A SPECTRAL DAMPING METHOD FOR NONLINEAR CONSERVATION LAWS

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Abstract. Although spectral methods provide highly accurate approximations to smooth solutions, they typically perform poorly for nonlinear conservation laws when shock discontinuities develop. In this work, motivated by the numerical damping of discontinuous Galerkin methods [J. Lu, Y. Liu and C.-W. Shu, *SIAM J. Numer. Anal.*, 59 (2021), pp. 1299-1324], we introduce a novel numerical damping term to stabilize the Fourier spectral method for nonlinear conservation laws. With this damping, the Fourier coefficients of the numerical solution decay rapidly. Consequently, we rigorously prove that the spectrally damped approximation is uniformly bounded, has bounded total variation, and satisfies the entropy condition in the limit, thereby guaranteeing convergence to the unique entropy solution. Furthermore, we demonstrate that the method retains spectral accuracy for smooth solutions. Compared with the classical spectral viscosity method [E. Tadmor, *SIAM J. Numer. Anal.*, 26 (1989), pp. 30-44], the proposed method shows improved performance in our numerical experiments.

1. Introduction

In this paper, we consider the following scalar conservation law

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = 0, \tag{1.1}$$

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subject to given initial data $u(x,0)$ with 2π-periodic boundary conditions. It is well known that solutions of nonlinear conservation laws may develop discontinuities in finite time, even when the initial data are smooth. Hence, the class of weak solutions must be introduced and eq. (1.1) should be augmented with an additional entropy condition that requires for all convex U,

$$\frac{\partial}{\partial t}U(u)+\frac{\partial}{\partial x}F(u)\leq 0, \quad F(u)\equiv\int^{u}U'(\xi)f'(\xi)d\xi. \tag{1.2}$$

The inequality is understood in the sense of distributions.

We aim to solve the 2π-periodic initial value problem eq. (1.1) by spectral methods. Let

$$S_N=\operatorname{span}\{e^{ikx}:|k|\leq N\},$$

and let P_N denote the L^2-orthogonal projection onto S_N. The standard Fourier method is given by, e.g., [9, 2, 12, 24]

$$\frac{\partial}{\partial t}u_N+\frac{\partial}{\partial x}P_N f\left(u_N\right)=0, \tag{1.3}$$

where $u_N(x,t)=\sum_{|k|\leq N}\hat{u}_k(t)e^{ikx}$. It is well known that the standard spectral method provides highly accurate approximations to the solutions of partial differential equations, provided the exact solution is sufficiently smooth [9, 2, 24, 27]. However, for nonlinear conservation laws, the direct Fourier spectral approximation may fail to converge to the entropy solution once shock discontinuities develop. Due to the presence of shock discontinuities, standard spectral approximations of eq. (1.1) suffer from spurious Gibbs oscillations, which in turn lead to two related difficulties: loss of accuracy in the overall computational domain and, in the nonlinear case, loss of convergence. Indeed, counterexamples provided in [28, 29] show that the spectral solution $u_N(x,t)$ in eq. (1.3) lacks entropy dissipation and does not necessarily converge to the corresponding entropy solution. This failure is closely related to the fact that the approximation does not satisfy the entropy condition eq. (1.2) in the limit. To restore stability and enforce entropy dissipation, the spectral viscosity (SV) method was introduced by Tadmor [28] for the Fourier approximation of nonlinear conservation laws. The main ingredient of the SV method is the use of high-frequency diffusion which stabilizes the spectral computation without sacrificing spectral accuracy, which amounts to

$$\begin{aligned}\frac{\partial u_N}{\partial t}+\frac{\partial}{\partial x}P_N f\left(u_N\right)&=\varepsilon_N(-1)^{s+1}\frac{\partial^s}{\partial x^s}\left[\mathcal{Q}_m(x)*\frac{\partial^s u_N}{\partial x^s}\right]\\&=-\varepsilon_N\sum_{m<|k|\leq N}k^{2s}\widehat{\mathcal{Q}}_k\widehat{u}_k(t)e^{ikx},\end{aligned} \tag{1.4}$$

where $s\geq 1$, $0<\widehat{\mathcal{Q}}_k\leq 1$, $\varepsilon_N\sim\frac{1}{N^{2s-1}}$. The classical SV method corresponds to $s=1$. Further results on the periodic SV method can be found in [31, 23, 3, 1]. A more robust method based on "super-viscosity" regularization, e.g., $s>1$ in eq. (1.4), was introduced in [30]. For non-periodic problems, one may consult [19, 21, 17, 18]. The theoretical analyses in [31, 3] show that the spectral viscosity method is total-variation bounded, uniformly bounded and satisfies an entropy inequality, which implies convergence to the entropy solution. Another important issue is accuracy. Numerical experiments reported in [29, 19, 25] have demonstrated that spectral accuracy can be recovered from the SV method via a postprocessing procedure such as filter or reconstruction techniques.

Recently, an oscillation-free discontinuous Galerkin (OFDG) method was developed in [14, 16] for hyperbolic conservation laws. The main idea is to add damping terms to the high-order coefficients of the basis expansion. As demonstrated in [16], the added damping terms not only suppress spurious oscillations but also preserve desirable theoretical properties of the original DG method such as mass conservation, optimal error estimates, superconvergence, etc. Further developments in OFDG methods can be found in [15, 6, 22]. Inspired by the work of [14, 16], we introduce a novel numerical damping term to stabilize Fourier spectral approximations of nonlinear conservation laws, as presented in eq. (2.1). Compared with the SV method, the proposed spectral damping method exhibits improved accuracy in smooth regions. Our numerical experiments also indicate that the proposed method performs more robustly in long-time computations.

The paper is organized as follows. We first present the spectral damping scheme and basic L^2-type a priori energy estimates in Section 2. In Section 3, we study the spectral decay rate of the Fourier coefficients. These decay estimates enable us to obtain an L^∞ a priori estimate and to prove the total-variation bound for the spectral damping solution in Section 4. Based on the estimates established in Sections 3 and 4, we prove that our method converges to the unique entropy solution of eq. (1.1) in Section 5. In Section 6, we show that the damping method retains spectral accuracy for smooth solutions. Subsequently, numerical experiments are presented in Section 7 to evaluate the performance of the proposed method. Finally, conclusions are given in Section 8.

2. The spectral damping method

We consider the spectral damping approximation

$$\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}\left(P_N f(u_N(x,t))\right) = -N^\theta Q_N u_N(x,t). \tag{2.1}$$

Here Q_N is the spectral damping operator defined as a convolution

$$Q_N u_N(x,t) \equiv Q_N(x) * u_N(x,t) = \sum_{m<|k|\le N} \widehat{Q}_k \widehat{u}_k(t) e^{ikx}.$$

The parameter $\theta > 1$ is chosen so that the high-frequency modes decay sufficiently fast. $m := \lfloor N^\alpha \rfloor$ is the effective size of the damping-free spectrum, $0 < \alpha < \frac{1}{2}$. The increasing Fourier coefficients $\hat{Q}_k$, which are activated only on high wavenumbers, satisfy

$$\widehat{Q}_k = \begin{cases} 0, & |k| \le m, \\ 1 - \exp\left\{ -\dfrac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}} \left(\dfrac{|k|-m}{m}\right)^2 \right\}, & m < |k| \le 2m, \\ 1 - \exp\left\{ -\dfrac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}} \right\}, & 2m < |k| \le N. \end{cases} \tag{2.2}$$

For $\frac{4-2\alpha}{3} < \theta < 2-2\alpha$, it follows that $1+\alpha+\frac{\theta}{2} < 2$ and $1-\alpha-\frac{\theta}{2} > 0$. For example, let $\alpha = 0.48$ and $\theta = 1.02$, then we have $1+\alpha+\frac{\theta}{2} = 1.99$ and $1-\alpha-\frac{\theta}{2} = 0.01$.

Remark 2.1. To illustrate the basic idea, eq. (2.2) could also be defined in two pieces instead of the current three pieces, e.g.,

$$\widehat{Q}_k = \begin{cases} 0, & |k| \leq m, \\ 1 - \exp\left\{ -\dfrac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}} \right\}, & m < |k| \leq N. \end{cases}$$

In this case, the subsequent proof would even be simpler. However the definition in eq. (2.2) makes $\widehat{Q}_k$ continuous in k and gives better numerical performance in our numerical tests.

Remark 2.2. The oscillation-free discontinuous Galerkin (DG) method developed in [16, 14] is formulated as follows:

$$\begin{aligned} \int_{I_j} (u_h)_t v_h dx = & \int_{I_j} f(u_h)(v_h)_x dx - \hat{f}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} + \hat{f}_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} \\ & - \sum_{l=0}^{k} \frac{\sigma_j^l}{h_j} \int_{I_j} \left(u_h - P_h^{l-1} u_h \right) v_h dx, \quad \forall v_h \in V_h^k, \end{aligned}$$

where u_h is the DG numerical solution in the piecewise k-th degree polynomial DG space V_h^k, I_j is a local cell with cell size h_j, $\hat{f}_{j+\frac{1}{2}}$ is the numerical flux, P_h^{l-1} is the L^2 projection operator to the subspace of V_h^k consisting of piecewise polynomials of degree $\max(0, l-1)$, and σ_j^l is the damping coefficient. Motivated by this idea, one may choose a damping term eq. (2.1) analogous to that in the oscillation-free discontinuous Galerkin method:

$$\begin{aligned} - \sum_{l=0}^{N-m-1} b_l (I - P_{m+l}) u_N(x,t) &= - \sum_{l=0}^{N-m-1} b_l \sum_{|k|>m+l} \hat{u}_k e^{ikx} \\ &= - \sum_{m<|k|\leq N} \left(\sum_{l=0}^{|k|-m-1} b_l \right) \hat{u}_k e^{ikx} \\ &= - \sum_{m<|k|\leq N} A_k \hat{u}_k e^{ikx}, \end{aligned} \tag{2.3}$$

where $A_k := \sum_{l=0}^{|k|-m-1} b_l$, and $b_l > 0$ are the damping coefficients. This yields a Fourier-multiplier damping term whose damping strength increases with wavenumber. Thus, it has the same high-mode damping philosophy as eq. (2.1)-eq. (2.2), although the precise damping rates may be different.

Next, we derive some basic L^2-type a priori energy estimates. We multiply eq. (2.1) by u_N,

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} u_N^2 \right) + u_N \frac{\partial}{\partial x} (f(u_N)) = & u_N \frac{\partial}{\partial x} [(I - P_N) f(u_N)] \\ & - u_N N^\theta \sum_{|k|>m} \hat{Q}_k \hat{u}_k(t) e^{ikx}, \end{aligned} \tag{2.4}$$

and integrate over a 2π-period: the integrals of the second and third terms vanish by periodicity and orthogonality. We are left with

$$\frac{1}{2}\frac{d}{dt}\|u_N(\cdot,t)\|^2_{L^2(x)} + N^\theta \sum_{|k|>m} \left|\sqrt{\hat{Q}_k}\hat{u}_k(t)\right|^2 = 0. \tag{2.5}$$

Temporal integration of eq. (2.5) yields

$$\frac{1}{2}\|u_N(\cdot,t)\|^2_{L^2(x)} + N^\theta \int_{\tau=0}^{t} \sum_{|k|>m} \left|\sqrt{\hat{Q}_k}\hat{u}_k(\tau)\right|^2 d\tau = \frac{1}{2}\|u_N(\cdot,0)\|^2_{L^2(x)}. \tag{2.6}$$

Hence for $u_N(x,t) \equiv \sum_{|k|\le N} \hat{u}_k(t)e^{ikx}$ we have

$$\|u_N(\cdot,t)\|^2_{L^2(x)} = \sum_{|k|\le N} |\hat{u}_k(t)|^2 \le E_0^2, \qquad E_0 := \|u(\cdot,0)\|_{L^2(x)}, \tag{2.7}$$

which implies that its spatial L^2 norm decays in time. Equality eq. (2.6) also gives us the second a priori estimate

$$N^\theta \int_{\tau=0}^{t} \sum_{|k|>m} \left|\sqrt{\hat{Q}_k}\hat{u}_k(\tau)\right|^2 d\tau \le \frac{E_0^2}{2}. \tag{2.8}$$

3. The decay rate of the Fourier coefficients

In this section, we follow the approach of [20, 13, 23], to show that, due to the high-frequency damping term, the Fourier coefficients $|\hat{u}_k(t)|$ decay rapidly for $|k| > 2m$, and therefore, the discretization error $\partial_x(I - P_N)f(u_N)$ is spectrally small. Before presenting the main results, we first recall some lemmas.

Lemma 3.1. [23] *For all u, w in L^∞,*

$$\|(I-P_k)\,uw\|_{L^2(x)} \le 2\|w\|_{L^\infty}\|\left(I-P_{[k/8]}\right)u\|_{L^2(x)} + 2\|u\|_{L^\infty}\|\left(I-P_{[k/8]}\right)w\|_{L^2(x)}. \tag{3.1}$$

Lemma 3.2. [23] *If f is in C^s with $s \ge 2$ and $u \in L^\infty$, then for any positive integer p*

$$\begin{aligned} &\|(I-P_k)\,f(u)\|_{L^2(x)} \\ &\le C_*\left[\sum_{1\le j\le p}\left(j^{3-s}\left\|\left(I-P_{\lfloor k/(32j^2)\rfloor}\right)u\right\|_{L^2(x)}\right) + p^{1-s}\right], \end{aligned} \tag{3.2}$$

where

$$C_* := C_s \cdot K^s\|f\|_{C^s([-K,K])}, \quad K := \|u\|_{L^\infty},$$

and

$$\|f\|_{C^s([-K,K])} := \max_{0\le n\le s}\sup_{|z|\le K}\left|f^{(n)}(z)\right|.$$

First, we assume that

$$\|u_N(\cdot,t)\|_{L^\infty(x)} \le M, \quad t\in[0,T].$$

This a priori assumption will be justified later. The notation $A \lesssim B$ implies that there exists a constant $c_0 > 0$ independent of N, M such that $A \le c_0 B$. Then, Let

$$\tilde{g}(M) := \|f\|_{C^s([-M,M])}.$$

Using eq. (3.1) and eq. (3.2), we obtain the following theorem.

Theorem 3.3. *Let u_N be the solution of the approximation eq. (2.1)-eq. (2.2) with $m = \lfloor N^\alpha \rfloor$, $0 < \alpha < \frac{1}{2}$, $\frac{4-2\alpha}{3} < \theta < 2 - 2\alpha$. Suppose that f is in C^s, and*

$$\|(I - P_k)\, u_N(x,0)\|_{L^2(x)} \lesssim k^{-r}, \quad r \geq 0. \tag{3.3}$$

L is a positive integer which satisfies

$$L > \frac{4-\theta}{\theta - 1},$$

and

$$s \geq s_0 \equiv \max\left\{4L + 2r + 7, \frac{5(\theta-1)(L+1)^2}{1-\alpha} + 2\right\}. \tag{3.4}$$

Then we have

$$\begin{aligned} &\left\|(I - P_N)\, \partial_x^2 f(u_N(\cdot,t))\right\|_{L^2(x)} \\ &\lesssim \left(1 + M^{s+1}\tilde{g}^2(M)\right)^{L+2} \left(N^{(1-\theta)(L+1)+2} + N^{2-r} e^{-ctN^{\frac{\theta}{2}}}\right) \\ &\lesssim g(M) \left(N^{(1-\theta)(L+1)+2} + N^{2-r} e^{-ctN^{\frac{\theta}{2}}}\right), \end{aligned} \tag{3.5}$$

where $g(M) := \left(1 + M^{s+1}\tilde{g}^2(M)\right)^{L+2}$.

Proof. We begin by taking the $I - P_k$ projection of eq. (2.1). For $k > 2m$ we have, by eq. (2.1) and eq. (2.2)

$$\frac{\partial}{\partial t}\left[(I - P_k)u_N\right] + (I - P_k)\frac{\partial}{\partial x}\left[P_N f(u_N)\right] = -N^\theta (I - P_k) Q_N u_N, \quad 2m < k \leq N. \tag{3.6}$$

Multiplying by $(I - P_k)u_N$ and integrating over a 2π-period, we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(I - P_k)u_N(\cdot,t)\|^2_{L^2(x)} = &-\frac{1}{2\pi}\int_0^{2\pi} (I - P_k)u_N \cdot (I - P_k)\frac{\partial}{\partial x} P_N f(u_N) dx \\ &- \frac{N^\theta}{2\pi}\int_0^{2\pi} (I - P_k) Q_N u_N \cdot (I - P_k) u_N dx. \end{aligned} \tag{3.7}$$

The first integral on the right-hand side is bounded in absolute value by

$$\begin{aligned} &\left|\frac{1}{2\pi}\int_0^{2\pi} (I - P_k)u_N \cdot (I - P_k)\frac{\partial}{\partial x} P_N f(u_N) dx\right| \\ &\leq \|(I - P_k)u_N(\cdot,t)\|_{L^2(x)} \cdot \left\|(I - P_k)\frac{\partial}{\partial x} P_N f(u_N)\right\|_{L^2(x)} \\ &\leq \|(I - P_k)u_N(\cdot,t)\|_{L^2(x)} \cdot \left\|(I - P_k)\frac{\partial}{\partial x} f(u_N)\right\|_{L^2(x)} \\ &\leq \|(I - P_k)u_N(\cdot,t)\|_{L^2(x)} \cdot \|f'(u_N)\partial_x u_N\|_{L^2(x)} \\ &\leq N\, \|(I - P_k)u_N(\cdot,t)\|_{L^2(x)} \cdot \|f\|_{C^s} \|u_N\|_{L^2(x)} \\ &\leq \tilde{g}(M) N\, \|(I - P_k)u_N(\cdot,t)\|_{L^2(x)}. \end{aligned} \tag{3.8}$$

To estimate the second term of eq. (3.7), since for $a > 0$, $1 - e^{-a} > \frac{a}{1+a}$, we find that for $k > 2m$

$$
\begin{aligned}
&\frac{N^\theta}{2\pi}\int_0^{2\pi}(I-P_k)Q_N u_N\cdot(I-P_k)u_N dx \\
&= N^\theta \sum_{k<|j|\le N}\left(1-e^{-\frac{j^2}{N^{(1+\alpha+\frac{\theta}{2})}}}\right)|\hat{u}_j|^2 \\
&\ge N^\theta\left(1-e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}}\right)\sum_{k<|j|\le N}|\hat{u}_j|^2 \ge N^\theta \frac{\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}}{1+\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}}\sum_{k<|j|\le N}|\hat{u}_j|^2 \\
&\qquad\qquad = N^\theta\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}+k^2}\left\|(I-P_k)u_N(\cdot,t)\right\|^2_{L^2(x)}.
\end{aligned}
\tag{3.9}
$$

Combining eq. (3.8) and eq. (3.9), we return to eq. (3.7) to find that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|(I-P_k)u_N(\cdot,t)\right\|^2_{L^2(x)} \\
&\quad\le -N^\theta\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}+k^2}\left\|(I-P_k)u_N(\cdot,t)\right\|^2_{L^2(x)} + N\tilde{g}(M)\left\|(I-P_k)u_N(\cdot,t)\right\|_{L^2(x)}.
\end{aligned}
\tag{3.10}
$$

After removing the common factor of $\|(I-P_k)u_N(\cdot,t)\|_{L^2(x)}$, applying Gronwall's lemma along with eq. (3.3), we obtain

$$
\begin{aligned}
&\left\|(I-P_k)\,u_N(\cdot,t)\right\|_{L^2(x)} \\
&\quad\le \frac{\tilde{g}(M)N^{1-\theta}}{\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}+k^2}} + \exp\left(-tN^\theta\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}+k^2}\right)\cdot\left\|(I-P_k)\,u_N(x,0)\right\|_{L^2(x)} \\
&\quad\le \tilde{g}(M)N^{1-\theta}\frac{N^{(1+\alpha+\frac{\theta}{2})}+k^2}{k^2} + k^{-r}\exp\left(-tN^\theta\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}+k^2}\right), \quad k > 2m.
\end{aligned}
\tag{3.11}
$$

From eq. (3.7) and eq. (3.9), we also obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\|(I-P_k)\,u_N\right\|_{L^2(x)} + N^\theta\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}+k^2}\left\|(I-P_k)\,u_N\right\|_{L^2(x)} \\
&\quad\le \left\|(I-P_k)\,\partial_x P_N f(u_N)\right\|_{L^2(x)} \\
&\quad\le \left\|(I-P_k)\,\partial_x f(u_N)\right\|_{L^2(x)}.
\end{aligned}
\tag{3.12}
$$

According to eq. (3.1) and eq. (3.2), we can estimate the high-frequency tail of $f(u_N)$ from that of u_N. Then, we can use eq. (3.11) in eq. (3.12) to improve the decay rate of the high-frequency tail of u_N, and repeat this process to obtain successively higher rates of decay. Such estimates were obtained in a similar manner in [20, 23]. Specifically, let L be a fixed nonnegative integer and choose $p = \lfloor N^\delta \rfloor$ in eq. (3.2), $\delta = \frac{1-\alpha}{4(L+1)}$. We can proceed by induction and obtain the following estimate: for each $0 \le l \le L$ and for every integer k satisfying $k \gtrsim 2m(32N^{2\delta})^l$, there holds

$$\begin{aligned}
&\|(I-P_k)\,u_N(\cdot,t)\|_{L^2(x)} \\
&\lesssim \left(C_* M\tilde{g}(M)N^{1-\theta}\frac{\left(N^{(1+\alpha+\frac{\theta}{2})}+k^2\right)}{k^2}\right)^{l+1} \\
&\quad+\left(1+C_*M\tilde{g}(M)\frac{N^2}{k^2}\right)^{l}k^{-r}\exp\left(-ctN^{\theta}\frac{k^2}{N^{4\delta l+(1+\alpha+\frac{\theta}{2})}+k^2}\right),
\end{aligned} \tag{3.13}$$

where $s \geq \max\left\{\frac{5(\theta-1)(L+1)^2}{1-\alpha}+2,\ 4L+2r+7\right\}$, $0<c<1$. Verification of the induction step is deferred to Appendix A. Now we set $l=L$ in eq. (3.13). For sufficiently large N, we obtain that for $k=\lfloor\frac{N}{8}\rfloor \gtrsim 2m\cdot(32N^{2\delta})^L$,

$$\begin{aligned}
&\left\|\left(I-P_{\lfloor\frac{N}{8}\rfloor}\right)u_N(\cdot,t)\right\|_{L^2(x)} \\
&\lesssim \left(C_*M\tilde{g}(M)N^{1-\theta}\frac{\left(N^{(1+\alpha+\frac{\theta}{2})}+N^2\right)}{N^2}\right)^{L+1} \\
&\quad+(1+C_*M\tilde{g}(M))^L N^{-r}e^{-ctN^{\theta}\left(\frac{N^2}{N^{4\delta L+(1+\alpha+\frac{\theta}{2})}+N^2}\right)} \\
&\lesssim \left(C_*M\tilde{g}(M)N^{1-\theta}\right)^{L+1}+\left(1+C_*\tilde{g}(M)M\right)^L N^{-r}e^{-ctN^{\frac{\theta}{2}}},
\end{aligned} \tag{3.14}$$

where $4\delta L+1+\alpha<2$. Next, let $N_j := \left\lfloor\frac{N}{32j^2}\right\rfloor$. By eq. (3.2) and *eq.* (3.13), we have

$$\begin{aligned}
&\|(I-P_N)\,f(u_N)\|_{L^2(x)} \\
&\lesssim C_*\left[\sum_{1\leq j\leq p} j^{3-s}\left\|\left(I-P_{N_j}\right)u_N\right\|_{L^2(x)}+p^{1-s}\right] \\
&\lesssim C_*\sum_{1\leq j\leq p} j^{3-s}\left(C_*M\widetilde{g}(M)N^{1-\theta}\frac{N^{1+\alpha+\frac{\theta}{2}}+N_j^2}{N_j^2}\right)^{L+1}+C_*p^{1-s} \\
&\quad+C_*\sum_{1\leq j\leq p} j^{3-s}\left(1+C_*M\widetilde{g}(M)\frac{N^2}{N_j^2}\right)^{L}N_j^{-r}\exp\left(-ctN^{\theta}\frac{N_j^2}{N^{4\delta L+1+\alpha+\frac{\theta}{2}}+N_j^2}\right) \\
&\lesssim C_*\left(C_*M\widetilde{g}(M)N^{1-\theta}\right)^{L+1}\sum_{1\leq j\leq p} j^{3-s}\left(1+j^4\right)^{L+1} \\
&\quad+C_*\left(1+C_*M\widetilde{g}(M)\right)^L N^{-r}\exp\left(-ctN^{\frac{\theta}{2}}\right)\sum_{1\leq j\leq p} j^{3-s+4L+2r}+C_*N^{(1-s)\delta} \\
&\lesssim C_*\left(C_*M\widetilde{g}(M)N^{1-\theta}\right)^{L+1}+C_*\left(1+C_*M\widetilde{g}(M)\right)^L N^{-r}\exp\left(-ctN^{\frac{\theta}{2}}\right).
\end{aligned} \tag{3.15}$$

Note that for $N\geq 2m\cdot(32N^{2\delta})^{L+1}$, $\lfloor N/32j^2\rfloor \geq 2m\cdot(32N^{2\delta})^L$ for $j\leq\lfloor N^{\delta}\rfloor$ so that the use of eq. (3.13) in *eq.* (3.2) with $l=L$ is justified. In fact, in obtaining estimate eq. (3.15) we have only used the fact that $s\geq s_0-2$, where s_0 is defined in eq. (3.4), and so estimate eq. (3.15) that we have deduced is also valid for f'

and f''. According to eq. (3.14) and by the same argument as in eq. (3.15), with f replaced by f' and N replaced by $\frac{N}{8}$, we have

(3.16)
$$\begin{aligned}
&\|(I-P_N)\left[f'(u_N)\partial_x u_N\right]\|_{L^2(x)}\\
&\lesssim \|f'(u_N)\|_{L^\infty}\|\left(I-P_{\lfloor N/8\rfloor}\right)\partial_x u_N\|_{L^2(x)} + \|\partial_x u_N\|_{L^\infty}\|\left(I-P_{\lfloor N/8\rfloor}\right)f'(u_N)\|_{L^2(x)}\\
&\lesssim N\tilde{g}(M)\|\left(I-P_{\lfloor N/8\rfloor}\right)u_N\|_{L^2(x)} + NM\|\left(I-P_{\lfloor N/8\rfloor}\right)f'(u_N)\|_{L^2(x)}\\
&\lesssim (C_*\tilde{g}(M)M)^{L+1}N^{(1-\theta)(L+1)+1} + (1+C_*\tilde{g}(M)M)^{L+1}N^{1-r}e^{-ctN^{\frac{\theta}{2}}}.
\end{aligned}$$

Applying Lemma 3.1 repeatedly, together with the estimates obtained in eq. (3.14)-eq. (3.16), and the corresponding bounds with f replaced by f' and f'', we get

(3.17)
$$\begin{aligned}
&\left\|(I-P_N)\,\partial_x^2 f(u_N)\right\| = \left\|(I-P_N)\left[f''(u_N)(\partial_x u_N)^2 + f'(u_N)\partial_x^2 u_N\right]\right\|\\
&\lesssim \|f''(u_N)\|_{L^\infty}\|\left(I-P_{\lfloor N/8\rfloor}\right)(\partial_x u_N)^2\| + \|(\partial_x u_N)^2\|_{L^\infty}\|\left(I-P_{\lfloor N/8\rfloor}\right)f''(u_N)\|\\
&\qquad + \|f'(u_N)\|_{L^\infty}\|\left(I-P_{\lfloor N/8\rfloor}\right)\partial_x^2 u_N\| + \|\partial_x^2 u_N\|_{L^\infty}\|\left(I-P_{\lfloor N/8\rfloor}\right)f'(u_N)\|\\
&\lesssim N^2\tilde{g}(M)\|\left(I-P_{\lfloor N/8\rfloor}\right)u_N\|_{L^2(x)} + (NM)^2\|\left(I-P_{\lfloor N/8\rfloor}\right)f''(u_N)\|_{L^2(x)}\\
&\lesssim (1+C_*M\tilde{g}(M))^{L+2}\left(N^{(1-\theta)(L+1)+2} + N^{2-r}e^{-ctN^{\frac{\theta}{2}}}\right)\\
&\lesssim \left(1+M^{s+1}\tilde{g}^2(M)\right)^{L+2}\left(N^{(1-\theta)(L+1)+2} + N^{2-r}e^{-ctN^{\frac{\theta}{2}}}\right)\\
&\lesssim g(M)\left(N^{(1-\theta)(L+1)+2} + N^{2-r}e^{-ctN^{\frac{\theta}{2}}}\right).
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Remark 3.4. As noted in [23, §3], the smoothness requirement eq. (3.3) is given only for the sake of concreteness and is by no means optimal. Improved estimates of the amount of smoothness required can be obtained by a more precise estimate of the kernel's L^1 norm, or more delicate estimates of the j-dependence in the high-frequency cutoff in eq. (3.2).

Remark 3.5. According to eq. (3.16)-eq. (3.17), and the Gagliardo-Nirenberg inequality
$$\|w_x\|_{L^\infty} \le c\,\|w_{xx}\|_{L^2(x)}^{1/2}\,\|w_x\|_{L^2(x)}^{1/2},$$
(see [7]) with $w=(I-P_N)\,f(u_N)$, we also have

$$\|(I-P_N)\,\partial_x f(u_N)\|_{L^\infty(x)} \lesssim g(M)\left(N^{(1-\theta)(L+1)+2} + N^{2-r}e^{-ctN^{\frac{\theta}{2}}}\right). \tag{3.18}$$

4. Total-variation boundedness and L^∞ a priori estimate

Equipped with Theorem 3.1, one can show that the spectral damping approximation eq. (2.1) is total-variation bounded and remains uniformly bounded. In the following lemmas we collect the necessary estimates in this direction.

Lemma 4.1. *If $\sigma(x)\ge 0$, $\|\sigma\|_{L^1}=1$, and $w\in L^1(0,2\pi)$ is a real-valued function, then we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\left(w(x)-(\sigma * w)(x)\right)\cdot \operatorname{sgn}(w(x))\,dx \ge 0. \tag{4.1}$$

Proof. With the normalized periodic convolution convention, we have

$$(\sigma * w)(x) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(y) w(x-y)\, dy.$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} (\sigma * w)(x)\cdot \operatorname{sgn}(w(x))\, dx \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\left(\int_0^{2\pi} \sigma(y) w(x-y)\, dy\right)\cdot \operatorname{sgn}(w(x))\, dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sigma(y)\left(\frac{1}{2\pi}\int_0^{2\pi} w(x-y)\cdot \operatorname{sgn}(w(x))\, dx\right) dy \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \sigma(y)\left(\frac{1}{2\pi}\int_0^{2\pi} |w(x-y)|\, dx\right) dy \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi} \sigma(y)\, dy\right)\cdot\left(\frac{1}{2\pi}\int_0^{2\pi} |w(x)|\, dx\right) \\
&= \|\sigma\|_{L^1}\|w\|_{L^1} = \frac{1}{2\pi}\int_0^{2\pi} |w(x)|\, dx,
\end{aligned} \tag{4.2}$$

where we have utilized the assumption $\|\sigma\|_{L^1} = 1$ and the periodicity of $w(x)$. Noting that $\frac{1}{2\pi}\int_0^{2\pi} w(x)\cdot\operatorname{sgn}(w(x))\, dx = \frac{1}{2\pi}\int_0^{2\pi} |w(x)|\, dx$, substituting this relation back into eq. (4.2) confirms eq. (4.1). $\square$

Lemma 4.2.

$$\tilde{\sigma}(x) = \sum_{k=-\infty}^{+\infty} e^{-\frac{k^2}{c_0}} e^{ikx} > 0, \quad c_0 > 0. \tag{4.3}$$

Proof. Let $f(y) = e^{-y^2/c_0} e^{ixy}$, then we have

$$\tilde{\sigma}(x) = \sum_{k\in\mathbb{Z}} f(k). \tag{4.4}$$

It is easy to know that

$$f(x) \in \mathcal{S}(\mathbb{R}) \equiv \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left| x^m f^{(n)}(x) \right| < \infty \quad \forall m, n \in \mathbb{N} \right\},$$

then the Poisson summation formula holds [26]. Namely, we have

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{m\in\mathbb{Z}} \hat{f}(m), \tag{4.5}$$

by Cauchy's Integral Theorem, we know that $\int_{-\infty}^{\infty} e^{-\frac{1}{c_0}\left(y-\frac{ic_0(x-2\pi m)}{2}\right)^2} dy = \sqrt{\pi c_0}$, and

$$\begin{aligned}\hat{f}(m) = \int_{-\infty}^{\infty} f(y)e^{-2\pi iym}dy &= \int_{-\infty}^{\infty} e^{-y^2/c_0}e^{ixy}e^{-2\pi iym}dy \\ &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{c_0}+iy(x-2\pi m)}dy \\ &= e^{-\frac{c_0(x-2\pi m)^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{1}{c_0}\left(y-\frac{ic_0(x-2\pi m)}{2}\right)^2} dy \\ &= e^{-\frac{c_0(x-2\pi m)^2}{4}}\sqrt{\pi c_0}.\end{aligned} \tag{4.6}$$

Therefore, we have

$$\tilde{\sigma}(x) = \sum_{k=-\infty}^{+\infty} e^{-\frac{k^2}{c_0}}e^{ikx} = \sqrt{\pi c_0}\sum_{m=-\infty}^{\infty} e^{-\frac{c_0}{4}(x-2\pi m)^2} > 0. \tag{4.7}$$

□

Lemma 4.3. *Let* $R_m(x) = \sum_{|k|\le m}\hat{R}(k)e^{ikx} = \hat{R}_0 + 2\cdot\sum_{k=1}^{m}\hat{R}(k)\cos kx$, *where*

$$\hat{R}(k) = \hat{R}(-k), \quad k = 1, 2, \cdots, m.$$

The Fourier coefficients $\{\widehat{R}(k)\}_{k=0}^{m}$ *are monotonically increasing and* $\hat{R}(k) \in (0,1)$. *Then we have*

$$\|R_m(x)\|_{L^1} \lesssim \log m. \tag{4.8}$$

Proof. If $\hat{R}(k)$ is monotonically decreasing, then the conclusion follows from [20, Lemma A.1]. If $\{\widehat{R}(k)\}_{k=0}^{m}$ is monotonically increasing, define

$$a_k = \hat{R}(m) - \hat{R}(k).$$

Then $\{a_k\}$ is monotonically decreasing and $a_k \in (0,1)$. Hence, by [20, Lemma A.1] applied to

$$A_m(x) := \sum_{|k|\le m} a_k e^{ikx} = \hat{R}(m)C_m(x) - R_m(x), \quad C_m(x) := \sum_{|k|\le m} e^{ikx}.$$

Since $\|C_m\|_{L^1} \lesssim \log m$, we get

$$\|A_m(x)\|_{L^1} \lesssim \log m.$$

It follows that

$$\|R_m(x)\|_{L^1} \lesssim \log m.$$

This completes the proof. Moreover, by the triangle inequality, we find that for monotonic coefficient $\hat{R}(k) \in (0,1)$, and $0 \le p < q$,

$$\left\|\sum_{p\le|k|\le q}\hat{R}(k)e^{ikx}\right\|_{L^1} \lesssim \log q. \tag{4.9}$$

□

Using the above lemma and taking into account the spectral decay of the truncation error eq. (3.17), we can show now that the spectral damping method is total-variation bounded. We conclude the following theorem.

Theorem 4.4. *Under the conditions of Theorem 3.3 and if* $r > 2 - \frac{\theta}{2}$, *then the spectral damping approximation eq.* (2.1)*-eq.* (2.2) *has a bounded variation, and the following estimate holds:*

$$\left\| \frac{\partial}{\partial x} u_N(\cdot, t) \right\|_{L^1} \equiv \| u_N(\cdot, t) \|_{\mathrm{BV}} \lesssim e^{tN^{-\frac{1}{4}(1-\alpha-\frac{\theta}{2})}} \left[\left\| \frac{\partial}{\partial x} u_N(\cdot, 0) \right\|_{L^1} + g(M) \left(N^{(1-\theta)(L+1)+2+\frac{1}{4}(1-\alpha-\frac{\theta}{2})} + N^{2-r-\frac{\theta}{2}} \right) \right], \tag{4.10}$$

where $0 < 1 - \alpha - \frac{\theta}{2} < \frac{1}{2}$.

Proof. In order to prove the total-variation boundedness, we rewrite eq. (2.1) in the equivalent form

$$\begin{aligned} &\frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x} f(u_N(x,t)) \\ &\quad = \frac{\partial}{\partial x} \left[(I - P_N) f(u_N(x,t)) \right] - N^{\theta} \left[u_N(x,t) - \sigma(x) * u_N(x,t) \right] \\ &\qquad - N^{\theta} [\sigma(x) - V_N(x)] * u_N(x,t), \end{aligned} \tag{4.11}$$

where $V_N(x) := \sum_{|k| \le N} \hat{V}_k e^{ikx}$, $\sigma(x) := \sum_{k=-\infty}^{+\infty} e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}} e^{ikx}$, and $\hat{V}_k$ satisfies

$$\widehat{V}_k = \begin{cases} 1, & |k| \le m, \\ \exp\left\{ -\frac{k^2}{N^{1+\alpha+\frac{\theta}{2}}} \left(\frac{|k| - m}{m} \right)^2 \right\}, & m < |k| \le 2m, \\ \exp\left\{ -\frac{k^2}{N^{1+\alpha+\frac{\theta}{2}}} \right\}, & 2m < |k| \le N. \end{cases}$$

We find that

$$\begin{aligned} &\sigma(x) - V_N(x) \\ &= - \sum_{|k| \le m} (1 - e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}}) e^{ikx} + \sum_{m < |k| \le 2m} \left(e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}} - e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}} (\frac{|k|-m}{m})^2} \right) e^{ikx} \\ &\quad + \sum_{|k| > N} e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}} e^{ikx} := \Delta_1 + \Delta_2 + \Delta_3. \end{aligned} \tag{4.12}$$

With eq. (4.12), spatial differentiation of eq. (4.11) yields

$$\begin{aligned} &\frac{\partial}{\partial t} \frac{\partial}{\partial x} u_N + \frac{\partial^2}{\partial x^2} f(u_N) \\ &= - N^{\theta} \left(\frac{\partial}{\partial x} u_N - \sigma(x) * \frac{\partial}{\partial x} u_N \right) - N^{\theta} (\sigma(x) - V_N(x)) * \frac{\partial}{\partial x} u_N \\ &\qquad + \frac{\partial^2}{\partial x^2} (I - P_N) f(u_N) \\ &= - N^{\theta} \left(\frac{\partial}{\partial x} u_N - \sigma(x) * \frac{\partial}{\partial x} u_N \right) - N^{\theta} \Delta_1 * \frac{\partial}{\partial x} u_N \\ &\quad - N^{\theta} \Delta_2 * \frac{\partial}{\partial x} u_N - N^{\theta} \Delta_3 * \frac{\partial}{\partial x} u_N + \frac{\partial^2}{\partial x^2} (I - P_N) f(u_N). \end{aligned} \tag{4.13}$$

We integrate against $\mathrm{sgn}(\frac{\partial}{\partial x}u_N)$, and in view of eq. (4.1) , we have

$$
\begin{aligned}
&\frac{d}{dt}\left\|\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^1}\\
&\leq -N^\theta\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial x}u_N-\sigma(x)*\frac{\partial}{\partial x}u_N\right)\cdot\mathrm{sgn}\left(\frac{\partial}{\partial x}u_N\right)\,dx\\
&\quad+N^\theta\left\|\Delta_1\right\|_{L^1}\left\|\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^1}+N^\theta\left\|\Delta_2\right\|_{L^1}\left\|\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^1}\\
&\quad+N^\theta\left\|\Delta_3\right\|_{L^1}\left\|\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^1}+\left\|\frac{\partial^2}{\partial x^2}\left(I-P_N\right)f(u_N(\cdot,t))\right\|_{L^1}\\
&\leq N^\theta\left\|\Delta_1\right\|_{L^1}\left\|\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^1}+N^\theta\left\|\Delta_2\right\|_{L^1}\left\|\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^1}\\
&\quad+N^\theta\left\|\Delta_3\right\|_{L^1}\left\|\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^1}+\left\|\frac{\partial^2}{\partial x^2}\left(I-P_N\right)f(u_N(\cdot,t))\right\|_{L^1},
\end{aligned}
\tag{4.14}
$$

where we have used Young's inequality $\|f*g\|_{L^1}\leq\|f\|_{L^1}\|g\|_{L^1}$. In what follows we bound the terms in the right-hand side of eq. (4.14). First, rewriting the first term and then applying eq. (4.8), we find that

(4.15)

$$
\begin{aligned}
&N^\theta\left\|\Delta_1\right\|_{L^1}:=N^\theta\left\|\sum_{|k|\leq m}(1-e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}})e^{ikx}\right\|_{L^1}\\
&\leq N^\theta\left\|\sum_{|k|\leq m}\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}e^{ikx}\right\|_{L^1}+N^\theta\left\|\sum_{|k|\leq m}\sum_{l=2}^{\infty}\frac{1}{l!}\left(\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}\right)^l e^{ikx}\right\|_{L^1}\\
&\lesssim N^\theta\left\|\sum_{|k|\leq m}\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}e^{ikx}\right\|_{L^1}+N^\theta\sum_{|k|\leq m}\sum_{l=2}^{\infty}\frac{1}{l!}\left(\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}\right)^l\\
&\lesssim N^{-\frac{1}{2}(1-\alpha-\frac{\theta}{2})}\left\|\sum_{|k|\leq m}\frac{k^2}{N^{\frac{1}{2}(1+3\alpha-\frac{\theta}{2})}}e^{ikx}\right\|_{L^1}+N^{-\theta}\\
&\leq N^{-\frac{1}{2}(1-\alpha-\frac{\theta}{2})}\log N^\alpha+N^{-\theta}\lesssim N^{-\frac{1}{4}(1-\alpha-\frac{\theta}{2})},\quad m=N^\alpha,\quad 1-\alpha-\frac{\theta}{2}>0.
\end{aligned}
$$

For the second term, by eq. (4.9) and using $e^{-x} = 1 - x + O\left(x^2\right)$, we have

$$
\begin{aligned}
N^\theta \left\|\Delta_2\right\|_{L^1} &:= N^\theta \left\| \sum_{m<|k|\le 2m} \left(e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}(\frac{|k|-m}{m})^2} - e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}}\right) e^{ikx} \right\|_{L^1} \\
&\lesssim N^{-\frac{1}{2}(1-\alpha-\frac{\theta}{2})} \left\| \sum_{m<|k|\le 2m} \frac{k^2}{N^{\frac{1}{2}(1+3\alpha-\frac{\theta}{2})}} \left[1 - (\frac{|k|-m}{m})^2\right] e^{ikx} \right\|_{L^1} \\
&\quad + N^\theta \sum_{m<|k|\le 2m} \left(\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}\right)^2 \\
&\lesssim N^{-\frac{1}{2}(1-\alpha-\frac{\theta}{2})} \left\| \sum_{m<|k|\le 2m} \frac{k^2}{N^{\frac{1}{2}(1+3\alpha-\frac{\theta}{2})}} \left[1 - (\frac{|k|-m}{m})^2\right] e^{ikx} \right\|_{L^1} \\
&\quad + N^\theta \sum_{m<|k|\le 2m} N^{-2+\alpha-\theta} \\
&\lesssim N^{-\frac{1}{2}(1-\alpha-\frac{\theta}{2})} \left\| \sum_{m<|k|\le \frac{3m}{2}} \frac{k^2}{N^{\frac{1}{2}(1+3\alpha-\frac{\theta}{2})}} \left[1 - (\frac{|k|-m}{m})^2\right] e^{ikx} \right\|_{L^1} \\
&\quad + N^{-\frac{1}{2}(1-\alpha-\frac{\theta}{2})} \left\| \sum_{\frac{3m}{2}<|k|\le 2m} \frac{k^2}{N^{\frac{1}{2}(1+3\alpha-\frac{\theta}{2})}} \left[1 - (\frac{|k|-m}{m})^2\right] e^{ikx} \right\|_{L^1} \\
&\quad + \sum_{m<|k|\le 2m} N^{-2+\alpha} \\
&\lesssim N^{-\frac{1}{4}(1-\alpha-\frac{\theta}{2})}.
\end{aligned}
\tag{4.16}
$$

For the third term, let $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-t^2} dt$. We know that $\operatorname{erfc}(z) \sim \frac{e^{-z^2}}{\sqrt{\pi} z} \quad (z \to +\infty)$. Therefore, we have

$$
\begin{aligned}
N^\theta \left\|\Delta_3\right\|_{L^1} &:= N^\theta \left\| \sum_{|k|>N} e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}} e^{ikx} \right\|_{L^1} \\
&\le N^\theta \sum_{|k|>N} e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}} \le N^\theta \int_N^{+\infty} e^{-\frac{x^2}{N^{(1+\alpha+\frac{\theta}{2})}}} dx \\
&\le \frac{1}{2} N^{(\alpha+\frac{3\theta}{2})} e^{-N^{2-(1+\alpha+\frac{\theta}{2})}} \to 0, \quad 1+\alpha+\frac{\theta}{2} < 2.
\end{aligned}
\tag{4.17}
$$

By eq. (3.5), equipped with eq. (4.15)-eq. (4.17), we return to eq. (4.14) to find that

$$
\begin{aligned}
&\frac{d}{dt} \left\| \frac{\partial}{\partial x} u_N(\cdot,t) \right\|_{L^1} \\
&\quad \lesssim N^{-\frac{1}{4}(1-\alpha-\frac{\theta}{2})} \|\frac{\partial}{\partial x} u_N(\cdot,t)\|_{L^1} + g(M) \left(N^{(1-\theta)(L+1)+2} + N^{2-r} e^{-ctN^{\frac{\theta}{2}}} \right).
\end{aligned}
\tag{4.18}
$$

Gronwall's lemma yields

$$
\begin{aligned}
&\left\| \frac{\partial}{\partial x} u_N(\cdot,t) \right\|_{L^1} \\
&\lesssim e^{tN^{-\frac{1}{4}(1-\alpha-\frac{\theta}{2})}} \left[\left\| \frac{\partial}{\partial x} u_N(\cdot,0) \right\|_{L^1} + g(M) \left(N^{(1-\theta)(L+1)+2+\frac{1}{4}(1-\alpha-\frac{\theta}{2})} + N^{2-r-\frac{\theta}{2}} \right) \right].
\end{aligned} \tag{4.19}
$$

□

Remark 4.5. To ensure that $r > 2 - \frac{\theta}{2}$, one might need a slightly stronger assumption regarding the initial smoothness and we shall not explore this issue here since, as noted above, the smoothness requirements for eq. (3.5) to hold are not optimal to begin with. On the other hand, to guarantee the BV-boundedness of $u_N(\cdot, 0)$, without sacrificing spectral accuracy, one can preprocess the exact initial data $u_N(\cdot, 0)$ by the frequency filter and we refer the reader to [32, 25, 31] for further details.

Remark 4.6. The total-variation bound indicated above implies the usual decay

$$
|\hat{u}_k(t)| \le \frac{\|u_N(\cdot,t)\|_{\mathrm{BV}}}{|k|+1}. \tag{4.20}
$$

We now turn to the justification of the a priori assumption of the L^∞-boundedness. For even p, we multiply eq. (4.11) by pu_N^{p-1} and integrate over the 2π-period. By the Hölder inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt} \|u_N(\cdot,t)\|^p_{L^p(x)} \\
&\quad = p \cdot \frac{1}{2\pi} \int_0^{2\pi} u_N^{p-1} \frac{\partial}{\partial x} \left[(I - P_N) f(u_N) \right] dx \\
&\qquad - N^\theta p \cdot \frac{1}{2\pi} \int_0^{2\pi} u_N^{p-1} \left[(u_N - \sigma * u_N) + (\sigma - V_N) * u_N \right] dx \\
&\quad \le p \cdot \|u_N(\cdot,t)\|^{p-1}_{L^p(x)} \left\| \frac{\partial}{\partial x} \left[(I - P_N) f(u_N) \right] \right\|_{L^p(x)} \\
&\qquad - N^\theta p \cdot \frac{1}{2\pi} \int_0^{2\pi} u_N^{p-1} \left(u_N - \sigma * u_N \right) dx \\
&\qquad\quad - N^\theta p \cdot \frac{1}{2\pi} \int_0^{2\pi} u_N^{p-1} \left(\sigma - V_N \right) * u_N \, dx.
\end{aligned} \tag{4.21}
$$

We need to bound the terms on the right-hand side of eq. (4.21). First, it is easy to know that $\|\sigma\|_{L^1(x)} = 1$ and we find that

$$
\begin{aligned}
& - N^\theta p \cdot \frac{1}{2\pi} \int_0^{2\pi} u_N^{p-1} \cdot \left(u_N - \sigma * u_N \right) dx \\
& = -N^\theta p \|u_N\|^p_{L^p(x)} + N^\theta p \cdot \frac{1}{2\pi} \int_0^{2\pi} u_N^{p-1} \cdot (\sigma * u_N) \, dx \\
& \le -N^\theta p \|u_N\|^p_{L^p(x)} + N^\theta p \|u_N\|^{p-1}_{L^p(x)} \|\sigma * u_N\|_{L^p(x)} \\
& \le -N^\theta p \|u_N\|^p_{L^p(x)} + N^\theta p \|u_N\|^{p-1}_{L^p(x)} \|\sigma\|_{L^1(x)} \|u_N\|_{L^p(x)} \\
& \le 0.
\end{aligned} \tag{4.22}
$$

Using the kernel L^1- bounds obtained in eq. (4.15)-eq. (4.17), together with Hölder's inequality and Young's inequality, we obtain

$$
\begin{aligned}
&N^{\theta} p \cdot \frac{1}{2\pi} \int_0^{2\pi} u_N^{p-1} \left(\sigma - V_N\right) * u_N \, dx \\
&\lesssim N^{\theta} p \left\| u_N \right\|_{L^p(x)}^{p-1} \left\| (\sigma - V_N) * u_N \right\|_{L^p(x)} \\
&\lesssim N^{\theta} p \left\| u_N \right\|_{L^p(x)}^{p} \left\| (\sigma - V_N) \right\|_{L^1(x)} \\
&\lesssim N^{-\frac{1}{4}\left(1-\alpha-\frac{\theta}{2}\right)} p \left\| u_N \right\|_{L^p(x)}^{p} .
\end{aligned} \tag{4.23}
$$

By plugging eq. (4.22), eq. (4.23) into eq. (4.21), we obtain

$$
\begin{aligned}
&\frac{d}{dt} \left\| u_N(\cdot,t) \right\|_{L^p(x)}^{p} \\
&\quad \leq p \cdot N^{-\frac{1}{4}\left(1-\alpha-\frac{\theta}{2}\right)} \left\| u_N(\cdot,t) \right\|_{L^p(x)}^{p} + p \cdot \left\| u_N(\cdot,t) \right\|_{L^p(x)}^{p-1} \left\| \frac{\partial}{\partial x} \left[\left(I - P_N\right) f(u_N) \right] \right\|_{L^p(x)} .
\end{aligned} \tag{4.24}
$$

Dividing both sides of eq. (4.24) by $p\|u_N(\cdot,t)\|_{L^p}^{p-1}$ and taking the limit as $p \to \infty$ through even integers, we obtain, in conjunction with eq. (3.18), that

$$
\begin{aligned}
&\frac{d}{dt} \left\| u_N(\cdot,t) \right\|_{L^\infty(x)} \\
&\quad \lesssim N^{-\frac{1}{4}\left(1-\alpha-\frac{\theta}{2}\right)} \left\| u_N(\cdot,t) \right\|_{L^\infty(x)} + g(M) \left(N^{(1-\theta)(L+1)+2} + N^{2-r} e^{-ctN^{\frac{\theta}{2}}} \right),
\end{aligned} \tag{4.25}
$$

which in turn implies that, for N large enough,

$$
\begin{aligned}
&\left\| u_N(\cdot,t) \right\|_{L^\infty(x)} \\
&\lesssim \left\| u_N(\cdot,0) \right\|_{L^\infty(x)} \cdot e^{N^{-\frac{1}{4}\left(1-\alpha-\frac{\theta}{2}\right)} t} \\
&\qquad + e^{N^{-\frac{1}{4}\left(1-\alpha-\frac{\theta}{2}\right)} t} \cdot g(M) \left(N^{(1-\theta)(L+1)+2+\frac{1}{4}\left(1-\alpha-\frac{\theta}{2}\right)} + N^{2-r-\frac{\theta}{2}} \right) \\
&\lesssim \frac{M}{4} + \frac{g(M)}{N^{(\theta-1)(L+1)-2-\frac{1}{4}\left(1-\alpha-\frac{\theta}{2}\right)}} + \frac{g(M)}{N^{r+\frac{\theta}{2}-2}} < \frac{M}{2}, \quad 0 < t \leq T,
\end{aligned} \tag{4.26}
$$

provided that $\left\| u_N(\cdot,0) \right\|_{L^\infty(x)} < \frac{M}{4}$. This justifies that the spectral damping approximation eq. (2.1)–eq. (2.2) is uniformly bounded by bootstrapping. Furthermore, from eq. (4.26), one can see that the numerical solution remains bounded by Const $\left\| u_N(\cdot,0) \right\|_{L^\infty(x)}$. We conclude the following theorem.

Theorem 4.7. *(L^∞-stability). The spectral damping approximation eq.* (2.1)-*eq.* (2.2) *is uniformly bounded, in the sense that there exists a constant such that*

$$
\left\| u_N(\cdot,t) \right\|_{L^\infty(x)} \leq Const \left\| u_N(\cdot,0) \right\|_{L^\infty(x)} , \quad 0 < t \leq T. \tag{4.27}
$$

5. Convergence to the entropy solution

The total-variation boundedness of the solution implies that (a subsequence of) $u_N(x,t)$ converges strongly to a limit $u(x,t)$, which is a weak solution of eq. (1.1). To conclude that this limit is the unique entropy solution of eq. (1.1), it remains to verify that u satisfies the entropy condition

$$
\frac{\partial}{\partial t} U(u) + \frac{\partial}{\partial x} F(u) \leq 0, \quad F(u) \equiv \int^u U'(\xi) f'(\xi) d\xi. \tag{5.1}
$$

In particular, when f is convex, according to the result of DiPerna (see [4]), the weak solution that satisfies eq. (5.1) for only one entropy pair is unique. This unique solution is physically meaningful and is called the entropy solution. The main result of this section asserts the following theorem.

Theorem 5.1. *Consider the spectral damping approximation eq.* (2.1)*-eq.* (2.2) *with* $0 < \alpha < \frac{1}{2}$, $\frac{4-2\alpha}{3} < \theta < 2 - 2\alpha$ *and f is convex. Then, the sequence of approximate solutions u_N converges strongly to the unique entropy solution of the conservation law eq.* (1.1)*.*

Proof. We first multiply eq. (2.1) by u_N:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{2}u_N^2\right) + \partial_x \int^{u_N} \omega f'(\omega)\mathrm{d}\omega \\
&= u_N \frac{\partial}{\partial x}\left[\left(I - P_N\right) f(u_N)\right] - N^\theta u_N(x,t) Q_N * u_N(x,t) \\
&= \frac{\partial}{\partial x}\left[u_N \left(I - P_N\right) f(u_N)\right] - \frac{\partial}{\partial x} u_N \left[\left(I - P_N\right) f(u_N)\right] \\
&\quad - N^\theta \left(\tilde{Q}_N * u_N\right)^2 + N^\theta \left[\left(\tilde{Q}_N * u_N\right)^2 - u_N \left(Q_N * u_N\right)\right],
\end{aligned} \tag{5.2}$$

where

$$\tilde{Q}_N := \sum_{|k| \le N} \sqrt{\hat{Q}_k} e^{ikx}.$$

To this end, we multiply eq. (5.2) by a test function $\psi \in C_0^\infty(\mathbb{R} \times [0,T))$, $\psi \ge 0$:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}\left(\frac{1}{2}u_N^2\right), \psi\right) + \left(\partial_x \int^{u_N} \omega f'(\omega)\mathrm{d}\omega, \psi\right) \\
&= \left(\frac{\partial}{\partial x}\left[u_N \left(I - P_N\right) f(u_N)\right], \psi\right) - \left(\frac{\partial}{\partial x} u_N \left[\left(I - P_N\right) f(u_N)\right], \psi\right) \\
&\quad - N^\theta \left(\left|\tilde{Q}_N * u_N\right|^2, \psi\right) + N^\theta \left(\left[\left|\tilde{Q}_N * u_N\right|^2 - u_N \left(Q_N * u_N\right)\right], \psi\right) \\
&\equiv I + II + III + IV.
\end{aligned} \tag{5.3}$$

Since $III \le 0$, it suffices to demonstrate that the remaining terms I, II, and IV on the right-hand side of eq. (5.3) vanish as $N \to \infty$. From eq. (3.15) and eq. (4.26) we have

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial x}\left[u_N \left(I - P_N\right) f(u_N)\right], \psi\right)\right| &\le \left\|\frac{\partial \psi}{\partial x}\right\|_{L^\infty_{\mathrm{loc}}(x,t)} \left\|u_N\right\|_{L^\infty} \left\|\left(I - P_N\right) f\left(u_N\right)\right\|_{L^2_{\mathrm{loc}}(x,t)} \\
&\to 0.
\end{aligned} \tag{5.4}$$

By Hölder's inequality and the inverse estimate, we have

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial x} u_N \left[\left(I - P_N\right) f(u_N)\right], \psi\right)\right| \\
&\le \left\|\psi\right\|_{L^\infty_{\mathrm{loc}}(x,t)} \left\|\left(I - P_N\right) f\left(u_N\right)\right\|_{L^2_{\mathrm{loc}}(x,t)} \left\|\partial_x u_N\right\|_{L^2_{\mathrm{loc}}(x,t)} \\
&\le N \left\|\psi\right\|_{L^\infty_{\mathrm{loc}}(x,t)} \left\|\left(I - P_N\right) f\left(u_N\right)\right\|_{L^2_{\mathrm{loc}}(x,t)} \left\|u_N\right\|_{L^2_{\mathrm{loc}}(x,t)} \to 0.
\end{aligned} \tag{5.5}$$

Then, we deal with the fourth term term. Let $\psi = \sum_{k=-\infty}^{\infty} \hat{\psi}_k(t)e^{ikx}$, $v_N := \tilde{Q}_N * u_N = \sum_{k=-N}^{N} \hat{v}_k(t)e^{ikx}$, and we find that

$$
\begin{aligned}
& N^\theta \left(\left| \tilde{Q}_N * u_N \right|^2, \psi \right) \\
& = N^\theta \int_0^t \sum_{k,p} \hat{\psi}_p(\tau)\hat{v}_{k-p}(\tau)\hat{v}_{-k}(\tau)d\tau \\
& = N^\theta \int_0^t \sum_{k,p} \hat{\psi}_p(\tau)\sqrt{\hat{Q}_{k-p}}\hat{u}_{k-p}(\tau)\sqrt{\hat{Q}_{-k}}\hat{u}_{-k}(\tau)d\tau \\
& = N^\theta \int_0^t \sum_{j,p} \hat{\psi}_p(\tau)\sqrt{\hat{Q}_j}\sqrt{\hat{Q}_{j+p}}\hat{u}_j(\tau)\hat{u}_{-j-p}(\tau)d\tau, \quad j = k-p.
\end{aligned}
\tag{5.6}
$$

Next, let $w_N := Q_N * u_N = \sum_{k=-N}^{N} \hat{w}_k(t)e^{ikx}$. similarly, we also have

$$
\begin{aligned}
& N^\theta \left(u_N \left(Q_N * u_N \right), \psi \right) \\
& = \int_0^t N^\theta \sum_{k,p} \hat{\psi}_p(\tau)\hat{u}_{k-p}(\tau)\hat{w}_{-k}(\tau)d\tau \\
& \quad = N^\theta \int_0^t \sum_{k,p} \hat{\psi}_p(\tau)\hat{u}_{k-p}(\tau)\hat{Q}_{-k}\hat{u}_{-k}(\tau)d\tau \\
& \quad = N^\theta \int_0^t \sum_{j,p} \hat{\psi}_p(\tau)\hat{Q}_{j+p}\hat{u}_j(\tau)\hat{u}_{-j-p}(\tau)d\tau \\
& \quad = N^\theta \int_0^t \sum_{j',p} \hat{\psi}_p(\tau)\hat{Q}_{-j'}\hat{u}_{-j'-p}(\tau)\hat{u}_{j'}(\tau)d\tau \\
& \quad = N^\theta \int_0^t \sum_{j,p} \hat{\psi}_p(\tau)\hat{Q}_j\hat{u}_{-j-p}(\tau)\hat{u}_j(\tau)d\tau,
\end{aligned}
\tag{5.7}
$$

$j = k - p, j' = -j - p$. Averaging the two equivalent representations in eq. (5.7), we obtain

$$
N^\theta \left(u_N \left(Q_N * u_N \right), \psi \right) = N^\theta \int_0^t \sum_{j,p} \hat{\psi}_p \hat{u}_j(\tau)\hat{u}_{-j-p}(\tau)\frac{1}{2}(\hat{Q}_{j+p} + \hat{Q}_j)d\tau. \tag{5.8}
$$

According to eq. (5.6) and eq. (5.8) we find that

$$
\begin{aligned}
IV & = -\frac{N^\theta}{2} \int_0^t \sum_{j,p} \hat{\psi}_p(\tau)\hat{u}_j(\tau)\hat{u}_{-j-p}(\tau) \left(\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_{j+p}} \right)^2 d\tau \\
& = -\frac{N^\theta}{2} \int_0^t \sum_{j=-N}^{N} \sum_{p=-N-j}^{N-j} \hat{\psi}_p(\tau)\hat{u}_j(\tau)\hat{u}_{-j-p}(\tau) \left(\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_{j+p}} \right)^2 d\tau.
\end{aligned}
\tag{5.9}
$$

Then, we obtain

$$
|IV| \leq \int_0^t \sum_{|p| \leq 2N} |\hat{\psi}_p(\tau)| A_p \left(\sum_{j=-N}^{N} |\hat{u}_j(\tau)||\hat{u}_{-j-p}(\tau)| \right) d\tau, \tag{5.10}
$$

where $A_p := \max_{\substack{|j|\le N\\ |j+p|\le N}} \frac{N^\theta}{2}\left(\sqrt{\widehat{Q}_j} - \sqrt{\widehat{Q}_{j+p}}\right)^2$. By the Cauchy-Schwarz inequality and eq. (4.20), we have

(5.11)
$$\begin{aligned}\sum_j |\hat{u}_j|\,|\hat{u}_{-j-p}| &\le \|u_N(\cdot,t)\|^2_{\mathrm{BV}} \sum_j \frac{1}{(|j|+1)(|-j-p|+1)}\\ &\le \|u_N(\cdot,t)\|^2_{\mathrm{BV}} \left(\sum_j \frac{1}{(|j|+1)^2}\right)^{1/2}\left(\sum_j \frac{1}{(|j+p|+1)^2}\right)^{1/2}\\ &\le C.\end{aligned}$$

To estimate $|IV|$, we use the following lemma, whose proof is postponed to the Appendix B.

Lemma 5.2. *Consider the* $A_p := \max_{\substack{|j|\le N\\ |j+p|\le N}} \frac{N^\theta}{2}\left(\sqrt{\widehat{Q}_j} - \sqrt{\widehat{Q}_{j+p}}\right)^2$ *subject to the* $\hat{Q}_k$ *in eq.* (2.2), *then we have*

$$A_p \le \frac{C}{2N^{1+\alpha-\frac{\theta}{2}}}p^2. \tag{5.12}$$

Hence, according to eq. (5.10)-eq. (5.12), we have

$$\begin{aligned}|IV| &\le \int_0^t \sum_{|p|\le 2N} \left|\hat{\psi}_p(\tau)\right| A_p \left(\sum_{j=-N}^{N} |\hat{u}_j(\tau)|\,|\hat{u}_{-j-p}(\tau)|\right) d\tau\\ &\le C\int_0^t \sum_{|p|\le 2N} \left|\hat{\psi}_p(\tau)\right| A_p\, d\tau\\ &\le \frac{C}{N^{1+\alpha-\theta/2}}\int_0^t \sum_{|p|\le 2N} \left|\hat{\psi}_p(\tau)\right| p^2\, d\tau\\ &\le \frac{C_s}{N^{1+\alpha-\theta/2}} \to 0.\end{aligned} \tag{5.13}$$

where $s>3$, $\quad \left|\hat{\psi}_p\right| \le \frac{C_s}{(1+|p|)^s}$, $\quad 1+\alpha-\frac{\theta}{2} > 0$.

In summary, for $0<\alpha<\frac{1}{2}$, $\frac{4-2\alpha}{3} < \theta < 2-2\alpha$, it follows that the limit $u = \lim_{N\to\infty} u_N$ satisfies the entropy inequality eq. (5.1). Consequently, by the uniqueness of the entropy solution, the entire sequence $\{u_N\}$ converges strongly to the unique entropy solution. □

6. Convergence for smooth solutions

In this section, we will show that as long as the solution of eq. (1.1) remains smooth, the spectral damping method is stable and enjoys spectral convergence. Before presenting the final results, we need to give the following lemma.

Lemma 6.1. [27] *Assume* $w(x)$ *is* 2π*-periodic analytic.* $-\rho_0 < \operatorname{Im} z < \rho_0$ *is the strip of analyticity where* $w(z)$ *admits the absolutely convergent expansion*

$$w(z) = \sum_{p=-\infty}^{\infty} \hat{w}(p)e^{ipz}, \quad |\operatorname{Im} z| \leqq \rho < \rho_0.$$

Denoting

$$M(\rho) = \max_{|\operatorname{Im} z|\leq\rho} |w(z)|,$$

then we have that

$$\|w(x) - P_N w(x)\|_{H^\sigma} \leqq \textit{ Const }_\sigma \cdot \frac{M(\rho)}{\sinh(\rho)} N^\sigma e^{-N\rho}. \tag{6.1}$$

We conclude that the following result holds.

Theorem 6.2. *Assume that for $0 < t \leq T_c$, the solution of eq.* (1.1) *is analytic. Then, the spectral method eq.* (2.1)*-eq.* (2.2) *converges in* $L^\infty\left([0,T_c], L^2[0,2\pi]\right)$,

$$||u(\cdot,t) - u_N(\cdot,t)||_{L^2(x)} \lesssim e^{ct}\left(Ne^{-N\rho} + N^{1+\frac{\theta}{2}-\alpha}e^{-N^\alpha\rho}\right). \tag{6.2}$$

Proof. Denote $u^* = P_N u$, $u^* = \sum_{|k|\leq N} \tilde{u}_k e^{ikx}$. Let $e(t) = u_N(t) - u^*(t)$. From eq. (1.1) and eq. (2.1), we obtain

$$\partial_t e(t) + \partial_x P_N\left[f\left(u_N(t)\right) - f(u(t))\right] = -N^\theta Q_N u_N(t). \tag{6.3}$$

Furthermore,

$$\partial_t e(t) = \partial_x P_N\left[f\left(u^*(t)\right) - f\left(u_N(t)\right)\right] + \partial_x P_N\left[f(u(t)) - f\left(u^*(t)\right)\right] - N^\theta Q_N u_N(t). \tag{6.4}$$

We multiply eq. (6.4) by $e(t)$ and integrate over the 2π-period,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{L^2(x)}\\
&= \left(\partial_x P_N\left[f\left(u^*(t)\right) - f\left(u_N(t)\right)\right], e\right) + \left(\partial_x P_N\left[f(u(t)) - f\left(u^*(t)\right)\right], e\right)\\
&\qquad - N^\theta (Q_N u_N(t), e(t))\\
&\lesssim \left(\partial_x\left[f\left(u^*(t)\right) - f\left(u_N(t)\right)\right], e\right) + \left\|f(u(t)) - f\left(u^*(t)\right)\right\|_{L^2(x)} \left\|\partial_x e(t)\right\|_{L^2(x)}\\
&\qquad - N^\theta\left(Q_N\left(e(t) + u^*(t)\right), e(t)\right)\\
&\leq \left(e(t)\int_0^1 f'(u^*(t) + \eta e(t))d\eta, \partial_x e(t)\right)\\
&\qquad + \left\|f(u(t)) - f\left(u^*(t)\right)\right\|_{L^2(x)} \left\|\partial_x e(t)\right\|_{L^2(x)} - N^\theta\left(Q_N u^*(t), e(t)\right).
\end{aligned} \tag{6.5}$$

In what follows we bound the terms in the right-hand side of eq. (6.5). Consider $0 \leq s \leq t$. For given $t \in [0,T]$, we assume

$$\max_{0\leq s\leq t} \|e(s)\|_{L^2} \lesssim N^{-\frac{3}{2}}, \tag{6.6}$$

and this a priori assumption will also be justified later. Next, we have

$$
\begin{aligned}
&\left|\left(e(s)\int_0^1 f'(u^*(s)+\eta e(s))d\eta, \partial_x e(s)\right)\right| \\
&= \left|\left(\partial_x \int_0^1 f'(u^*(s)+\eta e(s))d\eta, \frac{e^2(s)}{2}\right)\right| \\
&= \left|\left(\int_0^1 f''(u^*(s)+\eta e(s))\partial_x(u^*(s)+\eta e(s))d\eta, \frac{e^2(s)}{2}\right)\right| \\
&\lesssim \max_{|\xi|\le \max(||u^*||_{L^\infty},||u_N||_{L^\infty})} |f''(\xi)| \left(||\partial_x u^*(s)||_{L^\infty} + ||\partial_x e(s)||_{L^\infty}\right) \|e(s)\|^2_{L^2(x)} \\
&\lesssim \max_{|\xi|\le \max(||u^*||_{L^\infty},||u_N||_{L^\infty})} |f''(\xi)| \left(||\partial_x u^*(s)||_{L^\infty} + N^{\frac{3}{2}}||e(s)||_{L^2}\right) \|e(s)\|^2_{L^2(x)} \\
&\lesssim c\,\|e(s)\|^2_{L^2(x)}\,.
\end{aligned}
\tag{6.7}
$$

According to eq. (6.1), we know that

$$
\begin{aligned}
&\|f(u(\cdot,s)) - f\left(u^*(\cdot,s)\right)\|_{L^2(x)} \\
&\quad \le \max_{|\xi|\le \max(||u^*||_{L^\infty},||u||_{L^\infty})} |\partial_u f(\xi)| \sqrt{\sum_{|k|>N} \tilde{u}_k(t)^2} \lesssim e^{-N\rho}.
\end{aligned}
\tag{6.8}
$$

For the third term in eq. (6.5), since $1-e^{-z}\le z (z\ge 0)$ and $1-e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}(\frac{|k|-m}{m})^2} \le 1-e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}}$, $m<|k|\le 2m$. We have

$$
\begin{aligned}
N^\theta \|Q_N u^*(\cdot,s)\|_{L^2(x)} &\le N^\theta \sqrt{\sum_{m<|k|\le N} (1-e^{-\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}}})^2 |\tilde{u}_k(s)|^2} \\
&\le N^\theta \sqrt{\sum_{m<|k|\le N} (\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}})^2 |\tilde{u}_k(s)|^2} \\
&\le N^{1+\frac{\theta}{2}-\alpha} \sqrt{\sum_{m<|k|\le N} |\tilde{u}_k(s)|^2} \\
&\le N^{1+\frac{\theta}{2}-\alpha} e^{-N^\alpha \rho}.
\end{aligned}
\tag{6.9}
$$

By plugging eq. (6.6)-eq. (6.9) into eq. (6.5), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e(s)\|^2_{L^2(x)} \\
&\quad \lesssim c\,\|e(s)\|^2_{L^2(x)} + N\cdot e^{-N\rho}\,\|e(s)\|_{L^2(x)} + N^{1+\frac{\theta}{2}-\alpha} e^{-N^\alpha\rho}\,\|e(s)\|_{L^2(x)}\,.
\end{aligned}
\tag{6.10}
$$

Dividing both sides by $\|e(s)\|_{L^2}$ and applying Gronwall's lemma, we obtain

$$
\begin{aligned}
\|e(t)\|_{L^2(x)} &\lesssim e^{ct}\|e(0)\|_{L^2(x)} + e^{ct}\left(Ne^{-N\rho} + N^{1+\frac{\theta}{2}-\alpha}e^{-N^\alpha\rho}\right) \\
&\lesssim e^{ct}\left(Ne^{-N\rho} + N^{1+\frac{\theta}{2}-\alpha}e^{-N^\alpha\rho}\right),
\end{aligned}
\tag{6.11}
$$

and this justifies eq. (6.6) by bootstrapping. Finally, we obtain from the above estimates, and triangle inequality that

$$||u(\cdot,t) - u_N(\cdot,t)||_{L^2(x)} \lesssim e^{ct}\left(Ne^{-N\rho} + N^{1+\frac{\theta}{2}-\alpha}e^{-N^\alpha \rho}\right). \tag{6.12}$$

□

7. Numerical experiments

In this section, we present numerical results to validate the theoretical convergence and stability estimates established in the previous sections. We employ the classical fourth-order Runge–Kutta method for temporal integration. The time step is chosen to be sufficiently small so that the temporal discretization error is negligible compared with the spatial discretization error. Unless otherwise specified, the spectral damping method is implemented with the following parameter configuration:

$$m \sim 3\sqrt{N}, \qquad \alpha = 0.5, \qquad \theta = 1.2.$$

We consider the inviscid Burgers' equation

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left(\frac{u^2(x,t)}{2}\right) = 0, \tag{7.1}$$

subject to the smooth initial condition $u(x,0) = \sin(x)$ under 2π-periodic boundary conditions. Different final times T are selected to evaluate the performance of our algorithm.

7.1. Smooth case.

For the given initial data, the classical shock discontinuity first forms at the critical time $T_c = 1$. Consequently, the exact entropy solution remains smooth at $T = 0.5$. We report the resulting L^1 errors and corresponding numerical convergence orders in Table 1. The numerical results demonstrate that the introduced numerical damping term preserves the spectral accuracy of the method for smooth solutions, without undermining the convergence rate.

Table 1. $L^1([0,2\pi])$ errors and convergence orders for the inviscid Burgers' equation at $T = 0.5$.

	Damped (Proposed)			Undamped (Standard)	
$2N$	L^1 error	Order	$2N$	L^1 error	Order
16	3.9153e-04	–	16	6.7049e-04	–
32	1.4449e-05	4.76	32	6.6970e-06	6.65
64	2.6587e-07	5.76	64	1.7636e-09	11.89
128	7.3360e-10	8.50	128	1.8131e-13	13.25
256	2.5399e-13	11.49	256	1.8147e-13	–

7.2. Non-smooth case.

When evaluating the solution at the final time $T = 1.5$ in eq. (7.1), a shock discontinuity has already developed within the computational domain. Figure 1 displays the numerical profiles obtained by the proposed spectral damping method using $2N = 128$ modes. These results are compared against those obtained via the classical spectral viscosity (SV) method eq. (1.4), where the underlying SV parameters have been optimized.

To further enhance convergence behavior in the smooth regions, we incorporate a high-frequency filter to stabilize the spectral damping framework following the

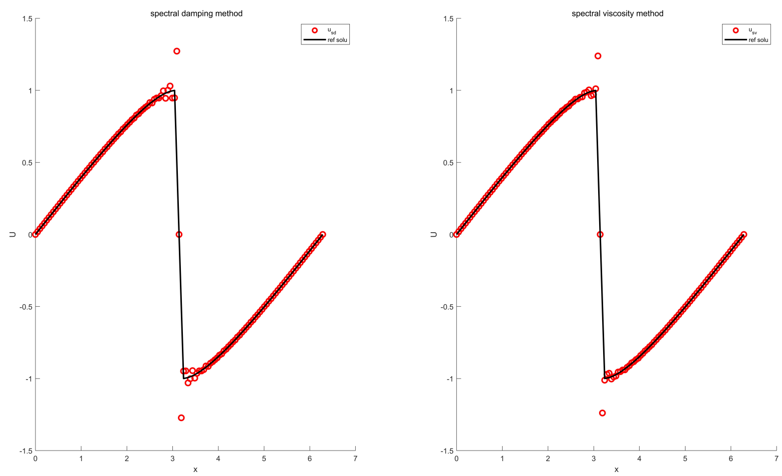

FIGURE 1. Numerical solutions of the inviscid Burgers' equation at $T = 1.5$ with $2N = 128$ modes. Left: proposed spectral damping method. Right: classical spectral viscosity method.

techniques discussed in [25, 5]. Specifically, after each Runge-Kutta time step, the numerical solution is postprocessed using the exponential filter

$$\sigma(\xi) = e^{-\epsilon|\xi|^{\gamma}}, \quad \xi = \frac{|k|}{N} \in [0, 1], \tag{7.2}$$

where the parameter γ increases with N and dictates the order of the filter, while ϵ is determined such that $e^{-\epsilon}$ matches the machine precision.

In Table 2, we present the L^1 errors of the filtered numerical solution restricted to the shock-free subdomains $[0, \pi/2] \cup [3\pi/2, 2\pi]$. The results clearly indicate that the formal spectral convergence is successfully recovered within the smooth intervals of the entropy solution. To enhance the convergence up to the location of the shock, the numerical solution can also be postprocessed using the Gegenbauer reconstruction [10], which can remove the Gibbs phenomena completely, and recover pointwise exponential accuracy in any interval of analyticity. We refer the reader to [10, 8, 11] for further details.

TABLE 2. $L^1([0, \pi/2] \cup [3\pi/2, 2\pi])$ errors of the filtered spectral damping solution at $T = 1.5$.

	Spectral Damping Method	
$2N$	L^1 error	Order
32	1.6142e-03	–
64	1.7245e-05	6.55
128	3.1963e-08	9.08
256	8.9350e-12	11.80

Finally, to evaluate the long-time stability and robustness of the proposed framework, we present the numerical profiles computed at an final time $T = 100$ in

Figure 2. The long-time simulations verify that the spectral damping method exhibits enhanced structural robustness and stability compared with the classical SV method.

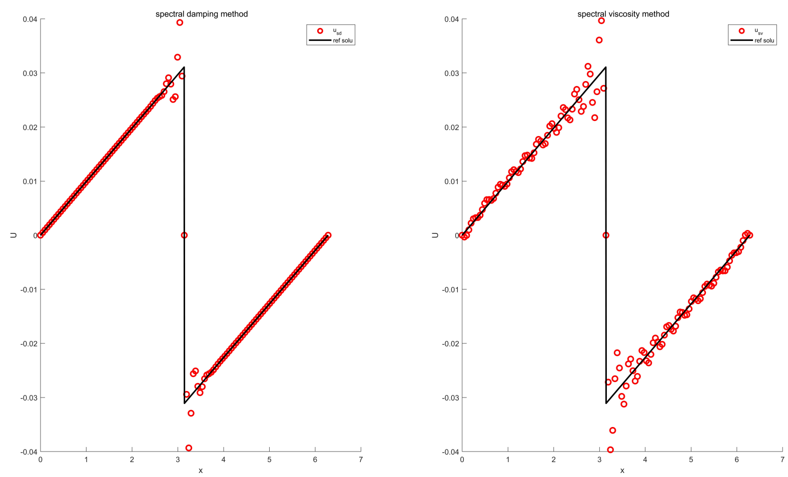

FIGURE 2. Long-time numerical solutions of the inviscid Burgers' equation at $T = 100$ with $2N = 128$ modes. Left: proposed spectral damping method. Right: classical spectral viscosity method.

8. CONCLUDING REMARKS

In this paper, we introduce a novel numerical damping term to stabilize the Fourier spectral approximations of nonlinear conservation laws. The proposed damping mechanism acts exclusively on high-frequency modes, inducing a rapid decay of the corresponding Fourier coefficients. Consequently, we prove that the resulting approximation is uniformly bounded and has bounded total variation. Together with the entropy consistency of the approximation, these estimates guarantee convergence to the unique entropy solution. We also demonstrate that the method retains spectral accuracy for smooth solutions. Also, numerical experiments validate the stability and long-time robustness of the proposed approach.

Furthermore, numerical experiments (not reported in this paper) suggest that the proposed method can be efficiently implemented for non-periodic problems, a rigorous analysis of which is deferred to future work. Additionally, extending this framework to systems of conservation laws, such as the inviscid compressible Euler equations, as well as to multidimensional cases, is currently under active investigation.

Appendix A. The decay rate of the Fourier coefficients revisited

In Section 3, we claimed that, for $k \gtrsim 2m\cdot(32N^{2\delta})^l$, the following estimate holds:

$$
\begin{aligned}
&\|(I-P_k)\,u_N(\cdot,t)\|_{L^2(x)} \\
&\lesssim \left(C_* M\tilde{g}(M)N^{1-\theta}\frac{\left(N^{(1+\alpha+\frac{\theta}{2})}+k^2\right)}{k^2}\right)^{l+1} \\
&\qquad + \left(1+C_* M\tilde{g}(M)\frac{N^2}{k^2}\right)^{l} k^{-r}\exp\left(-ctN^{\theta}\frac{k^2}{N^{4\delta l+(1+\alpha+\frac{\theta}{2})}+k^2}\right).
\end{aligned}
\tag{A.1}
$$

We set $\delta = \frac{1-\alpha}{4(L+1)}$ and $p = \lfloor N^{\delta} \rfloor$ as in Section 3. Here we give the details of the induction step, and obtain that for $0 \le l \le L-1$ and $k \gtrsim 2m\cdot(32N^{2\delta})^{l+1}$ we have

$$
\begin{aligned}
&\|(I-P_k)\,u_N(\cdot,t)\|_{L^2(x)} \\
&\lesssim \left(C_* M\tilde{g}(M)N^{1-\theta}\frac{\left(N^{(1+\alpha+\frac{\theta}{2})}+k^2\right)}{k^2}\right)^{l+2} \\
&\qquad + \left(1+C_* M\tilde{g}(M)\frac{N^2}{k^2}\right)^{l+1} k^{-r}\exp\left(-ctN^{\theta}\frac{k^2}{N^{4\delta(l+1)+(1+\alpha+\frac{\theta}{2})}+k^2}\right),
\end{aligned}
\tag{A.2}
$$

i.e., eq. (3.13) holds. For $l = 0$, eq. (A.1) reduces to eq. (3.11). Now assume that eq. (A.1) is valid for $k \gtrsim 2m\cdot(32N^{2\delta})^l$. In particular, for $k \gtrsim 2m\cdot(32N^{2\delta})^{l+1}$,

again use eq. (3.2), we get
(A.3)
$$
\begin{aligned}
&\|(I-P_k)\, f(u_N)\|_{L^2(x)} \\
&\le C_* \sum_{1\le j\le p} j^{3-s}\left(C_* M\tilde{g}(M) N^{1-\theta}\frac{N^{1+\alpha+\frac{\theta}{2}}+k_j^2}{k_j^2}\right)^{l+1} \\
&\quad + C_* \sum_{1\le j\le p} j^{3-s}\left(1+C_*\tilde{g}(M)M\frac{N^2}{k_j^2}\right)^{l} k_j^{-r}\exp\left(-cN^{\theta}\frac{k_j^2}{N^{4\delta l+1+\alpha+\frac{\theta}{2}}+k_j^2}t\right) \\
&\quad + C_* N^{(1-s)\delta} \\
&\lesssim C_*\left(C_*\tilde{g}(M)M\frac{N^{2+\alpha-\frac{\theta}{2}}}{k^2}\right)^{l+1}\sum_{1\le j\le p} j^{4l+7-s} \\
&\quad + C_* \sum_{1\le j\le p} j^{3-s}\left(C_*\tilde{g}(M)MN^{1-\theta}\right)^{l+1} \\
&\quad + \Bigg[C_*\left(1+C_*\tilde{g}(M)M\frac{N^2}{k^2}\right)^{l} k^{-r} \\
&\qquad\quad \times \sum_{1\le j\le p} j^{4l+2r+3-s}\exp\left(-ctN^{\theta}\frac{k^2}{N^{4\delta(l+1)+1+\alpha+\frac{\theta}{2}}+k^2}\right)\Bigg] \\
&\quad + C_* N^{(1-s)\delta} \\
&\lesssim C_*\left(C_*\tilde{g}(M)M\frac{N^{2+\alpha-\frac{\theta}{2}}}{k^2}\right)^{l+1} + C_*\left(C_*\tilde{g}(M)MN^{1-\theta}\right)^{l+1} \\
&\quad + C_*\left(1+C_*\tilde{g}(M)M\frac{N^2}{k^2}\right)^{l} k^{-r}\exp\left(-ctN^{\theta}\frac{k^2}{N^{4\delta(l+1)+1+\alpha+\frac{\theta}{2}}+k^2}\right),
\end{aligned}
$$

where $k_j := \left\lfloor \frac{k}{32j^2}\right\rfloor$, $0<c<1$, $2m\cdot(32N^{2\delta})^{l+1}\lesssim k\le \frac{N}{8}$. Furthermore, eq. (3.4) ensures that $N^{(1-s)\delta} < N^{(1-\theta)(l+1)}$. By eq. (A.1), eq. (A.3) and eq. (3.1), we have
(A.4)
$$
\begin{aligned}
&\|(I-P_k)\,[f'(u_N)\partial_x u_N]\|_{L^2(x)} \\
&\le 2\|f'(u_N)\|_{L^\infty}\|\left(I-P_{[k/8]}\right)\partial_x u_N\|_{L^2(x)} + \|\partial_x u_N\|_{L^\infty}\|\left(I-P_{[k/8]}\right) f'(u_N)\|_{L^2(x)} \\
&\lesssim N\tilde{g}(M)\|\left(I-P_{[k/8]}\right)u_N\|_{L^2(x)} + NM\|\left(I-P_{[k/8]}\right)f'(u_N)\|_{L^2(x)} \\
&\lesssim C_*\tilde{g}(M)MN\left(C_*M\tilde{g}(M)N^{1-\theta}\frac{\left(N^{(1+\alpha+\frac{\theta}{2})}+k^2\right)}{k^2}\right)^{l+1} \\
&\quad + NC_*\tilde{g}(M)M\left(1+C_*\tilde{g}(M)M\frac{N^2}{k^2}\right)^{l} k^{-r}\exp\left(-ctN^{\theta}\frac{k^2}{N^{4\delta(l+1)+(1+\alpha+\frac{\theta}{2})}+k^2}\right).
\end{aligned}
$$

Similar to eq. (3.12), we can again get
(A.5)

$$\frac{d}{dt}\left\|(I-P_k)\,u_N\right\|_{L^2(x)} + N^{\theta}\frac{k^2}{N^{(1+\alpha+\frac{\theta}{2})}+k^2}\left\|(I-P_k)\,u_N\right\|_{L^2(x)}$$
$$\lesssim C_*\tilde{g}(M)MN\left(C_*M\tilde{g}(M)N^{1-\theta}\frac{\left(N^{(1+\alpha+\frac{\theta}{2})}+k^2\right)}{k^2}\right)^{l+1}$$
$$+NC_*\tilde{g}(M)M\left(1+C_*\tilde{g}(M)M\frac{N^2}{k^2}\right)^{l}k^{-r}\exp\left(-ctN^{\theta}\frac{k^2}{N^{4\delta(l+1)+(1+\alpha+\frac{\theta}{2})}+k^2}\right).$$

Now, set

$$X_k(t) := \|(I-P_k)u_N(\cdot,t)\|_{L^2}, \qquad \gamma := 1+\alpha+\frac{\theta}{2}, \qquad a := 4\delta(l+1),$$

and

$$G := C_*\tilde{g}(M)M, \qquad \lambda_k := N^{\theta}\frac{k^2}{N^{\gamma}+k^2}, \qquad \mu_k := cN^{\theta}\frac{k^2}{N^{a+\gamma}+k^2}.$$

Since $a>0$, for N sufficiently large one has

$$\mu_k < \lambda_k.$$

By Gronwall's lemma applied to eq. (A.5), we get

$$X_k(t) \lesssim e^{-\lambda_k t}X_k(0) + GN\left(GN^{1-\theta}\frac{N^{\gamma}+k^2}{k^2}\right)^{l+1}\int_0^t e^{-\lambda_k(t-s)}\,ds$$
$$+GN\left(1+G\frac{N^2}{k^2}\right)^{l}k^{-r}\int_0^t e^{-\lambda_k(t-s)}e^{-\mu_k s}\,ds.$$

The first integral is bounded by $1/\lambda_k$, and therefore

$$GN\left(GN^{1-\theta}\frac{N^{\gamma}+k^2}{k^2}\right)^{l+1}\frac{1}{\lambda_k} = \left(GN^{1-\theta}\frac{N^{\gamma}+k^2}{k^2}\right)^{l+2}.$$

For the second integral, since $\lambda_k>\mu_k$, we have

$$\int_0^t e^{-\lambda_k(t-s)}e^{-\mu_k s}\,ds = \frac{e^{-\mu_k t}-e^{-\lambda_k t}}{\lambda_k-\mu_k} \le \frac{e^{-\mu_k t}}{\lambda_k-\mu_k}.$$

Moreover,

$$\frac{1}{\lambda_k-\mu_k} = N^{-\theta}\frac{(N^{\gamma}+k^2)(N^{a+\gamma}+k^2)}{k^2\left[N^{\gamma}(N^a-c)+(1-c)k^2\right]}.$$

Since $a>0$ and $0<c<1$, for N sufficiently large,

$$\begin{aligned}\frac{N}{\lambda_k-\mu_k} &= N^{1-\theta}\frac{\left(N^{\gamma}+k^2\right)\left(N^{a+\gamma}+k^2\right)}{k^2\left[N^{\gamma}\left(N^a-c\right)+(1-c)k^2\right]}\\ &\lesssim N^{1-\theta}\frac{\left(N^{\gamma}+k^2\right)\left(N^{a+\gamma}+k^2\right)}{k^2N^{a+\gamma}}\\ &= N^{1-\theta}\left(\frac{N^{\gamma}}{k^2}+1+N^{-a}+\frac{k^2}{N^{a+\gamma}}\right)\\ &= \frac{N^{1-\theta+\gamma}}{k^2}+N^{1-\theta}+N^{1-\theta-a}+N^{1-\theta}\frac{k^2}{N^{a+\gamma}}.\end{aligned}$$

Using $k \leq N$, $\theta > 1$, and $\theta > 2\alpha$, the first three terms are bounded by CN^2/k^2. For the last term, we have

$$N^{1-\theta}\frac{k^2}{N^{a+\gamma}} \leq N^{3-\theta-a-\gamma} = N^{2-\alpha-\frac{3}{2}\theta-a}.$$

By the parameter condition

$$\frac{4-2\alpha}{3} < \theta,$$

this term is uniformly bounded. In summary, we have

$$\frac{N}{\lambda_k - \mu_k} \lesssim \frac{N^2}{k^2}.$$

Therefore,

$$\begin{aligned} &GN\left(1+G\frac{N^2}{k^2}\right)^l k^{-r}\int_0^t e^{-\lambda_k(t-s)}e^{-\mu_k s}\,ds \\ &\lesssim G\frac{N^2}{k^2}\left(1+G\frac{N^2}{k^2}\right)^l k^{-r}e^{-\mu_k t} \\ &\lesssim \left(1+G\frac{N^2}{k^2}\right)^{l+1} k^{-r}e^{-\mu_k t}. \end{aligned}$$

Combining the above estimates yields

$$\begin{aligned} \|(I-P_k)u_N(\cdot,t)\|_{L^2} \lesssim &\left(GN^{1-\theta}\frac{N^\gamma+k^2}{k^2}\right)^{l+2} \\ &+\left(1+G\frac{N^2}{k^2}\right)^{l+1} k^{-r}\exp\left(-ctN^\theta\frac{k^2}{N^{a+\gamma}+k^2}\right). \end{aligned}$$

Returning to the definitions of G, γ and a, we get

$$\begin{aligned} &\|(I-P_k)\,u_N(\cdot,t)\|_{L^2(x)} \\ &\lesssim \left(C_*M\tilde{g}(M)N^{1-\theta}\frac{\left(N^{(1+\alpha+\frac{\theta}{2})}+k^2\right)}{k^2}\right)^{l+2} \\ &\quad+\left(1+C_*M\tilde{g}(M)\frac{N^2}{k^2}\right)^{l+1} k^{-r}\exp\left(-ctN^\theta\frac{k^2}{N^{4\delta(l+1)+(1+\alpha+\frac{\theta}{2})}+k^2}\right), \end{aligned} \tag{A.6}$$

which completes the induction proof of eq. (A.2).

Appendix B. Proof of Lemma 5.2

Divide the interval $[0,N]$ into

$$I_0 = [0,m], \quad I_1 = (m,2m], \quad I_2 = (2m,N].$$

We show below that $\sqrt{\hat{Q}_j}$, viewed as a function of $|j|$, satisfies a uniform Lipschitz estimate on $[0,N]$. We first verify this estimate in the following three cases.

Case 1. if $|j|, |j+p| \in I_0$. In this case, $\sqrt{\hat{Q}_j} = \sqrt{\hat{Q}_{j+p}} = 0$. Therefore,

$$\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_{j+p}} = 0. \tag{B.1}$$

Case 2. if $|j|$, $|j+p| \in I_1$. Define

$$\sqrt{\hat{Q}_j} := g(f(|j|)), \quad g(y) = \sqrt{1-e^{-y^2}}, \quad f(x) = \frac{x(x-m)}{mN^{0.5(1+\alpha+\frac{\theta}{2})}}, \quad x \in I_1. \tag{B.2}$$

We know that

$$g'(y) = \frac{ye^{-y^2}}{\sqrt{1-e^{-y^2}}}, \quad |g'(y)| \leq C, \quad y > 0.$$

Also,

$$|f'(x)| = \left|\frac{2x-m}{mN^{\frac{1}{2}(1+\alpha+\frac{\theta}{2})}}\right| \leq \left|\frac{3m}{mN^{\frac{1}{2}(1+\alpha+\frac{\theta}{2})}}\right| \leq \frac{3}{N^{\frac{1}{2}(1+\alpha+\frac{\theta}{2})}}.$$

Therefore,

$$\left|\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_{j+p}}\right| = |g(f(|j|)) - g(f(|j+p|))| \leq \frac{C}{N^{\frac{1}{2}(1+\alpha+\frac{\theta}{2})}} \left||j| - |j+p|\right|. \tag{B.3}$$

Case 3. if $|j|$, $|j+p| \in I_2$. By the same estimates as in Case 2, we also have,

$$\begin{aligned}\left|\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_{j+p}}\right| &= \left|g\left(\frac{|j|}{N^{\frac{1}{2}(1+\alpha+\theta/2)}}\right) - g\left(\frac{|j+p|}{N^{\frac{1}{2}(1+\alpha+\theta/2)}}\right)\right| \\ &\leq \frac{C}{N^{\frac{1}{2}(1+\alpha+\frac{\theta}{2})}} \left||j| - |j+p|\right|.\end{aligned}$$

Next, it remains to consider the cases where $|j|$ and $|j+p|$ lie in different intervals. Although the decomposition is taken as $I_1 = (m, 2m]$, and $I_2 = (2m, N]$, the same estimates remain valid on the closed intervals $[m, 2m]$ and $[2m, N]$ by the continuity of $\hat{Q}_k$ at $|k| = m$ and $|k| = 2m$. Hence, for instance, $|j| \in I_0$, $|j+p| \in I_2$,

$$\begin{aligned}\left|\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_{j+p}}\right| &\leq \left|\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_m}\right| + \left|\sqrt{\hat{Q}_m} - \sqrt{\hat{Q}_{2m}}\right| + \left|\sqrt{\hat{Q}_{2m}} - \sqrt{\hat{Q}_{j+p}}\right| \\ &\leq \frac{C}{N^{\frac{1}{2}(1+\alpha+\frac{\theta}{2})}} \left||j| - |j+p|\right|.\end{aligned} \tag{B.4}$$

Other cases can be deduced similarly. In summary, for $\forall |j|, |j+p| \in [0, N]$, we have

$$\left|\sqrt{\hat{Q}_j} - \sqrt{\hat{Q}_{j+p}}\right| \leq \frac{C}{N^{\frac{1}{2}(1+\alpha+\frac{\theta}{2})}} \left||j| - |j+p|\right|. \tag{B.5}$$

Therefore,

$$A_p := \max_{\substack{|j| \leq N \\ |j+p| \leq N}} \frac{N^\theta}{2} \left(\sqrt{\widehat{Q}_j} - \sqrt{\widehat{Q}_{j+p}}\right)^2 \leq \frac{C}{2N^{1+\alpha-\frac{\theta}{2}}} p^2, \tag{B.6}$$

which completes the proof of Lemma 5.2.

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