\chi_1\|_\infty \leq (\frac{\alpha}{2c_3})^{\frac{1}{2}}$, inequality (66) holds, where C_1 just depends on the initial data.

□

Combining Lemma 8 and Lemma 9 gives the following a-priori estimates

$$\|u_h^n\|_\infty \leq C\exp(\frac{4n\tau}{r_\diamond})\|u_0\|_\infty, \tag{67}$$

$$\|v_h^n\chi_2\|_\infty \leq C\|v_0\|_\infty \exp\left(3c\|u_0\|_\infty^2(-1+e^{\frac{4n\tau}{r_\diamond}})\right), \tag{68}$$

where $C, c > 0$ only depend on the initial data.

We need the following auxiliary Lemma 10 to prove the error estimate.

Lemma 10. *Define*

$$\begin{aligned}e &= u - u_h,\\ \Delta &= g - g_h,\\ \Delta_r &= g_r - g_h^{'},\\ \tilde{\Delta}_r &= \tilde{g}_r - \tilde{g}_h^{'}.\end{aligned}$$

where

$$\begin{aligned}g_h^{'} &:= \frac{1}{r}g_h(u_h - \varphi_h),\\ \tilde{g}_h^{'} &:= \frac{1}{r}(g_h - \tilde{g}_h).\end{aligned}$$

Then, we have the estimate

$$\begin{aligned}\|\Delta\|_\infty &\leq \|e\|_\infty C_1(\|u_r\|_\infty + \|D^+u_h\|_\infty),\\ \|\tilde{\Delta}\|_\infty &\leq \|\Delta\|_\infty,\\ \|\Delta_r\|_\infty &\leq \|e\|_\infty C_3(\|u_r\|_\infty, \|D^+u_h\|_\infty),\\ \|\tilde{\Delta}_r\|_\infty &\leq \|\Delta_r\|_\infty.\end{aligned}$$

226 The proof of this lemma will be given in Appendix A.2.

Theorem 5. *Define the error* $e^n := u^n - u_h^n, u_h \in V_h^0$, *then*

$$\|e^n\|_\infty \lesssim \left(n\tau + \int_0^{n\tau} Cse^{sC}\ ds\right)(h+\tau).$$

Proof. Setting $u_i^n = u(t_n, r_i)$, where $u(t,r)$ is the exact solution, and

$$g'_{h,i+\frac{1}{2}} := \frac{1}{r_{i+\frac{1}{2}}}(g_{h,i+\frac{1}{2}} - \tilde{g}_{h,i+\frac{1}{2}}),$$
$$g'_{i+\frac{1}{2}} := \frac{1}{r_{i+\frac{1}{2}}}(g_{i+\frac{1}{2}} - \tilde{g}_{i+\frac{1}{2}}),$$

227 The exact solution satisfies

$$\frac{1}{\tau}(u_i^{n+1} - u_i^n) - \frac{1}{2}\tilde{g}^n_{i+\frac{1}{2}} D^+ u_i^n = \frac{1}{2}(\tilde{g}'_{i+\frac{1}{2}})^n (u_i^n - \varphi_i^n) + T_i^n, \tag{69}$$

228 where $T_i^n = O(h+\tau)$ is truncation error. Then we have the error equation

$$\frac{1}{\tau}(e_i^{n+1} - e_i^n) - \frac{1}{2}(\tilde{g}^n_{i+\frac{1}{2}} D^+ u_i^n - \tilde{g}^n_{h,i+\frac{1}{2}} D^+ u^n_{h,i}) = \frac{1}{2}(\tilde{g}'_{i+\frac{1}{2}})^n (u_i^n - \varphi_i^n) - \frac{1}{2}(\tilde{g}'_{h,i+\frac{1}{2}})^n (u^n_{h,i} - \varphi^n_{h,i}) + T_i^n, \tag{70}$$

then

$$\begin{aligned}
\frac{1}{\tau}(e_i^{n+1} - e_i^n) - \frac{1}{2}(\tilde{g}^n_{i+\frac{1}{2}} D^+ e_i^n + (\tilde{g}^n_{i+\frac{1}{2}} - \tilde{g}^n_{h,i+\frac{1}{2}}) D^+ u^n_{h,i}) &= \frac{1}{2}((\tilde{g}'_{i+\frac{1}{2}})^n - (\tilde{g}'_{h,i+\frac{1}{2}})^n)(u_i^n - \varphi_i^n) \\
&\quad + \frac{1}{2}(\tilde{g}'_{h,i+\frac{1}{2}})^n (e_i^n - \tilde{e}_i^n) + T_i^n, \\
\frac{1}{\tau}(e_i^{n+1} - e_i^n) - \frac{1}{2}(\tilde{g}^n_{i+\frac{1}{2}} D^+ e_i^n + \tilde{\Delta}_i D^+ u^n_{h,i}) &= \frac{1}{2}\tilde{\Delta}_{r,i}(u_i^n - \varphi_i^n) + \frac{1}{2}(\tilde{g}'_{h,i+\frac{1}{2}})^n (e_i^n - \tilde{e}_i^n) + T_i^n,
\end{aligned}$$

where we define

$$\tilde{\Delta}_i = \tilde{g}_{i+\frac{1}{2}} - \tilde{g}_{h,i+\frac{1}{2}},$$
$$\tilde{\Delta}_{r,i} = \tilde{g}'_{i+\frac{1}{2}} - \tilde{g}'_{h,i+\frac{1}{2}}.$$

Then we have

$$\frac{1}{\tau}(e_i^{n+1} - e_i^n) - \frac{1}{2}\tilde{g}^n_{i+\frac{1}{2}} D^+ e^n = \frac{1}{2}\tilde{\Delta}_i D^+ u^n_{h,i} + \frac{1}{2}\tilde{\Delta}_{r,i}(u_i^n - \varphi_i^n) + \frac{1}{2}(\tilde{g}'_{h,i+\frac{1}{2}})^n (e_i^n - \tilde{e}_i^n) + T_i^n,$$

by Lemma 10, we have

$$\begin{aligned}
\frac{1}{\tau}(e_i^{n+1} - e_i^n) - \frac{1}{2}\tilde{g}^n_{i+\frac{1}{2}} D^+ e_i^n &\le C\|e^n\|_\infty + T_i^n \\
e_i^{n+1} &\le (1 - \frac{\tau}{2h}\tilde{g}^n_{i+\frac{1}{2}}) + \frac{\tau}{2h}\tilde{g}^n_{i+\frac{1}{2}} e^n_{i+1} + \tau C\|e^n\|_\infty + \tau|T_i^n|, \\
\|e^{n+1}\|_\infty &\le (1 + C\tau)\|e^n\|_\infty + \tau\|T_i^n\|_\infty,
\end{aligned}$$

then

$$\|e^n\|_\infty \lesssim (n\tau + \int_0^{n\tau} Cse^{Cs}\, ds)(h+\tau).$$

Taking h, τ small enough such that $(n\tau + \int_0^{n\tau} Cse^{Cs}\, ds)(h+\tau) \le h^{\frac{1}{2}}$, then we close the bootstrap assumption (58). □

8 Numerical test: black hole formation

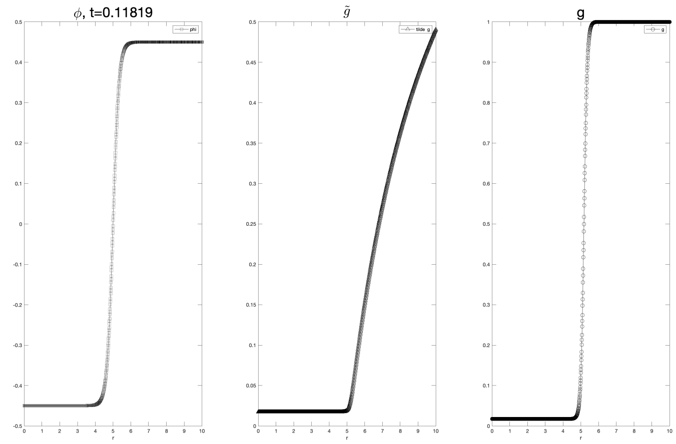
In this section we present numerical experiments to simulate the black hole formation. We take the initial data $\varphi_0(r) = 0.45\tanh(3(r-5))$, and $u_0(r) = (r\varphi_0(r))_r$ and boundary condition $U_b = u_0(b)$, where $b = 10$. For the time discretization, we use the fourth order explicit Runge-Kutta (RK4) scheme. To test the numerical accuracy, we take P^3 elements and use the numerical solution (u_h, g_h) with $N = 3200$ and $T = 0.5$ as the "exact" (reference) solution. We show that the method with P^3 elements gives fourth order of accuracy in both L^2 and L^∞ norms in Table 1. As $t \to \infty$, by the estimate of Theorem 2, $g = 0$ for $r < 2M_1$, where M_1 is the final Bondi mass and $2M_1 \approx 5$. This theory is consistent with the results given in Figure 1. In fact, according to [14], we also know

$$g \to g_1 := \begin{cases} 1 & r \ge 2M_1, \\ 0 & r < 2M_1, \end{cases} \quad \text{as } t \to \infty.$$

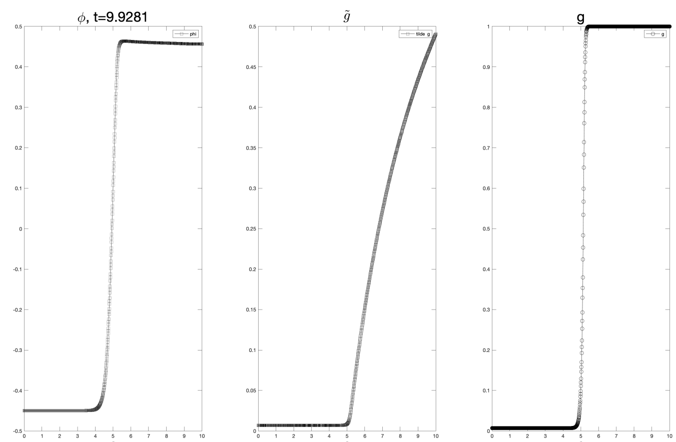
This behavior can also be seen in Figure 1.

N	u_h				g_h			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
100	3.35E-04	–	1.15E-03	–	1.67E-04	–	1.01E-03	–
200	2.56E-05	3.71	9.16E-05	3.65	1.24E-05	3.75	8.33E-05	3.60
400	1.71E-06	3.90	6.22E-06	3.88	8.43E-07	3.88	6.02E-06	3.79
800	1.06E-07	4.02	3.91E-07	3.99	5.27E-08	4.00	3.81E-07	3.98

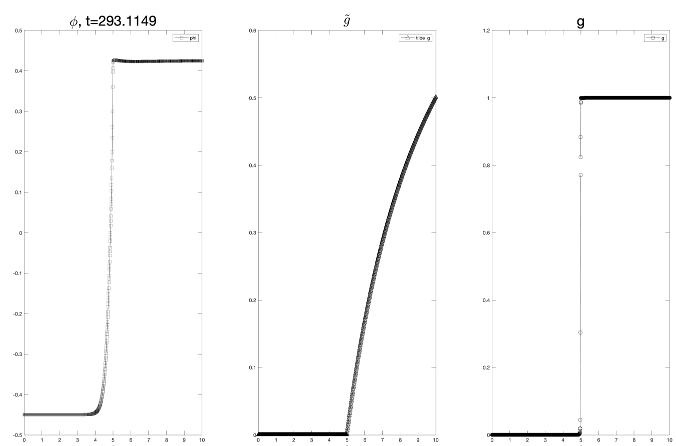
Table 1: Error table of numerical solution for Einstein-Scalar equation with P^3 polynomials, $T = 0.5$.



(a) $t = 0.11819$,from left to right φ,$\tilde{g}$,g



(b) $t = 9.9281$,from left to right φ,$\tilde{g}$,g



(c) $t = 293.1149$,from left to right φ,$\tilde{g}$,g

Figure 1: Solutions of Einstein-Scalar equation, the φ,$\tilde{g}$,g.

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A Appendix

A.1 Proof of Lemma 5

In order to reduce the trouble caused by the sub-index and simplify the expression, we use mathring to label the numerical solutions, namely, define

$$
\begin{aligned}
\mathring{u} &= u_h, \\
\mathring{g} &= g_h, \quad \tilde{\mathring{g}} = \tilde{g}_h, \\
e &= u - \mathring{u}, \\
\Delta &= g - \mathring{g}, \\
\Delta_r &= g_r - \mathring{g}_r, \\
\tilde{\Delta} &= \tilde{g} - \tilde{\mathring{g}}, \\
\tilde{\Delta}_r &= \tilde{g}_r - \tilde{\mathring{g}}_r = \frac{1}{r}(\Delta - \tilde{\Delta}).
\end{aligned}
$$

Proof. We consider the estimate (34)

$$|\tilde{u}| \le \frac{1}{r}\int_0^r |u| dr \le \|u\|_\infty.$$

Using similar method, we have

$$
\begin{aligned}
\|\tilde{e}\|_\infty \le \|e\|_\infty& \\
&\le \|\eta\|_\infty + \|\xi\|_\infty, \quad \text{(using } \|\eta\|_\infty \le Ch^{k+1} \text{ and } \|\xi\|_\infty \le Ch^{-\frac{1}{2}}\|\xi\|) \\
&\lesssim h^{k+1} + h^{-\frac{1}{2}}\|\xi\|.
\end{aligned}
$$

This finishes the proof of (34) and (39). Next, we consider the proof of the inequality (35).

$$
\begin{aligned}
\int_0^b \tilde{u}^2 \, dr &= \int_0^b \frac{1}{r^2}(\int_0^r uds)^2 \, dr, \\
&= \int_0^b (-\frac{1}{r})_r(\int_0^r uds)^2 \, dr, \\
&= -\frac{1}{r}\left(\int_0^r u\right)^2 |_0^b + \int_0^b 2u\frac{1}{r}\int_0^r uds \, dr, \\
&= -\frac{1}{b}(\int_0^r u)^2 + \int_0^b \frac{2}{r}u\int_0^r uds \, dr, \\
&\le 2\int_0^b u\tilde{u} \, dr, \\
&\le 2\|u\|\|\tilde{u}\|.
\end{aligned}
$$

Using same method, we have

$$\|\tilde{e}\| \le 2\|e\|.$$

This finishes the proof of (35).

In the next step we consider the estimate (36): $\|\tilde{u}_r\|_\infty \le \|u_r\|_\infty$.

$$\begin{aligned} r\tilde{u}_r &= u - \tilde{u}, \\ &= \frac{1}{r}\int_0^r u(r) - u(s) ds, \\ &= \frac{1}{r}\int_0^r \int_r^s u_r(\theta) d\theta \ ds, \\ &\le \frac{1}{r}\int_0^r r\|u_r\|_\infty ds, \\ &\le r\|u_r\|_\infty, \\ \|\tilde{u}_r\|_\infty &\le \|u_r\|_\infty. \end{aligned}$$

This finishes the proof of (36).

We also have the (38) by the same method,

$$\|\tilde{g}_r\|_\infty \le \|g_r\|_\infty.$$

Moreover, since $g_r = g\frac{1}{r}(u-\tilde{u})^2 = g\tilde{u}_r(u-\tilde{u})$, then

$$\begin{aligned} \|g_r\|_\infty &\le 2\|\tilde{u}_r\|_\infty \|u\|_\infty \\ &\le 2\|u_r\|_\infty \|u\|_\infty. \end{aligned}$$

Using the a-priori estimate (24)-(25), we have

$$\|g_r\|_\infty \le Ce^{c_3 t},$$

where C, c_3 are positive constants depending only on the initial data. This finishes the proof of (38).

We will next show the proof of (40): $\|\tilde{e}_r\|_\infty \lesssim h^{-\frac{3}{2}}\|\xi\| + h^k$. In the first cell I_1, using the same method as above we have

$$\sup_{r\in I_1} |\tilde{\xi}_r| \le \sup_{r\in I_1} |\xi_r| \le Ch^{-\frac{3}{2}}\|\xi\|. \tag{71}$$

For $i \ge 2, \forall r \ge r_{\frac{3}{2}}$,

$$\begin{aligned} |r\tilde{\xi}_r| &= |\xi - \tilde{\xi}| \\ &= |\frac{1}{r}\int_0^r \xi(r) - \xi(s) \ ds| \\ &\le \frac{1}{r}\int_0^r |\xi(r) - \xi(s)| \ ds \\ &\le \sup_{r_{\frac{3}{2}} \le s \le r} |\xi(r) - \xi(s)| \le 2\|\xi\|_\infty, \\ &\le Ch^{-\frac{1}{2}}\|\xi\|, \end{aligned}$$

then

$$|\tilde{\xi}_r| \le \frac{C}{r_{\frac{3}{2}}} h^{-\frac{1}{2}} \|\xi\| \le C h^{-\frac{3}{2}} \|\xi\|. \tag{72}$$

Collecting (71) and (72) we have

$$\|\tilde{\xi}_r\|_\infty \le C h^{-\frac{3}{2}} \|\xi\|. \tag{73}$$

Similarly, we have

$$\|\tilde{\eta}_r\|_\infty \le C h^k. \tag{74}$$

Using (73)-(74), we have

$$\|\tilde{e}_r\|_\infty \lesssim h^k + h^{-\frac{3}{2}} \|\xi\|.$$

This finishes the proof of (40).

Finally, we give the estimate (37): $\|\tilde{u}_r\| \le 2\|u_r\|$.

$$\begin{aligned}
\int_0^b \tilde{u}_r^2 \; dr &= \int_0^b \frac{1}{r^2}(u-\tilde{u})^2 \; dr \\
&= \int_0^b (-\frac{1}{r})_r (u-\tilde{u})^2 \; dr \\
&= -\frac{1}{r}(u-\tilde{u})^2|_0^T + \int_0^b \frac{2}{r}(u-\tilde{u})(u_r-\tilde{u}_r) \; dr \\
&= -\frac{1}{T}(u-\tilde{u})^2|_T + \int_0^b 2\tilde{u}_r(u_r-\tilde{u}_r) \; dr \\
&\le 2\int_0^b \tilde{u}_r u_r \; dr \\
&\le 2\|\tilde{u}_r\|\|u_r\| \\
\|\tilde{u}_r\| &\le 2\|u_r\|.
\end{aligned}$$

This finishes the proof of (37).

Using the same method, we can prove the inequality (41): $\|\tilde{e}_r\| \lesssim h^{-1}\|\xi\| + h^k$. Since $e = \eta - \xi, \xi \in V_h$ are piecewise polynomial, in the first cell I_1, using the same method as above, we have

$$\left(\int_{I_1} \tilde{\xi}_r^2 \; dr\right)^{\frac{1}{2}} \le 2\left(\int_{I_1} \xi_r^2 \; dr\right)^{\frac{1}{2}} \le C h^{-1}\|\xi\|. \tag{75}$$

For $i \ge 2$, integration by parts, we have

$$\int_{I_i} \tilde{\xi}_r^2 \; dr = -\frac{1}{r_{i+\frac{1}{2}}}(\xi_{i+\frac{1}{2}}^- - \tilde{\xi}_{i+\frac{1}{2}}^-)^2 + \frac{1}{r_{i-\frac{1}{2}}}(\xi_{i-\frac{1}{2}}^+ - \tilde{\xi}_{i-\frac{1}{2}}^+)^2 + \int_{I_i} \frac{2}{r}(\xi - \tilde{\xi})(\xi_r - \tilde{\xi}_r) \; dr,$$

Summing over i, using trace inverse inequality (29), we have

$$\begin{aligned}
\sum_{i=2}^{N}\int_{I_i}\tilde{\xi}_r^2\ dr &\le \frac{C}{r_{\frac{3}{2}}}h^{-1}\|\xi\|^2+\sum_{i=2}^{N}2\int_{I_i}\tilde{\xi}_r(\xi_r-\tilde{\xi}_r)\ dr\\
&\le Ch^{-2}\|\xi\|^2+\sum_{i=2}^{N}2\int_{I_i}\tilde{\xi}_r\xi_r\ dr\\
&\le Ch^{-2}\|\xi\|^2+\sum_{i=2}^{N}\int_{I_i}\frac{1}{2}\tilde{\xi}_r^2+2\xi_r^2\ dr,\qquad (\text{using } ab\le\frac{1}{4}a^2+b^2)\\
\frac{1}{2}\sum_{i=2}^{N}\int_{I_i}\tilde{\xi}_r^2\ dr &\le Ch^{-2}\|\xi\|^2+\sum_{i=2}^{N}\int_{I_i}2\xi_r^2\ dr\\
&\le Ch^{-2}\|\xi\|^2+C_1h^{-2}\|\xi\|^2,\qquad (\text{using the inverse inequality } \|\xi_r\|\lesssim h^{-1}\|\xi\|.\ )
\end{aligned}$$

365 then

$$\|\tilde{\xi}_r\|^2\le C_2h^{-2}\|\xi\|^2. \tag{76}$$

Collecting (75) and (76), we have

$$\|\tilde{\xi}_r\|\le Ch^{-1}\|\xi\|.$$

Using $\|\eta_r\|\le Ch^k$ and $\|\eta\|\le Ch^{k+1}$, the similar method as above, we have

$$\|\tilde{\eta}_r\|\le Ch^k,$$

then

$$\|\tilde{e}_r\|\le C(h^k+h^{-1}\|\xi\|).$$

366 This finishes the proof of (41).

We will give an estimate for $|\Delta|\le c_2\|e\|$.

$$\begin{aligned}
\Delta &= g-\mathring{g}\\
&=\exp(-\int_r^b\frac{1}{s}(u-\tilde{u})^2ds)-\exp(-\int_r^b\frac{1}{s}(\mathring{u}-\tilde{\mathring{u}})^2ds)\\
&=e^{\xi}\int_r^b\frac{1}{s}((\mathring{u}-\tilde{\mathring{u}})^2-(u-\tilde{u})^2)ds,\qquad (\text{where } \xi \text{ is between } -\int_r^b\frac{1}{s}(u-\tilde{u})^2\ ds \text{ and } -\int_r^b\frac{1}{s}(\mathring{u}-\tilde{\mathring{u}})^2\ ds)\\
&=e^{\xi}\int_r^b\frac{1}{s}(e-\tilde{e})^2+\frac{2}{s}(u-\tilde{u})(e-\tilde{e})ds\\
&=e^{\xi}\int_r^b(\tilde{e}_r+2\tilde{u}_r)(e-\tilde{e})ds,\qquad (\text{where } \xi<0, \text{ then } e^{\xi}\le 1)\\
&\le\|\tilde{e}_r+2\tilde{u}_r\|\|e+\tilde{e}\|\\
&\le(\|\tilde{e}_r\|+2\|\tilde{u}_r\|)\,3\|e\|.
\end{aligned}$$

Under the bootstrap assumption (31) we have $\|\tilde{e}_r\| \lesssim h^{\frac{1}{2}}$, then using (35)-(25) we have

$$|\Delta| \leq c_2 \|e\|, \tag{77}$$

then

$$\|\tilde{\Delta}\|_\infty \leq \|\Delta\|_\infty \leq c_2 \|e\|.$$

367 This finishes the proof of (43).

We shall give the proof of (44) and (45).

$$\begin{aligned}
\Delta_r &= g_r - \mathring{g}_r \\
&= \frac{1}{r}\left(g(u-\tilde{u})^2 - \mathring{g}(\mathring{u} - \tilde{\mathring{u}})^2\right) \\
&= \frac{\mathring{g}}{r}\left((u-\tilde{u})^2 - (\mathring{u} - \tilde{\mathring{u}})^2\right) + \frac{1}{r}(u-\tilde{u})^2(g - \mathring{g}),
\end{aligned}$$

here, we define the first part

$$\begin{aligned}
I_1 &= \frac{\mathring{g}}{r}\left((u-\tilde{u})^2 - (\mathring{u} - \tilde{\mathring{u}})^2\right) \\
&= -\frac{\mathring{g}}{r}(e-\tilde{e})^2 - \frac{\mathring{g}}{r}(u-\tilde{u})(e-\tilde{e}) \\
&= -\tilde{e}_r(e-\tilde{e})\mathring{g} - 2\tilde{u}_r(e-\tilde{e})\mathring{g}, \\
|I_1| &\leq (|\tilde{e}_r| + 2|\tilde{u}_r|)\,|e-\tilde{e}|.
\end{aligned}$$

Using the triangle inequality, we have

$$\|I_1\| \leq 3(\|\tilde{e}_r\|_\infty + 2\|\tilde{u}_r\|_\infty)\|e\|.$$

Under the bootstrap assumption (31) we have the inequality (40) $\|\tilde{e}_r\|_\infty \leq C$. Using the a-priori estimate (25) and $\|\tilde{u}_r\|_\infty \leq \|u_r\|_\infty \lesssim e^{ct}$ we have

$$\|I_1\| \leq C\|e\|.$$

The second part is defined as

$$\begin{aligned}
I_2 &= \frac{1}{r}(u-\tilde{u})^2(g-\mathring{g}) \\
&= r\tilde{u}_r^2\Delta, \\
|I_2| &\leq \|u_r\|_\infty^2 b|\Delta|.
\end{aligned}$$

Then, using (77) we have

$$\|I_2\| \leq c_2 b^{\frac{3}{2}}\|u_r\|_\infty^2\|e\|.$$

Using the a-prior estimate (25), we have

$$\|I_2\| \leq C\|e\|.$$

Then,

$$\|\Delta_r\| \leq \|I_1\| + \|I_2\| \leq C\|e\|.$$

Using the same method of the proof for (37), we have $\|\tilde{\Delta}_r\| \leq \|\Delta_r\|$. Finally, we also have the following estimate

$$\|\tilde{\Delta}_r\| \leq \|\Delta_r\| \leq C\|e\|.$$

This finishes the proof of (44) and (45). □

A.2 Proof of Lemma 10

Proof.

$$\begin{aligned}
\Delta &= g - g_h \\
&= \exp(-\int_r^b \frac{1}{s}(u-\varphi)^2 ds)^2 - \exp(-\int_r^b \frac{1}{s}(u_h-\varphi_h)^2 ds) \\
&= \exp(\xi)\int_r^b \frac{1}{s}(u_h-\varphi_h)^2 - \frac{1}{s}(u-\varphi)^2 \; ds \\
&= \exp(\xi)\int_r^b \frac{1}{s}(u_h-\varphi_h+u-\varphi)(u_h-u+\varphi-\varphi_h) \; ds,
\end{aligned}$$

where ξ is between $-\int_r^b \frac{1}{s}(u_h-\varphi_h)^2 \; ds$ and $-\int_r^b \frac{1}{s}(u-\varphi)^2 \; ds$, so $e^{\xi} \leq 1$. Using Lemma 19, Eq(117), $\frac{1}{r}|u_h-\varphi_h| \leq 2\|D^+u_h\|_\infty$, we have

$$\frac{1}{r}|u_h-\varphi_h+u-\varphi| \leq \|u_r\|_\infty + 2\|D^+u_h\|_\infty.$$

Moreover,

$$\begin{aligned}
\varphi - \varphi_h &= \frac{1}{r}\int_0^r u - u_h \; ds, \\
|\varphi - \varphi_h| &\leq \frac{1}{r}\int_0^r |u-u_h| \; ds \\
&\leq \|e\|_\infty,
\end{aligned}$$

then

$$|u_h - u + \varphi - \varphi_h| \leq |u-u_h| + |\varphi-\varphi_h| \leq 2\|e\|_\infty.$$

Therefore

$$|\Delta| \leq 2b\|e\|_\infty(\|u_r\|_\infty + 2\|D^+u\|_\infty),$$

where b is the length of the computation domain. Since $\tilde{\Delta} = \frac{1}{r}\int_0^r \Delta \; ds$, then

$$|\tilde{\Delta}| \leq \|\Delta\|_\infty \leq 2b\|e\|_\infty(\|u_r\|_\infty + 2\|D^+u_h\|_\infty).$$

Next, we estimate Δ_r,

$$\begin{aligned}\Delta_r &= \frac{1}{r}(g(u-\varphi)^2 - g_h(u_h-\varphi_h)^2)\\ &= \frac{1}{r}(g(u-\varphi)^2 - g(u_h-\varphi_h)^2) + \frac{1}{r}(g-g_h)(u_h-\varphi_h)^2\\ &= \frac{g}{r}(u-\varphi-u_h+\varphi_h)(u-\varphi+u_h-\varphi_h) + \frac{1}{r}(g-g_h)(u_h-\varphi_h)^2.\end{aligned}$$

The second term, by Lemma 19, $\frac{1}{r}|u_h-\varphi_h| \le 2\|D^+u_h\|_\infty$,

$$\begin{aligned}\frac{1}{r}(u_h-\varphi_h)^2 &= r(\frac{u_h-\varphi_h}{r})^2\\ &\le 4r\|D^+u_h\|_\infty^2\end{aligned}$$

and

$$|u-u_h+\varphi_h-\varphi| \le 2\|e\|_\infty,$$

then

$$\begin{aligned}|\Delta_r| &\le 2\|e\|_\infty(\|u_r\|_\infty + 2\|D^+u_h\|_\infty) + 8b^2\|D^+u_h\|_\infty^2\|e\|_\infty(\|u_r\|_\infty + 2\|D^+u_h\|_\infty)\\ &\le \|e\|_\infty C_3(\|D^+u_h\|_\infty, \|u_r\|_\infty).\end{aligned}$$

Finally,

$$\begin{aligned}\tilde{\Delta}_r &= \frac{1}{r}(\Delta - \tilde{\Delta}),\\ |\tilde{\Delta}_r| &\le \frac{1}{r}|\Delta - \tilde{\Delta}|\\ &\le \|\Delta_r\|_\infty\\ &\le \|e\|_\infty C_3(\|D^+u_h\|_\infty, \|u_r\|_\infty).\end{aligned}$$

□

A.3 Einstein field equation

We give the details of the derivation of the Einstein equation. The space-time metric is given by

$$ds^2 = -g(t,r)\tilde{g}(t,r)dt^2 - 2g(t,r)dtdr + r^2 d\Omega^2.$$

First step, we need compute the nonvanishing components of Christoffel symbols using

$$\Gamma^c_{ab} = \frac{1}{2}g^{ci}(\partial_a g_{bi} + \partial_b g_{ia} - \partial_i g_{ab}).$$

In practice, we just use the professional software package xAct to calculate these components. There are 11 nonvanishing components of Christoffel symbols, but we only need four of these components.

$$\Gamma^t_{\theta\theta} = \frac{r}{g},$$
$$\Gamma^r_{tr} = \frac{1}{2g}(g\tilde{g})_r,$$
$$\Gamma^r_{rr} = \frac{g_r}{g},$$
$$\Gamma^r_{\theta\theta} = -\frac{r\tilde{g}}{g}.$$

Second step, we compute the Ricci tensor. There are five nonvanishing components of $R_{\mu\nu} : R_{tt}, R_{tr}, R_{rr}, R_{\theta\theta}, R_{\phi\phi}$, but we only require two of them.

$$R_{rr} = \frac{2}{gr}g_r,$$
$$R_{\theta\theta} = 1 - \frac{(r\tilde{g})_r}{g}.$$

Last step, we derive the Einstein equation and the wave equation for φ.
The wave equation for φ is given by

$$\nabla^\mu \partial_\mu \varphi = 0,$$

that is

$$-\frac{2}{g}(\partial_t - \Gamma^r_{rt})\varphi_r + \frac{\tilde{g}}{g}(\partial_r - \Gamma^r_{rr})\varphi_r - \frac{2\Gamma^r_{\theta\theta}}{r^2}\partial_r\varphi - \frac{2}{r^2}\Gamma^t_{\theta\theta}\partial_t\varphi = 0,$$

then

$$-2(\varphi_{tr} + \frac{1}{r}\varphi_t) + \frac{1}{r}\left(\tilde{g}(r\varphi)_r\right)_r - \frac{1}{r}\varphi\tilde{g}_r = 0. \tag{78}$$

Taking the component $R_{rr} = 8\pi\partial_r\varphi\partial_r\varphi$, we have

$$\frac{2g_r}{rg} = 8\pi(\varphi_r)^2, \tag{79}$$

and the $R_{\theta\theta} = 0$, then

$$(r\tilde{g})_r = g. \tag{80}$$

A.4 Proof of Theorem 2 and Theorem 3

A.4.1 The main ideas

Intuitively, our initial conditions satisfy the following conditions (20)-(21)-(22)-(23), as illustrated in Figure 2, which make $\int_{r_0}^b su_0^2\, ds$ sufficiently large, which results in a sufficiently small wave velocity $\frac{1}{2}\tilde{g}$ at r_0. This will also cause an exponential decay of g along the characteristic Γ which begin from r_0. Moving along the characteristic Γ, as illustrated in Figure 3, we find that Γ halts at $2M_1$. Utilizing the exponential decay of g along the characteristic,

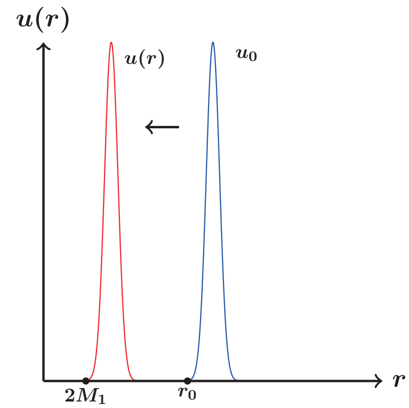

Figure 2: This is a diagram to illustrate the initial data. The initial data make $\int_{r_0}^{b} su_0^2\, ds$ sufficiently large. However, it is not necessary for the initial data to have a compact support set. This picture is just a simple picture drawn for ease of illustration.

we can establish the uniform a-priori estimate for u and u_r within the region $r \leq 2M_1$, which is the most crucial observation in this context.

The proofs are rather technical, so we proceed in several steps to clarify the main ideas.

Step 1: We define a characteristic $\Gamma : \dfrac{dr}{dt} = -\dfrac{1}{2}\tilde{g}, r(0) = r_0$, as illustrated in Figure 3, and demonstrate that it will terminate at $2M_1 := \lambda r_0$ for a given set of large initial data satisfying (20)-(21)-(22)-(23). This will be proven in Lemma 14. In Lemma 15, we will control the terms that are utilized in the proof of Lemma 13.

Step 2: In Lemma 13, we define $F(t,r) := \displaystyle\int_r^b su^2(t,s)\, ds$. We will prove that, if the initial data is sufficiently large such that conditions (20)-(21)-(22)-(23) are met, then along Γ, we derive $D(F) \geq c_0$. Subsequently, through a straightforward calculation, we obtain the decay estimate for g along Γ, $g \leq C\exp(-ct)$. This represents a black hole formation estimate, indicating that as $t \to \infty$, a black hole will form. Thus, we conclude the proof of Theorem 2.

Step 3: Since $g(t,r)$ is monotonically increasing with respect r, from the Theorem 2 we can derive $g \leq C\exp(-ct), \forall r \leq 2M_1$. This is a key idea. We can use this conclusion to show that in the region $r \leq 2M_1$, there are the following a-prior estimates

$$\sup_{r \leq 2M_1} |u| \leq C_1, \ \sup_{r \leq 2M_1} |u_r| \leq C_2,$$

for the initial data satisfying condition (20)-(21)-(22)-(23). These proofs are written in Lemma 16.

Step 4: It is not difficult to prove that ,$\forall r > 2M_1$,

$$\sup_{r > 2M_1} |u| \leq C\exp(c_1 t), \ \sup_{r > 2M_1} |u_r| \leq C\exp(-1 + e^{\frac{2t}{M_1}}).$$

We show this in Lemma 17 and Lemma 18. Combing Step 3 and 4, we finish the proof of Theorem 3.

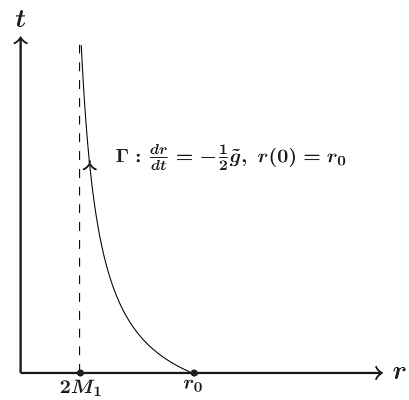

Figure 3: The characteristic Γ will stop at $r = 2M_1 := \lambda r_0$.

A.4.2 The details

We list some lemmas in [14] here.

Lemma 11. *[14] For each $r_1 > 2M_1$ there are constants C and C_1 such that*

$$\sup_{r\geq r_1} r^2 |u(t,r)| \leq C,$$
$$\sup_{r\geq r_1} r^3 |u_r(t,r)| \leq C_1,$$
$$|\int_0^b u\ dr| \leq B,$$
$$|\varphi(r_1)| \leq \frac{B}{r_1},$$

where M_1 is final Bondi mass and B is a constant.

We define

$$D := \partial_t - \frac{1}{2}\tilde{g}\partial_r,$$

then we have the evolution equation of F :

Lemma 12. *[14]*

$$D\left(F\right) = -\frac{1}{2}b\tilde{g}(b)\varphi^2(b) + \frac{1}{2}r\tilde{g}\varphi^2 + \frac{r}{2}(1-g) + \frac{1}{2}\int_r^b (1-g)\ ds - r\tilde{g}\log(\frac{1}{g}) - \int_r^b g\log(\frac{1}{g})\ ds. \quad (81)$$

Define a characteristic

$$\frac{dr}{dt} = -\frac{1}{2}\tilde{g}(t, r(t)), \qquad r(0) = r_0. \tag{82}$$

Along the characteristic (82), we define

$$r_* := r_0 - \frac{1}{2}\int_0^t \tilde{g}(s, r(s))\ ds.$$

We shall show the following lemma.

Lemma 13. *For large initial data that satisfy (20), (21), (22) and (23), then, along the characteristic (82) we have*

$$D\left(F(t, r_*)\right) \geq c_0,$$

where $c_0 = \frac{1}{2}\theta r_0$ only depends on the initial data.

Proof. We give a bootstrap assumption

$$D(F(t, r_*)) \geq c_0, \tag{83}$$

where c_0 is a constant satisfying $0 < 2c_0 < r_0$, and we attempt to use the condition of large initial data to improve the lower bound c_0 to $2c_0$. Let $r_2 > 2M_0$, then for $\forall r \leq r_*$,

$$\begin{aligned}
&\int_r^{r_2} (u - \varphi)^2\ ds \\
=& \int_r^{r_2} u^2\ ds - \int_r^{c_2} \partial_s(s\varphi^2)\ ds \\
=& \int_r^{r_2} u^2\ ds - r_2\varphi^2(r_2) + r\varphi^2(r) \\
\geq& \int_r^{r_2} u^2\ ds - \frac{B^2}{r_2},
\end{aligned}$$

where we use Lemma 11 and Lemma 14. Therefore,

$$\begin{aligned}
\int_r^{r_2} \frac{1}{s}(u - \varphi)^2\ ds &\geq \frac{1}{r_2}\int_r^{r_2} (u - \varphi)^2\ ds \\
&\geq \frac{1}{r_2}\left(\int_r^{r_2} u^2\ ds - \frac{B^2}{r_2}\right) \\
&= \frac{1}{r_2}\int_r^{r_2} u^2\ ds - \frac{B^2}{r_2^2} \\
&\geq \frac{1}{r_2^2}\left(\int_r^{r_2} su^2\ ds - B^2\right).
\end{aligned}$$

Using Lemma 11 and Lemma 14,

$$\begin{aligned}\int_r^{r_2} su^2\ ds &= \int_r^b su^2\ ds - \int_{r_2}^b su^2\ ds \\ &\geq \int_r^b su^2\ ds - \int_{r_2}^b \frac{C^2}{s^3}\ ds \\ &= \int_r^b su^2\ ds - \frac{C^2}{2r_2^2}.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{r_2^2}\left(\int_r^{r_2} su^2\ ds - B^2\right) \\ \geq& \frac{1}{r_2^2}\left(\int_r^b su^2\ ds - B^2 - \frac{C^2}{2r_2^2}\right).\end{aligned}$$

409 Define

$$\gamma = \frac{1}{r_2^2}(B^2 + \frac{C^2}{2r_2^2}), \tag{84}$$

then

$$\int_r^b \frac{1}{s}(u-\varphi)^2\ ds \geq \int_r^{r_2} \frac{1}{s}(u-\varphi)^2\ ds \geq \frac{1}{r_2^2}\int_r^b su^2\ ds - \gamma.$$

So, we have

$$\begin{aligned}g &= \exp\left(-\int_r^b \frac{1}{s}(u-\varphi)^2\ ds\right) \\ &\leq \exp(\gamma - \frac{1}{r_2^2}\int_r^b su^2\ ds).\end{aligned} \tag{85}$$

By the bootstrap assumption (83),

$$D(F) \geq c_0.$$

Integration along the characteristic (82) from r_0 to arbitrary r, we obtain

$$\begin{aligned}F\left(t, r(t)\right) - F(0, r_0) &\geq c_0 t \\ F\left(t, r(t)\right) &\geq F(0, r_0) + c_0 t \\ -\frac{1}{r_2^2}F\left(t, r(t)\right) &\leq -\frac{F(0, r_0)}{r_2^2} - \frac{c_0 t}{r_2^2},\end{aligned} \tag{86}$$

410 then, combing (85) and (86), we get

$$g(t, r) \leq \exp\left(\gamma - \frac{F_0}{r_2^2} - \frac{c_0}{r_2^2}t\right), \forall r < r_*. \tag{87}$$

Moreover, since

$$r(1-g)+\int_r^b (1-g)\ ds \geq r\left(1-g(t,r)\right), \tag{88}$$

and

$$r\tilde{g}\log(\frac{1}{g})+\int_r^b g\log\frac{1}{g}\ ds \leq rg\log\frac{1}{g}+\int_r^b g\log\frac{1}{g}\ ds. \tag{89}$$

Using (81)-(83)-(88)-(89), we have

$$\begin{aligned}&\frac{r}{2}(1-g)+\frac{1}{2}\int_r^b (1-g)\ ds - r\tilde{g}\log(\frac{1}{g})-\int_r^b g\log(\frac{1}{g})\ ds\\ \geq &\frac{1}{2}r(1-g(r))-\int_r^b g\log\frac{1}{g}\ ds - rg\log\frac{1}{g}.\end{aligned}$$

Using (81) , we have

$$D(F)\geq -\frac{1}{2}b\tilde{g}(b)\varphi^2(b)+\frac{1}{2}r\tilde{g}\varphi^2+\frac{r}{2}(1-g)+\frac{1}{2}\int_r^b (1-g)\ ds - r\tilde{g}\log(\frac{1}{g})-\int_r^b g\log(\frac{1}{g})\ ds.$$

Using Lemma 11, we have

$$|\varphi|\leq \frac{B}{r},\qquad r\varphi^2\leq\frac{B^2}{r},\qquad b\varphi^2(b)\leq\frac{B^2}{b}.$$

Then,

$$D(F)\geq\frac{1}{2}r(1-g)-\int_r^b g\log\frac{1}{g}\ ds - rg\log\frac{1}{g}-\frac{1}{2}\frac{B^2}{b},$$

and by Lemma 15, the right hand side can be controlled, so we obtain

$$D(F)\geq 2c_0.$$

So, we improved the bootstrap assumption (83). Using (87) again, we finishes the proof of Theorem 2. The proof of Lemma 14, 15 will be given in below. □

Lemma 14. *For large initial data such that*

$$F(0,r)>\left(\gamma-\log\left(2c_0(r_0-\lambda c_0)\frac{1}{r_2^2}\right)\right)r_2^2, \tag{90}$$

then, along the characteristic (82) we have

$$r=r_0-\int_0^t\frac{1}{2}\tilde{g}\ ds\geq\lambda r_0,$$

where r_0 is defined in (18).

417 *Proof.* Along the characteristic (82), we have

$$\frac{dr}{dt} = -\frac{1}{2}\tilde{g} \geq -\frac{1}{2}g, r(0) = r_0, \tag{91}$$

and by (87), we obtain

$$\begin{aligned} r &\geq r_0 - \int_0^t \frac{1}{2}g\ ds \\ &\geq r_0 - \int_0^\infty \frac{1}{2}g\ ds, \\ &\geq r_0 - \frac{1}{2}\int_0^\infty exp\left(\gamma - \frac{F_0}{r_2^2} - \frac{c_0}{r_2^2}s\right)\ ds \\ &= r_0 - \frac{1}{2}\frac{r_2^2}{c_0}e^\Gamma, \end{aligned}$$

where $\Gamma = \gamma - \frac{F_0}{r_2^2}$. We hope the lower bound of $r_0 - \frac{1}{2}\frac{r_2^2}{c_0}e^\Gamma$ is λr_0, then

$$\begin{aligned} r_0 - \frac{1}{2}\frac{r_2^2}{c_0}e^\Gamma &\geq \lambda r_0, \\ 2c_0(r_0 - \lambda r_0)\frac{1}{r_2^2} &\geq e^\Gamma, \\ \log\left(\frac{2c_0}{r_2^2}(r_0 - \lambda r_0)\right) &\geq \Gamma = \gamma - \frac{F_0}{r_2^2}, \\ F_0 &\geq \left(\gamma - \log\left(\frac{2c_0}{r_2^2}(r_0 - \lambda r_0)\right)\right) r_2^2. \end{aligned} \tag{92}$$

418 So, if (92) hold, $r_0 - \frac{1}{2}\frac{r_2^2}{c_0}e^\Gamma \geq \lambda r_0$. This finish the proof of Lemma 14. □

419 **Lemma 15.** *For large initial data satisfying*

$$g(0, r) \leq g_0 \leq g_0 \log(\frac{1}{g_0}) \leq \frac{1}{100}, \forall r < r_0, \tag{93}$$

420 *and*

$$r_0 \geq \frac{b}{2}, \tag{94}$$

421 *and (90) , then*

$$\frac{1}{2}r(1-g) - \int_r^b g\log(\frac{1}{g})\ ds - rg\log(\frac{1}{g}) - \varepsilon \geq \theta r_0 = 2c_0. \tag{95}$$

Proof. We mention that

$$\sup_{0\leq g\leq 1} g\log\frac{1}{g} = \frac{1}{e}.$$

We shall show that

$$\int_{r_*}^{b} g\log(\frac{1}{g})\ ds \le (2M_0 - r_0)\frac{1}{e} + C_3(\frac{1}{2M_0} - \frac{1}{b}).$$

First,

$$\begin{aligned}\int_{r_*}^{b} g\log(\frac{1}{g})\ ds &= \int_{r_*}^{2M_0} g\log(\frac{1}{g})\ ds + \int_{2M_0}^{b} g\log(\frac{1}{g})\ ds \\ &\le (2M_0 - r_*)\frac{1}{e} + \int_{2M_0}^{b} g\log(\frac{1}{g})\ ds.\end{aligned}$$

Next we shall give the bound of $\int_{2M_0}^{b} g\log(\frac{1}{g})\ ds$. Since

$$\begin{aligned}\log(\frac{1}{g}) &= \log(exp\int_{r}^{b}\frac{1}{s}(u-\varphi)^2\ ds) \\ &= \int_{r}^{b}\frac{1}{s}(u-\varphi)^2\ ds,\end{aligned}$$

and

$$(u-\varphi)^2 \le 2(u^2+\varphi^2).$$

Using Lemma 11, $\forall r \ge 2M_0$,

$$|\varphi| \le \frac{B}{r}, \qquad |u| \le \frac{C}{r^2},$$

then we have

$$u^2+\varphi^2 \le \frac{B^2}{r^2} + \frac{C^2}{r^4} \le \frac{B^2}{r^2}(1+\frac{C^2}{4B^2M_0^2}) := \frac{C_3}{r^2}.$$

For any $r \ge 2M_0$, we have

$$\begin{aligned}\int_{r}^{b}\frac{1}{s}(u-\varphi)^2\ ds &\le \int_{r}^{b}\frac{2C_3}{s^3}\ ds \\ &\le C_3(\frac{1}{r^2} - \frac{1}{b^2}),\end{aligned}$$

then

$$\begin{aligned}\int_{2M_0}^{b} g\log\frac{1}{g}\ ds &\le \int_{2M_0}^{b}\log\frac{1}{g}\ ds \\ &\le C_3\int_{2M_0}^{b}\frac{1}{r^2} - \frac{1}{b^2}\ ds \\ &\le C_3(\frac{1}{2M_0} - \frac{1}{b}) - C_3(b-2M_0)\frac{1}{b^2} \\ &\le C_3(\frac{1}{2M_0} - \frac{1}{b}). \qquad (96)\end{aligned}$$

We can take the initial data large enough such that

$$g(r_0) \le g(r_0)\log(\frac{1}{g(r_0)}) \le \varepsilon_0,$$

then

$$rg\log(\frac{1}{g}) \le r_0\varepsilon_0, \quad \forall r \le r_0.$$

For any $r \le r_0, g(r_0, t_0) := g_0 \ge g(r,t)$ then

$$\begin{aligned}
&\frac{1}{2}r(1-g) - \int_r^b g\log(\frac{1}{g})\ ds - rg\log\frac{1}{g} - \varepsilon \\
&\ge \frac{1}{2}r(1-g_0) - \frac{1}{e}(2M_0 - r) - \frac{C_3}{2M_0} - r_0\varepsilon_0 - \varepsilon \\
&\ge r(\frac{1}{2}(1-g_0) + \frac{1}{e}) - \frac{2M_0}{e} - \frac{C_3}{2M_0} - r_0\varepsilon_0 - \varepsilon.
\end{aligned}$$

Using Lemma 14, the characteristic (91) will stop at λr_0. Then, along the characteristic, we have

$$\begin{aligned}
&r(\frac{1}{2}(1-g_0) + \frac{1}{e}) - \frac{2M_0}{e} - \frac{C_3}{2M_0} - r_0\varepsilon_0 - \varepsilon \\
&\ge \lambda r_0(\frac{1}{2}(1-g_0) + \frac{1}{e}) - r_0\varepsilon_0 - \frac{2}{e}(1+\delta)r_0 - \frac{C_3}{2(1+\delta)r_0} - \varepsilon.
\end{aligned}$$

Since $g_0 \le g_0\log(\frac{1}{g_0}) \le \varepsilon_0$, then

$$\begin{aligned}
&\lambda r_0(\frac{1}{2}(1-g_0) + \frac{1}{e}) - r_0\varepsilon_0 - \frac{2}{e}(1+\delta)r_0 - \frac{C_3}{2(1+\delta)r_0} - \varepsilon \\
&\ge \lambda r_0(\frac{1}{2}(1-\varepsilon_0) + \frac{1}{e}) - r_0\varepsilon_0 - \frac{2}{e}(1+\delta)r_0 - \frac{C_3}{2(1+\delta)r_0} - \varepsilon.
\end{aligned}$$

422 We hope

$$\lambda r_0(\frac{1}{2}(1-\varepsilon_0) + \frac{1}{e}) - r_0\varepsilon_0 - \frac{2}{e}(1+\delta)r_0 - \frac{C_3}{2(1+\delta)r_0} - \varepsilon \ge \theta r_0. \tag{97}$$

Then

$$\frac{\lambda}{2}(1-\varepsilon_0) - \varepsilon_0 + \frac{\lambda}{e} - \frac{2}{e}(1+\delta) - \frac{C_3}{2(1+\delta)r_0^2} - \frac{\varepsilon}{r_0} \ge \theta.$$

Taking $\varepsilon_0 = \frac{1}{100}, \lambda = \frac{9}{10}$, we hope

$$\frac{\lambda}{2}(1-\varepsilon_0) - \varepsilon_0 + \frac{\lambda}{e} - \frac{2}{e}(1+\delta) \ge 0.$$

Solving this inequality, we get

$$\delta \leq 0.04.$$

So, we let $\delta = \frac{1}{100}$. Since $\varepsilon = \frac{B^2}{2b}$ only depends on b, we can take r_0 and b large enough such that

$$\frac{C_3}{2(1+\delta)r_0^2} + \frac{\varepsilon}{r_0} \leq \frac{1}{100}\left(\frac{\lambda}{2}(1-\varepsilon_0) - \varepsilon_0 + \frac{\lambda}{e} - \frac{2}{e}(1+\delta)\right).$$

Then inequality (97) becomes

$$\frac{99}{100}\left(\frac{\lambda}{2}(1-\varepsilon_0) - \varepsilon_0 + \frac{\lambda}{e} - \frac{2}{e}(1+\delta)\right) \geq \theta,$$

then

$$\theta \leq \frac{2}{100}.$$

So, if we taking $\theta = \frac{2}{100}$, the inequality(95) holds.

$\square$

Next, we shall show some *a priori* estimate for $\|u\|_\infty$ and $\|u_r\|_\infty$. Set

$$\chi(r) = \begin{cases} 1, & r < 2M_1, \\ 0, & r \geq 2M_1. \end{cases} \tag{98}$$

Lemma 16.

$$\|u(r)\chi(r)\|_\infty \leq \|u_0\chi\|_\infty + \frac{c_5}{\alpha},$$

$$\|u_r\chi(r)\|_\infty \leq \left(\|u_r(0,r)\chi\|_\infty^{-2} - \frac{12M_0}{\alpha}e^{-\beta^2}\right)^{-\frac{1}{2}},$$

where $\alpha := \frac{c_0}{r_2^2}$ *and*

$$c_5 := e^{-\beta^2}\left(\frac{1}{\|v(0,r)\chi\|_\infty^2} - \frac{12M_0}{\alpha}e^{-\beta^2}\right)^{-\frac{1}{2}}.$$

Proof. For $r \leq r_*$, since

$$g_r = \frac{1}{r}g(u-\varphi)^2 = rg(\varphi_r)^2,$$

and by Theorem 5, we have

$$\begin{aligned} |g_r| &\leq 2M_0 g|\varphi_r|^2 \\ &\leq 2M_0 g\|u_r\chi(r)\|_\infty^2 \\ &\leq 2M_0 \exp\left(\gamma - \frac{F_0}{r_2^2} - \frac{c_0}{r_2^2}t\right)\|u_r\chi(r)\|_\infty^2 \\ &\leq 2M_0 \exp\left(-\beta^2 - \alpha t\right)\|u_r\chi(r)\|_\infty^2, \end{aligned}$$

where

$$-\beta^2 := \gamma - \frac{F_0}{r_2^2},$$

then

$$\|g_r u_r \chi\|_\infty \leq 2M_0 \exp\left(-\beta^2 - \alpha t\right) \|u_r \chi(r)\|_\infty^3.$$

Set $v := u_r$. Recall the evolution equation of v

$$v_t - \frac{1}{2}\tilde{g} v_r = \frac{1}{2}\tilde{g}_{rr}(u - \varphi) + \frac{1}{2}\tilde{g}_r(2v - \varphi_r),$$

where

$$\begin{aligned}\tilde{g}_{rr} &= -\frac{1}{r^2}(g - \tilde{g}) + \frac{1}{r}(g_r - \tilde{g}_r)\\ &= -\frac{1}{r}\tilde{g}_r + \frac{1}{r}(g_r - \tilde{g}_r)\\ &= -\frac{2}{r}\tilde{g}_r + \frac{1}{r}g_r,\end{aligned}$$

then

$$\begin{aligned}\tilde{g}_{rr}(u - \varphi) &= (-\frac{2}{r}\tilde{g}_r + \frac{1}{r}g_r)(u - \varphi)\\ &= (-2\tilde{g}_r + g_r)\varphi_r\\ |\tilde{g}_{rr}(u - \varphi)| &\leq 3\|g_r\chi\|_\infty \|v\chi\|_\infty\\ |\tilde{g}_r(2v - \varphi_r)| &\leq 3\|g_r\chi\|_\infty \|v\chi\|_\infty.\end{aligned}$$

We define a characteristic

$$\frac{dr}{dt} = -\frac{1}{2}\tilde{g},, \qquad r(0) = r_5 \leq 2M_1. \tag{99}$$

Hence, along the characteristic (99), (59) becomes

$$\begin{aligned}|D(v)| &\leq 3\|g_r\chi\|_\infty \|v\chi\|_\infty\\ &\leq 6M_0 \exp(-\beta^2 - \alpha t)\|v\chi\|_\infty^3.\end{aligned} \tag{100}$$

Integrating (100) along the characteristic (99), we have

$$\begin{aligned}|v(t)| &\leq |v(0)| + \int_0^t 6M_0 e^{-\beta^2 - \alpha s}\|v\chi\|_\infty^3 \, ds\\ \|v(t)\|_\infty &\leq \|v(0)\|_\infty + \int_0^t 6M_0 e^{-\beta^2 - \alpha s}\|v\chi\|_\infty^3 \, ds.\end{aligned}$$

Setting $q := \|v(0)\|_\infty + \int_0^t 6M_0 e^{-\beta^2 - \alpha s}\|v\chi\|_\infty^3 \, ds$, then, if the initial data satisfy

$$\|v(0, r)\chi\|_\infty < \left(\frac{\alpha}{12M_0}\right)^{\frac{1}{2}} e^{\frac{1}{2}\beta^2}, \tag{101}$$

we have

$$\begin{aligned} q_t &\leq 6M_0 e^{-\beta^2-\alpha s}\|v\chi\|_\infty^3 \leq 6M_0 e^{-\beta^2-\alpha s} q^3, \\ \frac{d}{dt}(-\frac{1}{2}q^{-2}) &\leq 6M_0 e^{-\beta^2-\alpha t}, \\ q &\leq \left(q^{-2}(0) - \frac{12M_0}{\alpha}e^{-\beta^2}\right)^{-\frac{1}{2}}, \end{aligned}$$

then

$$\|v\chi\|_\infty \leq \left(\frac{1}{\|v(0,r)\chi\|_\infty^2} - \frac{12M_0}{\alpha}e^{-\beta^2}\right)^{-\frac{1}{2}} := \kappa. \tag{102}$$

427 Thus the inequality (102) holds.

Consider the evolution equation of u

$$\begin{aligned} u_t - \frac{1}{2}\tilde{g}u_r &= \frac{1}{2}\tilde{g}_r(u-\varphi) \\ &= \frac{1}{2}\frac{1}{r}(g-\tilde{g})(u-\varphi) \\ &= \frac{1}{2}(g-\tilde{g})\varphi_r, \end{aligned} \tag{103}$$

where

$$|\frac{1}{2}(g-\tilde{g})\varphi_r| \leq \|g\chi\|_\infty \|v\chi\|_\infty.$$

Then, integrating (103) along the characteristic (99) and using (102), we obtain

$$\begin{aligned} |D(u)| &\leq \|g\chi\|_\infty \|v\chi\|_\infty \\ &\leq \exp(-\beta^2-\alpha t)\|v\chi\|_\infty \\ &\leq \exp(-\beta^2-\alpha t)\left(\frac{1}{\|v(0,r)\chi\|_\infty^2} - \frac{12M_0}{\alpha}e^{-\beta^2}\right)^{-\frac{1}{2}} \\ &= \exp(-\alpha t)c_5. \\ \|u\chi\|_\infty &\leq \|u_0\chi\|_\infty + \frac{1}{\alpha}\left(c_5 - c_5 e^{-\alpha t}\right), \\ &\leq \|u_0\chi\|_\infty + \frac{c_5}{\alpha}. \end{aligned}$$

428 □

Lemma 17. $\forall r > 2M_1$, *for large initial data satisfying (90), (93) and (94), then*

$$\|u(t,r)(1-\chi)\|_\infty \leq \|u(0,r)\|_\infty \exp\left(\frac{1}{M_1}t\right).$$

Proof. Since

$$\tilde{g}_r = \frac{1}{r}(g - \tilde{g}),$$

we have, for $r \geq r_* \geq 2M_1$,

$$\|\tilde{g}_r\|_\infty \leq \frac{1}{2M_1}\|g - \tilde{g}\|_\infty \leq \frac{1}{M_1}. \tag{104}$$

On the other hand, since

$$\|u - \varphi\|_\infty \leq \|u\|_\infty + \|\varphi\|_\infty \leq 2\|u\|_\infty, \tag{105}$$

and

$$u_t - \frac{1}{2}\tilde{g}u_r = \frac{1}{2}\tilde{g}_r(u - \varphi).$$

Define a characteristic : $\frac{dr}{dt} = -\frac{1}{2}\tilde{g}(t,r), r(0) = r_3$,(where $r_3 > 2M_1$), along this characteristic,we have

$$\begin{aligned} \frac{d}{dt}u &= \frac{1}{2}\tilde{g}_r(u - \varphi), \\ |\frac{d}{dt}u| &\leq \frac{1}{2}\frac{1}{M_1}2\|u\|_\infty, \quad (\text{ using } (104).) \end{aligned}$$

Integrating this inequality along the characteristic, we obtain

$$\begin{aligned} |\int_0^t \frac{du}{ds}\, ds| \leq \int_0^t |\frac{du}{ds}\, ds| \leq \frac{1}{M_1}\int_0^t \|u\|_\infty\, ds, \\ \|u(t,r)\|_\infty \leq \|u_0\|_\infty + \frac{1}{M_1}\int_0^t \|u\|_\infty\, ds. \end{aligned}$$

By Gronwall's inequality, we have

$$\|u(t,r)\|_\infty \leq \|u_0\|_\infty \exp\left(\frac{t}{M_1}\right),$$

then

$$\|u(1-\chi)\|_\infty \leq \|u\|_\infty \leq \|u_0\|_\infty \exp(\frac{t}{M_1}).$$

∎

Lemma 18. *$\forall r > 2M_1$, for large initial data satisfying (90), (93) and (94), then*

$$\|u_r(1-\chi)\|_\infty \lesssim \|u_r(0,r)\|_\infty \exp(-1 + e^{\frac{2t}{M_1}}).$$

Proof. First step, we want to define a characteristic

$$\frac{dr}{dt} = -\frac{1}{2}\tilde{g}, r(0) = r_5, \forall r_5 > 2M_1. \tag{106}$$

Next, we know that

$$\begin{aligned}\tilde{g}_{rr} &= -\frac{1}{r^2}(g - \tilde{g}) + \frac{1}{r}(g_r - \tilde{g}_r) \\ &= -\frac{2}{r}\tilde{g}_r + \frac{1}{r}g_r,\end{aligned}$$

and

$$\begin{aligned}\|\tilde{g}_r\|_\infty &\le \|g_r\|_\infty \\ &\le \|g\frac{1}{r}(u-\varphi)^2\|_\infty \\ &\le \frac{1}{2M_1}\|(u-\varphi)^2\|_\infty \\ &\le \frac{1}{2M_1}\|2u^2 + 2\varphi^2\|_\infty \\ &\le \frac{1}{2M_1}2(\|u\|_\infty^2 + \|\varphi\|_\infty^2) \\ &\le \frac{2}{M_1}\|u\|_\infty^2.\end{aligned}$$

Since

$$\tilde{g}_{rr}(u-\varphi) = (-2\tilde{g}_r + g_r)\varphi_r,$$

and $|\varphi_r| \le \|v\|_\infty$, then, collecting above we obtain

$$|\tilde{g}_{rr}(u-\varphi)| \le \frac{6}{M_1}\|u\|_\infty^2\|v\|_\infty, \tag{107}$$

$$|\tilde{g}_r(2v-\varphi_r)| \le \frac{6}{M_1}\|u\|_\infty^2\|v\|_\infty. \tag{108}$$

Denote $v := u_r$, the evolution equation of v is

$$v_t - \frac{1}{2}\tilde{g}v_r = \frac{1}{2}\tilde{g}_{rr}(u-\varphi) + \frac{1}{2}\tilde{g}_r(2v-\varphi_r),$$

then, along the characteristic (106), we have

$$|\frac{dv}{dt}| \le \frac{6}{M_1}\|u\|_\infty^2\|v\|_\infty, \forall r > 2M_1.$$

Integrating this along the characteristic, we obtain

$$|\int_0^t \frac{dv}{ds}\, ds| \le \int_0^t |\frac{dv}{ds}|\, ds \le \frac{6}{M_1}\int_0^t \|u\|_\infty^2\|v\|_\infty\, ds,$$

$$|v(t)| \leq |v(0)| + \frac{6}{M_1}\int_0^t \|u\|_\infty^2 \|v\|_\infty \, ds.$$

By Gronwall's inequality

$$\|v\|_\infty \leq \|v(0)\|_\infty \exp\left(\frac{6}{M_1}\int_0^t \|u\|_\infty^2 \, ds\right), \forall r > 2M_1.$$

Using Lemma 17, $\|u(t,r)\|_\infty \lesssim e^{\frac{t}{M_1}}$, we get

$$\int_0^t \|u\|_\infty^2 \, ds \lesssim \frac{1}{2}M_1(-1 + e^{\frac{2t}{M_1}}),$$

then

$$\|v(1-\chi)\|_\infty \leq \|v\|_\infty \lesssim \|v_0\|_\infty \exp(-1 + e^{\frac{2t}{M_1}}).$$

□

Combining Lemmas 18, 17 and 16, we can get

$$\|u(t,r)\|_\infty \leq C_1 \exp(\frac{t}{M_1}),$$
$$\|u_r(t,r)\|_\infty \leq C_2 \exp\left(-1 + e^{\frac{2t}{M_1}}\right),$$

where C_1, C_2 only depend on the initial data. By the standard method, we can prove there is a unique solution $u \in C^1([0,\infty) \times [0,b])$. This finishes the proof of Theorem 3.

A.5 Lemma 19

Lemma 19. *Assume $u_h \in P^0$, u_h is a piecewise constant function and h is the mesh size, then*

$$\frac{1}{h}|\varphi_h(r+h) - \varphi_h(r)| \leq \|D^+u\|_\infty, \tag{109}$$
$$\frac{1}{r}(g_h(r) - \tilde{g}_h(r)) \leq \|g_{h,r}\|_\infty, \tag{110}$$
$$\frac{1}{h}|\tilde{g}_{h,r}(r+h) - \tilde{g}_{h,r}(r)| \leq 8\|D^+u\|_\infty^2, \tag{111}$$
$$\frac{1}{h}|\tilde{g}_r(r+h) - \tilde{g}_r(r)| \leq \frac{3}{r+h}\|g_r\|_\infty, \tag{112}$$
$$\frac{1}{h}|\tilde{g}_h(r+h) - \tilde{g}_h(r)| \leq 2(r+h)\|D^+u\|_\infty^2, \tag{113}$$
$$\frac{1}{h}|\tilde{g}_h(r+h) - \tilde{g}_h(r)| \leq \|g_{h,r}\|_\infty, \tag{114}$$
$$\frac{1}{h}|g_h(r+h) - g_h(r)| \leq 4(r+h)\|D^+u\|_\infty^2, \tag{115}$$
$$|\varphi_{h,r}| \leq \|D^+u\|_\infty, \tag{116}$$
$$|u_h - \varphi_h| \leq 2r\|D^+u\|_\infty, \tag{117}$$
$$|\tilde{g}_r| \leq 2r\|D^+u\|_\infty^2, \tag{118}$$
$$\|\tilde{g}_{h,r}\|_\infty \leq \|g_{h,r}\|_\infty \leq 4b\|D^+u\|_\infty^2. \tag{119}$$

Proof. In the first step, we prove the Eq(109). To simplify our expression, we denote $\varphi = \varphi_h$, $u(r) = u_h(r)$.

$$\begin{aligned}
\varphi(r+h) - \varphi(r) &= \frac{1}{r+h}\int_0^{r+h} u(s)\ ds - \frac{1}{r}\int_0^r u(s)\ ds \\
&= \frac{1}{r+h}\int_0^{r+h} u(s)\ ds - \frac{1}{r+h}\int_0^r u(s)\ ds + \frac{1}{r+h}\int_0^r u(s)\ ds - \frac{1}{r}\int_0^r u(s)\ ds \\
&= \frac{1}{r+h}\left(\int_r^{r+h} u(s)\ ds - \frac{h}{r}\int_0^r u(s)\ ds\right).
\end{aligned}$$

Denote

$$\bar{u}(r) := \frac{1}{h}\int_r^{r+h} u(s)\ ds,$$

then

$$\begin{aligned}
\frac{1}{h}\left(\varphi(r+h) - \varphi(r)\right) &= \frac{1}{r+h}\left(\frac{1}{h}\int_r^{r+h} u(s)\ ds - \frac{1}{r}\int_0^r u(s)\ ds\right) \\
&= \frac{1}{r+h}\left(\bar{u}(r) - \varphi(r)\right).
\end{aligned}$$

We decompose $r = ih + \varepsilon, 0 \le \varepsilon < h, i \in \mathbb{Z}$, then

$$\varphi(r) = \frac{1}{r}\int_0^r u(s)\ ds = \frac{1}{r}\left(\int_0^{\varepsilon} u(s)\ ds + \int_{\varepsilon}^r u(s)\ ds\right),$$

hence

$$\begin{aligned}
\varphi(r) &= \frac{1}{r}\left(\varepsilon u_1 + \sum_{j=0}^{i-1} h\frac{1}{h}\int_{\varepsilon+jh}^{\varepsilon+(j+1)h} u(s)ds\right) \\
&= \frac{1}{r}\left(\varepsilon u_1 + \sum_{j=0}^{i-1} h\bar{u}(\varepsilon+jh)\right),
\end{aligned} \tag{120}$$

where

$$\bar{u}(\varepsilon+jh) = \frac{h-\varepsilon}{h}u_{j+1} + \frac{\varepsilon}{h}u_{j+2},$$

this is because

$$\bar{u}(\varepsilon+jh) = \frac{1}{h}\int_{\varepsilon+jh}^{\varepsilon+(j+1)h} u(s)\ ds = \frac{1}{h}\left((h-\varepsilon)u_{j+1} + \varepsilon u_{j+2}\right),$$

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$$\bar{u}(r) = \frac{1}{r}\int_0^r \bar{u}(r)\ ds = \frac{1}{r}\left(\varepsilon\bar{u}(r) + \sum_{j=0}^{i-1} h\bar{u}(r)\right). \tag{121}$$

By (121)-(120), we have

$$\bar{u}(r) - \varphi(r) = \frac{1}{r}\left(\varepsilon(\bar{u}(r) - u_1) + \sum_{j=0}^{i-1} h(\bar{u}(r) - \bar{u}(\varepsilon+jh))\right),$$

where

$$\begin{aligned}
&\bar{u}(r) - \bar{u}(\varepsilon + jh) \\
&= \sum_{k=j}^{i-1} \bar{u}(\varepsilon + (k+1)h) - \bar{u}(\varepsilon + kh) \\
|\bar{u}(r) - \bar{u}(\varepsilon + jh)| &\leq \sum_{k=j}^{i-1} h\frac{1}{h}|\bar{u}(\varepsilon + (k+1)h) - \bar{u}(\varepsilon + kh)|,
\end{aligned}$$

since

$$\begin{aligned}
&\bar{u}(\varepsilon + (k+1)h) - \bar{u}(\varepsilon + kh) \\
&= \frac{h-\varepsilon}{h}u_{k+2} + \frac{\varepsilon}{h}u_{k+3} - \frac{h-\varepsilon}{h}u_{k+1} - \frac{\varepsilon}{h}u_{k+2} \\
&= \frac{h-\varepsilon}{h}(u_{k+2} - u_{k+1}) + \frac{\varepsilon}{h}(u_{k+3} - u_{k+2}),
\end{aligned}$$

then

$$|\bar{u}(\varepsilon + (k+1)h) - \bar{u}(\varepsilon + kh)| \leq h \max\left(|D^+u_{k+1}|, |D^+u_{k+2}|\right),$$

and

$$\begin{aligned}
&|\bar{u}(r) - \bar{u}(\varepsilon + jh)| \\
&\leq \sum_{k=j}^{i-1} h\|D^+u\|_\infty \\
&= (i-j)h\|D^+u\|_\infty. \qquad (122)
\end{aligned}$$

Since

$$\begin{aligned}
\bar{u}(r) - u_1 &= \bar{u}(\varepsilon) - u_1 + \sum_{k=1}^{i} \bar{u}(\varepsilon + kh) - \bar{u}(\varepsilon + (k-1)h) \\
&= \frac{h-\varepsilon}{h}u_1 + \frac{\varepsilon}{h}u_2 - u_1 + \sum_{k=1}^{i} \bar{u}(\varepsilon + kh) - \bar{u}(\varepsilon + (k-1)h) \\
&= \frac{\varepsilon}{h}(u_2 - u_1) + \sum_{k=1}^{i} \bar{u}(\varepsilon + kh) - \bar{u}(\varepsilon + (k-1)h),
\end{aligned}$$

then

$$\begin{aligned}
|\bar{u}(r) - u_1| &\leq \varepsilon\|D^+u\|_\infty + ih\|D^+u\|_\infty, \\
&= (\varepsilon + ih)\|D^+u\|_\infty, \\
&= r\|D^+u\|_\infty. \qquad (123)
\end{aligned}$$

Using Eq(122) and Eq(123), we have

$$\begin{aligned}
|\bar{u}(r) - \varphi(r)| &\le \frac{1}{r}\left(\varepsilon r\|D^+u\|_\infty + ih\|D^+u\|_\infty\right) \\
&\le \frac{1}{r}(\varepsilon r + ihr)\|D^+u\|_\infty, \\
&= r\|D^+u\|_\infty, \\
\frac{1}{r+h}|\bar{u}(r) - \varphi(r)| &\le \|D^+u\|_\infty.
\end{aligned}$$

Then we have

$$\frac{1}{h}|\varphi(r+h) - \varphi(r)| \le \|D^+u\|_\infty.$$

This finish the proof of (109).

Next, we prove Eq(110). Since $g_{h,r} = g_h\frac{1}{r}(u_h - \varphi_h)^2$, and u_h is a piecewise constant function, then $g_{h,r}$ is discontinuous. To simplify our expression, we denote $g := g_h, \tilde{g} := \tilde{g}_h$. We assume

$$r = ih + \varepsilon, i \in \mathbb{Z}, 0 \le \varepsilon < h.$$

Then, we have

$$\begin{aligned}
g(r) - \tilde{g}(r) &= g(r) - \frac{1}{r}\int_0^r g(s)\ ds \\
&= \frac{1}{r}\int_0^r g(r) - g(s)\ ds \\
&= \frac{1}{r}\int_0^r g(r) - g(r-\varepsilon) + \sum_{k=0}^{i-1} g(h(k+1)) - g(kh)\ ds \\
&\le \frac{1}{r}\int_0^r \varepsilon\|g_r\|_\infty + ih\|g_r\|_\infty\ ds \\
&= \frac{1}{r}\int_0^r r\|g_r\|_\infty\ ds.
\end{aligned}$$

Then,

$$\frac{1}{r}\left(g(r) - \tilde{g}(r)\right) \le \|g_r\|_\infty.$$

This finish the proof of Eq(110).

Next, we will prove Eq(115). Since $g_r = \frac{1}{r}g(u-\varphi)^2$ and using Eq(117), $|u - \varphi| \le 2r\|D^+u\|_\infty$, we have

$$|g_r| \le 4r\|D^+u\|_\infty^2, \tag{124}$$

then,

$$
\begin{aligned}
g(r+h)-g(r) &= \int_r^{r+h} g_r(s)\ ds \\
&\le \int_r^{r+h} 4s\|D^+u\|_\infty^2\ ds \\
&= 2h(2r+h)\|D^+u\|_\infty^2 \\
&\le 4h(r+h)\|D^+u\|_\infty^2.
\end{aligned}
$$

Finally, we have

$$
\frac{1}{h}\left(g(r+h)-g(r)\right) \le 4(r+h)\|D^+u\|_\infty^2.
$$

This finish the proof of Eq(115).

Next step, we will prove Eq(117),

$$
u(r)-\varphi(r) = \frac{1}{r}\int_0^r u(r)-u(s)\ ds.
$$

Assume $r \in (r_{q-\frac{1}{2}}, r_{q+\frac{1}{2}}]$, $s \in (r_{j-\frac{1}{2}}, r_{j+\frac{1}{2}}]$, $1 \le j \le q \le N, q, j \in \mathbb{Z}$,

$$
\begin{aligned}
u(r)-u(s) &= \sum_{i=j}^{q-1} u_{i+1}-u_i, \\
&= \sum_{i=j}^{q-1} h\frac{1}{h}(u_{i+1}-u_i), \\
|u(r)-u(s)| &\le \sum_{i=j}^{q-1} h|D^+u_i|, \\
&\le (q-j)h\|D^+u\|_\infty,
\end{aligned}
$$

then, we have

$$
\begin{aligned}
|u(r)-\varphi(r)| &\le \frac{1}{r}\int_0^r (q-j)h\|D^+u\|_\infty\ ds \\
&\le \frac{1}{r} rqh\|D^+u\|_\infty, \quad \text{(where } qh \ge r, \text{)} \\
&\le (r+h)\|D^+u\|_\infty.
\end{aligned}
$$

Then, if $r > h$,

$$
\begin{aligned}
\frac{1}{r}|u(r)-\varphi(r)| &\le \frac{r+h}{r}\|D^+u\|_\infty, \\
&\le 2\|D^+u\|_\infty.
\end{aligned}
$$

If $r \le h$, we have $\varphi(r) = u_1 = u$, then

$$\frac{1}{r}(u - \varphi) = 0,$$

so,

$$\frac{1}{r}|u - \varphi| \le 2\|D^+u\|_\infty.$$

Next, we will prove Eq(113) and Eq(114).

$$\begin{aligned}
&\tilde{g}(r+h) - \tilde{g}(r) \\
&= \int_r^{r+h} \tilde{g}_r(s)\ ds \\
&= \int_r^{r+h} \frac{1}{s}\left(g(s) - \tilde{g}(s)\right)\ ds \\
&\le \int_r^{r+h} \frac{1}{s}\left(g(s) - g(0)\right)\ ds,
\end{aligned}$$

where we use

$$\begin{aligned}
\tilde{g}(s) &= \frac{1}{s}\int_0^s g(\theta)\ d\theta \\
&\ge \frac{1}{s}\int_0^s g(0) d\theta, \text{ since } g_r \ge 0, \\
&\ge g(0).
\end{aligned}$$

Then

$$\begin{aligned}
&\tilde{g}(r+h) - \tilde{g}(r) \\
&\le \int_r^{r+h} \frac{1}{s}\left(g(s) - g(0)\right)\ ds \\
&\le \int_r^{r+h} \frac{1}{s}\int_0^s g_r(\theta)\ d\theta \le \|g_r\|_\infty h
\end{aligned} \tag{125}$$

$$\begin{aligned}
&\tilde{g}(r+h) - \tilde{g}(r) \\
&\le \int_r^{r+h} \frac{1}{s}\left(g(s) - g(0)\right)\ ds \\
&\le \int_r^{r+h} \frac{1}{s}\int_0^s g_r(\theta)\ d\theta \\
&\le \int_r^{r+h} \frac{1}{s}\int_0^s 4\theta\|D^+u\|_\infty^2\ d\theta\ ds \\
&= \int_r^{r+h} 2s\|D^+u\|_\infty^2\ ds \\
&\le 2h\|D^+u\|_\infty(r+h).
\end{aligned}$$

Hence

$$\frac{1}{h}\left(\tilde{g}(r+h)-\tilde{g}(r)\right) \leq 2\|D^+u\|_\infty (r+h).$$

Next, we will prove (118) and (119). Since

$$r\tilde{g}_r(r) = g(r) - \tilde{g}(r) = \frac{1}{r}\int_0^r g(r) - g(s)\ ds.$$

Assume $r \in (r_{q-\frac{1}{2}}, r_{q+\frac{1}{2}}], s \in (r_{j-\frac{1}{2}}, r_{j+\frac{1}{2}}], 1 \leq j \leq q \leq N, q, j \in \mathbb{Z}$, since

$$\begin{aligned} g(r) - g(s) &= g(r) - g(r_{q-\frac{1}{2}}) + \sum_{k=j+1}^{q-1} g(r_{k+\frac{1}{2}}) - g(r_{k-\frac{1}{2}}) + g(r_{j+\frac{1}{2}}) - g(s) \\ &\leq g_r(\theta)(r - r_{q-\frac{1}{2}}) + \sum_{k=j+1}^{q-1} g_r(\theta_k)h + (r_{j+\frac{1}{2}} - s)g_r(\theta_1) \\ &\leq 4r(r-s)\|D^+u\|_\infty^2, \end{aligned}$$

then

$$\begin{aligned} r\tilde{g}_r &\leq \frac{1}{r}\int_0^r g(r) - g(s)\ ds \\ &\leq \frac{1}{r}\|D^+u\|_\infty^2 \int_0^r 4r(r-s)\ ds \\ &= 2r^2\|D^+u\|_\infty^2, \\ \tilde{g}_r &\leq 2r\|D^+u\|_\infty^2. \end{aligned}$$

Moreover,

$$\begin{aligned} r\tilde{g}_r &\leq \frac{1}{r}\int_0^r g(r) - g(s)\ ds \\ &\leq \frac{1}{r}\|g_r\|_\infty \int_0^r r - s\ ds \\ &\leq r\|g_r\|_\infty, \\ \|\tilde{g}_r\|_\infty &\leq \|g_r\|_\infty. \end{aligned}$$

Using (124), we have

$$\|\tilde{g}_r\|_\infty \leq \|g_r\|_\infty \leq 4b\|D^+u\|_\infty^2.$$

Finally, we prove Eq(111),

$$\begin{aligned} \tilde{g}_r(r+h) - \tilde{g}_r(r) &= \frac{1}{r+h}\left(g(r+h) - \tilde{g}(r+h)\right) - \frac{1}{r}\left(g(r) - \tilde{g}(r)\right), \\ &= \frac{1}{r+h}\left(g(r+h) - g(r) + \tilde{g}(r) - \tilde{g}(r+h)\right) - \frac{h}{r(r+h)}\left(g(r) - \tilde{g}(r)\right), \end{aligned}$$

where

$$
\begin{aligned}
\frac{1}{r+h}\left(g(r+h)-g(r)\right) &\le 4h\|D^+u\|_\infty^2,\\
\frac{1}{r+h}\left(\tilde{g}(r+h)-\tilde{g}(r)\right) &\le 2h\|D^+u\|_\infty^2,\\
\frac{1}{r+h}\left(g(r+h)-g(r)\right) &\le \frac{h}{r+h}\|g_r\|_\infty,\\
\frac{1}{r+h}\left(\tilde{g}(r+h)-\tilde{g}(r)\right) &\le \frac{h}{r+h}\|g_r\|_\infty.
\end{aligned}
$$

Since $|\tilde{g}_r| \le 2r\|D^+u\|_\infty^2$, then

$$\frac{h}{(r+h)r}\left(g(r)-\tilde{g}(r)\right) = \frac{h}{r+h}\tilde{g}_r \le 2h\|D^+u\|_\infty^2.$$

So

$$\frac{1}{h}|\tilde{g}_r(r+h)-\tilde{g}_r(r)| \le 8\|D^+u\|_\infty^2.$$

Similarly, we have

$$\frac{1}{h}|\tilde{g}_r(r+h)-\tilde{g}_r(r)| \le \frac{3}{r+h}\|g_r\|_\infty.$$

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