L^2-stability of explicit second order Runge-Kutta discontinuous Galerkin method for nonlinear conservation laws*

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Abstract

We prove L^2-stability of an explicit second order Runge-Kutta discontinuous Galerkin (RKDG2) method for two classes of nonlinear convex scalar conservation laws in one space dimension under the Courant-Friedrichs-Lewy (CFL) condition $\Delta t \sim \Delta x$. We only consider uniform mesh and periodic boundary conditions.

1 Introduction

In this paper, we consider the nonlinear scalar conservation law in one space dimension (with I being any open interval)

$$u_t + f(u)_x = 0, \quad x \in I, \quad t \geq 0, \tag{1}$$

subject to periodic boundary conditions, which possesses the property that the L^2-norm of the entropy solution $u(x,t)$ is nonincreasing through time (see, e.g. [5]):

$$\|u(\cdot,t_2)\|_{L^2(I)} \leq \|u(\cdot,t_1)\|_{L^2(I)}, \quad \forall t_2 \geq t_1 \geq 0.$$

A numerical method for solving (1) is generally called L^2-*stable* if this structure is preserved, that is, if it produces a numerical solution $\tilde{u}^n(x) = \tilde{u}(x,t_n)$ (which approximates u at time $t_n = n\Delta t$, $n \geq 0$) satisfying:

$$\|\tilde{u}^{n+1}\|_{L^2(I)} \leq \|\tilde{u}^n\|_{L^2(I)}, \quad \forall n \geq 0, \tag{2}$$

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with or without the standard Courant-Friedrichs-Lewy (CFL) condition

$$C \cdot \max_{x \in I} |f'(\tilde{u}^n(x))| \frac{\Delta t}{\Delta x} \leq 1, \tag{3}$$

where $C > 0$ is a constant. Such numerical schemes are particularly desirable since they allow us to compute the numerical solution for an arbitrary long time, among which the most notable one is the discontinuous Galerkin (DG) method.

The DG method for scalar conservation laws (1) is a semi-discrete framework for spatial discretization established by Cockburn and Shu [2] in 1989, where it was combined with Runge-Kutta method for time discretization, resulting in the fully-discrete scheme known as the Runge-Kutta discontinuous Galerkin (RKDG) method. In 1994, Jiang and Shu [7] proved that for arbitrary high-order DG method and any nonlinear $f(u)$, the semi-discrete version of (2) holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \tilde{u}(\cdot,t) \|_{L^2(I)} \leq 0, \quad \forall t \geq 0,$$

which immediately implies (2) for high-order RKDG with (implicit) algebraically stable Runge–Kutta methods [1]. However, implicit time-marching schemes are more computationally expensive than explicit ones. On the other hand, works on proving (2) for explicit RKDG methods with linear $f(u) = c \cdot u$ and CFL condition (3) are abundant in the literature (see [10] and the references therein), but most applications deal with nonlinear $f(u)$ instead of linear ones, and it remains an open problem whether (2) holds for general nonlinear $f(u)$ and high-order explicit RKDG methods under the standard CFL condition (3).

In this paper, we attempt to tackle this open problem by showing (2) for two general classes of convex nonlinear $f(u)$ and a second order RKDG method (RKDG2) under the CFL condition

$$C_f \left(\min_{x \in I} \{ \tilde{u}^n(x) \}, \max_{x \in I} \{ \tilde{u}^n(x) \} \right) \frac{\Delta t}{\Delta x} \leq 1,$$

where $C_f(\cdot,\cdot) \geq 0$ is a nonnegative continuous function on $\mathbb{R}^2$ depending on f. See Theorems 2 and 3. In particular, the RKDG2 scheme is obtained by discretizing the space dimension using DG method for 1D P^1 element, and using the second order strong stability preserving Runge-Kutta method [9, 6] (Heun's method) for explicit time marching. For simplicity, only uniform mesh will be considered. We assume f can be extended to an entire function on $\mathbb{C}$ and satisfies one of the following two hypotheses:

(H1) $f^{2n}(0) \geq 0$ and $f(0) = f^{(2n-1)}(0) = 0$ for all $n \geq 1$. (even and convex)

(H2) $f''(u) > 0$ on $\mathbb{R}$. (strictly convex)

2 The RKDG2 scheme

In what follows, we change the notation of the numerical solution from $\tilde{u}$ to u for simplicity and define the semi-discrete DG method following [2]: Let the interval I be divided into N disjoint subintervals of length Δx, labeled $I_1, I_2, \cdots, I_N$, $I = \cup_{i=1}^N I_i$, and the left and right end points of I_j

be denoted by $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$ respectively. For any function $v(x)$ that is continuous in the interior of I_j, define $v^+_{j-\frac{1}{2}}$ and $v^-_{j+\frac{1}{2}}$ to be the right and left limit values of v at the points $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$ respectively. Define

$$V := \{v \in L^2(I) : v\big|_{I_j} \in P^1(I_j), 1 \le j \le N\}$$

as the space of all piecewise linear functions on I that are linear on each cell I_j. The semi-discrete DG method for 1D P^1 element (with periodic boundary condition) can be stated as follows: Seek $u \in V$ such that

$$\int_{I_j} u_t v \,\mathrm{d}x - \int_{I_j} u v_x \,\mathrm{d}x + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \tag{4}$$

holds for all $v \in V$ and all $j = 1, 2, \cdots, N$, where the numerical flux $\hat{f}_{j+\frac{1}{2}}$ is taken to be the global Lax-Friedrich flux:

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) := \frac{f(u^+_{j+\frac{1}{2}}) + f(u^-_{j+\frac{1}{2}})}{2} - \frac{\alpha}{2}(u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}). \tag{5}$$

Set

$$u_{\max} := \max_j u^{\pm}_{j+\frac{1}{2}}, \quad u_{\min} := \min_j u^{\pm}_{j+\frac{1}{2}}, \quad \alpha := 2 \max_{u_{\min} \le u \le u_{\max}} |f'(u)|$$

so that

$$\begin{aligned}
&\min_{u_{\min} \le u \le u_{\max}} \frac{\partial \hat{f}}{\partial u}(u,v) = \min_{u_{\min} \le u \le u_{\max}} \frac{1}{2} f'(u) + \frac{\alpha}{2} \ge \frac{\alpha}{4}, \\
&\min_{u_{\min} \le v \le u_{\max}} \left(-\frac{\partial \hat{f}}{\partial v}(u,v)\right) = \min_{u_{\min} \le v \le u_{\max}} \left(-\frac{1}{2} f'(v)\right) + \frac{\alpha}{2} \ge \frac{\alpha}{4}.
\end{aligned} \tag{6}$$

The DG scheme (4) can be written as the following ODE system:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x) = \frac{1}{\Delta x} F[u](x) \tag{7}$$

where $F[u](x)$ is a piecewise linear (possibly discontinuous) function of $x \in I$ whose coefficients depend on u. In particular,

$$\begin{aligned}
&F^+_{j+\frac{1}{2}}[u] := F[u](x^+_{j+\frac{1}{2}}) = 4\hat{f}_{j+\frac{1}{2}} + 2\hat{f}_{j+\frac{3}{2}} - 6\bar{f}(u^+_{j+\frac{1}{2}}, u^-_{j+\frac{3}{2}}) \\
&F^-_{j+\frac{1}{2}}[u] := F[u](x^-_{j+\frac{1}{2}}) = -2\hat{f}_{j-\frac{1}{2}} - 4\hat{f}_{j+\frac{1}{2}} + 6\bar{f}(u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}) \\
&F[u](\beta x_{j+\frac{1}{2}} + (1-\beta) x_{j-\frac{1}{2}}) = \beta F[u](x^-_{j+\frac{1}{2}}) + (1-\beta) F[u](x^+_{j-\frac{1}{2}}), \quad 0 < \beta < 1,
\end{aligned} \tag{8}$$

where

$$\bar{f}(u,v) := \begin{cases} \frac{1}{u-v}\int_v^u f(t)\,\mathrm{d}t & \text{if } u \ne v, \\ f(u) & \text{if } u = v. \end{cases}$$

Define

$$\lambda := \frac{\Delta t}{\Delta x}.$$

The fully-discrete RKDG2 scheme is obtained by using the second order Heun's method to solve the above ODE system (7) numerically:

$$
\begin{aligned}
y_1(x) &= u(x) \\
y_2(x) &= u(x) + \lambda F[u](x) \\
u^{n+1}(x) &= u(x) + \frac{\lambda}{2} F[y_1](x) + \frac{\lambda}{2} F[y_2](x).
\end{aligned}
\tag{RKDG2}
$$

In the above notation, u (without the upper index) represents the solution at time level n. We assume that the same $\alpha = 2\max_{u_{\min} \le u \le u_{\max}} |f'(u)|$ is used in both steps of Heun's method. With the scheme properly defined, we now proceed to some preparations for the proof.

3 Preparations for the proof

Set:

$$
\sigma := u_{\max} - u_{\min}, \quad M_n := \max_{u_{\min} \le u \le u_{\max}} |f^{(n)}(u)|, \quad \forall n \ge 1; \quad P(z) := \sum_{k=1}^{\infty} \frac{M_{k+1}}{k!} (\sigma z)^k.
$$

Define the inner product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ as

$$
\langle u, v \rangle := \int_I u(x) v(x) \, \mathrm{d}x, \quad \|u\|^2 := \int_I u(x)^2 \, \mathrm{d}x.
$$

Define the jumps of u at cell interfaces:

$$
[\![u]\!]_{j+\frac{1}{2}} := u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}.
$$

We have assumed that f can be extended to an entire function on $\mathbb{C}$. Let its extension also be denoted by f. It follows from Cauchy's integral formula [4] that

$$
\left| \frac{M_{k+1}}{k!} \right| \le \frac{R \max_{|z|=R} |f'(z)|}{(R - \|u\|_\infty)^{k+1}}, \quad \forall R > \|u\|_\infty
$$

By picking R large enough in the above, one can show that $P(z)$ is also entire.

Lemma 1. *If*

$$
|x| \cdot \left(2M_1 + 8 \max_{|z| = 2\|u\|_\infty} |f'(z)| \right) \le 1,
$$

then

$$
|P(xM_1)| \le \frac{M_1}{2}.
$$

Proof. The statement is true if $\|u\|_\infty = 0$, since in which case we have $0 \le M_1/2$. Thus, suppose $\|u\|_\infty > 0$. By Cauchy's integral formula [4]: for all $R > \|u\|_\infty$,

$$
\left| \frac{M_{k+1}}{k!} \right| \le \frac{R \max_{|z|=R} |f'(z)|}{(R - \|u\|_\infty)^{k+1}}.
$$

Pick $R = 2\|u\|_\infty$ in the above and use $\sigma \le 2\|u\|_\infty$ to obtain for all $|x| \le \frac{1}{2M_1 + 8\max_{|z|=2\|u\|_\infty} |f'(z)|}$:

$$\begin{aligned}
|P(xM_1)| &\le \sum_{k=1}^{\infty} 2\|u\|_\infty^{-k} \max_{|z|=2\|u\|_\infty} |f'(z)| \cdot (2\|u\|_\infty |x| M_1)^k \\
&= \left(2 \max_{|z|=2\|u\|_\infty} |f'(z)|\right) \sum_{k=1}^{\infty} (2|x|M_1)^k \\
&= \left(2 \max_{|z|=2\|u\|_\infty} |f'(z)|\right) \frac{2|x|M_1}{1-2|x|M_1} \le \frac{M_1}{2}.
\end{aligned}$$

■

Lemma 2.

$$\begin{aligned}
|3\bar{f}(a,b) - f(a) - 2f(b)| &\le \frac{5}{6}|b-a| \max |f'(t)| \\
|3\bar{f}(a,b) - 2f(a) - f(b)| &\le \frac{5}{6}|b-a| \max |f'(t)|,
\end{aligned}$$

where the maximum is taken over all t between a and b.

Proof. If $a = b$ then both sides of the above inequality are zero. Since $\bar{f}(a,b) = \bar{f}(b,a)$, we can assume without loss of generality that $b > a$. Then by Peano Kernel Theorem [3] we obtain:

$$\begin{aligned}
3\int_a^b f(t)\,\mathrm{d}t - (f(a) + 2f(b))(b-a) &= \int_a^b f'(t)(2a+b-3t)\,\mathrm{d}t \\
3\int_a^b f(t)\,\mathrm{d}t - (2f(a) + f(b))(b-a) &= \int_a^b f'(t)(a+2b-3t)\,\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|3\int_a^b f(t)\,\mathrm{d}t - (f(a) + 2f(b))(b-a)\right| &\le \max|f'(t)| \int_a^b |2a+b-3t|\,\mathrm{d}t = \frac{5}{6}|b-a|^2 \max|f'(t)| \\
\left|3\int_a^b f(t)\,\mathrm{d}t - (2f(a) + f(b))(b-a)\right| &\le \max|f'(t)| \int_a^b |a+2b-3t|\,\mathrm{d}t = \frac{5}{6}|b-a|^2 \max|f'(t)|.
\end{aligned}$$

Dividing both sides by $|b-a|$ and the proof is complete. ■

Lemma 3.

$$|F^{\pm}_{j+\frac{1}{2}}[u]| \le \frac{32}{3}\sigma M_1$$

Proof. Use Lemma 2 and Mean Value Theorem:

$$\begin{aligned}
|F^{+}_{j+\frac{1}{2}}[u]| =& \left|6\bar{f}(u^{+}_{j+\frac{1}{2}}, u^{-}_{j+\frac{3}{2}}) - 4\left(\frac{f(u^{+}_{j+\frac{1}{2}}) + f(u^{-}_{j+\frac{1}{2}})}{2}\right) - 2\left(\frac{f(u^{+}_{j+\frac{3}{2}}) + f(u^{-}_{j+\frac{3}{2}})}{2}\right) + 2\alpha[\![u]\!]_{j+\frac{1}{2}} + \alpha[\![u]\!]_{j+\frac{3}{2}}\right| \\
\le& \left|6\bar{f}(u^{+}_{j+\frac{1}{2}}, u^{-}_{j+\frac{3}{2}}) - 4f(u^{+}_{j+\frac{1}{2}}) - 2f(u^{-}_{j+\frac{3}{2}})\right| + 2\left|f(u^{+}_{j+\frac{1}{2}}) - f(u^{-}_{j+\frac{1}{2}})\right| + \left|f(u^{+}_{j+\frac{3}{2}}) - f(u^{-}_{j+\frac{3}{2}})\right| \\
&+ 2\alpha|[\![u]\!]_{j+\frac{1}{2}}| + \alpha|[\![u]\!]_{j+\frac{3}{2}}| \\
\le& \frac{5}{3}\left|u^{+}_{j+\frac{1}{2}} - u^{-}_{j+\frac{3}{2}}\right| M_1 + 2\sigma M_1 + \sigma M_1 + 4\sigma M_1 + 2\sigma M_1 \le \frac{32}{3}\sigma M_1,
\end{aligned}$$

where in the last line Lemma 2 is used. The proof for $|F^{-}_{j+\frac{1}{2}}[u]|$ is the same. ■

Proposition 1. *Under the CFL condition*

$$\frac{64}{3}\left(M_1+4\max_{|z|=2\|u\|_\infty}|f'(z)|\right)\lambda \le 1, \tag{CFL1}$$

we have

$$\|u^{n+1}\|^2-\|u\|^2 \le -\frac{\alpha\Delta x}{8}\lambda\big(\sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2+\sum_j |[\![y_2]\!]_{j+\frac{1}{2}}|^2\big)+\frac{\lambda^2}{4}\|F[y_2]-F[y_1]\|^2. \tag{9}$$

Proof. Following the computation in [8], we have

$$\|u^{n+1}\|^2-\|u\|^2=\lambda\big(\langle y_1,F[y_1]\rangle+\langle y_2,F[y_2]\rangle\big)+\frac{\lambda^2}{4}\|F[y_2]-F[y_1]\|^2. \tag{10}$$

In [7], it was shown that

$$\langle u,\frac{1}{\Delta x}F[u]\rangle=\sum_j(u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})\hat{f}_{j+\frac{1}{2}}-\int_{u^-_{j+\frac{1}{2}}}^{u^+_{j+\frac{1}{2}}}f(t)\,\mathrm{d}t.$$

Direct computation yields

$$\begin{aligned}
\langle y_1,\frac{1}{\Delta x}F[y_1]\rangle &= \sum_j(u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})\hat{f}_{j+\frac{1}{2}}-\int_{u^-_{j+\frac{1}{2}}}^{u^+_{j+\frac{1}{2}}}f(t)\,\mathrm{d}t\\
&= \sum_j(u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})\hat{f}_{j+\frac{1}{2}}-(u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})f(\tilde{u}_{j+\frac{1}{2}}) \quad (\text{where } \tilde{u}_{j+\frac{1}{2}} \text{ is between } u^-_{j+\frac{1}{2}} \text{ and } u^+_{j+\frac{1}{2}})\\
&= \sum_j(u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})(\hat{f}(u^-_{j+\frac{1}{2}},u^+_{j+\frac{1}{2}})-f(\tilde{u}_{j+\frac{1}{2}}))\\
&= \sum_j(u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})(\hat{f}(u^-_{j+\frac{1}{2}},u^+_{j+\frac{1}{2}})-\hat{f}(u^-_{j+\frac{1}{2}},\tilde{u}_{j+\frac{1}{2}})+\hat{f}(u^-_{j+\frac{1}{2}},\tilde{u}_{j+\frac{1}{2}})-\hat{f}(\tilde{u}_{j+\frac{1}{2}},\tilde{u}_{j+\frac{1}{2}}))\\
&= \sum_j(u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})\left(\frac{\partial\hat{f}}{\partial v}(u^-_{j+\frac{1}{2}},\xi_1)(u^+_{j+\frac{1}{2}}-\tilde{u}_{j+\frac{1}{2}})-\frac{\partial\hat{f}}{\partial u}(\xi_2,\tilde{u}_{j+\frac{1}{2}})(\tilde{u}_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}})\right)\\
&\quad (\text{where } \xi_1 \text{ is between } u^+_{j+\frac{1}{2}} \text{ and } \tilde{u}_{j+\frac{1}{2}},\ \xi_2 \text{ is between } \tilde{u}_{j+\frac{1}{2}} \text{ and } u^-_{j+\frac{1}{2}})\\
&\le -\frac{\alpha}{4}\sum_j|[\![u]\!]_{j+\frac{1}{2}}|^2,
\end{aligned}\tag{11}$$

where inequalities (6) have been used in the last step.

It would not be as simple to estimate $\langle y_2,\frac{1}{\Delta x}F[y_2]\rangle$ as above, since it is assumed that the same α is used in both steps of Heun's method, while $\alpha=2\max|f'(y_1)|$ is not the maximum over y_2. The resolution is to impose CFL condition (CFL1) on λ, so that for any $0\le s\le\lambda$:

$$\begin{aligned}
\left|f'(u^\pm_{j+\frac{1}{2}}+sF^\pm_{j+\frac{1}{2}}[u])-f'(u^\pm_{j+\frac{1}{2}})\right| &= \left|\sum_{k=1}^{\infty}\frac{f^{(k+1)}(u^\pm_{j+\frac{1}{2}})}{k!}\left(sF^\pm_{j+\frac{1}{2}}[u]\right)^k\right|\\
&\le \sum_{k=1}^{\infty}\frac{M_{k+1}}{k!}\left(\lambda\frac{32}{3}\sigma M_1\right)^k \quad (\text{by Lemma 3})\\
&= P(\lambda\frac{32}{3}M_1)\le\frac{\alpha}{4} \quad (\text{by Lemma 1 and (CFL1)}).
\end{aligned}\tag{12}$$

Let J stand for the closed interval $[\min_j y^{\pm}_{2,j+1/2}, \max_j y^{\pm}_{2,j+1/2}]$. It follows from (12) that

$$\max_{u\in J}|f'(u)| \le \max_{u_{\min}\le u\le u_{\max}}|f'(u)| + \frac{\alpha}{4} \le \frac{3\alpha}{4},$$

so that

$$\begin{aligned}
\min_{u\in J}\frac{\partial \hat{f}}{\partial u}(u,v) &= \min_{u\in J}\frac{1}{2}f'(u) + \frac{\alpha}{2} \ge \frac{\alpha}{8},\\
\min_{v\in J}\left(-\frac{\partial \hat{f}}{\partial v}(u,v)\right) &= \min_{v\in J}\left(-\frac{1}{2}f'(v)\right) + \frac{\alpha}{2} \ge \frac{\alpha}{8}.
\end{aligned}$$

The same proof for $\langle y_1, \frac{1}{\Delta x}F[y_1]\rangle$ now works for $\langle y_2, \frac{1}{\Delta x}F[y_2]\rangle$ with $\alpha/4$ replaced by $\alpha/8$, which gives us

$$\langle y_2, \frac{1}{\Delta x}F[y_2]\rangle \le -\frac{\alpha}{8}\sum_j |[\![y_2]\!]_{j+\frac{1}{2}}|^2. \tag{13}$$

Applying (11) and (13) to inequality (10) yields:

$$\begin{aligned}
\|u^{n+1}\|^2 - \|u\|^2 &= \lambda(\langle y_1, F[y_1]\rangle + \langle y_2, F[y_2]\rangle) + \frac{\lambda^2}{4}||F[y_2]-F[y_1]||^2\\
&\le -\frac{\alpha\Delta x}{8}\lambda(\sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2 + \sum_j |[\![y_2]\!]_{j+\frac{1}{2}}|^2) + \frac{\lambda^2}{4}\|F[y_2]-F[y_1]\|^2.
\end{aligned}$$

■

In Proposition 1, let us separate the right hand side of (9) into

$$\begin{aligned}
\textbf{First Term} &:= -\frac{\alpha\Delta x}{8}\lambda(\sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2 + \sum_j |[\![y_2]\!]_{j+\frac{1}{2}}|^2),\\
\textbf{Second Term} &:= \frac{\lambda^2}{4}\|F[y_2]-F[y_1]\|^2.
\end{aligned}$$

In order to show the right hand side of (9) is non-positive, we need to prove, roughly speaking, that the First Term is large in absolute value (Section 4) while the Second Term is small (Section 5). In [8] it was shown that, for a general Runge-Kutta scheme of order p, the Second Term is of order $\mathscr{O}(\lambda^{p+1})$, but the constant in front of $\mathscr{O}(\lambda^{p+1})$ would still depend on the dimension N, which implicitly depends on Δx and makes the estimate very difficult. In order to eliminate this dependency on N, we propose a novel cyclic inequality in Theorem 1 (proved in Appendix B), then use it to compare the First Term with the Second Term (Section 6), eventually proving the L^2-stability (2).

4 Estimating the First Term

Define the average of jumps on the interfaces:

$$\bar{u}_{j+\frac{1}{2}} := \frac{u^+_{j+\frac{1}{2}} + u^-_{j+\frac{1}{2}}}{2},$$

and also define

$$
\begin{aligned}
&D_{j+\frac{1}{2}} := 3\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}) - f(\bar{u}_{j-\frac{1}{2}}) - 4f(\bar{u}_{j+\frac{1}{2}}) - f(\bar{u}_{j+\frac{3}{2}}) + 3\bar{f}(\bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{3}{2}}) \\
&D^2 := \sum_j D^2_{j+\frac{1}{2}}, \quad E^2 := \frac{1}{\lambda^2}\sum_j |[\![u]\!]_{j+\frac{1}{2}}|^2.
\end{aligned}
$$

The following Lemma provides bounds for all partial derivatives of $\bar{f}(u,v)$:

Lemma 4. *For all $n \geq 0$ and $i + j = n$, we have*

$$
\frac{\partial^n \bar{f}}{\partial u^i \partial v^j}(u,v) = \frac{i!j!}{(n+1)!} f^{(n)}(\xi),
$$

where ξ is between u and v. As a direct consequence, we have

$$
\left|\frac{\partial^n \bar{f}}{\partial u^i \partial v^j}(u,v)\right| \leq \frac{i!j!}{(n+1)!} M_n.
$$

if $u_{\min} \leq u, v \leq u_{\max}$.

Proof. Without loss of generality we can assume $u > v$. Fix such u and v. By direct computation, we have

$$
\begin{aligned}
\frac{\partial^n \bar{f}}{\partial u^i \partial v^j}(u,v) = \frac{(-1)^i n!}{(u-v)^{n+1}} \Bigg[\int_v^u f(t)\,\mathrm{d}t + \sum_{k=1}^{i} (-1)^k \frac{(n-k)!}{n!} \binom{i}{k} (u-v)^k f^{(k-1)}(u) + \\
+ \sum_{k=1}^{j} (-1)^k \frac{(n-k)!}{n!} \binom{j}{k} (u-v)^k f^{(k-1)}(v) \Bigg]
\end{aligned}
$$

If $i = n, j = 0$, we have

$$
\begin{aligned}
\frac{\partial^n \bar{f}}{\partial u^n}(u,v) &= \frac{(-1)^n n!}{(u-v)^{n+1}} \left[\int_v^u f(t)\,\mathrm{d}t + \sum_{k=1}^{n} \frac{(v-u)^k}{k!} f^{(k-1)}(u) \right] \\
&= \frac{(-1)^n n!}{(u-v)^{n+1}} \left[-\sum_{k=1}^{n} \frac{(v-u)^k}{k!} f^{(k-1)}(u) - \frac{(v-u)^{n+1}}{(n+1)!} f^{(n)}(\xi) + \sum_{k=1}^{n} \frac{(v-u)^k}{k!} f^{(k-1)}(u) \right] \\
&= \frac{1}{n+1} f^{(n)}(\xi).
\end{aligned}
$$

where in the second line the Taylor expansion has been used:

$$
-\int_v^u f(t)\,\mathrm{d}t = \int_u^v f(t)\,\mathrm{d}t = \sum_{k=1}^{n} \frac{(v-u)^k}{k!} f^{(k-1)}(u) + \frac{(v-u)^{n+1}}{(n+1)!} f^{(n)}(\xi).
$$

The proof is the same for the case $i = 0, j = n$. It remains to show for the case when $n \geq 2, i \geq 1, j \geq 1$ (so that $i \leq n-1, j \leq n-1$ since $i + j = n$). Fix $u > v$, define the following linear operator $L : C^{n-2}([v,u]) \to \mathbb{R}$ as

$$
\begin{aligned}
L[f] := \frac{(-1)^i n!}{(u-v)^{n+1}} \Bigg[\int_v^u f(t)\,\mathrm{d}t + \sum_{k=1}^{i} (-1)^k \frac{(n-k)!}{n!} \binom{i}{k} (u-v)^k f^{(k-1)}(u) + \\
+ \sum_{k=1}^{j} (-1)^k \frac{(n-k)!}{n!} \binom{j}{k} (u-v)^k f^{(k-1)}(v) \Bigg].
\end{aligned}
$$

It is clear that $L[f]=0$ for all $f\in\mathbb{P}^{n-1}$. If $f\in C^n([v,u])$, By Peano Kernel Theorem [3] we have

$$L[f]=\frac{1}{(n-1)!}\int_v^u K(\theta)f^{(n)}(\theta)\,\mathrm{d}\theta.$$

where

$$K(\theta)=L[t\to(t-\theta)_+^{n-1}]\quad\text{for }\theta\in[v,u].$$

Furthermore, if $K(\theta)$ does not change sign in $[v,u]$, we have

$$L[f]=\frac{1}{(n-1)!}\left[\int_v^u K(\theta)\,\mathrm{d}\theta\right]f^{(n)}(\xi).$$

Now we compute the expression of $K(\theta)$ (note that $v\le\theta\le u$):

$$\begin{aligned}K(\theta)&=L[t\to(t-\theta)_+^{n-1}]\\&=\frac{(-1)^i n!}{(u-v)^{n+1}}\left[\int_v^u(t-\theta)_+^{n-1}\,\mathrm{d}t+\frac{1}{n}\sum_{k=1}^{i}(-1)^k\binom{i}{k}(u-v)^k(u-\theta)_+^{n-k}\right]\\&=\frac{(n-1)!}{(u-v)^{n+1}}(u-\theta)^{n-i}(\theta-v)^i\ge 0\end{aligned}$$

so that $K(\theta)$ does not change sign in $[v,u]$. Moreover,

$$\begin{aligned}\int_v^u K(\theta)\,\mathrm{d}\theta&=\frac{(n-1)!}{(u-v)^{n+1}}\int_v^u(u-\theta)^{n-i}(\theta-v)^i\,\mathrm{d}\theta\\&=(n-1)!\frac{i!(n-i)!}{(n+1)!}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}L[f]&=\frac{1}{(n-1)!}\left[\int_v^u K(\theta)\,\mathrm{d}\theta\right]f^{(n)}(\xi)\\&=\frac{i!(n-i)!}{(n+1)!}f^{(n)}(\xi)\\&=\frac{i!j!}{(n+1)!}f^{(n)}(\xi).\end{aligned}$$

■

We are now ready to compute the time derivatives of jumps:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\![u]\!]_{j+\frac12}=&\frac{\mathrm{d}}{\mathrm{d}t}u^+_{j+\frac12}-\frac{\mathrm{d}}{\mathrm{d}t}u^-_{j+\frac12}\\=&\frac{4\hat f_{j+\frac12}+2\hat f_{j+\frac32}-6\bar f(u^+_{j+\frac12},u^-_{j+\frac32})}{\Delta x}+\frac{2\hat f_{j-\frac12}+4\hat f_{j+\frac12}-6\bar f(u^+_{j-\frac12},u^-_{j+\frac12})}{\Delta x}\\=&-2\frac{3\bar f(\bar u_{j-\frac12},\bar u_{j+\frac12})-f(\bar u_{j-\frac12})-4f(\bar u_{j+\frac12})-f(\bar u_{j+\frac32})+3\bar f(\bar u_{j+\frac12},\bar u_{j+\frac32})}{\Delta x}+R_{j+\frac12}.\end{aligned}$$

where the remainder is obtained by using Mean Value Theorem. By Lemma 4, it satisfies

$$|R_{j+\frac{1}{2}}|^2 \le C\frac{M_1^2}{\Delta x^2}\left(|[\![u]\!]_{j-\frac{1}{2}}|^2+|[\![u]\!]_{j+\frac{1}{2}}|^2+|[\![u]\!]_{j+\frac{3}{2}}|^2\right)$$

where $C > 0$ represents a constant independent of u and f, which can vary in the subsequent proofs.

Rewrite Jensen's inequality in the following way:

$$(a+b_1+\cdots+b_{k-1})^2 \ge \frac{1}{k}a^2-b_1^2-\cdots-b_{k-1}^2. \tag{14}$$

Using it with $k=2$ to get:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}[\![u]\!]_{j+\frac{1}{2}}\right|^2 \ge \frac{2}{\Delta x^2}D_{j+\frac{1}{2}}^2 - C\frac{M_1^2}{\Delta x^2}\left(|[\![u]\!]_{j-\frac{1}{2}}|^2+|[\![u]\!]_{j+\frac{1}{2}}|^2+|[\![u]\!]_{j+\frac{3}{2}}|^2\right)$$

Summing j over all intervals to get:

$$\sum_j\left|\frac{\mathrm{d}}{\mathrm{d}t}[\![u]\!]_{j+\frac{1}{2}}\right|^2 \ge \sum_j\frac{2}{\Delta x^2}D_{j+\frac{1}{2}}^2 - C\frac{M_1^2}{\Delta x^2}\sum_j|[\![u]\!]_{j+\frac{1}{2}}|^2.$$

Recall Heun's method:

$$\begin{aligned}
y_1 &= u\\
y_2 &= u+\lambda F[u]\\
u^{n+1} &= u+\frac{\lambda}{2}F[y_1]+\frac{\lambda}{2}F[y_2].
\end{aligned}$$

Therefore,

$$\sum_j|[\![y_1]\!]_{j+\frac{1}{2}}|^2 = \sum_j|[\![u]\!]_{j+\frac{1}{2}}|^2$$

and

$$\begin{aligned}
\sum_j|[\![y_2]\!]_{j+\frac{1}{2}}|^2 &= \sum_j\left|[\![u]\!]_{j+\frac{1}{2}}+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}[\![u]\!]_{j+\frac{1}{2}}\right|^2\\
&\ge \sum_j\frac{\Delta t^2}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}[\![u]\!]_{j+\frac{1}{2}}\right|^2-\sum_j|[\![u]\!]_{j+\frac{1}{2}}|^2\\
&\ge \frac{\Delta t^2}{\Delta x^2}\underbrace{\sum_j D_{j+\frac{1}{2}}^2}_{D^2} - C\frac{M_1^2\Delta t^2}{\Delta x^2}\sum_j|[\![u]\!]_{j+\frac{1}{2}}|^2-\sum_j|[\![u]\!]_{j+\frac{1}{2}}|^2\\
&= \lambda^2D^2-CM_1^2\lambda^2\sum_j|[\![y_1]\!]_{j+\frac{1}{2}}|^2-\sum_j|[\![y_1]\!]_{j+\frac{1}{2}}|^2.
\end{aligned} \tag{15}$$

Now we arrive at the important

Lemma 5. *There exists a constant $r_0 > 0$ independent of u and f such that under the CFL condition*

$$\lambda := \frac{\Delta t}{\Delta x} \leq \frac{r_0}{M_1}, \tag{CFL2}$$

we have

$$\max\{\sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2, \sum_j |[\![y_2]\!]_{j+\frac{1}{2}}|^2\} \geq \frac{\lambda^2}{4} D^2.$$

Proof. If $\sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2 \geq \frac{\lambda^2}{4} D^2$ then we are done, so suppose $\sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2 < \frac{\lambda^2}{4} D^2$.

Fix the constant C in (15) and set

$$r_0 = \sqrt{\frac{2}{C}},$$

then, if $\lambda \leq \frac{r_0}{M_1}$ we have

$$\sum_j |[\![y_2]\!]_{j+\frac{1}{2}}|^2 \geq \lambda^2 D^2 - CM_1^2 \frac{2}{CM_1^2} \frac{\lambda^2}{4} D^2 - \frac{\lambda^2}{4} D^2 = \frac{\lambda^2}{4} D^2.$$

■

Proposition 2. *Under the CFL condition*

$$\lambda := \frac{\Delta t}{\Delta x} \leq \frac{r_0}{M_1}, \tag{CFL2}$$

we have the estimate for the First Term of inequality (9):

$$-\frac{\alpha \Delta x \lambda}{8} (\sum_j |[y_1]_{j+\frac{1}{2}}|^2 + \sum_j |[y_2]_{j+\frac{1}{2}}|^2) \leq -\frac{\alpha \Delta x \lambda^3}{32} \max\{D^2, E^2\}. \tag{16}$$

Proof. It follows directly from Lemma 5 and the trivial inequality:

$$\max\{\sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2, \sum_j |[\![y_2]\!]_{j+\frac{1}{2}}|^2\} \geq \sum_j |[\![y_1]\!]_{j+\frac{1}{2}}|^2 = \lambda^2 E^2.$$

■

5 Estimating the Second Term

We now estimate the Second Term of inequality (9):

$$\begin{aligned}
\|F[y_2] - F[y_1]\|^2 =& \sum_j \int_{I_j} \Big(F[y_2](x) - F[y_1](x)\Big)^2 \mathrm{d}x \\
=& \Delta x \sum_j \int_0^1 \Big(\beta \big(F^-_{j+\frac{1}{2}}[y_2] - F^-_{j+\frac{1}{2}}[y_1]\big) + (1-\beta)\big(F^+_{j-\frac{1}{2}}[y_2] - F^+_{j-\frac{1}{2}}[y_1]\big)\Big)^2 \mathrm{d}\beta \\
=& \frac{\Delta x}{3} \sum_j \Big(\big(F^-_{j+\frac{1}{2}}[y_2] - F^-_{j+\frac{1}{2}}[y_1]\big)^2 + \big(F^-_{j+\frac{1}{2}}[y_2] - F^-_{j+\frac{1}{2}}[y_1]\big)\big(F^+_{j-\frac{1}{2}}[y_2] - F^+_{j-\frac{1}{2}}[y_1]\big) + \\
& + \big(F^+_{j-\frac{1}{2}}[y_2] - F^+_{j-\frac{1}{2}}[y_1]\big)^2 \Big) \\
\leq& \frac{\Delta x}{2} \Big(\sum_j \big(F^-_{j+\frac{1}{2}}[y_2] - F^-_{j+\frac{1}{2}}[y_1]\big)^2 + \sum_j \big(F^+_{j+\frac{1}{2}}[y_2] - F^+_{j+\frac{1}{2}}[y_1]\big)^2 \Big).
\end{aligned} \tag{17}$$

Proposition 3. *Under the CFL condition*

$$\frac{64}{3}\left(M_1+4\max_{|z|=2\|u\|_\infty}|f'(z)|\right)\lambda \leq 1, \tag{CFL1}$$

we have the estimate for the Second Term of inequality (9):

$$\begin{aligned}\frac{1}{\Delta x}\|F[y_2]-F[y_1]\|^2 \leq & 8\lambda^2\sum_j\left(f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2+f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2\right)(\bar{u}_{j+\frac{1}{2}}-\bar{u}_{j-\frac{1}{2}})^4 \\ & +2\lambda^2\sum_j|\bar{u}_{j+\frac{1}{2}}-\bar{u}_{j-\frac{1}{2}}|^6 H(\bar{u}_{j-\frac{1}{2}},\bar{u}_{j+\frac{1}{2}},\lambda)+C\lambda^2 M_1^2(E^2+D^2),\end{aligned} \tag{18}$$

where $C>0$ is a constant and $H(u,v,\lambda)$ is a nonnegative smooth function on $\mathbb{R}^3$. Moreover, if $f(u)=Cu^p+\mathscr{O}(u^{p+1})$ for some $p>2$ is analytic near $u=0$, then

$$\lim_{(u,v)\to(0,0)}\frac{H(u,v,\lambda)}{(\sqrt{u^2+v^2})^{3p-6}}=0. \tag{19}$$

Proof. Proved in Appendix A. ■

6 Proof of L^2-stability (2)

We first establish a few Lemmas that would enable us to compare the Second Term with the First Term later on:

Lemma 6. *Let*

$$q(x,y,z)=3\bar{f}\left(\frac{x+y}{2}+z,z\right)-f\left(\frac{x+y}{2}+z\right)-4f(z)-f\left(\frac{x-y}{2}+z\right)+3\bar{f}\left(\frac{x-y}{2}+z,z\right)$$

Then

$$q(0,0,z)=0$$

and for all $n\geq 1$ and $i+j=n$, we have

$$\frac{\partial^n q}{\partial x^i\partial y^j}(0,0,z)=\frac{(2-n)(1+(-1)^j)}{2^n(n+1)}f^{(n)}(z).$$

In particular,

$$\begin{aligned}&\frac{\partial q}{\partial x}(0,0,z)=\frac{f'(z)}{2},\quad \frac{\partial q}{\partial y}(0,0,z)=0,\\ &\frac{\partial^2 q}{\partial x^2}(0,0,z)=\frac{\partial^2 q}{\partial x\partial y}(0,0,z)=\frac{\partial^2 q}{\partial y^2}(0,0,z)=0.\end{aligned}$$

Proof. $q(0,0,z)=0$ is clear by using $\bar{f}(z,z)=f(z)$. Applying Lemma 4 for $u=v=z$, we have for $n\geq 1$

$$\begin{aligned}\frac{\partial^n q}{\partial x^i\partial y^j}(0,0,z)=&\frac{3}{n+1}f^{(n)}(z)\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^j-f^{(n)}(z)\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^j-f^{(n)}(z)\left(\frac{1}{2}\right)^i\left(-\frac{1}{2}\right)^j+\\&+\frac{3}{n+1}f^{(n)}(z)\left(\frac{1}{2}\right)^i\left(-\frac{1}{2}\right)^j\\=&\frac{(2-n)(1+(-1)^j)}{2^n(n+1)}f^{(n)}(z).\end{aligned}$$

■

Lemma 7. *Assume f is an entire function, then*

$$D_{j+\frac{1}{2}}=\frac{f'(\bar{u}_{j+\frac{1}{2}})}{2}(\bar{u}_{j-\frac{1}{2}}-2\bar{u}_{j+\frac{1}{2}}+\bar{u}_{j+\frac{3}{2}})+R(\bar{u}_{j-\frac{1}{2}},\bar{u}_{j+\frac{1}{2}},\bar{u}_{j+\frac{3}{2}}),\tag{20}$$

where

$$\left|R(\bar{u}_{j-\frac{1}{2}},\bar{u}_{j+\frac{1}{2}},\bar{u}_{j+\frac{3}{2}})\right|\leq|\bar{u}_{j+\frac{1}{2}}-\bar{u}_{j-\frac{1}{2}}|^3T^-(\bar{u}_{j-\frac{1}{2}},\bar{u}_{j+\frac{1}{2}})+|\bar{u}_{j+\frac{3}{2}}-\bar{u}_{j+\frac{1}{2}}|^3T^+(\bar{u}_{j+\frac{1}{2}},\bar{u}_{j+\frac{3}{2}}),\tag{21}$$

and $T^{\pm}(u,v)$ are nonnegative continuous functions on $\mathbb{R}^2$. Moreover, if $f(u)=Cu^p+\mathscr{O}(u^{p+1})$ for some $p>2$ near $u=0$, then

$$\sup_{(u,v)\to(0,0)}\frac{T^{\pm}(u,v)^2}{(\sqrt{u^2+v^2})^{2p-6}}<\infty.\tag{22}$$

Proof. Set

$$\begin{aligned}x&=\bar{u}_{j-\frac{1}{2}}-2\bar{u}_{j+\frac{1}{2}}+\bar{u}_{j+\frac{3}{2}}\\y&=\bar{u}_{j+\frac{3}{2}}-\bar{u}_{j-\frac{1}{2}}\\z&=\bar{u}_{j+\frac{1}{2}},\end{aligned}$$

then

$$\begin{aligned}\bar{u}_{j-\frac{1}{2}}&=\frac{x-y}{2}+z\\\bar{u}_{j+\frac{1}{2}}&=z\\\bar{u}_{j+\frac{3}{2}}&=\frac{x+y}{2}+z\end{aligned}$$

and

$$D_{j+\frac{1}{2}}=q(x,y,z)$$

where $q(x,y,z)$ is defined as same as in Lemma 6. Fix z and expand $q(x,y,z)$ into Taylor series at $x=0$ and $y=0$:

$$q(x,y,z)=\frac{f'(z)}{2}x+R(x,y,z)$$

where

$$\begin{aligned}
|R(x,y,z)| &= \left| \sum_{n=3}^{\infty} \frac{1}{n!} \sum_{0 \le i \le n} \binom{n}{i} \frac{\partial^n q}{\partial x^i \partial y^{n-i}}(0,0,z) x^i y^{n-i} \right| \\
&\le \sum_{n=3}^{\infty} \frac{1}{n!} \sum_{0 \le i \le n} \binom{n}{i} \left| \frac{\partial^n q}{\partial x^i \partial y^{n-i}}(0,0,z) \right| |x|^i |y|^{n-i} \\
&\le \sum_{n=3}^{\infty} \frac{1}{n!} \frac{|f^{(n)}(z)|}{2^{n-1}} \sum_{0 \le i \le n} \binom{n}{i} |x|^i |y|^{n-i} \quad \text{(Apply Lemma 6)} \\
&= 2 \sum_{n=3}^{\infty} \frac{|f^{(n)}(z)|}{n!} \left(\frac{|x|+|y|}{2} \right)^n \\
&\le 2 \sum_{n=3}^{\infty} \frac{|f^{(n)}(\bar{u}_{j+\frac{1}{2}})|}{n!} \left(|\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}| + |\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}}| \right)^n \\
&\le \sum_{n=3}^{\infty} \frac{2^n |f^{(n)}(\bar{u}_{j+\frac{1}{2}})|}{n!} |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^n + \sum_{n=3}^{\infty} \frac{2^n |f^{(n)}(\bar{u}_{j+\frac{1}{2}})|}{n!} |\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}}|^n
\end{aligned}$$

Let

$$\begin{aligned}
T^-(u,v) &= \sum_{n=3}^{\infty} \frac{2^n |f^{(n)}(v)|}{n!} |u-v|^{n-3} \\
T^-(u,v) &= \sum_{n=3}^{\infty} \frac{2^n |f^{(n)}(u)|}{n!} |u-v|^{n-3}.
\end{aligned}$$

Since $f(u)$ is entire, by Cauchy's integral formula [4]: for all $R > r$ and $|u-u_0|, |v-v_0| < r$,

$$\left| \frac{f^{(n)}(u)}{n!} \right| \le \frac{R \max_{|z-u_0|=R} |f(z)|}{(R-r)^{n+1}}, \quad \left| \frac{f^{(n)}(v)}{n!} \right| \le \frac{R \max_{|z-v_0|=R} |f(z)|}{(R-r)^{n+1}}.$$

Thus, by picking $R = 3|u_0 - v_0| + 7r$ one can show that the two series converge uniformly in $|u-u_0|, |v-v_0| < r$. Thus, they converge locally uniformly around any point $(u_0, v_0) \in \mathbb{R}^2$, which implies $T^{\pm}(u,v)$ is continuous since every summand in the series is a continuous function of (u,v). Moreover, we can take the limit inside the series. Now, set $u = r\cos(\theta)$, $v = r\sin(\theta)$, then as $r \to 0$ we have:

$$f^{(n)}(u) = \mathscr{O}(r^{p-n}), \quad f^{(n)}(v) = \mathscr{O}(r^{p-n}),$$

and

$$\begin{aligned}
\sup_{r \to 0} \frac{|T^+(u,v)|}{r^{p-3}} &= \sup_{r \to 0} \frac{4|f'''(u)|}{3r^{p-3}} = \mathscr{O}(1) < \infty \\
\sup_{r \to 0} \frac{|T^-(u,v)|}{r^{p-3}} &= \sup_{r \to 0} \frac{4|f'''(u)|}{3r^{p-3}} = \mathscr{O}(1) < \infty.
\end{aligned}$$

■

Now, we introduce a beautiful cyclic inequality:

Theorem 1. *For any real numbers $b_1, b_2, \cdots, b_{2N}$ satisfying:*

$$b_{2i-1} - 2b_{2i} + b_{2i+1} \geq 0, \quad \forall i = 1, 2, \cdots, N$$

with $b_{2N+1} = b_1, b_{2N+2} = b_2$, we have

$$\sum_{i=1}^{N} (b_{2i-1} - 2b_{2i} + b_{2i+1})^3 \leq 1024 \|\mathbf{b}\|_\infty \sum_{i=1}^{N} (b_{2i} - 2b_{2i+1} + b_{2i+2})^2.$$

Proof. The proof, which relies on geometry rather than computation, is postponed until Appendix B. ■

Lemma 8. *Suppose f is a convex function on $\mathbb{R}$. Then, there exists a constant $C > 0$ such that*

$$\sum_j |f(\bar{u}_{j-\frac{1}{2}}) - 2\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}) + f(\bar{u}_{j+\frac{1}{2}})|^3 \leq C \|u\|_\infty M_1 D^2.$$

Proof. Since either side of the above inequality does not change if f is translated by a constant, we can assume $\int_{u_{\min}}^{u_{\max}} f(u)\,\mathrm{d}u = 0$. Then $\max_{u_{\min} \leq u \leq u_{\max}} |f(u)| \leq 2\|u\|_\infty M_1$. Let

$$\begin{aligned} b_{2j} &= -f(\bar{u}_{j-\frac{1}{2}}) + 3\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}) - f(\bar{u}_{j+\frac{1}{2}}), \\ b_{2j+1} &= f(\bar{u}_{j+\frac{1}{2}}). \end{aligned}$$

Then

$$\begin{aligned} &b_{2j} - 2b_{2j+1} + b_{2j+2} = 3\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}) - f(\bar{u}_{j-\frac{1}{2}}) - 4f(\bar{u}_{j+\frac{1}{2}}) - f(\bar{u}_{j+\frac{3}{2}}) + 3\bar{f}(\bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{3}{2}}) = D_{j+\frac{1}{2}}, \\ &b_{2j-1} - 2b_{2j} + b_{2j+1} = 3f(\bar{u}_{j-\frac{1}{2}}) - 6\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}) + 3f(\bar{u}_{j+\frac{1}{2}}) \geq 0. \end{aligned}$$

Apply Theorem 1 to get

$$\sum_j |f(\bar{u}_{j-\frac{1}{2}}) - 2\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}) + f(\bar{u}_{j+\frac{1}{2}})|^3 \leq \frac{1024}{27} \|\mathbf{b}\|_\infty D^2 \leq C\|u\|_\infty M_1 D^2,$$

since $\|\mathbf{b}\|_\infty \leq 5 \max_{u_{\min} \leq u \leq u_{\max}} |f(u)| \leq 10\|u\|_\infty M_1$ by assumption. ■

Recall that we have assumed $f(u)$ can be extended to an entire function on $\mathbb{C}$ and satisfies one of the following two hypotheses:

(H1) $f^{2n}(0) \geq 0$ and $f(0) = f^{(2n-1)}(0) = 0$ for all $n \geq 1$. (even and convex)

(H2) $f''(u) > 0$ on $\mathbb{R}$. (strictly convex)

6.1 $f(u)$ satisfies (H1)

In this section, we shall assume **(H1)** is satisfied. Then, f can be written in the form:

$$f(u) = \sum_{n=p}^{\infty} a_n u^{2n}$$

where $p \geq 1$, $a_p > 0$ and $a_n \geq 0$ for $n > p$. Since f is entire, the above series converges absolutely for all $u \in \mathbb{R}$. Note that in this case $M_n = |f^{(n)}(\|u\|_\infty)|$.

Lemma 9. *For $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$, we have*

$$\frac{1}{2} \leq \frac{x^{n+1}-1}{(x-1)(x^n+1)} \leq \frac{n+1}{2}, \quad \textit{if } n \textit{ is even}$$
$$\frac{n+1}{2n} \leq \frac{x^{n+1}-1}{(x-1)(x^n+1)} \leq \frac{n+1}{2}, \quad \textit{if } n \textit{ is odd},$$

where the left equality is attained when $x \to -1$, and the right equality is attained when $x \to 1$.

Proof. The proof is trivial for $n = 0, 1$. For $n \geq 2$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^{n+1}-1}{(x-1)(x^n+1)} = \frac{nx^{n+1}-nx^{n-1}+1-x^{2n}}{(x-1)^2(x^n+1)^2} = \frac{(1-x^2)(\sum_{k=0}^{n-1}x^{2k}-nx^{n-1})}{(x-1)^2(x^n+1)^2}.$$

By AM-GM inequality:

$$\sum_{k=0}^{n-1} x^{2k} \geq n\sqrt[n]{\prod_{k=0}^{n-1} x^{2k}} = n|x|^{n-1} \geq nx^{n-1},$$

where equality holds if and only if $x = 1$. Therefore, if we define

$$h(x) := \frac{x^{n+1}-1}{(x-1)(x^n+1)}, \quad x \neq \pm 1$$
$$h(1) := \lim_{x\to 1} h(x) = \frac{n+1}{2}$$
$$h(-1) := \lim_{x\to -1} h(x) = \frac{1}{2}, \quad \text{if } n \text{ is even}$$
$$h(-1) := \lim_{x\to -1} h(x) = \frac{n+1}{2n}, \quad \text{if } n \text{ is odd.}$$

Then, $h'(x) > 0$ if $|x| < 1$, $h'(x) < 0$ if $|x| > 1$, and $h'(x) = 0$ if $x = \pm 1$. Therefore, $h(x)$ is decreasing on $(-\infty, -1] \cup [1, \infty)$, and increasing on $[-1, 1]$. Since $h(x) \to 1$ as $x \to \pm\infty$, the maximum and minimum of $h(x)$ are attained at 1 and -1 respectively. ■

Lemma 10. *Suppose $n \geq 2$ is an even integer, $f(u) = u^n$, then*

$$f(a) - 2\bar{f}(a,b) + f(b) \geq \frac{1}{12}(a-b)^2(a^{n-2}+b^{n-2}).$$

Proof. If $a = \pm b$ then the proof is trivial. Without loss of generality assume $|a| > |b|$. Set $x = \frac{b}{a}$. Then, $|x| < 1$ and by Lemma 9 we have

$$\frac{x^{k+1}-1}{(x-1)} \geq \frac{1}{2}(x^k+1) > 0, \quad k \geq 0.$$

Then,

$$\begin{aligned}
f(a)-2\bar{f}(a,b)+f(b) =& a^n-\frac{2}{n+1}\sum_{k=0}^{n}a^kb^{n-k}+b^n\\
=&\frac{1}{n+1}\sum_{k=1}^{n-1}[a^k(a^{n-k}-b^{n-k})]-\frac{1}{n+1}\sum_{k=1}^{n-1}[b^k(a^{n-k}-b^{n-k})]\\
=&\frac{1}{n+1}\sum_{k=1}^{n-1}(a^k-b^k)(a^{n-k}-b^{n-k})\\
=&\frac{1}{n+1}\sum_{k=1}^{n-1}(a-b)^2a^{n-2}\frac{x^k-1}{x-1}\frac{x^{n-k}-1}{x-1}\\
\geq&\frac{1}{4n+4}(a-b)^2a^{n-2}\sum_{k=1}^{n-1}(x^{k-1}+1)(x^{n-k-1}+1) \quad \text{(by Lemma 9)}\\
\geq&\frac{1}{4n+4}(a-b)^2a^{n-2}\sum_{k=1}^{\frac{n}{2}}(x^{2k-2}+1)(x^{n-2k}+1)\\
\geq&\frac{1}{4n+4}(a-b)^2a^{n-2}\sum_{k=1}^{\frac{n}{2}}(x^{n-2}+1)\\
\geq&\frac{n}{8n+8}(a-b)^2(a^{n-2}+b^{n-2})\\
\geq&\frac{1}{12}(a-b)^2(a^{n-2}+b^{n-2}).
\end{aligned}$$

■

Lemma 11. *There exists a constant $C>0$ such that*

$$\sum_j|\bar{u}_{j+\frac{1}{2}}-\bar{u}_{j-\frac{1}{2}}|^6\left(\frac{f(\bar{u}_{j-\frac{1}{2}})}{\bar{u}^2_{j-\frac{1}{2}}}+\frac{f(\bar{u}_{j+\frac{1}{2}})}{\bar{u}^2_{j+\frac{1}{2}}}\right)^3\leq C\|u\|_\infty M_1D^2. \tag{23}$$

Proof. Using Lemma 10, we obtain for any real numbers u and v:

$$\begin{aligned}
f(u)-2\bar{f}(u,v)+f(v)\geq&\frac{1}{12}(u-v)^2(\sum_{n=p}^{\infty}a_nu^{2n-2}+\sum_{n=p}^{\infty}a_nv^{2n-2})\\
=&\frac{1}{12}(u-v)^2\left(\frac{f(u)}{u^2}+\frac{f(v)}{v^2}\right).
\end{aligned}$$

Combine the above inequality with Lemma 8 and the proof is complete. ■

For all $p\geq 1$, define for $M\geq 0$ the function

$$\gamma^{(1)}(M):=\begin{cases}\max_{|u|,|v|\leq M}\frac{u^2f''(u)^2T^+(u,v)^2+v^2f''(v)^2T^-(u,v)^2}{(f(u)/u^2+f(v)/v^2)^3} & \text{if } M>0,\\ 0 & \text{if } M=0,\end{cases} \tag{24}$$

where $T^{\pm}(u,v)$ are nonnegative continuous functions on $\mathbb{R}^2$ from Lemma 7. Clearly, $\gamma^{(1)}(M)$ is an increasing function on $[0,\infty)$. If we let $u = r\cos(\theta)$, $v = r\sin(\theta)$, then

$$\begin{aligned} f(u)/u^2 + f(v)/v^2 \geq & a_p r^{2p-2}[\sin^{2p-2}(\theta)+\cos^{2p-2}(\theta)] \\ \geq & \frac{a_p}{2^{p-2}} r^{2p-2}[\sin^2(\theta)+\cos^2(\theta)]^{p-1} \quad \text{(by Jensen's inequality)} \\ = & \frac{a_p}{2^{p-2}} r^{2p-2} \end{aligned}$$

so that if $p \geq 2$, as $r \to 0$:

$$\frac{u^2 f''(u)^2 T^+(u,v)^2 + v^2 f''(v)^2 T^-(u,v)^2}{(f(u)/u^2 + f(v)/v^2)^3} \overset{(22)}{\leq} C\frac{r^{8p-8}}{r^{6p-6}} = Cr^{2p-2} \to 0.$$

And if $p = 1$ the denominator cannot vanish, so that as $r \to 0$:

$$\frac{u^2 f''(u)^2 T^+(u,v)^2 + v^2 f''(v)^2 T^-(u,v)^2}{(f(u)/u^2 + f(v)/v^2)^3} \leq Cr^2 \to 0.$$

Therefore, $\gamma^{(1)}(M) \to 0$ as $M \to 0$; which implies $\gamma^{(1)}(M)$ is continuous on $[0,\infty)$ for any $p \geq 1$.

Lemma 12. *There exists a constant $C > 0$ such that*

$$\sum_j \left(f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 \right)(\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^4 \leq C\left(\|u\|_\infty^2 M_2^2 + \|u\|_\infty M_1 \gamma^{(1)}(\|u\|_\infty) \right) D^2. \tag{25}$$

Proof. Using summation by parts, we have

$$\begin{aligned} &\sum_j \left(f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 \right)(\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^4 \\ =& -\sum_j \bar{u}_{j+\frac{1}{2}} \Bigg[\left(f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{3}{2}})^2 f''(\bar{u}_{j+\frac{3}{2}})^2 \right)(\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 \\ &\qquad - \left(f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 \right)(\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \Bigg] \\ =& -\sum_j \bar{u}_{j+\frac{1}{2}} \left[f'(\bar{u}_{j+\frac{3}{2}})^2 f''(\bar{u}_{j+\frac{3}{2}})^2 (\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \right] \\ &- \sum_j \bar{u}_{j+\frac{1}{2}} \left[f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 (\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \right] \\ =& -\sum_j \bar{u}_{j+\frac{1}{2}} \left[f'(\bar{u}_{j+\frac{3}{2}})^2 f''(\bar{u}_{j+\frac{3}{2}})^2 - f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 \right](\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 \\ &- \sum_j \bar{u}_{j+\frac{1}{2}} \left[f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 - f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 \right](\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \\ &- 2\sum_j \bar{u}_{j+\frac{1}{2}} f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 \left[(\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \right] \\ =& -\sum_j \underbrace{(\bar{u}^2_{j+\frac{1}{2}} - \bar{u}^2_{j-\frac{1}{2}}) \left[f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 - f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 \right]}_{\geq 0 \text{ since } |f'(u)f''(u)|^2 \text{ is an increasing function of } |u|} (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^2 \end{aligned}$$

$$
\begin{aligned}
&-2\sum_j \bar u_{j+\frac12} f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^3-(\bar u_{j+\frac12}-\bar u_{j-\frac12})^3\Big]\\
\le&-2\sum_j \bar u_{j+\frac12} f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^3-(\bar u_{j+\frac12}-\bar u_{j-\frac12})^3\Big]\\
=&-2\sum_j \bar u_{j+\frac12} f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2(\bar u_{j+\frac32}-2\bar u_{j+\frac12}+\bar u_{j-\frac12})\times\\
&\qquad\times\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^2+(\bar u_{j+\frac32}-\bar u_{j+\frac12})(\bar u_{j+\frac12}-\bar u_{j-\frac12})+(\bar u_{j+\frac12}-\bar u_{j-\frac12})^2\Big]\\
\overset{(20)}{=}&-4\sum_j \bar u_{j+\frac12} f'(\bar u_{j+\frac12}) f''(\bar u_{j+\frac12})^2 D_{j+\frac12}\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^2+(\bar u_{j+\frac32}-\bar u_{j+\frac12})(\bar u_{j+\frac12}-\bar u_{j-\frac12})+(\bar u_{j+\frac12}-\bar u_{j-\frac12})^2\Big]\\
&+4\sum_j \bar u_{j+\frac12} f'(\bar u_{j+\frac12}) f''(\bar u_{j+\frac12})^2 R(\bar u_{j-\frac12},\bar u_{j+\frac12},\bar u_{j+\frac32})\times\\
&\qquad\times\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^2+(\bar u_{j+\frac32}-\bar u_{j+\frac12})(\bar u_{j+\frac12}-\bar u_{j-\frac12})+(\bar u_{j+\frac12}-\bar u_{j-\frac12})^2\Big]\\
\le&\,6\sqrt2\|u\|_\infty M_2\Big[\sum_j D^2_{j+\frac12}\Big]^{1/2}\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&+6\sqrt2\Big[\sum_j \bar u^2_{j+\frac12} f''(\bar u_{j+\frac12})^2 R(\bar u_{j-\frac12},\bar u_{j+\frac12},\bar u_{j+\frac32})^2\Big]^{1/2}\times\\
&\qquad\times\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&\qquad\text{(by Cauchy–Schwarz inequality)}\\
\overset{(21),(24)}{\le}&\,6\sqrt2\|u\|_\infty M_2 D\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&+12\sqrt{\gamma^{(1)}(\|u\|_\infty)}\Big[\sum_j\big(f(\bar u_{j-\frac12})/\bar u^2_{j-\frac12}+f(\bar u_{j+\frac12})/\bar u^2_{j+\frac12}\big)^3|\bar u_{j+\frac12}-\bar u_{j-\frac12}|^6\Big]^{1/2}\times\\
&\qquad\times\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
\overset{(23)}{\le}&\,6\sqrt2\|u\|_\infty M_2 D\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&+12\sqrt{C\|u\|_\infty M_1\gamma^{(1)}(\|u\|_\infty)}D\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}
\end{aligned}
$$

where Lemma 11 is used in the last step. Dividing by

$$
\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}
$$

and taking square on both sides, we finally obtain:

$$
\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\le C\Big(\|u\|^2_\infty M_2^2+\|u\|_\infty M_1\gamma^{(1)}(\|u\|_\infty)\Big)D^2.
$$

■

For all $p \geq 1$, define for $M \geq 0$ and $\lambda \in \mathbb{R}$ the function

$$\delta^{(1)}(M,\lambda) := \begin{cases} \max_{|u|,|v| \leq M} \frac{H(u,v,\lambda)}{(f(u)/u^2+f(v)/v^2)^3} & \text{if } M > 0, \\ \frac{1}{8a_1^3} H(0,0,\lambda) & \text{if } M = 0, p = 1 \\ 0 & \text{if } M = 0, p \geq 2. \end{cases} \tag{26}$$

where $H(u,v,\lambda)$ is a nonnegative smooth function on $\mathbb{R}^3$ from Proposition 3. Clearly, for a fixed $\lambda \in \mathbb{R}$, $\delta^{(1)}(M,\lambda)$ is an increasing function of $M \geq 0$. If we let $u = r\cos(\theta)$, $v = r\sin(\theta)$, then if $p \geq 2$ as $r \to 0$:

$$\frac{H(u,v,\lambda)}{(f(u)/u^2+f(v)/v^2)^3} \overset{(19)}{\leq} C\frac{H(u,v,\lambda)}{r^{6p-6}} \to 0.$$

And for $p = 1$ the denominator cannot vanish. Therefore, $\delta^{(1)}(M,\lambda)$ is continuous on $[0,\infty) \times \mathbb{R}$ for any $p \geq 1$.

Lemma 13. *There exists a constant $C > 0$ such that*

$$\sum_j |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^6 H(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \lambda) \leq C\|u\|_\infty M_1 \delta^{(1)}(\|u\|_\infty, \lambda) D^2. \tag{27}$$

Proof. Use Lemma 11 and the definition of δ in (26):

$$\begin{aligned} \sum_j |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^6 H(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \lambda) \leq & \delta^{(1)}(\|u\|_\infty, \lambda) \sum_j |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^6 \left(\frac{f(\bar{u}_{j-\frac{1}{2}})}{\bar{u}^2_{j-\frac{1}{2}}} + \frac{f(\bar{u}_{j+\frac{1}{2}})}{\bar{u}^2_{j+\frac{1}{2}}} \right)^3 \\ \leq & C\|u\|_\infty M_1 \delta^{(1)}(\|u\|_\infty, \lambda) D^2. \end{aligned}$$

∎

Now we have obtained all the estimates for comparing the Second Term with the First Term:

Theorem 2. *Suppose $f(u)$ satisfies **(H1)**. Then, there exists a continuous function $C_f(\cdot) \geq 0$ on $[0,\infty)$ depending on f such that under the CFL condition*

$$C_f(\|u\|_\infty) \frac{\Delta t}{\Delta x} \leq 1,$$

we have L^2-stability:

$$\|u^{n+1}\|^2 - \|u\|^2 \leq 0.$$

Proof. Putting inequalities (25) and (27) together into inequality (18) yields:

$$\frac{\lambda^2}{4} \|F[y_1] - F[y_2]\|^2 \leq C\Delta x \lambda^4 \left(M_1^2 + \|u\|_\infty^2 M_2^2 + \|u\|_\infty M_1 \gamma^{(1)}(\|u\|_\infty) + \|u\|_\infty M_1 \delta^{(1)}(\|u\|_\infty, \lambda) \right) \max\{D^2, E^2\}.$$

Combining the above with inequalities (9) and (16) and dividing by $\Delta x\lambda^3 \max\{D^2,E^2\}/16$, we obtain:

$$16\frac{\|u^{n+1}\|^2-\|u\|^2}{\Delta x\lambda^3\max\{D^2,E^2\}} \le -M_1+\lambda C\Big(M_1^2+\|u\|_\infty^2 M_2^2+\|u\|_\infty M_1\gamma^{(1)}(\|u\|_\infty)+\|u\|_\infty M_1\delta^{(1)}(\|u\|_\infty,\lambda)\Big)$$
$$= -f'(\|u\|_\infty)+\lambda C\Big(f'(\|u\|_\infty)^2+\|u\|_\infty^2 f''(\|u\|_\infty)^2$$
$$+\|u\|_\infty f'(\|u\|_\infty)\gamma^{(1)}(\|u\|_\infty)+\|u\|_\infty f'(\|u\|_\infty)\delta^{(1)}(\|u\|_\infty,\lambda)\Big)$$

where the last step is justified because $M_n = f^{(n)}(\|u\|_\infty)$ if $f(u)$ satisfies **(H1)**.

Now, $f'(\|u\|_\infty)=0$ if and only if $\|u\|_\infty = 0$. Moreover, as $r=\|u\|_\infty \to 0$ we have

$$\frac{r^2|f''(r)|^2}{|f'(r)|} \le Cr^{2p-1} \to 0. \tag{28}$$

Therefore, we can divide both sides by $f'(\|u\|_\infty)$ as well to get

$$16\frac{\|u^{n+1}\|^2-\|u\|^2}{f'(\|u\|_\infty)\Delta x\lambda^3\max\{D^2,E^2\}} \le -1+\lambda K^{(1)}(\|u\|_\infty), \quad \text{if } 0\le\lambda\le 1, \tag{29}$$

where

$$K^{(1)}(M) := \begin{cases} C\Big(f'(M)+M^2\frac{f''(M)^2}{f'(M)}+M\gamma^{(1)}(M)+M\max_{0\le\lambda\le 1}\delta^{(1)}(M,\lambda)\Big) & \text{if } M>0, \\ 0 & \text{if } M=0. \end{cases}$$

Recall that $\gamma^{(1)}(M)\ge 0$ is continuous on $[0,\infty)$ and $\delta^{(1)}(M,\lambda)\ge 0$ is continuous on $[0,\infty)\times\mathbb{R}$, by (28) we deduce $K^{(1)}(M)$ is also nonnegative and continuous on $[0,\infty)$. Taking (CFL1) and (CFL2) into account, the right hand side of (29) is nonpositive if

$$\max\left\{\frac{f'(\|u\|_\infty)}{r_0}, \frac{64}{3}\Big(f'(\|u\|_\infty)+f'(2\|u\|_\infty)\Big), K^{(1)}(\|u\|_\infty), 1\right\}\frac{\Delta t}{\Delta x} \le 1,$$

where $r_0>0$ is the constant from (CFL2). Thus, the proof is complete if we set

$$C_f(M) = \max\left\{\frac{f'(M)}{r_0}, \frac{64}{3}\Big(f'(M)+4f'(2M)\Big), K^{(1)}(M), 1\right\} \ge 0$$

which is a continuous function on $[0,\infty)$. ∎

6.2 $f(u)$ satisfies (H2)

In this section, we shall assume **(H2)** is satisfied. Then, $m_2 = \min_{u_{\min}\le u\le u_{\max}}|f''(u)|>0$.

Lemma 14. *There exists a constant $C>0$ such that*

$$\sum_j m_2^3|\bar{u}_{j+\frac{1}{2}}-\bar{u}_{j-\frac{1}{2}}|^6 \le C\|u\|_\infty M_1 D^2. \tag{30}$$

Proof. Use the error estimate for Trapezoidal Rule: for all $u, v \in [u_{\min}, u_{\max}]$,

$$f(u) - 2\bar{f}(u,v) + f(v) = \frac{f''(\xi)}{6}(u-v)^2 \geq \frac{m_2}{6}(u-v)^2,$$

where ξ lies between u and v. Then use Lemma 8 to complete the proof. ∎

Define for $M \geq 0$ the function

$$\gamma^{(2)}(M) := \max_{|u|,|v|\leq M} \Big(T^+(u,v)^2 + T^-(u,v)^2 \Big), \tag{31}$$

where $T^\pm(u,v)$ are nonnegative continuous functions on $\mathbb{R}^2$ from Lemma 7. Clearly, $\gamma^{(2)}(M) \geq 0$ is a continuous increasing function on $[0,\infty)$.

Lemma 15. *There exists a constant $C > 0$ such that*

$$\sum_j \Big(f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 \Big) (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^4 \leq C \Big(\frac{M_1^2 M_2^2}{m_2^2} + \frac{\|u\|_\infty M_1^3 M_2^2}{m_2^5} \gamma^{(2)}(\|u\|_\infty) \Big) D^2. \tag{32}$$

Proof. Using summation by parts, we have

$$\begin{aligned}
& m_2 \sum_j \Big(f'(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 \Big) (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^4 \\
\leq & \sum_j \Big(f'(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 \Big) \Big(f'(\bar{u}_{j+\frac{1}{2}}) - f'(\bar{u}_{j-\frac{1}{2}}) \Big) (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \quad \text{(Mean Value Theorem)} \\
= & -\sum_j f'(\bar{u}_{j+\frac{1}{2}}) \Big[\Big(f'(\bar{u}_{j+\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{3}{2}})^2 \Big) (\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - \Big(f'(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 \Big) (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \Big] \\
= & -\sum_j f'(\bar{u}_{j+\frac{1}{2}}) \Big[f'(\bar{u}_{j+\frac{3}{2}})^2 (\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - f'(\bar{u}_{j+\frac{1}{2}})^2 (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \Big] \\
& - \sum_j f'(\bar{u}_{j+\frac{1}{2}}) \Big[f'(\bar{u}_{j+\frac{1}{2}})^2 (\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - f'(\bar{u}_{j-\frac{1}{2}})^2 (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \Big] \\
= & -\sum_j f'(\bar{u}_{j+\frac{1}{2}}) \Big[f'(\bar{u}_{j+\frac{3}{2}})^2 - f'(\bar{u}_{j+\frac{1}{2}})^2 \Big] (\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 \\
& - \sum_j f'(\bar{u}_{j+\frac{1}{2}}) \Big[f'(\bar{u}_{j+\frac{1}{2}})^2 - f'(\bar{u}_{j-\frac{1}{2}})^2 \Big] (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \\
& - 2\sum_j f'(\bar{u}_{j+\frac{1}{2}})^3 \Big[(\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \Big] \\
= & -\sum_j \Big[f'(\bar{u}_{j+\frac{1}{2}}) + f'(\bar{u}_{j-\frac{1}{2}}) \Big]^2 \underbrace{\Big[f'(\bar{u}_{j+\frac{1}{2}}) - f'(\bar{u}_{j-\frac{1}{2}}) \Big] (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3}_{\geq 0 \text{ since } f'(u) \text{ is an increasing function of } u} \\
& - 2\sum_j f'(\bar{u}_{j+\frac{1}{2}})^3 \Big[(\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \Big] \\
\leq & -2\sum_j f'(\bar{u}_{j+\frac{1}{2}})^3 \Big[(\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 - (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 \Big]
\end{aligned}$$

$$
\begin{aligned}
&= -2\sum_j f'(\bar u_{j+\frac12})^3(\bar u_{j+\frac32}-2\bar u_{j+\frac12}+\bar u_{j-\frac12})\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^2+(\bar u_{j+\frac32}-\bar u_{j+\frac12})(\bar u_{j+\frac12}-\bar u_{j-\frac12})+(\bar u_{j+\frac12}-\bar u_{j-\frac12})^2\Big]\\
&\overset{(20)}{=} -4\sum_j f'(\bar u_{j+\frac12})^2 D_{j+\frac12}\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^2+(\bar u_{j+\frac32}-\bar u_{j+\frac12})(\bar u_{j+\frac12}-\bar u_{j-\frac12})+(\bar u_{j+\frac12}-\bar u_{j-\frac12})^2\Big]\\
&\quad +4\sum_j f'(\bar u_{j+\frac12})^2 R(\bar u_{j-\frac12},\bar u_{j+\frac12},\bar u_{j+\frac32})\Big[(\bar u_{j+\frac32}-\bar u_{j+\frac12})^2+(\bar u_{j+\frac32}-\bar u_{j+\frac12})(\bar u_{j+\frac12}-\bar u_{j-\frac12})+(\bar u_{j+\frac12}-\bar u_{j-\frac12})^2\Big]\\
&\le 6\sqrt2 M_1\Big[\sum_j D_{j+\frac12}^2\Big]^{1/2}\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&\quad +6\sqrt2 M_1\Big[\sum_j R(\bar u_{j-\frac12},\bar u_{j+\frac12},\bar u_{j+\frac32})^2\Big]^{1/2}\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&\qquad\qquad \text{(By Cauchy–Schwarz inequality)}\\
&\overset{(21),(31)}{\le} 6\sqrt2 M_1 D\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&\quad +\frac{12M_1}{(m_2)^{3/2}}\sqrt{\gamma^{(2)}(\|u\|_\infty)}\Big[\sum_j m_2^3|\bar u_{j+\frac12}-\bar u_{j-\frac12}|^6\Big]^{1/2}\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&\overset{(30)}{\le} 6\sqrt2 M_1 D\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}\\
&\quad +\frac{12M_1}{(m_2)^{3/2}}\sqrt{C\|u\|_\infty M_1\gamma^{(2)}(\|u\|_\infty)}D\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2},
\end{aligned}
$$

where Lemma 14 is used in the last step. Dividing by

$$
\Big[\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\Big]^{1/2}
$$

and taking square on both sides, we obtain:

$$
\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4 \le C\frac{M_1^2}{m_2^2}\Big(1+\frac{\|u\|_\infty M_1}{m_2^3}\gamma^{(2)}(\|u\|_\infty)\Big)D^2.
$$

Therefore,

$$
\begin{aligned}
&\sum_j\Big(f'(\bar u_{j-\frac12})^2 f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2 f''(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\\
\le{}& M_2^2\sum_j\Big(f'(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2\Big)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4 \le C\Big(\frac{M_1^2M_2^2}{m_2^2}+\frac{\|u\|_\infty M_1^3M_2^2}{m_2^5}\gamma^{(2)}(\|u\|_\infty)\Big)D^2.
\end{aligned}
$$

■

Define for $M \ge 0$ and $\lambda \in \mathbb{R}$ the function

$$
\delta^{(2)}(M,\lambda) := \max_{|u|,|v|\le M} H(u,v,\lambda), \tag{33}
$$

where $H(u,v,\lambda)$ is a nonnegative smooth function on $\mathbb{R}^3$ from Proposition 3. Clearly, $\delta^{(2)}(M,\lambda)$ is continuous on $[0,\infty)\times\mathbb{R}$. For a fixed $\lambda \in \mathbb{R}$, $\delta^{(2)}(M,\lambda)$ is also an increasing function of $M \ge 0$.

Lemma 16. *There exists a constant $C > 0$ such that*

$$\sum_j |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^6 H(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \lambda) \le C \frac{\|u\|_\infty M_1}{m_2^3} \delta^{(2)}(\|u\|_\infty, \lambda) D^2. \tag{34}$$

Proof. Use Lemma 14 and the definition of δ in (33):

$$\begin{aligned}\sum_j |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^6 H(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \lambda) \le& \frac{1}{m_2^3} \delta^{(2)}(\|u\|_\infty, \lambda) \sum_j m_2^3 |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^6 \\ \le& C \frac{\|u\|_\infty M_1}{m_2^3} \delta^{(2)}(\|u\|_\infty, \lambda) D^2.\end{aligned}$$

■

Now we have obtained all the estimates for comparing the Second Term with the First Term:

Theorem 3. *Suppose $f(u)$ satisfies **(H2)**. Then, there exists a continuous function $C_f(\cdot,\cdot) \ge 0$ on $\mathbb{R}^2$ depending on f such that under the CFL condition*

$$C_f(u_{\min}, u_{\max}) \frac{\Delta t}{\Delta x} \le 1,$$

we have L^2-stability:

$$\|u^{n+1}\|^2 - \|u\|^2 \le 0.$$

Proof. Putting inequalities (32) and (34) together into inequality (18) yields:

$$\frac{\lambda^2}{4} \|F[y_1] - F[y_2]\|^2 \le C \Delta x \lambda^4 \left(M_1^2 + \frac{M_1^2 M_2^2}{m_2^2} + \frac{\|u\|_\infty M_1^3 M_2^2}{m_2^5} \gamma^{(2)}(\|u\|_\infty) + \frac{\|u\|_\infty M_1}{m_2^3} \delta^{(2)}(\|u\|_\infty, \lambda) \right) \max\{D^2, E^2\}.$$

Combining the above with inequalities (9) and (16) and dividing by $\Delta x \lambda^3 \max\{D^2, E^2\}/16$, we obtain:

$$16 \frac{\|u^{n+1}\|^2 - \|u\|^2}{\Delta x \lambda^3 \max\{D^2, E^2\}} \le -M_1 + \lambda C \left(M_1^2 + \frac{M_1^2 M_2^2}{m_2^2} + \frac{\|u\|_\infty M_1^3 M_2^2}{m_2^5} \gamma^{(2)}(\|u\|_\infty) + \frac{\|u\|_\infty M_1}{m_2^3} \delta^{(2)}(\|u\|_\infty, \lambda) \right)$$

If $M_1 = 0$ then the right hand side of the above inequality is zero; if $M_1 > 0$, then divide both sides by M_1 to get

$$16 \frac{\|u^{n+1}\|^2 - \|u\|^2}{M_1 \Delta x \lambda^3 \max\{D^2, E^2\}} \le -1 + \lambda K^{(2)}(u_{\min}, u_{\max}), \quad \text{if } 0 \le \lambda \le 1 \tag{35}$$

where

$$K^{(2)}(u_{\min}, u_{\max}) := C \left(M_1 + \frac{M_1 M_2^2}{m_2^2} + \frac{\|u\|_\infty M_1^2 M_2^2}{m_2^5} \gamma^{(2)}(\|u\|_\infty) + \frac{\|u\|_\infty}{m_2^3} \max_{0 \le \lambda \le 1} \delta^{(2)}(\|u\|_\infty, \lambda) \right).$$

Recall that $\gamma^{(2)}(M) \ge 0$ is continuous on $[0,\infty)$ and $\delta^{(2)}(M,\lambda) \ge 0$ is continuous on $[0,\infty) \times \mathbb{R}$. Moreover, $M_n = M_n(u_{\min}, u_{\max})$, $n \ge 1$, $m_2 = m_2(u_{\min}, u_{\max}) > 0$, and $\|u\|_\infty = \max\{|u_{\min}|, |u_{\max}|\}$

are continuous functions on $\mathbb{R}^2$, so that $K^{(2)}(u_{\min}, u_{\max})$ is also nonnegative and continuous on $\mathbb{R}^2$. Taking (CFL1) and (CFL2) into account, the right hand side of (35) is nonpositive if

$$\max\left\{\frac{M_1}{r_0}, \frac{64}{3}\left(M_1 + 4\max_{|z|=2\|u\|_\infty}|f'(z)|\right), K^{(2)}(u_{\min}, u_{\max}), 1\right\}\frac{\Delta t}{\Delta x} \leq 1,$$

where $r_0 > 0$ is the constant from (CFL2). Thus, the proof is complete if we set

$$C_f(u_{\min}, u_{\max}) = \max\left\{\frac{M_1}{r_0}, \frac{64}{3}\left(M_1 + 4\max_{|z|=2\|u\|_\infty}|f'(z)|\right), K^{(2)}(u_{\min}, u_{\max}), 1\right\} \geq 0$$

which is a continuous function on $\mathbb{R}^2$. ■

7 Concluding remarks

In this paper we have shown L^2-stability of an explicit RKDG2 scheme approximating the nonlinear scalar conservation law $u_t + f(u)_x = 0$ for two types of convex $f(u)$ in one space dimension under the CFL condition $\Delta t \sim \Delta x$, with uniform mesh and periodic boundary conditions. For future work, it would be extremely desirable to extend the result to higher-order explicit RKDG methods and non-uniform mesh, in which case the computation would be much more complicated and might involve new types of cyclic inequalities like Theorem 1, which is designed for the specific algebraic structure of RKDG2 that varies greatly between different RKDG methods. It would also be interesting to generalize the result to non-convex f, but since the proof relies heavily on the cyclic inequality in Lemma 8 which does not always hold for non-convex f, new techniques need to be discovered.

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Appendix A Proof of Proposition 3

Lemma A.1. *Fix any $1 \le j \le N$. Let*

$$g^+(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}, u^-_{j+\frac{3}{2}}, u^+_{j+\frac{3}{2}}, u^-_{j+\frac{5}{2}}, u^+_{j+\frac{5}{2}}, \lambda) := F^+_{j+\frac{1}{2}}[u+\lambda F[u]] - F^+_{j+\frac{1}{2}}[u].$$

Then, there exists a constant $C > 0$ such that

$$\begin{aligned} g^+(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}, u^-_{j+\frac{3}{2}}, u^+_{j+\frac{3}{2}}, u^-_{j+\frac{5}{2}}, u^+_{j+\frac{5}{2}}, \lambda) =& g^+(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{3}{2}}, \bar{u}_{j+\frac{3}{2}}, \bar{u}_{j+\frac{5}{2}}, \bar{u}_{j+\frac{5}{2}}, \lambda) \\ &+ R^+_{j+\frac{1}{2}}, \end{aligned}$$

where

$$|R^+_{j+\frac{1}{2}}| \le CM_1 \left(|[\![u]\!]_{j-\frac{1}{2}}| + |[\![u]\!]_{j+\frac{1}{2}}| + |[\![u]\!]_{j+\frac{3}{2}}| + |[\![u]\!]_{j+\frac{5}{2}}| \right)$$

whenever (CFL1) is satisfied.

Proof. Set

$$\begin{aligned} x_1 &= u^-_{j-\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}, \quad x_2 = u^+_{j-\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}} \\ x_3 &= u^-_{j+\frac{1}{2}} - \bar{u}_{j+\frac{1}{2}}, \quad x_4 = u^+_{j+\frac{1}{2}} - \bar{u}_{j+\frac{1}{2}} \\ x_5 &= u^-_{j+\frac{3}{2}} - \bar{u}_{j+\frac{3}{2}}, \quad x_6 = u^+_{j+\frac{3}{2}} - \bar{u}_{j+\frac{3}{2}} \\ x_7 &= u^-_{j+\frac{5}{2}} - \bar{u}_{j+\frac{5}{2}}, \quad x_8 = u^+_{j+\frac{5}{2}} - \bar{u}_{j+\frac{5}{2}}. \end{aligned}$$

$$u_1 = u_2 = \bar{u}_{j-\frac{1}{2}}, \quad u_3 = u_4 = \bar{u}_{j+\frac{1}{2}}, \quad u_5 = u_6 = \bar{u}_{j+\frac{3}{2}}, \quad u_7 = u_8 = \bar{u}_{j+\frac{5}{2}}.$$

Fix λ, u_i's and x_i's. Let $\mathbf{u} = (u_1, u_2, \cdots, u_8)$, $\mathbf{x} = (x_1, x_2, \cdots, x_8)$. Let s to be a real variable ranging from $[0,1]$. Define a single variable function:

$$\begin{aligned} h(s) :=& g^+(u_1 + sx_1, u_2 + sx_2, u_3 + sx_3, u_4 + sx_4, u_5 + sx_5, u_6 + sx_6, u_7 + sx_7, u_8 + sx_8, \lambda) \\ =& g^+(\mathbf{u} + s\mathbf{x}, \lambda). \end{aligned}$$

Then,

$$\begin{aligned} h(0) &= g^+(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{3}{2}}, \bar{u}_{j+\frac{3}{2}}, \bar{u}_{j+\frac{5}{2}}, \bar{u}_{j+\frac{5}{2}}, \lambda) \\ h(1) &= g^+(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}, u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}, u^-_{j+\frac{3}{2}}, u^+_{j+\frac{3}{2}}, u^-_{j+\frac{5}{2}}, u^+_{j+\frac{5}{2}}, \lambda). \end{aligned}$$

By Mean Value Theorem, we have

$$h(1) = h(0) + R^+_{j+\frac{1}{2}},$$

where

$$R^+_{j+\frac{1}{2}} = \frac{\mathrm{d}h}{\mathrm{d}s}(s_0) = \sum_{i=1}^{8} \partial_i g^+(\mathbf{u} + s_0\mathbf{x}, \lambda) x_i.$$

where $0 \le s_0 \le 1$. Fix such a s_0. Define

$$\begin{aligned}
\tilde{u}^-_{j-\frac{1}{2}} &= u_1 + s_0 x_1, \quad & \tilde{u}^+_{j-\frac{1}{2}} &= u_2 + s_0 x_2\\
\tilde{u}^-_{j+\frac{1}{2}} &= u_3 + s_0 x_3, \quad & \tilde{u}^+_{j+\frac{1}{2}} &= u_4 + s_0 x_4\\
\tilde{u}^-_{j+\frac{3}{2}} &= u_5 + s_0 x_5, \quad & \tilde{u}^+_{j+\frac{3}{2}} &= u_6 + s_0 x_6\\
\tilde{u}^-_{j+\frac{5}{2}} &= u_7 + s_0 x_7, \quad & \tilde{u}^+_{j+\frac{5}{2}} &= u_8 + s_0 x_8.
\end{aligned}$$

Clearly, each $\tilde{u}^\pm$ of the above still lies in $[m,M]$. Since in our current situation the only meaningful values of $u^\pm$ are those at the cell boundaries $j-\frac{1}{2}, j+\frac{1}{2}, j+\frac{3}{2}, j+\frac{5}{2}$, we can view $\tilde{u}$ as a replacement of the original numerical solution u. Thus,

$$|\tilde{u}_1 - \tilde{u}_2| \le \sigma, \quad |f^{(n)}(\tilde{u}_1)| \le M_n; \tag{A.1}$$

by Lemma 2,

$$|F^\pm_{j+\frac{1}{2}}[\tilde{u}]|, |F^\pm_{j+\frac{3}{2}}[\tilde{u}]| \le \frac{32}{3}\sigma M_1; \tag{A.2}$$

by Lemma 4,

$$\left|\frac{\partial^n \bar{f}}{\partial u^i \partial v^j}(\tilde{u}_1, \tilde{u}_2)\right| \le \frac{i!j!}{(n+1)!}M_n, \quad \forall n \ge 0 \text{ and } i+j=n,$$

where $\tilde{u}_1, \tilde{u}_2$ stand for any one of the 8 boundary values of $\tilde{u}$ defined above.

Now, we are able to estimate:

$$\begin{aligned}
\left|f'(\tilde{u}^\pm_{j+\frac{1}{2}} + \lambda F^\pm_{j+\frac{1}{2}}[\tilde{u}]) - f'(\tilde{u}^\pm_{j+\frac{1}{2}})\right| &= \left|\sum_{k=1}^{\infty} \frac{f^{(k+1)}(\tilde{u}^\pm_{j+\frac{1}{2}})}{k!}\left(\lambda F^\pm_{j+\frac{1}{2}}[\tilde{u}]\right)^k\right|\\
&\le \sum_{k=1}^{\infty} \frac{M_{k+1}}{k!}\left(\lambda \frac{32}{3}\sigma M_1\right)^k \quad \text{(using (A.1) and (A.2))}\\
&= P(\lambda \frac{32}{3} M_1) \le \frac{M_1}{2} \quad \text{(using Lemma 1 and (CFL1))}
\end{aligned}$$

so that

$$\left|f'(\tilde{u}^\pm_{j+\frac{1}{2}} + \lambda F^\pm_{j+\frac{1}{2}}[\tilde{u}])\right| \le \left|f'(\tilde{u}^\pm_{j+\frac{1}{2}})\right| + \frac{M_1}{2} \overset{(A.1)}{=} \frac{3}{2}M_1.$$

Similarly we have

$$\left|f'(\tilde{u}^\pm_{j+\frac{3}{2}} + \lambda F^\pm_{j+\frac{3}{2}}[\tilde{u}])\right| \le \frac{3}{2}M_1.$$

The above two inequalities clearly also hold if λ is replaced by any number between 0 and λ. Again, by Lemma 4:

$$\begin{aligned}
\left|\frac{\partial \bar{f}}{\partial u}(\tilde{u}^+_{j+\frac{1}{2}} + \lambda F^+_{j+\frac{1}{2}}[\tilde{u}], \tilde{u}^-_{j+\frac{3}{2}} + \lambda F^-_{j+\frac{3}{2}}[\tilde{u}])\right| &\le \frac{1}{2} \cdot \frac{3}{2}M_1\\
\left|\frac{\partial \bar{f}}{\partial v}(\tilde{u}^+_{j+\frac{1}{2}} + \lambda F^+_{j+\frac{1}{2}}[\tilde{u}], \tilde{u}^-_{j+\frac{3}{2}} + \lambda F^-_{j+\frac{3}{2}}[\tilde{u}])\right| &\le \frac{1}{2} \cdot \frac{3}{2}M_1
\end{aligned}$$

Up to now, we have obtained the estimates for all the terms involved in $\partial_i g^+$, which yields:

$$
\begin{aligned}
|R^+_{j+\frac{1}{2}}| &\le \sum_{i=1}^{8} |x_i||\partial_i g^+(\mathbf{u}+s_0\mathbf{x},\lambda)| \\
&\le C\sum_{i=1}^{8} |x_i| M_1(1+\lambda M_1) \\
&\overset{(CFL1)}{\le} CM_1\left(|[\![u]\!]_{j-\frac{1}{2}}|+|[\![u]\!]_{j+\frac{1}{2}}|+|[\![u]\!]_{j+\frac{3}{2}}|+|[\![u]\!]_{j+\frac{5}{2}}|\right).
\end{aligned}
$$

∎

In the same way, we can prove

Lemma A.2. *Fix any $1 \le j \le N$. Let*

$$
g^-(u^-_{j-\frac{3}{2}},u^+_{j-\frac{3}{2}},u^-_{j-\frac{1}{2}},u^+_{j-\frac{1}{2}},u^-_{j+\frac{1}{2}},u^+_{j+\frac{1}{2}},u^-_{j+\frac{3}{2}},u^+_{j+\frac{3}{2}},\lambda) := F^-_{j+\frac{1}{2}}[u+\lambda F[u]] - F^-_{j+\frac{1}{2}}[u].
$$

Then, there exists a constant $C>0$ such that

$$
\begin{aligned}
g^-(u^-_{j-\frac{3}{2}},u^+_{j-\frac{3}{2}},u^-_{j-\frac{1}{2}},u^+_{j-\frac{1}{2}},u^-_{j+\frac{1}{2}},u^+_{j+\frac{1}{2}},u^-_{j+\frac{3}{2}},u^+_{j+\frac{3}{2}},\lambda) =& g^-(\bar u_{j-\frac{3}{2}},\bar u_{j-\frac{3}{2}},\bar u_{j-\frac{1}{2}},\bar u_{j-\frac{1}{2}},\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{3}{2}},\bar u_{j+\frac{3}{2}},\lambda) \\
&+ R^-_{j+\frac{1}{2}},
\end{aligned}
$$

where

$$
|R^-_{j+\frac{1}{2}}| \le CM_1\left(|[\![u]\!]_{j-\frac{3}{2}}|+|[\![u]\!]_{j-\frac{1}{2}}|+|[\![u]\!]_{j+\frac{1}{2}}|+|[\![u]\!]_{j+\frac{3}{2}}|\right)
$$

whenever (CFL1) is satisfied.

Proof. The proof is the same as that of Lemma A.1. ∎

Lemma A.3. *Fix any $1 \le j \le N$. Assume g^+ is defined to be the same as in Lemma A.1. Let*

$$
\begin{aligned}
G^+_{j+\frac{1}{2}} :=& 4f\left(\bar u_{j+\frac{1}{2}}+\lambda\left[4f(\bar u_{j+\frac{1}{2}})+2f(\bar u_{j+\frac{3}{2}})-6\bar f(\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{3}{2}})\right]\right) \\
&+2f\left(\bar u_{j+\frac{3}{2}}+\lambda\left[-2f(\bar u_{j+\frac{1}{2}})-4f(\bar u_{j+\frac{3}{2}})+6\bar f(\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{3}{2}})\right]\right) \\
&-6\bar f\Big(\bar u_{j+\frac{1}{2}}+\lambda\left[4f(\bar u_{j+\frac{1}{2}})+2f(\bar u_{j+\frac{3}{2}})-6\bar f(\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{3}{2}})\right], \\
&\qquad \bar u_{j+\frac{3}{2}}+\lambda\left[-2f(\bar u_{j+\frac{1}{2}})-4f(\bar u_{j+\frac{3}{2}})+6\bar f(\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{3}{2}})\right]\Big) \\
&-4f(\bar u_{j+\frac{1}{2}})-2f(\bar u_{j+\frac{3}{2}})+6\bar f(\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{3}{2}}).
\end{aligned}
$$

Then,

$$
g^+(\bar u_{j-\frac{1}{2}},\bar u_{j-\frac{1}{2}},\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{1}{2}},\bar u_{j+\frac{3}{2}},\bar u_{j+\frac{3}{2}},\bar u_{j+\frac{5}{2}},\bar u_{j+\frac{5}{2}},\lambda) = G^+_{j+\frac{1}{2}} + \tilde R^+_{j+\frac{1}{2}},
$$

where

$$
|\tilde R^+_{j+\frac{1}{2}}| \le 7\lambda M_1\left(2|D_{j+\frac{1}{2}}|+|D_{j+\frac{3}{2}}|\right)
$$

whenever (CFL1) is satisfied.

Proof. By direct computation,

$$\begin{aligned}\tilde{R}^{+}_{j+\frac{1}{2}} =&\lambda\alpha\left(4D_{j+\frac{1}{2}}+2D_{j+\frac{3}{2}}\right)\\ &+2f\left(\bar{u}_{j+\frac{1}{2}}+\lambda\left[-2f(\bar{u}_{j-\frac{1}{2}})-4f(\bar{u}_{j+\frac{1}{2}})+6\bar{f}(\bar{u}_{j-\frac{1}{2}},\bar{u}_{j+\frac{1}{2}})\right]\right)\\ &-2f\left(\bar{u}_{j+\frac{1}{2}}+\lambda\left[4f(\bar{u}_{j+\frac{1}{2}})+2f(\bar{u}_{j+\frac{3}{2}})-6\bar{f}(\bar{u}_{j+\frac{1}{2}},\bar{u}_{j+\frac{3}{2}})\right]\right)\\ &+f\left(\bar{u}_{j+\frac{3}{2}}+\lambda\left[4f(\bar{u}_{j+\frac{3}{2}})+2f(\bar{u}_{j+\frac{5}{2}})-6\bar{f}(\bar{u}_{j+\frac{3}{2}},\bar{u}_{j+\frac{5}{2}})\right]\right)\\ &-f\left(\bar{u}_{j+\frac{3}{2}}+\lambda\left[-2f(\bar{u}_{j+\frac{1}{2}})-4f(\bar{u}_{j+\frac{3}{2}})+6\bar{f}(\bar{u}_{j+\frac{1}{2}},\bar{u}_{j+\frac{3}{2}})\right]\right).\end{aligned}$$

The first term is good. By Mean Value Theorem, the sum of the second and third term equals to:

$$4\lambda f'(\bar{u}_{j+\frac{1}{2}}+\xi)D_{j+\frac{1}{2}},$$

where ξ lies between $\lambda\left[-2f(\bar{u}_{j-\frac{1}{2}})-4f(\bar{u}_{j+\frac{1}{2}})+6\bar{f}(\bar{u}_{j-\frac{1}{2}},\bar{u}_{j+\frac{1}{2}})\right]$ and $\lambda\left[4f(\bar{u}_{j+\frac{1}{2}})+2f(\bar{u}_{j+\frac{3}{2}})-6\bar{f}(\bar{u}_{j+\frac{1}{2}},\bar{u}_{j+\frac{3}{2}})\right]$, which is further contained in the interval $[-\frac{32}{3}\sigma M_1\lambda, \frac{32}{3}\sigma M_1\lambda]$ by Lemma 3. Therefore,

$$\begin{aligned}|f'(\bar{u}_{j+\frac{1}{2}}+\xi)-f'(\bar{u}_{j+\frac{1}{2}})| =&\left|\sum_{k=1}^{\infty}\frac{f^{(k+1)}(\bar{u}_{j+\frac{1}{2}})}{k!}\xi^k\right|\\ &\leq\sum_{k=1}^{\infty}\frac{M_{k+1}}{k!}\left(\lambda\frac{32}{3}\sigma M_1\right)^k\\ &=P(\lambda\frac{32}{3}M_1)\leq\frac{M_1}{2}\quad\text{(using Lemma 1 and (CFL1))},\end{aligned}$$

so that

$$|f'(\bar{u}_{j+\frac{1}{2}}+\xi)|\leq|f'(\bar{u}_{j+\frac{1}{2}})|+\frac{M_1}{2}=\frac{3}{2}M_1,$$

which implies the sum of the second and third term is bounded by $6\lambda M_1|D_{j+\frac{1}{2}}|$. Perform the same argument to the fourth and fifth term, and we eventually obtain:

$$|\tilde{R}^{+}_{j+\frac{1}{2}}|\leq 7\lambda M_1\left(2|D_{j+\frac{1}{2}}|+|D_{j+\frac{3}{2}}|\right).$$

■

In the same way, we can prove:

Lemma A.4. *Fix any $1 \le j \le N$. Assume g^- is defined to be the same as in Lemma A.2. Let*

$$
\begin{aligned}
G^-_{j+\frac{1}{2}} := &-2f\Big(\bar{u}_{j-\frac{1}{2}} + \lambda\big[4f(\bar{u}_{j-\frac{1}{2}}) + 2f(\bar{u}_{j+\frac{1}{2}}) - 6\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}})\big]\Big) \\
&-4f\Big(\bar{u}_{j+\frac{1}{2}} + \lambda\big[-2f(\bar{u}_{j-\frac{1}{2}}) - 4f(\bar{u}_{j+\frac{1}{2}}) + 6\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}})\big]\Big) \\
&+6\bar{f}\Big(\bar{u}_{j-\frac{1}{2}} + \lambda\big[4f(\bar{u}_{j-\frac{1}{2}}) + 2f(\bar{u}_{j+\frac{1}{2}}) - 6\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}})\big], \\
&\qquad \bar{u}_{j+\frac{1}{2}} + \lambda\big[-2f(\bar{u}_{j-\frac{1}{2}}) - 4f(\bar{u}_{j+\frac{1}{2}}) + 6\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}})\big]\Big) \\
&+2f(\bar{u}_{j-\frac{1}{2}}) + 4f(\bar{u}_{j+\frac{1}{2}}) - 6\bar{f}(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}).
\end{aligned}
$$

Then,

$$
g^-(\bar{u}_{j-\frac{3}{2}}, \bar{u}_{j-\frac{3}{2}}, \bar{u}_{j-\frac{1}{2}}, \bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{3}{2}}, \bar{u}_{j+\frac{3}{2}}, \lambda) = G^-_{j+\frac{1}{2}} + \tilde{R}^-_{j+\frac{1}{2}},
$$

where

$$
|\tilde{R}^-_{j+\frac{1}{2}}| \le 7\lambda M_1\Big(2|D_{j+\frac{1}{2}}| + |D_{j-\frac{1}{2}}|\Big)
$$

whenever (CFL1) is satisfied.

Proof. The proof is the same as that of Lemma A.3. ■

Recall:

$$
E^2 = \frac{1}{\lambda^2}\sum_j |[\![u]\!]_{j+\frac{1}{2}}|^2, \quad D^2 = \sum_j D^2_{j+\frac{1}{2}}.
$$

Lemma A.5. *Assume (CFL1) is satisfied. Then, there exists a constant $C > 0$ such that*

$$
\begin{aligned}
\sum_j \big(F^-_{j+\frac{1}{2}}[y_2] - F^-_{j+\frac{1}{2}}[y_1]\big)^2 + \sum_j \big(F^+_{j+\frac{1}{2}}[y_2] - F^+_{j+\frac{1}{2}}[y_1]\big)^2 \le& 2\sum_j |G^-_{j+\frac{1}{2}}|^2 + 2\sum_j |G^+_{j+\frac{1}{2}}|^2 \\
&+ C\lambda^2 M_1^2(E^2 + D^2),
\end{aligned}
$$

where $G^{\pm}_{j+\frac{1}{2}}$ is the same as in Lemma A.3, A.4.

Proof. Combine Lemma A.1, A.2, A.3, A.4 and use Jensen's inequality. ■

Lemma A.6. *Fix any $1 \le j \le N$. Let $G^+_{j+\frac{1}{2}}$ to be the same as in Lemma A.3. Then,*

$$
G^+_{j+\frac{1}{2}} = 2\lambda f'(\bar{u}_{j+\frac{1}{2}}) f''(\bar{u}_{j+\frac{1}{2}})(\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^2 + \lambda(\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}})^3 H^+(\bar{u}_{j+\frac{1}{2}}, \bar{u}_{j+\frac{3}{2}}, \lambda).
$$

where $H^+(u, v, \lambda)$ is a smooth function on $\mathbb{R}^3$. Moreover, if $f(u) = Cu^p + \mathscr{O}(u^{p+1})$ for some $p > 2$ is analytic near $u = 0$, then

$$
\lim_{(u,v)\to(0,0)} \frac{|H^+(u,v,\lambda)|^2}{(\sqrt{u^2+v^2})^{3p-6}} = 0.
$$

Proof. Define

$$\begin{aligned}
g(x,y,\lambda) = & 4f\Big(y+\lambda\big[4f(y)+2f(x+y)-6\bar{f}(y,x+y)\big]\Big) \\
& +2f\Big(x+y+\lambda\big[-2f(y)-4f(x+y)+6\bar{f}(y,x+y)\big]\Big) \\
& -6\bar{f}\Big(y+\lambda\big[4f(y)+2f(x+y)-6\bar{f}(y,x+y)\big], x+y+\lambda\big[-2f(y)-4f(x+y)+6\bar{f}(y,x+y)\big]\Big) \\
& -4f(y)-2f(x+y)+6\bar{f}(y,x+y).
\end{aligned}$$

If we set

$$\begin{aligned}
x &= \bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}} \\
y &= \bar{u}_{j+\frac{1}{2}},
\end{aligned}$$

then

$$\begin{aligned}
\bar{u}_{j+\frac{1}{2}} &= y \\
\bar{u}_{j+\frac{3}{2}} &= x+y
\end{aligned}$$

and

$$g(x,y) = G^{+}_{j+\frac{1}{2}}.$$

Let

$$\begin{aligned}
h_1(x,y) &:= 4f(y)+2f(x+y)-6\bar{f}(y,x+y) \\
h_2(x,y) &:= -2f(y)-4f(x+y)+6\bar{f}(y,x+y).
\end{aligned}$$

Clearly,

$$g(0,y,\lambda) = g(x,y,0) = h_1(0,y) = h_2(0,y) = 0, \quad \forall x,y,\lambda \in \mathbb{R}$$

By direct computation, we have for $n \geq 1$:

$$\begin{aligned}
\frac{\partial^n h_1}{\partial x^n}(x,y) &= 2f^{(n)}(x+y) - 6\frac{\partial^n \bar{f}}{\partial v^n}(y,x+y) \\
\frac{\partial^n h_2}{\partial x^n}(x,y) &= -4f^{(n)}(x+y) + 6\frac{\partial^n \bar{f}}{\partial v^n}(y,x+y)
\end{aligned}$$

Here is a list of $\frac{\partial^n g}{\partial x^n}$ up to third order (for those who are interested):

$$\begin{aligned}
\frac{\partial g}{\partial x}(x,y,\lambda) = & 4f'\big(y+\lambda h_1\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right] + 2f'\big(x+y+\lambda h_2(x,y)\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right] \\
& -6\frac{\partial \bar{f}}{\partial u}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right] \\
& -6\frac{\partial \bar{f}}{\partial v}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right] - \frac{\partial h_1}{\partial x}.
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 g}{\partial x^2}(x,y,\lambda) =& 4f''\big(y+\lambda h_1\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]^2 + 4f'\big(y+\lambda h_1\big)\cdot\left[\lambda\frac{\partial^2 h_1}{\partial x^2}\right] \\
&+2f''\big(x+y+\lambda h_2\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^2 + 2f'\big(x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial^2 h_2}{\partial x^2}\right] \\
&-6\frac{\partial^2 \bar f}{\partial u^2}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]^2 \\
&-12\frac{\partial^2 \bar f}{\partial u\,\partial v}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right] \\
&-6\frac{\partial^2 \bar f}{\partial v^2}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^2 \\
&-6\frac{\partial \bar f}{\partial u}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial^2 h_1}{\partial x^2}\right] \\
&-6\frac{\partial \bar f}{\partial v}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial^2 h_2}{\partial x^2}\right] - \frac{\partial^2 h_1}{\partial x^2}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 g}{\partial x^3}(x,y,\lambda) =& 4f'''\big(y+\lambda h_1\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]^3 + 12f''\big(y+\lambda h_1\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]\cdot\left[\lambda\frac{\partial^2 h_1}{\partial x^2}\right] \\
&+4f'\big(y+\lambda h_1\big)\cdot\left[\lambda\frac{\partial^3 h_1}{\partial x^3}\right] + 2f'''\big(x+y+\lambda h_2\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^3 \\
&+6f''\big(x+y+\lambda h_2\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right]\cdot\left[\lambda\frac{\partial^2 h_2}{\partial x^2}\right] + 2f'\big(x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial^3 h_2}{\partial x^3}\right] \\
&-6\frac{\partial^3 \bar f}{\partial u^3}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]^3 \\
&-18\frac{\partial^3 \bar f}{\partial u^2\,\partial v}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]^2\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right] \\
&-18\frac{\partial^3 \bar f}{\partial u\,\partial v^2}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^2 \\
&-6\frac{\partial^3 \bar f}{\partial v^3}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^3 \\
&-18\frac{\partial^2 \bar f}{\partial u^2}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]\cdot\left[\lambda\frac{\partial^2 h_1}{\partial x^2}\right] \\
&-18\frac{\partial^2 \bar f}{\partial u\,\partial v}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial^2 h_1}{\partial x^2}\right]\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right] \\
&-18\frac{\partial^2 \bar f}{\partial u\,\partial v}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial h_1}{\partial x}\right]\cdot\left[\lambda\frac{\partial^2 h_2}{\partial x^2}\right] \\
&-18\frac{\partial^2 \bar f}{\partial v^2}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[1+\lambda\frac{\partial h_2}{\partial x}\right]\cdot\left[\lambda\frac{\partial^2 h_2}{\partial x^2}\right] \\
&-6\frac{\partial \bar f}{\partial u}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial^3 h_1}{\partial x^3}\right] \\
&-6\frac{\partial \bar f}{\partial v}\big(y+\lambda h_1, x+y+\lambda h_2\big)\cdot\left[\lambda\frac{\partial^3 h_2}{\partial x^3}\right] - \frac{\partial^3 h_1}{\partial x^3}.
\end{aligned}
$$

We also need:

$$
\begin{aligned}
\frac{\partial^4 g}{\partial x^3 \partial \lambda}(x,y,\lambda) =& 4\lambda^3 f^{(4)}(y+\lambda h_1)h_1\left[\frac{\partial h_1}{\partial x}\right]^3 + 12\lambda^2 f^{(3)}(y+\lambda h_1)\left[\frac{\partial h_1}{\partial x}\right]^3 \\
&+ 12\lambda^2 f^{(3)}(y+\lambda h_1)h_1\left[\frac{\partial h_1}{\partial x}\right]\left[\frac{\partial^2 h_1}{\partial x^2}\right] + 24\lambda f^{(2)}(y+\lambda h_1)\left[\frac{\partial h_1}{\partial x}\right]\left[\frac{\partial^2 h_1}{\partial x^2}\right] \\
&+ 4\lambda f^{(2)}(y+\lambda h_1)h_1\left[\frac{\partial^3 h_1}{\partial x^3}\right] + 4f^{(1)}(y+\lambda h_1)\left[\frac{\partial^3 h_1}{\partial x^3}\right] \\
&+ 2f^{(4)}(x+y+\lambda h_2)h_2\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^3 + 6f^{(3)}(x+y+\lambda h_2)\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^2\left[\frac{\partial h_2}{\partial x}\right] \\
&+ 6f^{(3)}(x+y+\lambda h_2)h_2\left[1+\lambda\frac{\partial h_2}{\partial x}\right]\left[\lambda\frac{\partial^2 h_2}{\partial x^2}\right] + 6f^{(2)}(x+y+\lambda h_2)\left[1+2\lambda\frac{\partial h_2}{\partial x}\right]\left[\frac{\partial^2 h_2}{\partial x^2}\right] \\
&+ 2\lambda f^{(2)}(x+y+\lambda h_2)h_2\left[\frac{\partial^3 h_2}{\partial x^3}\right] + 2f^{(1)}(x+y+\lambda h_2)\left[\frac{\partial^3 h_2}{\partial x^3}\right] \\
&- 6\lambda^3\left[\frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^4}h_1 + \frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^3 \partial v}h_2\right]\left[\frac{\partial h_1}{\partial x}\right]^3 \\
&- 18\lambda^2\frac{\partial^3 \bar f}{\partial u^3}(y+\lambda h_1, x+y+\lambda h_2)\left[\frac{\partial h_1}{\partial x}\right]^3 \\
&- 18\left[\frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^3 \partial v}h_1 + \frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^2 \partial v^2}h_2\right]\left[\lambda\frac{\partial h_1}{\partial x}\right]^2\left[1+\lambda\frac{\partial h_2}{\partial x}\right] \\
&- 18\frac{\partial^4 \bar f}{\partial u^3 \partial v}(y+\lambda h_1, x+y+\lambda h_2)\left[\frac{\partial h_1}{\partial x}\right]^2\left[2\lambda+3\lambda^2\frac{\partial h_2}{\partial x}\right] \\
&- 18\left[\frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^2 \partial v^2}h_1 + \frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u \partial v^3}h_2\right]\left[\lambda\frac{\partial h_1}{\partial x}\right]\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^2 \\
&- 18\frac{\partial^3 \bar f}{\partial u \partial v^2}(y+\lambda h_1, x+y+\lambda h_2)\left[\frac{\partial h_1}{\partial x}\right]\left[1+3\lambda\frac{\partial h_2}{\partial x}\right]\left[1+\lambda\frac{\partial h_2}{\partial x}\right] \\
&- 6\left[\frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u \partial v^3}h_1 + \frac{\partial^4 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial v^4}h_2\right]\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^3 \\
&- 18\frac{\partial^3 \bar f}{\partial v^3}(y+\lambda h_1, x+y+\lambda h_2)\left[1+\lambda\frac{\partial h_2}{\partial x}\right]^2\left[\frac{\partial h_2}{\partial x}\right] \\
&- 18\lambda^2\left[\frac{\partial^3 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^3}h_1 + \frac{\partial^3 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^2 \partial v}h_2\right]\left[\frac{\partial h_1}{\partial x}\right]\left[\frac{\partial^2 h_1}{\partial x^2}\right] \\
&- 36\lambda\frac{\partial^2 \bar f}{\partial u^2}(y+\lambda h_1, x+y+\lambda h_2)\left[\frac{\partial h_1}{\partial x}\right]\left[\frac{\partial^2 h_1}{\partial x^2}\right] \\
&- 18\left[\frac{\partial^3 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^2 \partial v}h_1 + \frac{\partial^3 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u \partial v^2}h_2\right]\left[\lambda\frac{\partial^2 h_1}{\partial x^2}\right]\left[1+\lambda\frac{\partial h_2}{\partial x}\right] \\
&- 18\frac{\partial^2 \bar f}{\partial u \partial v}(y+\lambda h_1, x+y+\lambda h_2)\left[\frac{\partial^2 h_1}{\partial x^2}\right]\left[1+2\lambda\frac{\partial h_2}{\partial x}\right] \\
&- 18\lambda^2\left[\frac{\partial^3 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u^2 \partial v}h_1 + \frac{\partial^3 \bar f(y+\lambda h_1, x+y+\lambda h_2)}{\partial u \partial v^2}h_2\right]\left[\frac{\partial h_1}{\partial x}\right]\left[\frac{\partial^2 h_2}{\partial x^2}\right] \\
&- 36\lambda\frac{\partial^2 \bar f}{\partial u \partial v}(y+\lambda h_1, x+y+\lambda h_2)\left[\frac{\partial h_1}{\partial x}\right]\left[\frac{\partial^2 h_2}{\partial x^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-18\left[\frac{\partial^3 \bar f\big(y+\lambda h_1,x+y+\lambda h_2\big)}{\partial u\,\partial v^2}h_1+\frac{\partial^3 \bar f\big(y+\lambda h_1,x+y+\lambda h_2\big)}{\partial v^3}h_2\right]\left[1+\lambda\frac{\partial h_2}{\partial x}\right]\left[\lambda\frac{\partial^2 h_2}{\partial x^2}\right]\\
&-18\frac{\partial^2 \bar f}{\partial v^2}\big(y+\lambda h_1,x+y+\lambda h_2\big)\left[1+2\lambda\frac{\partial h_2}{\partial x}\right]\cdot\left[\frac{\partial^2 h_2}{\partial x^2}\right]\\
&-6\lambda\left[\frac{\partial^2 \bar f\big(y+\lambda h_1,x+y+\lambda h_2\big)}{\partial u^2}h_1+\frac{\partial^2 \bar f\big(y+\lambda h_1,x+y+\lambda h_2\big)}{\partial u\,\partial v}h_2\right]\left[\frac{\partial^3 h_1}{\partial x^3}\right]\\
&-6\frac{\partial \bar f}{\partial u}\big(y+\lambda h_1,x+y+\lambda h_2\big)\left[\frac{\partial^3 h_1}{\partial x^3}\right]\\
&-6\lambda\left[\frac{\partial^2 \bar f\big(y+\lambda h_1,x+y+\lambda h_2\big)}{\partial u\,\partial v}h_1+\frac{\partial^2 \bar f\big(y+\lambda h_1,x+y+\lambda h_2\big)}{\partial v^2}h_2\right]\left[\frac{\partial^3 h_2}{\partial x^3}\right]\\
&-6\frac{\partial \bar f}{\partial v}\big(y+\lambda h_1,x+y+\lambda h_2\big)\left[\frac{\partial^3 h_2}{\partial x^3}\right].
\end{aligned}
$$

By Lemma 4, we have for $\forall n\geq 1$ and $\forall x,y,\lambda\in\mathbb{R}$:

$$
\begin{aligned}
&\frac{\partial^n h_1}{\partial x^n}(0,y)=\frac{2n-4}{n+1}f^{(n)}(y)\\
&\frac{\partial^n h_2}{\partial x^n}(0,y)=\frac{-4n+2}{n+1}f^{(n)}(y)\\
&\frac{\partial^n \bar f}{\partial u^k\,\partial v^{n-k}}(y,y)=\frac{k!(n-k)!}{(n+1)!}f^{(n)}(y),\quad 0\leq k\leq n\\
&\frac{\partial g}{\partial x}(0,y,\lambda)\equiv 0\\
&\frac{\partial^2 g}{\partial x^2}(0,y,\lambda)=4\lambda f'(y)f''(y)\\
&\frac{\partial^n g}{\partial x^n}(x,y,0)\equiv 0,\quad \forall n\geq 0\quad \text{this is because } g(x,y,0)\equiv 0.
\end{aligned}
$$

Fix y,λ and expand $g(x,y,\lambda)$ into Taylor series of x at $x=0$ with integral remainder:

$$
\begin{aligned}
g(x,y,\lambda)=&2\lambda f'(y)f''(y)x^2+\frac{x^3}{2}\int_0^1(1-s)^2\frac{\partial^3 g}{\partial x^3}(sx,y,\lambda)\,\mathrm{d}s\\
=&2\lambda f'(y)f''(y)x^2+\frac{\lambda x^3}{2}\int_0^1(1-s)^2\left(\int_0^1\frac{\partial^4 g}{\partial x^3\,\partial\lambda}(sx,y,t\lambda)\,\mathrm{d}t\right)\mathrm{d}s\\
=&2\lambda f'(y)f''(y)x^2+\lambda x^3H^+(x,x+y,\lambda).
\end{aligned}
$$

where

$$
H^+(x,x+y,\lambda):=\frac{1}{2}\int_0^1(1-s)^2\left(\int_0^1\frac{\partial^4 g}{\partial x^3\,\partial\lambda}(sx,y,t\lambda)\,\mathrm{d}t\right)\mathrm{d}s
$$

is a smooth function of x,y,λ on $\mathbb{R}^3$ since g is smooth. Now, suppose $f(u)=Cu^p+\mathscr{O}(u^{p+1})$ for some $p>2$ is analytic near $u=0$. Set $x=r\cos(\theta)$, $x+y=r\sin(\theta)$, then using Lemma 4 we find as $r \to 0$:

$$\begin{aligned}
&f^{(n)}(x) = \mathscr{O}(r^{p-n}), \quad f^{(n)}(x+y) = \mathscr{O}(r^{p-n}) \\
&\frac{\partial^n \bar{f}}{\partial u^i \partial v^j}(x, x+y) = \mathscr{O}(r^{p-n}) \\
&h_{1,2}(x,y) = \mathscr{O}(r^p), \quad \frac{\partial^n h_{1,2}}{\partial x^n}(x,y) = \mathscr{O}(r^{p-n}) \\
&f^{(n)}(x+\lambda h_1) = \mathscr{O}(r^{p-n}), \quad f^{(n)}(x+y+\lambda h_2) = \mathscr{O}(r^{p-n}) \\
&\frac{\partial^n \bar{f}}{\partial u^i \partial v^j}(x+\lambda h_1, x+y+\lambda h_2) = \mathscr{O}(r^{p-n}).
\end{aligned}$$

If λ is fixed, direct computation yields:

$$H^+(x, x+y, \lambda) \leq Cr^{2p-4} \quad \text{as } r \to 0.$$

Therefore,

$$\frac{|H^+(x, x+y, \lambda)|^2}{r^{3p-6}} \leq Cr^{p-2} \to 0 \quad \text{as } r \to 0$$

and the lemma is proved. ■

Following the same argument we can also prove

Lemma A.7. *Fix any $1 \leq j \leq N$. Let $G^-_{j+\frac{1}{2}}$ to be the same as in Lemma A.4. Then,*

$$G^-_{j+\frac{1}{2}} = 2\lambda f'(\bar{u}_{j+\frac{1}{2}}) f''(\bar{u}_{j+\frac{1}{2}})(\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^2 + \lambda(\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^3 H^-(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \lambda).$$

where $H^-(u,v,\lambda)$ is a smooth function on $\mathbb{R}^3$. Moreover, if $f(u) = Cu^p + \mathscr{O}(u^{p+1})$ for some $p > 2$ is analytic near $u = 0$, then

$$\lim_{(u,v)\to(0,0)} \frac{|H^-(u,v,\lambda)|^2}{(\sqrt{u^2+v^2})^{3p-6}} = 0.$$

Proof. The proof is the same as that of Lemma A.6. ■

Lemma A.8. *Let $G^\pm_{j+\frac{1}{2}}$ to be the same as in Lemma A.3 and A.4. Then,*

$$\begin{aligned}
\sum_j |G^-_{j+\frac{1}{2}}|^2 + \sum_j |G^+_{j+\frac{1}{2}}|^2 \leq & 8\lambda^2 \sum_j \left(f'(\bar{u}_{j-\frac{1}{2}})^2 f''(\bar{u}_{j-\frac{1}{2}})^2 + f'(\bar{u}_{j+\frac{1}{2}})^2 f''(\bar{u}_{j+\frac{1}{2}})^2 \right) (\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}})^4 \\
& + 2\lambda^2 \sum_j |\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}|^6 H(\bar{u}_{j-\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \lambda).
\end{aligned}$$

where $H(u,v,\lambda)$ is a nonnegative smooth function on $\mathbb{R}^3$. Moreover, if $f(u) = Cu^p + \mathscr{O}(u^{p+1})$ for some $p > 2$ is analytic near $u = 0$, then

$$\lim_{(u,v)\to(0,0)} \frac{H(u,v,\lambda)}{(\sqrt{u^2+v^2})^{3p-6}} = 0. \tag{19}$$

Proof. Combine Lemma A.6 with A.7 and use Jensen's inequality. ■

Proposition 3. *Under the CFL condition*

$$\frac{64}{3}\left(M_1+4\max_{|z|=2\|u\|_\infty}|f'(z)|\right)\lambda\le 1, \tag{CFL1}$$

we have the estimate for the Second Term of inequality (9):

$$\begin{aligned}\frac{1}{\Delta x}\|F[y_2]-F[y_1]\|^2\le&8\lambda^2\sum_j\left(f'(\bar u_{j-\frac12})^2f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2f''(\bar u_{j+\frac12})^2\right)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\\&+2\lambda^2\sum_j|\bar u_{j+\frac12}-\bar u_{j-\frac12}|^6H(\bar u_{j-\frac12},\bar u_{j+\frac12},\lambda)+C\lambda^2M_1^2(E^2+D^2),\end{aligned} \tag{18}$$

where $C>0$ is a constant and $H(u,v,\lambda)$ is a nonnegative smooth function on $\mathbb{R}^3$. Moreover, if $f(u)=Cu^p+\mathscr{O}(u^{p+1})$ for some $p>2$ is analytic near $u=0$, then

$$\lim_{(u,v)\to(0,0)}\frac{H(u,v,\lambda)}{(\sqrt{u^2+v^2})^{3p-6}}=0. \tag{19}$$

Proof. Apply Lemma A.5 and Lemma A.8 to inequality (17) successively:

$$\begin{aligned}\frac{1}{\Delta x}\|F[y_2]-F[y_1]\|^2\overset{(17)}{\le}&\frac12\sum_j\left(F^-_{j+\frac12}[y_2]-F^-_{j+\frac12}[y_1]\right)^2+\frac12\sum_j\left(F^+_{j+\frac12}[y_2]-F^+_{j+\frac12}[y_1]\right)^2\\\le&\sum_j|G^-_{j+\frac12}|^2+\sum_j|G^+_{j+\frac12}|^2+C\lambda^2M_1^2(E^2+D^2)\quad\text{(by Lemma A.5)}\\\le&8\lambda^2\sum_j\left(f'(\bar u_{j-\frac12})^2f''(\bar u_{j-\frac12})^2+f'(\bar u_{j+\frac12})^2f''(\bar u_{j+\frac12})^2\right)(\bar u_{j+\frac12}-\bar u_{j-\frac12})^4\\&+2\lambda^2\sum_j|\bar u_{j+\frac12}-\bar u_{j-\frac12}|^6H(\bar u_{j-\frac12},\bar u_{j+\frac12},\lambda)+C\lambda^2M_1^2(E^2+D^2).\quad\text{(by Lemma A.8)}\end{aligned}$$

■

Appendix B Proof of Theorem 1

Lemma B.1. *For any real numbers $a_1,a_2,\cdots,a_N$ satisfying:*

$$a_1\ge a_2\ge a_3\ge\cdots\ge a_N\ge 0\quad or\quad 0\ge a_1\ge a_2\ge a_3\ge\cdots\ge a_N,$$

we have

$$\frac14(a_1-a_N)^3\le\left|\sum_{i=1}^N a_i\right|\sum_{i=1}^{N-1}(a_{i+1}-a_i)^2.$$

Proof. First suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_N \geq 0$. By Jensen's inequality, we have

$$\begin{aligned}\left[\frac{(a_1+a_N)(a_1-a_N)}{a_1+2a_2+2a_3+\cdots+2a_{N-1}+a_N}\right]^2 &= \left[\frac{a_1^2-a_N^2}{a_1+2a_2+2a_3+\cdots+2a_{N-1}+a_N}\right]^2 \\ &= \left[\sum_{i=1}^{N-1}\frac{a_i^2-a_{i+1}^2}{a_1+2a_2+2a_3+\cdots+2a_{N-1}+a_N}\right]^2 \\ &= \left[\sum_{i=1}^{N-1}\frac{(a_i+a_{i+1})}{a_1+2a_2+2a_3+\cdots+2a_{N-1}+a_N}(a_i-a_{i+1})\right]^2 \\ &\leq \frac{\sum_{i=1}^{N-1}(a_i+a_{i+1})(a_i-a_{i+1})^2}{a_1+2a_2+2a_3+\cdots+2a_{N-1}+a_N}.\end{aligned}$$

Therefore,

$$\begin{aligned}a_1^2(a_1-a_N)^2 &\leq (a_1+a_N)^2(a_1-a_N)^2 \\ &\leq (a_1+2a_2+2a_3+\cdots+2a_{N-1}+a_N)\sum_{i=1}^{N-1}(a_i+a_{i+1})(a_i-a_{i+1})^2 \\ &\leq 2\left|\sum_{i=1}^{N}a_i\right|\sum_{i=1}^{N-1}(a_i+a_{i+1})(a_i-a_{i+1})^2 \\ &\leq 4\left|\sum_{i=1}^{N}a_i\right|a_1\sum_{i=1}^{N-1}(a_i-a_{i+1})^2.\end{aligned}$$

Dividing both sides by a_1, we get

$$(a_1-a_N)^3 \leq a_1(a_1-a_N)^2 \leq 4\left|\sum_{i=1}^{N}a_i\right|\sum_{i=1}^{N-1}(a_i-a_{i+1})^2.$$

Replacing a_i by $-a_{N-i+1}$ gives the proof when $0 \geq a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_N$. ∎

Definition 1. *A* **lattice curve** γ *is the graph of a continuous piecewise linear function* $\gamma(x)$*:*

$$\gamma := \{(x,\gamma(x))|x_l \leq x \leq x_r\}, \quad x_l < x_r$$

such that $x_l, x_r \in \mathbb{Z}\cup\{\pm\infty\}$ *and* $\gamma(x)$ *is linear on each closed interval* $[i,i+1]$ *for all* $i \in \mathbb{Z}\cap[x_l,x_r-1]$*. If* $i \in \mathbb{Z}\cap[x_l,x_r]$*, then the point* $(i,\gamma(i)) \in \gamma$ *is called a* **link of** γ*.* γ *is called* **nonnegative (nonpositive)***, denoted by* $\gamma \geq (\leq) 0$*, if* $\gamma(x) \geq (\leq) 0$ *for all* $x_l \leq x \leq x_r$*. The set of all lattice curves is denoted by* $\mathscr{L}$*. The set of all nonnegative (nonpositive) lattice curves is denoted by* $\mathscr{L}^+$ *(*$\mathscr{L}^-$*). Define*

$$\begin{aligned}⨍\gamma &:= \sum_{i=x_l}^{x_r}\gamma(i), \\ I(\gamma) &:= \sum_{i=x_l}^{x_r-1}|\gamma(i+1)-\gamma(i)|^2, \\ J(\gamma) &:= \big(\gamma(x_r)-\gamma(x_l)\big)^3.\end{aligned}$$

Definition 2. *Suppose* $\gamma_1 = \{(x, \gamma_1(x))|x_l \leq x \leq x_r\}$ *and* $\gamma_2 = \{(x, \gamma_2(x))|x'_l \leq x \leq x'_r\}$ *are two lattice curves. We say* γ_1 **is contained in** γ_2*, denoted by* $\gamma_1 \subset \gamma_2$*, if* $[x_l, x_r] \subset [x'_l, x'_r]$ *and* $\gamma_1(x) \equiv \gamma_2(x)$*,* $\forall x \in [x_l, x_r]$*. We say* γ_1 **lies below (above)** γ_2*, if* $[x_l, x_r] \subset [x'_l, x'_r]$ *and* $\gamma_1(x) \leq \gamma_2(x)$*,* $\forall x \in [x_l, x_r]$.

Definition 1 implies that a lattice curve must be continuous and is a closed set in the topology of $\mathbb{R}^2$. Also, $x \in \mathbb{Z}$ is a necessary condition for a point (x, y) to be a link. Examples of the Definitions 1 and 2 are shown in Fig.1. The following two observations are immediate:

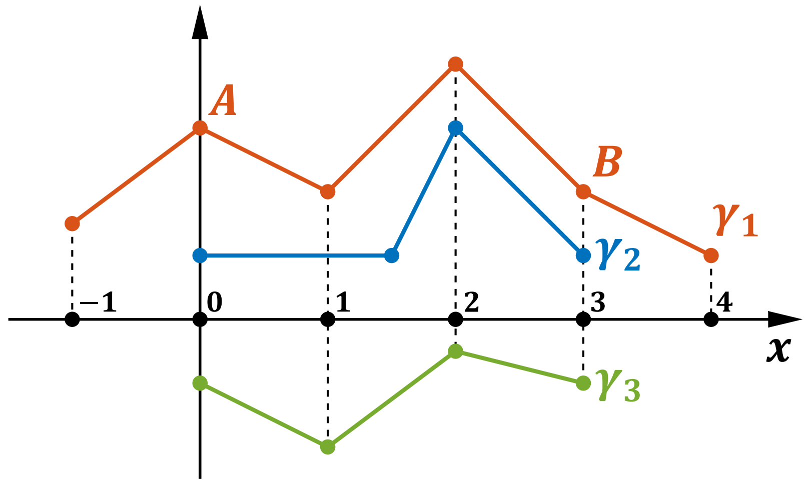


Figure 1: In this example, γ_1 and γ_3 (the orange and green curves) are lattice curves, while γ_2 (the blue one) is not since it is not linear on the interval $[1, 2]$. The orange and green dots are the links of γ_1 and γ_3 respectively. $\gamma_1 \in \mathscr{L}^+$ and $\gamma_3 \in \mathscr{L}^-$. γ_3 lies below γ_1. The part of γ_1 restricted to the interval $[0, 3]$ (the lattice curve between A and B) is contained in γ_1.

Property 1. *Suppose* $\gamma_1, \gamma_2 \in \mathscr{L}$ *and* γ_1 *lies below (above)* γ_2*, then* $\fint \gamma_1 \leq \fint \gamma_2$ *(*$\fint \gamma_1 \geq \fint \gamma_2$*).*

Property 2. *Suppose* $\gamma \in \mathscr{L}$ *and* $\gamma_1, \gamma_2 \subset \gamma$ *are two lattice curves which are disjoint except at their endpoints, then* $I(\gamma_1) + I(\gamma_2) \leq I(\gamma)$*. Moreover, if* $\gamma_1 \cup \gamma_2 = \gamma$*, then* $I(\gamma_1) + I(\gamma_2) = I(\gamma)$*.*

Definition 3. *Suppose* $\gamma = \{(x, \gamma(x))|x_l \leq x \leq x_r\}$ *is a lattice curve.* $i \in \mathbb{Z} \cap [x_l, x_r]$ *is called a* **local minimum of** γ *if one of the following is true:*

(1) $\gamma(i-k) = \gamma(i-k+1) = \cdots = \gamma(i)$ *with* $\gamma(i) < \gamma(i+1)$ *and* $\gamma(i-k) < \gamma(i-k-1)$ *for some* $k \geq 0$, $[i-k-1, i+1] \subset [x_l, x_r]$.

(2) $\gamma(x_l) = \gamma(x_l+1) = \cdots = \gamma(i)$ *with* $\gamma(i) < \gamma(i+1)$, $i+1 \leq x_r$.

(3) $i = x_r$ *and* $\gamma(x_r-k) = \gamma(x_r-k+1) = \cdots = \gamma(x_r)$ *for some* $k \geq 0$ *with* $\gamma(x_r-k) < \gamma(x_r-k-1)$, $x_r-k-1 \geq x_l$.

Definition 4. *Suppose* $\gamma = \{(x, \gamma(x))|x_l \leq x \leq x_r\}$ *is a lattice curve.* $i \in \mathbb{Z} \cap [x_l, x_r]$ *is called a* **local maximum of** γ *if one of the following is true:*

(1) $\gamma(i) = \gamma(i+1) = \cdots = \gamma(i+k)$ *with* $\gamma(i) > \gamma(i-1)$ *and* $\gamma(i+k) > \gamma(i+k+1)$ *for some* $k \geq 0$, $[i-1, i+k+1] \subset [x_l, x_r]$.

(2) $i = x_l$ *and* $\gamma(x_l) = \gamma(x_l+1) = \cdots = \gamma(x_l+k)$ *for some* $k \geq 0$ *with* $\gamma(x_l+k) > \gamma(x_l+k+1)$, $x_l+k+1 \leq x_r$.

(3) $\gamma(i) = \gamma(i+1) = \cdots = \gamma(x_r)$ *with* $\gamma(i) > \gamma(i-1)$, $i-1 \geq x_l$.

In less rigorous words, Definition 3 states that we always take the rightmost point on the bottom of a valley to be the local minimum; while Definition 4 states that we always take the leftmost point on the plateau of a peak to be the local maximum.

Definition 5. *Suppose* $\gamma = \{(x, \gamma(x)) | x_l \leq x \leq x_r\}$ *is a lattice curve. A subset of* γ *is called a* **decreasing (increasing) component**, *if it is of the form* $\{(x, \gamma(x)) | i \leq x \leq i+n\}$ *for some* $n \geq 1$, *where* $i \geq x_l$ *is a local maximum (minimum),* $i+n \leq x_r$ *is a local minimum (maximum), and* $\gamma(i+k) \leq (\geq) \gamma(i+k-1)$ $\forall k = 1, 2, \cdots, n$. γ *is called* **decreasing (increasing)** *if it is a decreasing (increasing) component of itself.*

Examples of the Definitions 3–5 are shown in Fig.2.

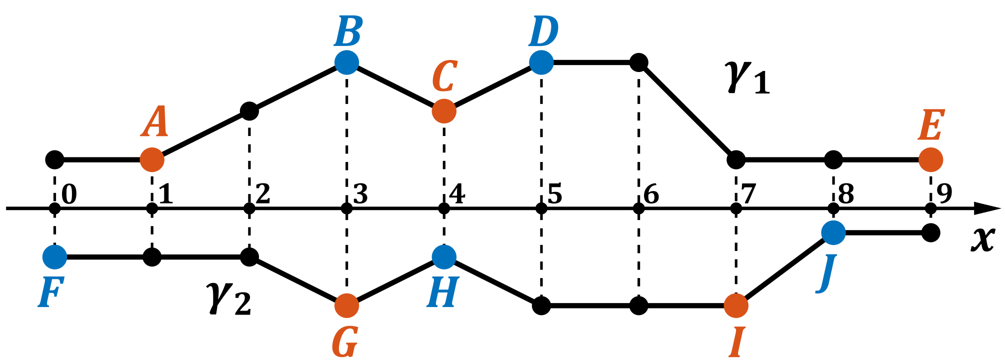


Figure 2: In this example, the red points A, C, E, G, I are local minima, while the blue points B, D, F, H, J are local maxima. The segment CD, and the polygonal line between A, B are increasing components of γ_1, while the polygonal lines between B, C and D, E are decreasing components; similarly, The segments GH and IJ are increasing components of γ_2, while the polygonal lines between F, G and H, I are decreasing components.

Now Lemma B.1 can be restated in the following way:

Property 3. *Suppose* $\gamma \in \mathscr{L}^+ \cup \mathscr{L}^-$ *is decreasing, then*

$$\frac{1}{4}|J(\gamma)| \leq \left| \fint \gamma \right| I(\gamma).$$

Definition 6. *A lattice curve* $\gamma = \{(x, \gamma(x)) | x_l \leq x \leq x_r\}$ *is called* **divisible** *if it satisfies the following conditions:*

(1) $\gamma(x)$ *does not change sign on* $[i, i+1]$ *for all* $x_l \leq i \leq x_r - 1$.

(2) *If i is a local minimum of γ and $\gamma(i) > 0$, then $\exists j \in (i, x_r] \cap \mathbb{Z}$ such that $\gamma(i) = \gamma(j)$ and $\gamma(x) > \gamma(i)$ for all $i < x < j$.*

(3) *If i is a local maximum of γ and $\gamma(i) < 0$, then $\exists j \in [x_l, i) \cap \mathbb{Z}$ such that $\gamma(i) = \gamma(j)$ and $\gamma(x) < \gamma(i)$ for all $j < x < i$.*

For every divisible γ which is decreasing or increasing, define

$$\gamma^+ := \gamma \cap \mathbb{R} \times [0, \infty) \in \mathscr{L}^+,$$
$$\gamma^- := \gamma \cap \mathbb{R} \times (-\infty, 0] \in \mathscr{L}^-.$$

An example of a divisible lattice curve is shown in Fig.3, and three lattice curves that are not divisible are shown in Fig.4.

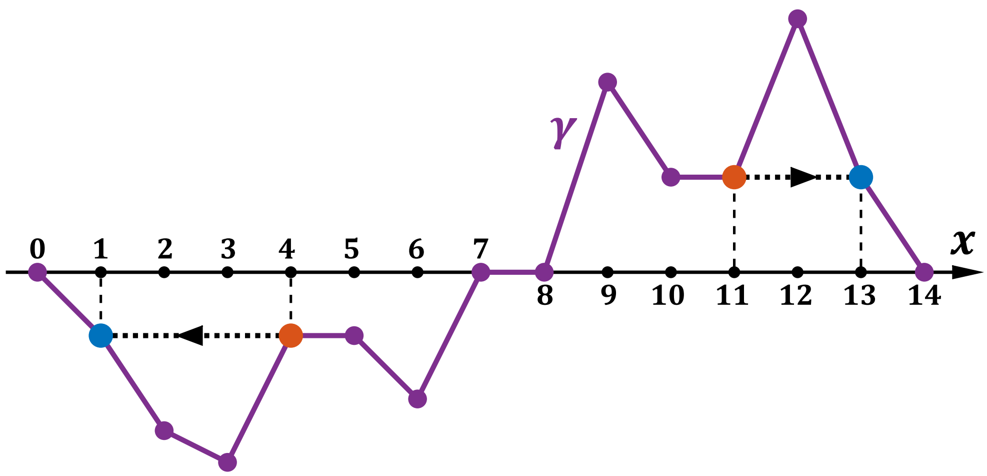


Figure 3: In this example, $\gamma \in \mathscr{L}$ is divisible. Note that $i = 11$ is a local minimum and $\gamma(11) > 0$ (red point on the right), so $\exists j = 13 > 11$ (blue point on the right) such that $\gamma(11) = \gamma(13)$ and $\gamma(x) > \gamma(11)$ for all $11 < x < 13$. Similarly, $i = 4$ is a local maximum and $\gamma(4) < 0$ (red point on the left), so $\exists j = 1 < 4$ (blue point on the left) such that $\gamma(4) = \gamma(1)$ and $\gamma(x) < \gamma(4)$ for all $1 < x < 4$.

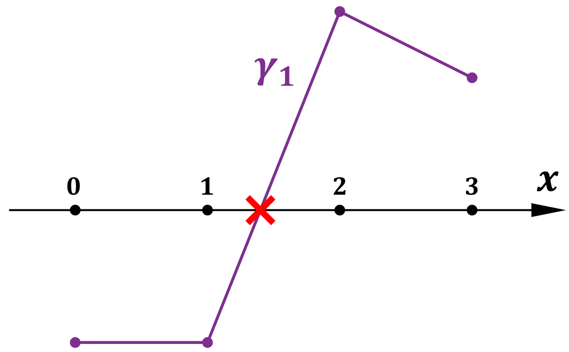
Definition 7. *A real infinite sequence $\{a_i\}_{i\in\mathbb{Z}}$ is called N-***periodic** *if there exists a smallest integer $N \geq 1$ such that $a_{i+N} = a_i$ $\forall i \in \mathbb{Z}$, in which case N is called the* **period** *of $\{a_i\}_{i\in\mathbb{Z}}$. We say* **a has two signs** *if $\exists i, j \in \mathbb{Z}$ such that $a_i > 0$ and $a_j < 0$. The class of all N-periodic sequences* **a** *having two signs is denoted by $\mathbb{S}_N$, and $\mathbb{S} := \cup_{N=2}^{\infty} \mathbb{S}_N$.*

In what follows the shorthand notation $\mathbf{a} = \{a_i\}_{i\in\mathbb{Z}}$ will be used whenever the context is clear.

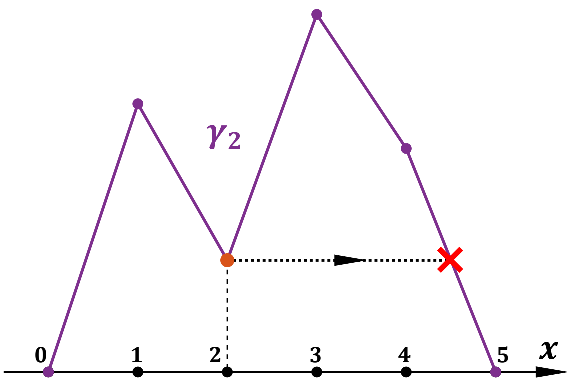
Definition 8. *We associate each real infinite sequence* **a** *with a lattice curve, called the* **graph of a** *and denoted by $\Gamma(\mathbf{a})$, which is defined as*

$$\Gamma(\mathbf{a}) := \{(x, a(x)) | x \in \mathbb{R}\},$$

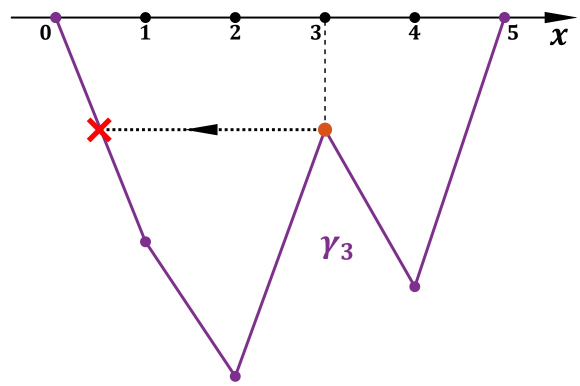
where $a(x)$ is a continuous piecewise linear function on $\mathbb{R}$ determined by $a(i) = a_i$ and linear on each interval $[i, i+1]$, $\forall i \in \mathbb{Z}$. See Fig.5. **a** *is called* **divisible** *if $\Gamma(\mathbf{a})$ is divisible. The class of all divisible N-periodic sequences* **a** *having two signs is denoted by $\hat{\mathbb{S}}_N$, and $\hat{\mathbb{S}} := \cup_{N=2}^{\infty} \hat{\mathbb{S}}_N$.*



(a) γ_1 violating condition (1) of Definition 6



(b) γ_2 violating condition (2) of Definition 6



(c) γ_3 violating condition (3) of Definition 6

Figure 4: Examples of lattice curves that are not divisible.

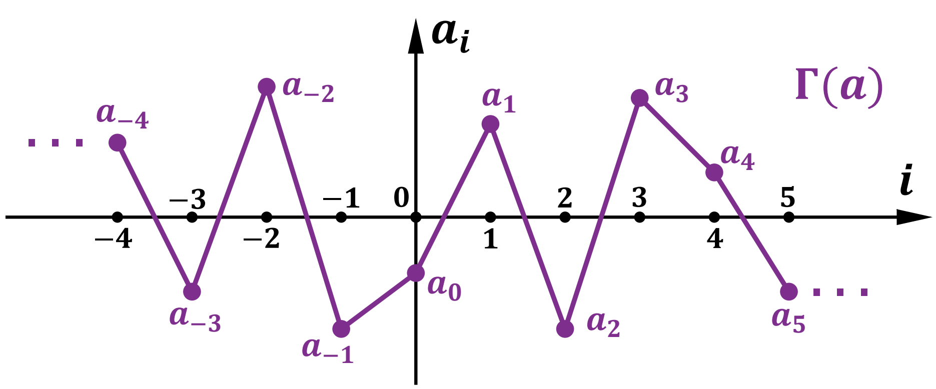


Figure 5: The graph $\Gamma(\mathbf{a})$.

Since if $\mathbf{a}$ is N-periodic, then the graph of $\mathbf{a}$ is invariant under translations by integer multiples of N, so we have the following

Definition 9. *Suppose* $\mathbf{a}$ *is* N*-periodic,* γ *is a decreasing (increasing) component of* $\Gamma(\mathbf{a})$ *and* N' *is an integer multiple of* N*. The* **congruence class of** γ **modulo** N'*, denoted by* $[\gamma]_{N'}$*, is the set of all decreasing (increasing) components of the form* $\gamma + k(N',0)$*, where* k *is any integer. An element of* $[\gamma]_{N'}$ *is called a* **representative** *of the congruence class. The set of all congruence classes of decreasing components of* $\Gamma(\mathbf{a})$ *modulo* N' *is denoted by* $\Gamma^{\flat}(\mathbf{a})_{N'}$*; the set of all congruence classes of increasing components of* $\Gamma(\mathbf{a})$ *modulo* N' *is denoted by* $\Gamma^{\sharp}(\mathbf{a})_{N'}$*.*

Clearly, if $\mathbf{a}$ is constant, then both $\Gamma^{\flat}(\mathbf{a})_{N'}$ and $\Gamma^{\sharp}(\mathbf{a})_{N'}$ are empty; however, if $\mathbf{a}$ is nonconstant (for instance, if $\mathbf{a}$ has two signs), then both $\Gamma^{\flat}(\mathbf{a})_{N'}$ and $\Gamma^{\sharp}(\mathbf{a})_{N'}$ are nonempty.

Because there is a unique decreasing component lying between two adjacent increasing components, any nonconstant N-periodic sequence $\mathbf{a}$ can be decomposed into an equal number of congruence classes of decreasing and increasing components modulo N that are mutually disjoint except at their endpoints.

Definition 10. *Define a transformation* $T : \mathbb{S} \to \hat{\mathbb{S}} \times \mathbb{Z}$*,* $(\hat{\mathbf{a}}, N') := T(\mathbf{a})$ *given by:*

(1) *If* $\mathbf{a} \in \hat{\mathbb{S}}_N$*, then* $\hat{\mathbf{a}} = \mathbf{a}$ *and* $N' = N$*.*

(2) *If* $\mathbf{a} \notin \hat{\mathbb{S}}$*,* $\mathbf{a} \in \mathbb{S}_N$*, then we construct* $\hat{\mathbf{a}}$ *and* N' *as follows (see Fig.6):*

Step 1 *For any local minimum* i *of* $\Gamma(\mathbf{a})$ *such that* $a_i > 0$ *but does not satisfy condition (2) of Definition 6, find the smallest index* $j > i$ *such that* $a_j > a_i > a_{j+1}$ *(this can be done since* $\exists j > i$ *s.t.* $a_j < 0$*). Insert* $b_j = a_i$ *between* a_j *and* a_{j+1}*.*

Step 2 *For any local maximum* i *of* $\Gamma(\mathbf{a})$ *such that* $a_i < 0$ *but does not satisfy condition (3) of Definition 6, find the smallest index* $j < i$ *such that* $a_j > a_i > a_{j+1}$ *(this can be done since* $\exists j < i$ *s.t.* $a_j > 0$*). Insert* $b_j = a_i$ *between* a_j *and* a_{j+1}*.*

Step 3 *For any* $j \in \mathbb{Z}$ *such that* $a_j > 0 > a_{j+1}$*, insert* $b_j = 0$ *between* a_j *and* a_{j+1}*.*

Step 4 *For any* $j \in \mathbb{Z}$ *such that* $a_j < 0 < a_{j+1}$*, insert* $c_j = 0$ *between* a_j *and* a_{j+1}*.*

Note that b_j*'s can only be inserted in decreasing components of* $\Gamma(\mathbf{a})$ *and* c_j*'s only in increasing components. If* $m \geq 1$ *number of* b_j*'s in total has been inserted between* a_j *and* a_{j+1}*, we put them in decreasing order and label them as* $a_j > b_j^{(1)} > b_j^{(2)} > \cdots > b_j^{(m)} > a_{j+1}$*. After the above steps, we would have an amended sequence of* a_j*'s interwoven with the additional* b_j*'s and* c_j*'s. Keep the order of this new sequence and relabel it* $\hat{\mathbf{a}}$ *with* $\hat{a}_0 = a_0$*. If we have in total inserted* p *number of* b_j*'s and* c_j*'s in one period of* $\mathbf{a}$ *(for* $0 \leq j \leq N-1$*), define* $N' = N + p$*. Clearly,* $\hat{\mathbf{a}} \in \hat{\mathbb{S}}$*.*

It is clear from the above definition that N' is an integer multiple of the period of $\hat{\mathbf{a}}$.

Example 1. *Suppose* $\mathbf{a} \notin \hat{\mathbb{S}}$*,* $\mathbf{a} \in \mathbb{S}_8$*, and the values of* a_i*'s in one period are:*

$$a_0 = -2, a_1 = 2, a_2 = 1, a_3 = 3, a_4 = 2, a_5 = 4, a_6 = -2, a_7 = -1.$$

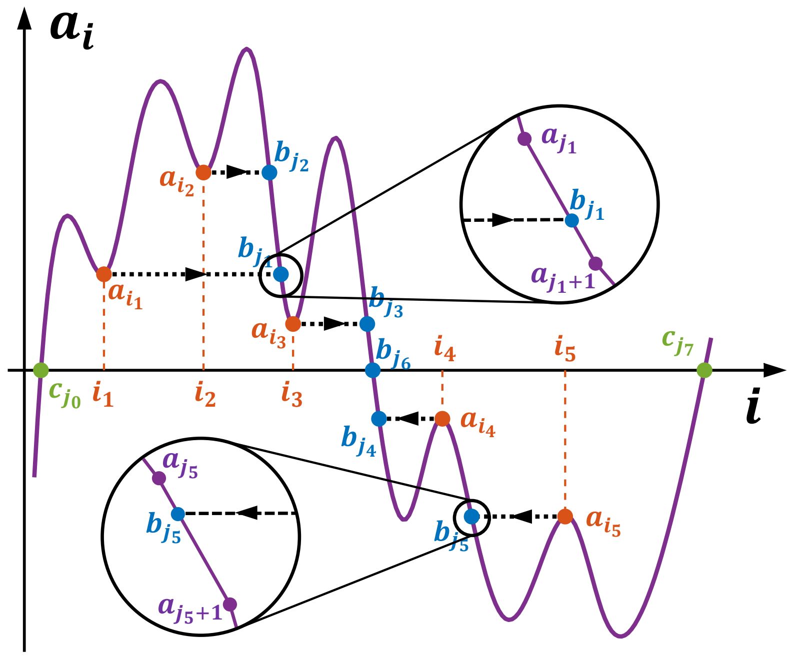


Figure 6: The transformation T inserts b_j's and c_j's (blue and green points) into the original sequence $\mathbf{a}$ where Definition 6 is violated. Here, for $k=1,2,3$, each local minimum $a_{i_k}>0$ gives rise to a b_{j_k} which is added in Step 1; for $k=4,5$, each local maximum $a_{i_k}<0$ gives rise to a b_{j_k} which is added in 2. b_{j_6} is added in Step 3, while c_{j_0} and c_{j_7} are added in Step 4.

Then after inserting b_j's and c_j's according to Steps 1–4 of Definition 10, the above sequence changes to

$$a_0=-2, c_0=0, a_1=2, a_2=1, a_3=3,$$
$$a_4=2, a_5=4, b_5^{(1)}=2, b_5^{(2)}=1, b_5^{(3)}=0, b_5^{(4)}=-1, a_6=-2, a_7=-1,$$

where in Step 1, we added $b_5^{(1)}=2$ and $b_5^{(2)}=1$ between $a_5=4$ and $a_6=-2$, which corresponds to local minima $a_4=2>0$ and $a_2=1>0$ respectively. In Step 2, $b_5^{(4)}=-1$ is added between $a_5=4$ and $a_6=-2$ corresponding to local maxima $a_7=-1>0$. In Step 3, $b_5^{(3)}=0$ is added between $a_5=4$ and $a_6=-2$. In Step 4, $c_0=0$ is added between $a_0=-2$ and $a_1=2$. In total we have inserted 5 b_j's and c_j's in one period of $\mathbf{a}$, so $N'=N+5=13$. Inserting b_j's and c_j's in the same way for all periods of $\mathbf{a}$ and relabeling it to $\hat{\mathbf{a}}$ with $\hat{a}_0=a_0$ yields:

$$\cdots, \hat{a}_0=-2, \hat{a}_1=0, \hat{a}_2=2, \hat{a}_3=1, \hat{a}_4=3, \hat{a}_5=2, \hat{a}_6=4,$$
$$\hat{a}_7=2, \hat{a}_8=1, \hat{a}_9=0, \hat{a}_{10}=-1, \hat{a}_{11}=-2, \hat{a}_{12}=-1, \cdots.$$

The diagram is shown in Fig.7. In this case, N' is the period of $\hat{\mathbf{a}}$.

The increasing components of $\Gamma(\mathbf{a})$ in Fig.7 are the segments a_0a_1, a_2a_3, a_4a_5 and a_6a_7, and their congruence classes modulo N comprise the whole $\Gamma^\sharp(\mathbf{a})_N$. Similarly, the decreasing components of $\Gamma(\mathbf{a})$ in Fig.7 are the segments a_1a_2, a_3a_4, a_5a_6 and a_7a_8, and their congruence classes modulo N comprise the whole $\Gamma^\flat(\mathbf{a})_N$.

The increasing components of $\Gamma(\hat{\mathbf{a}})$ in Fig.7 are the segments $\hat{a}_3\hat{a}_4$, $\hat{a}_5\hat{a}_6$, $\hat{a}_{11}\hat{a}_{12}$, and the polygonal line between $\hat{a}_0$ and $\hat{a}_2$. Their congruence classes modulo N' comprise the whole $\Gamma^\sharp(\hat{\mathbf{a}})_{N'}$. Similarly, the decreasing components of $\Gamma(\hat{\mathbf{a}})$ in Fig.7 are the segments $\hat{a}_2\hat{a}_3$, $\hat{a}_4\hat{a}_5$, $\hat{a}_{12}\hat{a}_{13}$, and the polygonal line between $\hat{a}_6$ and $\hat{a}_{11}$. Their congruence classes modulo N' comprise the whole $\Gamma^\flat(\mathbf{a})_{N'}$.

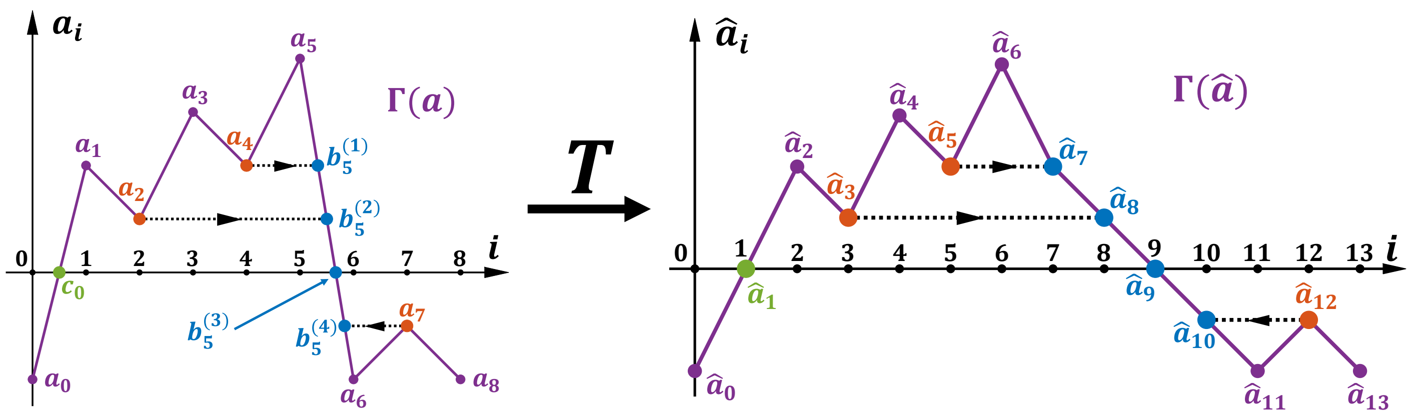


Figure 7: Diagram of example 1.

Example 2. *Suppose $\mathbf{a} \notin \hat{\mathbb{S}}$, $\mathbf{a} \in \mathbb{S}_6$, and the values of a_i's in one period are:*

$$a_0 = -1, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = -1, a_5 = 1.$$

Then after inserting b_j's and c_j's according to Steps 1–4 of Definition 10, the above sequence changes to

$$a_0 = -1, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = -1, c_4 = 0, a_5 = 1, b_5 = 0,$$

where in Step 3, we inserted $b_5 = 0$ between $a_5 = 1$ and $a_6 = -1$; in Step 4, $c_4 = 0$ is added between $a_4 = -1$ and $a_5 = 1$; and nothing is added in Step 1–2. In total we have inserted 2 b_j's and c_j's in one period of $\mathbf{a}$, so $N' = N + 2 = 8$. Inserting b_j's and c_j's in the same way for all periods of $\mathbf{a}$ and relabeling it to $\hat{\mathbf{a}}$ with $\hat{a}_0 = a_0$ yields:

$$\cdots, \hat{a}_0 = -1, \hat{a}_1 = 0, \hat{a}_2 = 1, \hat{a}_3 = 0, \hat{a}_4 = -1, \hat{a}_5 = 0, \hat{a}_6 = 1, \hat{a}_7 = 0, \cdots.$$

In this case, N' is twice the period of $\hat{\mathbf{a}}$. The increasing components $\hat{a}_0\hat{a}_1\hat{a}_2$ and $\hat{a}_4\hat{a}_5\hat{a}_6$ are the same modulo N, but different modulo N'.

Lemma B.2. *Suppose $\mathbf{a} \in \mathbb{S}_N$. Then,*

$$\sum_{[\gamma]_{N'} \in \Gamma^\flat(\hat{\mathbf{a}})_{N'}} I(\gamma) \leq \sum_{[\gamma]_N \in \Gamma^\flat(\mathbf{a})_N} I(\gamma),$$

$$\sum_{[\gamma]_{N'} \in \Gamma^\sharp(\hat{\mathbf{a}})_{N'}} J(\gamma^+) + J(\gamma^-) \geq \frac{1}{4} \sum_{[\gamma]_N \in \Gamma^\sharp(\mathbf{a})_N} J(\gamma).$$

where the summation on the left is over all congruence classes $[\gamma]_{N'}$ (each congruence class is summed only once) with γ being an arbitrary representative, and similarly for the sum on the right. If either $\gamma^+ = \emptyset$ or $\gamma^- = \emptyset$, we set $J(\emptyset) = 0$. Moreover, if

$$\left| \fint \gamma \right| \leq 1, \quad \forall \gamma \subset \Gamma(\mathbf{a}) \text{ such that } \gamma \in \mathscr{L}^+ \cup \mathscr{L}^-,$$

then

$$\left| \fint \gamma \right| \leq 2, \quad \forall \gamma \subset \Gamma(\hat{\mathbf{a}}) \text{ such that } \gamma \in \mathscr{L}^+ \cup \mathscr{L}^-.$$

Proof. The statement is obvious if $\mathbf{a} \in \hat{\mathbb{S}}$ (since we then have $\hat{\mathbf{a}} = \mathbf{a}$, $N' = N$), so assume $\mathbf{a} \notin \hat{\mathbb{S}}$ and perform Step 1–4 in Definition 10 following the same notation. Since b_j can only be added in a decreasing component (Step 1–3), let us pick any $[\gamma]_N \in \Gamma^\flat(\mathbf{a})_N$ with left and right endpoint of γ being (x_l, a_{x_l}) and (x_r, a_{x_r}) respectively, $x_l < x_r$. Suppose $b_{j_k}^{(1)}, b_{j_k}^{(2)}, \cdots, b_{j_k}^{(m_k)}$ are added in between a_{j_k} and a_{j_k+1} for $1 \leq k \leq n$, where $x_l \leq j_1 < j_2 < \cdots < j_n < x_r$. After all the b_{j_k}'s are inserted into γ, we would obtain a new decreasing sequence

$$\begin{aligned} a_{x_l}, \cdots, a_{j_1}, b_{j_1}^{(1)}, b_{j_1}^{(2)}, \cdots, b_{j_1}^{(m_1)}, a_{j_1+1}, \cdots \\ \cdots, a_{j_2}, b_{j_2}^{(1)}, b_{j_2}^{(2)}, \cdots, b_{j_2}^{(m_2)}, a_{j_2+1}, \cdots \\ \vdots \\ \cdots, a_{j_n}, b_{j_n}^{(1)}, b_{j_n}^{(2)}, \cdots, b_{j_n}^{(m_n)}, a_{j_n+1}, \cdots, a_{x_r} \end{aligned}$$

which corresponds to a decreasing component $\hat{\gamma}$ of $\Gamma(\hat{\mathbf{a}})$ after relabeling the sequence to $\hat{a}_i$'s. Then,

$$\begin{aligned} &\left| a_{j_k} - b_{j_k}^{(1)} \right|^2 + \sum_{l=1}^{m_k-1} \left| b_{j_k}^{(l)} - b_{j_k}^{(l+1)} \right|^2 + \left| b_{j_k}^{(m_k)} - a_{j_k+1} \right|^2 \\ \leq &\left(\left| a_{j_k} - b_{j_k}^{(1)} \right| + \sum_{l=1}^{m_k-1} \left| b_{j_k}^{(l)} - b_{j_k}^{(l+1)} \right| + \left| b_{j_k}^{(m_k)} - a_{j_k+1} \right| \right)^2 \\ = &\left(a_{j_k} - b_{j_k}^{(1)} + \sum_{l=1}^{m_k-1} \left(b_{j_k}^{(l)} - b_{j_k}^{(l+1)} \right) + b_{j_k}^{(m_k)} - a_{j_k+1} \right)^2 \\ = &|a_{j_k} - a_{j_k+1}|^2, \quad \forall 1 \leq k \leq n, \end{aligned}$$

which implies

$$I(\hat{\gamma}) \leq I(\gamma).$$

Note that for any $[\gamma]_N \in \Gamma^\flat(\mathbf{a})_N$ we can find a unique $[\hat{\gamma}]_{N'} \in \Gamma^\flat(\hat{\mathbf{a}})_{N'}$ such that the above holds, because the transformation T can only insert b_j's in the interior of a decreasing component of $\Gamma(\mathbf{a})$ and does not change its endpoints. Summing over all $[\gamma]_N$ and $[\hat{\gamma}]_{N'}$ yields:

$$\sum_{[\gamma]_{N'} \in \Gamma^\flat(\hat{\mathbf{a}})_{N'}} I(\gamma) \leq \sum_{[\gamma]_N \in \Gamma^\flat(\mathbf{a})_N} I(\gamma).$$

For the second part, note that only the c_j's can enter an increasing component (Step 4), and at most one c_j can be inserted into each increasing component. As before, let us pick any $[\gamma]_N \in \Gamma^\sharp(\mathbf{a})_N$ with left and right endpoints of γ being (x_l, a_{x_l}) and (x_r, a_{x_r}) respectively, $x_l < x_r$. Suppose $c_j = 0$ is added in between a_j and a_{j+1} in Step 4, where $x_l \le j < x_r$. After $c_j = 0$ is inserted into γ, we obtained a new sequence

$$a_{x_l}, \cdots, a_j, c_j, a_{j+1}, \cdots, a_{x_r}$$

which corresponds to an increasing component $\hat{\gamma}$ of $\Gamma(\hat{\mathbf{a}})$ after relabeling from a_i's to $\hat{a}_i$'s. Note that $a_{x_r} > c_j > a_{x_l}$, by Jensen's inequality:

$$J(\hat{\gamma}^+) + J(\hat{\gamma}^-) = (a_{x_r} - c_j)^3 + (c_j - a_{x_l})^3 \ge \frac{1}{4}(a_{x_r} - c_j + c_j - a_{x_l})^3 = \frac{1}{4}(a_{x_r} - a_{x_l})^3 = J(\gamma).$$

If c_j is not inserted into γ, then either $\gamma \ge 0$ or $\gamma \le 0$, in which case we have

$$J(\hat{\gamma}^+) + J(\hat{\gamma}^-) = J(\gamma).$$

The correspondence between γ and $\hat{\gamma}$ is one-to-one, because the transformation T can only insert c_j's in the interior of an increasing component of $\Gamma(\mathbf{a})$ and does not change its endpoints. Summing over all $[\gamma]_N$ and $[\hat{\gamma}]_{N'}$ yields:

$$\sum_{[\gamma]_{N'} \in \Gamma^\sharp(\hat{\mathbf{a}})_{N'}} J(\gamma^+) + J(\gamma^-) \ge \frac{1}{4} \sum_{[\gamma]_N \in \Gamma^\sharp(\mathbf{a})_N} J(\gamma).$$

For the third part, let us pick any $\hat{\gamma} \subset \Gamma(\hat{\mathbf{a}})$ such that $\hat{\gamma} \in \mathscr{L}^+$. Clearly, one can find $\gamma \subset \Gamma(\mathbf{a})$ and $\gamma \in \mathscr{L}^+$ such that after inserting b_j's and c_j's (Step 1–4) and relabeling it transforms into $\hat{\gamma}$. Then, there exists a largest $\tilde{\gamma} \in \mathscr{L}^+$ such that $\gamma \subset \tilde{\gamma} \subset \Gamma(\mathbf{a})$. Each $b_j > 0$ can only be added into γ (in Step 1) if it is a copy of some local minimum $0 < a_i \in \tilde{\gamma}$, and each $a_i > 0$ is duplicated at most once. Also note that adding $b_j = 0$ in Step 3 or $c_j = 0$ in Step 4 does not contribute to the sum. Therefore,

$$0 < ⨍ \hat{\gamma} \le 2 ⨍ \tilde{\gamma} \le 2.$$

Similarly, if $\hat{\gamma} \subset \Gamma(\hat{\mathbf{a}})$ and $\hat{\gamma} \in \mathscr{L}^-$ we have

$$0 > ⨍ \hat{\gamma} \ge 2 ⨍ \tilde{\gamma} \ge -2.$$

Combining the two cases yields

$$\left| ⨍ \gamma \right| \le 2, \quad \forall \gamma \subset \Gamma(\hat{\mathbf{a}}) \text{ such that } \gamma \in \mathscr{L}^+ \cup \mathscr{L}^-.$$

■

Let $E_{\mathbf{a}}$ be a set containing only positive local minima and negative local maxima of $\Gamma(\mathbf{a})$.

Definition 11. *For any $\mathbf{a} \in \hat{\mathbb{S}}$. Define a map $L_{\mathbf{a}} : E_{\mathbf{a}} \to \mathbb{Z}$ as:*

(1) *If i is a local minimum of $\Gamma(\mathbf{a})$ and $a_i > 0$, then $L_{\mathbf{a}}(i) = j$ where $j > i$, $a_i = a_j$ and $a_l > a_i$ for all $i < l < j$ (which is guaranteed by condition (2) of Definition 6).*

(2) *If i is a local maximum of $\Gamma(\mathbf{a})$ and $a_i < 0$, then $L_{\mathbf{a}}(i) = j$ where $j < i$, $a_i = a_j$ and $a_l < a_i$ for all $j < l < i$ (which is guaranteed by condition (3) of Definition 6).*

Lemma B.3. *Suppose $\mathbf{a} \in \mathbb{S}_N$ and*

$$\left| \fint \gamma \right| \le 1, \quad \forall \gamma \subset \Gamma(\mathbf{a}) \text{ such that } \gamma \in \mathscr{L}^+ \cup \mathscr{L}^-.$$

Then,

$$\frac{1}{32} \sum_{[\gamma]_N \in \Gamma^\sharp(\mathbf{a})_N} J(\gamma) \le \sum_{[\gamma]_N \in \Gamma^\flat(\mathbf{a})_N} I(\gamma).$$

Proof. Let $(\hat{\mathbf{a}}, N') = T(\mathbf{a})$. Assume $\hat{a}_{n_{1,0}} = 0$ (so that we also have $\hat{a}_{n_{1,0}+N'} = 0$ since N' is an integer multiple of the period of $\hat{\mathbf{a}}$). Denote the x-axis (which is also a lattice curve) by $l_x = \{(x,0) | x \in \mathbb{R}\}$. Let R denote the interior of the region enclosed by the lattice curve $\Gamma(\hat{\mathbf{a}})$ and l_x restricted to $x \in [n_{1,0}, n_{1,0} + N']$.

We now perform a series of steps to divide the interior R as follows:

Step 1 Firstly, note that l_x has already divided R naturally into simply connected open regions that are solely above or below l_x flanked by two points lying on l_x. Label them as $R_1, R_2, \cdots, R_M$, so that $R = \cup_{i=1}^{M} R_i$ and $R_i \cap R_j = \emptyset$ for any $i \ne j$. See Fig.8.

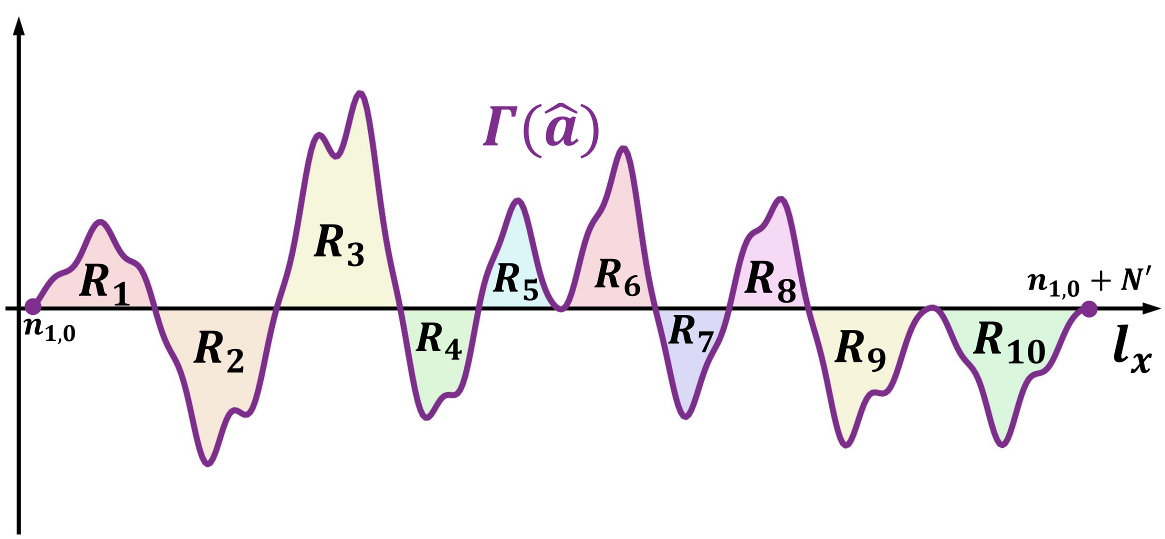


Figure 8: R is naturally divided by l_x into simply connected open regions R_i's. In this example, $M = 10$.

Step 2 For any R_i lying above l_x, denote the local minima of the lattice curve $\partial R_i \cap \Gamma(\hat{\mathbf{a}})$ (top boundary of R_i) from left to right as $n_{i,0}, n_{i,1}, \cdots, n_{i,s_i+1}$, where $\hat{a}_{n_{i,0}} = \hat{a}_{n_{i,s_i+1}} = 0$ and $a_k > 0$, $\forall n_{i,0} < k < n_{i,s_i+1}$. For each $1 \le j \le s_i$, connect the link $(n_{i,j}, \hat{a}_{n_{i,j}})$ and the link

$(L_{\hat{\mathbf{a}}}(n_{i,j}), \hat{a}_{L_{\hat{\mathbf{a}}}(n_{i,j})})$ with a horizontal line segment $l_{i,j}$ (the dashed lines in Fig.9), which lies entirely in the interior of R_i except for its endpoints by the definition of $L_{\hat{\mathbf{a}}}$. In this way, R_i is divided by s_i segments not intersecting each other into s_i+1 disjoint simply connected open subregions $R_{i,0}, R_{i,1}, \cdots, R_{i,s_i}$ with $\cup_{j=0}^{s_i} \overline{R_{i,j}} = \overline{R_i}$. Note that each one of the s_i+1 increasing components of $\partial R_i \cap \Gamma(\hat{\mathbf{a}})$ abuts on some $R_{i,j}$, and each $\overline{R_{i,j}}$ contains one and only one increasing component. Thus, we can label these subregions according to the order of occurrence as we travel along $\partial R_i \cap \Gamma(\hat{\mathbf{a}})$ from left to right. Let $\gamma_{i,0} = \overline{\partial R_{i,0} \backslash l_x}$ and $\gamma_{i,j} = \overline{\partial R_{i,j} \backslash l_{i,j}}$, $1 \le j \le s_i$, denote the top boundary of $R_{i,j}$.

Step 3 For any R_i lying below l_x, denote the local maxima of the lattice curve $\partial R_i \cap \Gamma(\hat{\mathbf{a}})$ (bottom boundary of R_i) from right to left as $n_{i,0}, n_{i,1}, \cdots, n_{i,s_i+1}$, where $\hat{a}_{n_{i,0}} = \hat{a}_{n_{i,s_i+1}} = 0$ and $a_k < 0$, $\forall n_{i,0} > k > n_{i,s_i+1}$. For each $0 \le j \le s_i$, connect the link $(n_{i,j}, \hat{a}_{n_{i,j}})$ and the link $(L_{\hat{\mathbf{a}}}(n_{i,j}), \hat{a}_{L_{\hat{\mathbf{a}}}(n_{i,j})})$ with a horizontal line segment $l_{i,j}$ (the dashed lines in Fig.9), which lies entirely in the interior of R_i except for its endpoints by the definition of $L_{\hat{\mathbf{a}}}$. In this way, R_i is divided by s_i segments not intersecting each other into s_i+1 disjoint simply connected open subregions $R_{i,0}, R_{i,1}, \cdots, R_{i,s_i}$ with $\cup_{j=0}^{s_i} \overline{R_{i,j}} = \overline{R_i}$. By the same reasoning as before, we can label these subregions according to the order of occurrence as we travel along $\partial R_i \cap \Gamma(\hat{\mathbf{a}})$ from right to left. Let $\gamma_{i,0} = \overline{\partial R_{i,0} \backslash l_x}$ and $\gamma_{i,j} = \overline{\partial R_{i,j} \backslash l_{i,j}}$, $1 \le j \le s_i$, denote the bottom boundary of $R_{i,j}$.

After all the above steps are done, we have obtained a division of the interior of R as $R = \cup_{i=1}^{M} \cup_{j=0}^{s_i} R_{i,j}$, where all the $R_{i,j}$'s are mutually disjoint simply connected open regions.

Let $\gamma_{i,j}^{\sharp}$ and $\gamma_{i,j}^{\flat}$ denote the increasing and decreasing component of $\gamma_{i,j}$ respectively (both are lattice curves). This is possible since $\gamma_{i,j}$ contains only one increasing component and one decreasing component. See Fig.10.

From Property 3, we have:

$$\frac{1}{4} J(\gamma_{i,j}^{\sharp}) = \frac{1}{4} |J(\gamma_{i,j}^{\flat})| \le \left| \fint \gamma_{i,j}^{\flat} \right| I(\gamma_{i,j}^{\flat}), \quad 0 \le j \le s_i, \quad 1 \le i \le M.$$

Recall our assumption:

$$\left| \fint \gamma \right| \le 1, \quad \forall \gamma \subset \Gamma(\mathbf{a}) \text{ such that } \gamma \in \mathscr{L}^+ \cup \mathscr{L}^-.$$

By the third statement in Lemma B.2, we then have

$$\left| \fint \gamma \right| \le 2, \quad \forall \gamma \subset \Gamma(\hat{\mathbf{a}}) \text{ such that } \gamma \in \mathscr{L}^+ \cup \mathscr{L}^-.$$

If $R_{i,j}$ lies above l_x, then the lattice curve $\gamma_{i,j}^{\flat} \ge 0$ always lies below the lattice curve $0 \le \partial R_{i,j} \cap \Gamma(\hat{\mathbf{a}}) \subset \Gamma(\hat{\mathbf{a}})$, and by Property 1 we have

$$0 \le \fint \gamma_{i,j}^{\flat} \le \fint \partial R_{i,j} \cap \Gamma(\hat{\mathbf{a}}) \le 2.$$

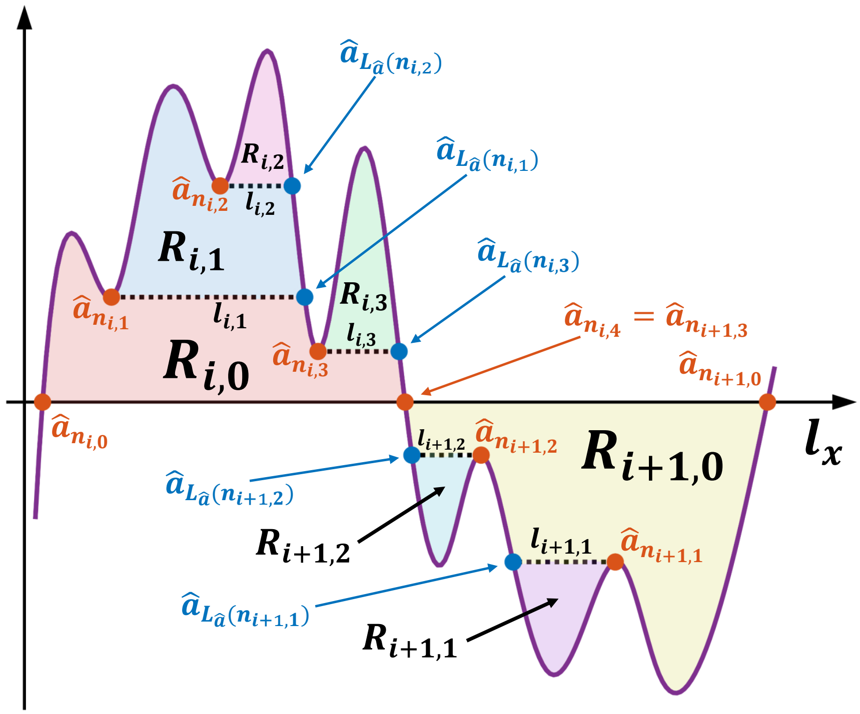


Figure 9: In this example, $s_i = 3$ and $s_{i+1} = 2$. The region R_i lies above l_x, and is separated into disjoint subregions $R_{i,0}, R_{i,1}, R_{i,2}, R_{i,3}$ by the dashed lines $l_{i,1}, l_{i,2}, l_{i,3}$; similarly, the region R_{i+1} lies below l_x, and is separated into disjoint subregions $R_{i+1,0}, R_{i+1,1}, R_{i+1,2}$ by the dashed lines $l_{i+1,1}, l_{i+1,2}$.

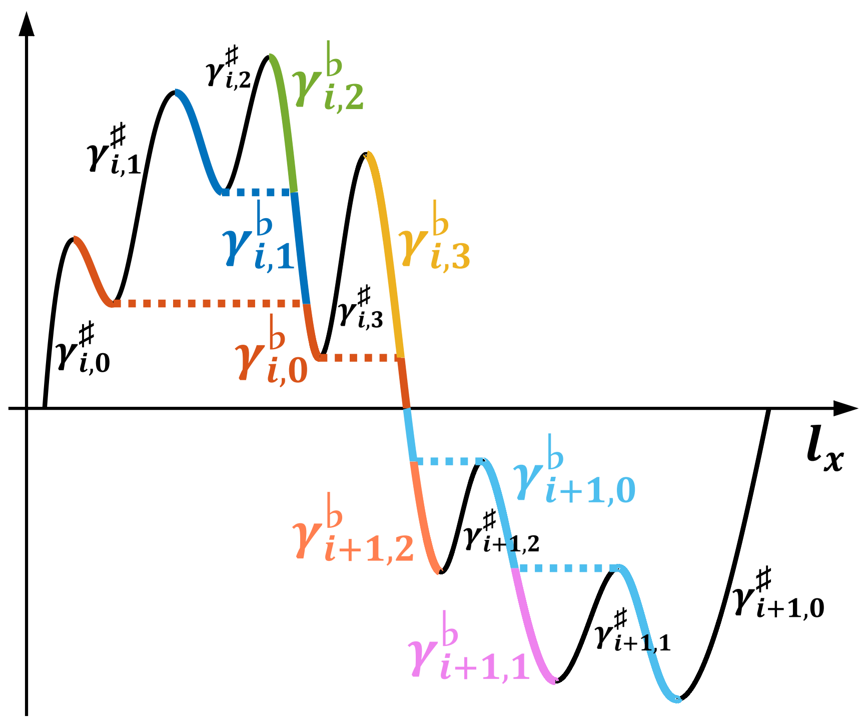


Figure 10: $\gamma^\sharp_{i,j}$ and $\gamma^\flat_{i,j}$ are the increasing and decreasing component of the top (bottom) boundary of $R_{i,j}$ lying above (below) l_x, all of which are lattice curves.

If $R_{i,j}$ lies below l_x, then the lattice curve $\gamma^{\flat}_{i,j} \leq 0$ always lies above the lattice curve $0 \geq \partial R_{i,j} \cap \Gamma(\hat{\mathbf{a}}) \subset \Gamma(\hat{\mathbf{a}})$, and again by Property 1 we have

$$0 \geq \fint \gamma^{\flat}_{i,j} \geq \fint \partial R_{i,j} \cap \Gamma(\hat{\mathbf{a}}) \geq -2.$$

Combining the two cases yields

$$\left| \fint \gamma^{\flat}_{i,j} \right| \leq 2.$$

Then,

$$\frac{1}{4} J(\gamma^{\sharp}_{i,j}) \leq \left| \fint \gamma^{\flat}_{i,j} \right| I(\gamma^{\flat}_{i,j}) \leq 2 I(\gamma^{\flat}_{i,j}), \quad 0 \leq j \leq s_i, \quad 1 \leq i \leq M.$$

Divide the above inequality by 2 and sum over all regions $R_{i,j}$:

$$\frac{1}{8} \sum_{i=0}^{M} \sum_{j=0}^{s_i} J(\gamma^{\sharp}_{i,j}) \leq \sum_{i=0}^{M} \sum_{j=0}^{s_i} I(\gamma^{\flat}_{i,j}).$$

Since all the horizontal segments $l_{i,j}$'s contained in $\gamma^{\flat}_{i,j}$ does not contribute to the sum on the right, by Property 2 we have

$$\sum_{i=0}^{M} \sum_{j=0}^{s_i} I(\gamma^{\flat}_{i,j}) = \sum_{[\gamma]_{N'} \in \Gamma^{\flat}(\hat{\mathbf{a}})_{N'}} I(\gamma) \leq \sum_{[\gamma]_N \in \Gamma^{\flat}(\mathbf{a})_N} I(\gamma),$$

where the first inequality from Lemma B.2 is used in the last step.

For the sum on the left, note that none of the $l_{i,j}$'s is contained in $\gamma^{\sharp}_{i,j}$, so we have

$$\frac{1}{8} \sum_{i=0}^{M} \sum_{j=0}^{s_i} J(\gamma^{\sharp}_{i,j}) = \frac{1}{8} \sum_{[\gamma]_{N'} \in \Gamma^{\sharp}(\hat{\mathbf{a}})_{N'}} \Big(J(\gamma^{+}) + J(\gamma^{-}) \Big) \geq \frac{1}{32} \sum_{[\gamma]_N \in \Gamma^{\sharp}(\mathbf{a})_N} J(\gamma)$$

where the second inequality from Lemma B.2 is used in the last step. Combining the above three inequalities eventually results in

$$\frac{1}{32} \sum_{[\gamma]_N \in \Gamma^{\sharp}(\mathbf{a})_N} J(\gamma) \leq \sum_{[\gamma]_N \in \Gamma^{\flat}(\mathbf{a})_N} I(\gamma).$$

■

Let $\mathbb{Z}_N = \{[0], [1], \cdots, [N]\}$ denote the set of (congruence classes of) integers modulo N, where $[i]$ denotes the congruence class of i modulo N.

Lemma B.4. *For any N-periodic sequences $\mathbf{x}$ and $\mathbf{y}$ satisfying:*

$$0 \leq x_i \leq 1, \quad 0 \leq y_i \leq 1, \quad \forall i \in \mathbb{Z},$$

we have

$$\sum_{i=1}^{N} x_i^3 \leq 64 \sum_{i=1}^{N} \left(\frac{y_{i-1}}{2} - y_i + \frac{y_{i+1}}{2} + x_{i-1} + x_i \right)^2.$$

Proof. Without loss of generality we prove the inequality when all the x_i's are positive, since then the inequality would also hold in the limit as x_i's tend to zero. Also assume $\mathbf{y}$ is not constant, since otherwise the proof is obvious. Let us introduce the N-periodic sequence $-\mathbf{y}$ lying in $[-1,0]$:

$$\cdots, -y_1, -y_2, -y_3, \cdots, -y_{N-1}, -y_N, \cdots$$

and another N-periodic sequence $\mathbf{a}$, which is the slope of $\Gamma(-\mathbf{y})$, defined by

$$a_i = -y_{i+1} + y_i, \quad i \in \mathbb{Z}.$$

First note that $\mathbf{a}$ has two signs because $\mathbf{y}$ is nonconstant and $\sum_{i=1}^N a_i = 0$, so that $\mathbf{a} \in \mathbb{S}$. Also, since $-1 \le -y_i \le 0$, $\forall i \in \mathbb{Z}$, the widths of monotone components of $-\mathbf{y}$ are within one as well, which immediately yields

$$\left| \fint \gamma \right| \le 1, \quad \forall \gamma \subset \Gamma(\mathbf{a}) \text{ such that } \gamma \in \mathscr{L}^+ \cup \mathscr{L}^-.$$

Set

$$A = \{[i] \in \mathbb{Z}_N \big| a_i > a_{i-1}\}, \quad B = \{[i] \in \mathbb{Z}_N \big| a_i \le a_{i-1}\}, \quad r_i = \frac{a_i - a_{i-1}}{2(x_{i-1} + x_i)}.$$

Further separate A into two disjoint subsets:

$$A_1 = \left\{ [i] \in A \middle| |r_i - 1| < \frac{1}{2} \right\}, \quad A_2 = \left\{ [i] \in A \middle| |r_i - 1| \ge \frac{1}{2} \right\}.$$

Clearly, $A_1 \cup A_2 = A$, $A \cup B = \mathbb{Z}_N$, $A_1 \cap A_2 = \emptyset$, $A \cap B = \emptyset$. Since $a_i - a_{i-1} = 2r_i(x_{i-1} + x_i)$, we have for any $m \le n$:

$$a_n - a_{m-1} = 2 \sum_{i=m}^{n} r_i (x_{i-1} + x_i).$$

Also note that $r_i > 0$ if $[i] \in A$.

Recall that any nonconstant N-periodic sequence $\mathbf{a}$ can be decomposed into an equal number of congruence classes of decreasing and increasing components modulo N that are mutually disjoint except at their endpoints. Thus, suppose $\Gamma^\sharp(\mathbf{a})_N = \{[\gamma_j^\sharp]_N\}_{j=1}^s$, $\Gamma^\flat(\mathbf{a})_N = \{[\gamma_j^\flat]_N\}_{j=1}^s$ for some $s \ge 1$. Further suppose for each $1 \le j \le s$ the left and right endpoint of $\gamma_j^\sharp$ is $(m_j - 1, a_{m_j - 1})$ and (n_j, a_{n_j}) respectively, where $m_j \le n_j$ and $a_{m_j - 1} < a_{n_j}$. Clearly, $[i] \in A$ implies i is a link of some increasing component of $\Gamma(\mathbf{a})$, so that $A \subset \cup_{j=1}^s \{[m_j], [m_j + 1], \cdots, [n_j]\}$ and we deduce the following

Observation 1

$$\begin{aligned}\sum_{[\gamma]_N \in \Gamma^\sharp(\mathbf{a})_N} J(\gamma) &= \sum_{j=1}^{s} J(\gamma_j^\sharp)\\ &= \sum_{j=1}^{s} (a_{n_j} - a_{m_j-1})^3\\ &= 8\sum_{j=1}^{s}\left(\sum_{i=m_j}^{n_j} r_i(x_{i-1}+x_i)\right)^3\\ &\geq 8\sum_{j=1}^{s}\sum_{i=m_j}^{n_j} r_i^3(x_{i-1}+x_i)^3\\ &\geq 8\sum_{[i]\in A} r_i^3(x_{i-1}+x_i)^3\\ &\geq 8\sum_{[i]\in A_1} r_i^3(x_{i-1}+x_i)^3\\ &> \sum_{[i]\in A_1}(x_{i-1}+x_i)^3.\end{aligned}$$

Observation 2

$$\sum_{[i]\in A_2}\left(\frac{y_{i-1}}{2} - y_i + \frac{y_{i+1}}{2} + x_{i-1} + x_i\right)^2 = \sum_{[i]\in A_2}(r_i-1)^2(x_{i-1}+x_i)^2 \geq \frac{1}{4}\sum_{[i]\in A_2}(x_{i-1}+x_i)^2.$$

Observation 3

$$\begin{aligned}\sum_{[i]\in B}\left(\frac{y_{i-1}}{2} - y_i + \frac{y_{i+1}}{2} + x_{i-1} + x_i\right)^2 &= \sum_{[i]\in B}\left(-\frac{a_i}{2} + \frac{a_{i-1}}{2} + x_{i-1} + x_i\right)^2\\ &\geq \frac{1}{4}\sum_{[i]\in B}(a_i - a_{i-1})^2 + \sum_{[i]\in B}(x_{i-1}+x_i)^2\\ &= \frac{1}{4}\sum_{[\gamma]_N\in\Gamma^\flat(\mathbf{a})_N} I(\gamma) + \sum_{[i]\in B}(x_{i-1}+x_i)^2.\end{aligned}$$

Using Lemma B.3 and the first observation, we have

$$\sum_{[\gamma]_N\in\Gamma^\flat(\mathbf{a})_N} I(\gamma) \geq \frac{1}{32}\sum_{[\gamma]_N\in\Gamma^\sharp(\mathbf{a})_N} J(\gamma) > \frac{1}{32}\sum_{[i]\in A_1}(x_{i-1}+x_i)^3.$$

Combining the above with the third observation yields

$$\sum_{[i]\in B}\left(\frac{y_{i-1}}{2} - y_i + \frac{y_{i+1}}{2} + x_{i-1} + x_i\right)^2 \geq \frac{1}{128}\sum_{[i]\in A_1}(x_{i-1}+x_i)^3 + \sum_{[i]\in B}(x_{i-1}+x_i)^2.$$

Taking the second observation into account would give us

$$\begin{aligned}\sum_{i=1}^{N}\left(\frac{y_{i-1}}{2} - y_i + \frac{y_{i+1}}{2} + x_{i-1} + x_i\right)^2 &\geq \sum_{[i]\in B\cup A_2}\left(\frac{y_{i-1}}{2} - y_i + \frac{y_{i+1}}{2} + x_{i-1} + x_i\right)^2\\ &\geq \frac{1}{128}\sum_{[i]\in A_1}(x_{i-1}+x_i)^3 + \sum_{[i]\in B}(x_{i-1}+x_i)^2 + \frac{1}{4}\sum_{[i]\in A_2}(x_{i-1}+x_i)^2\\ &\geq \frac{1}{64}\sum_{i=1}^{N} x_i^3\end{aligned}$$

where $(x_{i-1}+x_i)^2 \geq \frac{1}{2}(x_{i-1}+x_i)^3$ (since $0 < x_i, x_{i+1} \leq 1$) is used in the last step and the proof is complete.

■

Theorem 1. *For any real numbers $b_1, b_2, \cdots, b_{2N}$ satisfying:*

$$b_{2i-1} - 2b_{2i} + b_{2i+1} \geq 0, \quad \forall i = 1, 2, \cdots, N$$

with $b_{2N+1} = b_1, b_{2N+2} = b_2$, we have

$$\sum_{i=1}^{N} (b_{2i-1} - 2b_{2i} + b_{2i+1})^3 \leq 1024 \|\mathbf{b}\|_\infty \sum_{i=1}^{N} (b_{2i} - 2b_{2i+1} + b_{2i+2})^2.$$

Proof. First suppose $-1 \leq b_i \leq 0$. Let

$$x_i = \frac{b_{2i-1} - 2b_{2i} + b_{2i+1}}{2}$$
$$y_i = -b_{2i-1}.$$

Then,

$$\sum_{i=1}^{N} (b_{2i-1} - 2b_{2i} + b_{2i+1})^3 \leq 512 \sum_{i=1}^{N} (b_{2i} - 2b_{2i+1} + b_{2i+2})^2 \tag{B.1}$$

is equivalent to

$$8 \sum_{i=1}^{N} x_i^3 \leq 512 \sum_{i=1}^{N} \left(\frac{y_{i-1}}{2} - y_i + \frac{y_{i+1}}{2} + x_{i-1} + x_i \right)^2,$$

which is true by Lemma B.4 since $0 \leq x_i, y_i \leq 1$. By replacing b_i with $b_i + \frac{1}{2}$, we see that inequality (B.1) also holds for all $-\frac{1}{2} \leq b_i \leq \frac{1}{2}$. The general case is proved by rescaling b_i. ■