THE ERROR ESTIMATE OF ENTROPY-STABLE DISCONTINUOUS GALERKIN METHODS FOR HYPERBOLIC CONSERVATION LAWS *

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Abstract. Entropy inequalities are fundamental to the well-posedness of hyperbolic conservation laws, providing the essential criterion for selecting the physically admissible solution among infinitely many weak solutions. Chen and Shu [J. Comput. Phys. 345 (2017)] proposed a unified framework for constructing high-order discontinuous Galerkin (DG) methods that satisfy entropy inequalities for any given entropy via specific numerical quadrature; however, their accompanying error analysis was limited to the truncation error level, leaving a critical gap in the rigorous convergence theory for these entropy-stable schemes. This paper closes that gap by establishing rigorous a priori error estimates for semi-discrete entropy-stable DG (ESDG) methods on general unstructured meshes for hyperbolic conservation laws. The analysis is built upon a finite-difference-type consistency-stability argument carried out directly at the nodal level. Under a polynomial-reconstruction hypothesis and an L^∞ a priori bound, we prove an $O(h^k)$ error estimate in a quadrature-based norm, which is equivalent to the broken L^2 norm on the finite-dimensional reconstruction space. We further extend this framework to the entropy-stable oscillation-free DG (ESOFDG) method introduced by Liu, Lu, and Shu [SIAM J. Sci. Comput. 46 (2024)], demonstrating that the additional damping terms do not degrade the convergence order. Numerical experiments suggest that the observed convergence rates may exceed the theoretical prediction by up to half an order.

Key words. discontinuous Galerkin method, error estimate, hyperbolic conservation laws, entropy stability, summation-by-parts

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1. Introduction.

Hyperbolic conservation laws govern a wide variety of physical phenomena, among which gas dynamics in the continuum regime represents a canonical and extensively studied setting. The general form of hyperbolic conservation law systems is

$$\partial_t \mathbf{u} + \sum_{m=1}^{d} \partial_{x_m} \mathbf{f}_m(\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega \times [0, T], \tag{1.1a}$$

$$\mathbf{u} = \mathbf{g}, \quad (\mathbf{x}, t) \in \Omega \times \{0\}, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$ and $\mathbf{u}, \mathbf{f}_m \in \mathbb{R}^n, n \geq 1$ are vector-valued functions and flux functions respectively. Throughout this paper, the solution is considered to be either periodic or compactly supported. A hallmark of nonlinear hyperbolic conservation laws is the spontaneous formation of shock waves and contact discontinuities in finite time, irrespective of the smoothness of the initial or boundary data. This fundamental phenomenon necessitates the framework of weak solutions, within which unique-

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ness is generically lost. Consequently, to isolate the physically relevant solution, one must augment the governing equations with entropy admissibility conditions. The well-posedness theory is well established for scalar conservation laws ($n = 1$) and one-dimensional systems ($d = 1$) subject to initial data of sufficiently small total variation, for which the entropy solution exists and is unique. By stark contrast, for general multi-dimensional systems, the global existence and uniqueness of entropy solutions constitute outstanding open problems, and the theoretical foundations remain substantially incomplete. We direct the reader to [11, 18] for an exhaustive survey. In the computational realm, although entropy conditions alone do not preclude non-uniqueness, the construction of discrete approximations satisfying discrete analogues of entropy inequalities at the grid level remains a central desideratum. This requirement defines the notion of entropy stability, and schemes that satisfy it are commonly termed entropy-stable (ES) schemes.

For first- and second-order discretizations, the entropy-conservative and entropy-stable fluxes of Tadmor [32, 33] furnish a general framework. In the finite volume setting, Fjordholm, Mishra, and Tadmor [13] devised the TeCNO scheme, which combines high-order entropy-conservative fluxes [23] with the sign property of essentially non-oscillatory (ENO) reconstruction [14] to ensure entropy stability. More recently, attention has turned to entropy-stable quadrature-based discontinuous Galerkin (DG) methods. Chen and Shu [6] constructed such a method on unstructured simplex meshes using Gauss–Lobatto-type quadrature rules with collocated surface quadrature points and discrete operators endowed with the multi-dimensional summation-by-parts (SBP) property [12, 21]. Several related entropy-stable DG methods within the SBP framework have subsequently been proposed [3,4,9,10]; the reader is referred to [7] for a comprehensive survey. The paradigm has further been extended to encompass the Navier-Stokes equations [2, 16], magnetohydrodynamics (MHD) [1, 27], shallow water equations [15, 34], Euler equations with gravity [24, 25], gradient flows [31], two-phase flows [30], and stochastic problems [28].

Although entropy stability of the ESDG method is well established, rigorous error estimates for smooth solutions remain unavailable, to the best of our knowledge. For semi-discrete DG and Runge–Kutta DG schemes, such estimates have been extensively developed; see, for example, [37–39]. The effect of inexact quadrature on DG error analysis has also been addressed in [22]; however, the results therein do not extend to general entropy-conservative SBP flux-differencing schemes. Recently, Worku, Del Rey Fernández, and Zingg [35] established convergence of entropy-stable split-form discretizations of symmetric hyperbolic systems with homogeneous flux functions that possess globally bounded second derivatives, within the multi-dimensional continuous-SBP framework. Ranocha [29] subsequently extended this analysis to diagonal-norm SBP operators on curved meshes and derived an a priori h^p error estimate for entropy-conservative high-order SBP semi-discretizations of nonlinear symmetrizable systems via a discrete relative-entropy argument. Neither of these analyses, however, encompasses the Chen-Shu ESDG method [6] with general entropy-stable interface fluxes or the oscillation-free extension [26] analyzed below. The fundamental obstacle is that classical DG error analysis relies crucially on the assumption that the numerical solution resides in a prescribed space of piecewise polynomials of degree k. For the entropy-stable nodal DG method introduced in [6], this assumption is violated: in multiple space dimensions, the nodal values produced by the scheme do not, in general, correspond to a piecewise polynomial of degree k. Consequently, existing error estimates for DG methods with inexact quadrature are neither directly applicable nor adequate to establish high-order convergence for the ESDG method. Nevertheless, the nodal flux-differencing formulation and SBP operators impart a finite-difference-type structure to the ESDG method, thereby motivating a consistency-stability analysis formulated directly at the nodal level.

In this paper, we prove an a priori h^k error estimate in a quadrature-based norm equivalent to the broken L^2 norm on the reconstruction space for smooth solutions of the semi-discrete ESDG method applied to (1.1). We prove the main results under a polynomial reconstruction hypothesis and an L^∞ a priori bound, and extends to the entropy-stable oscillation-free DG (ESOFDG) method [26], where the additional damping terms are shown not to degrade the convergence rate. The proof follows a consistency–stability approach: we first derive a local truncation error by inserting the nodal values of the exact solution into the semi-discrete scheme, then formulate the error equation and bound the volume, interface, and truncation contributions separately. The leading volume terms are controlled through the skew-symmetric SBP structure combined with entropy symmetrization of the flux Jacobians; the remaining nonlinear terms are estimated via Taylor's expansions and mesh-scaled matrix-norm bounds; and the interface contributions are managed using the entropy stability of the numerical flux. The resulting differential inequality is closed by a Gronwall argument. We further justify the polynomial reconstruction hypothesis and a priori assumption by continuation arguments. For the ESOFDG method, estimates of the damping coefficients together with the projection structure of the damping terms ensure that the additional dissipation does not affect the convergence order. Numerical experiments, however, indicate that the observed convergence rates may be higher than the theoretical bound proved by up to half an order. This suggests that the present estimate, which is indeed derived from the truncation error, may not be sharp. Similar phenomena have been reported in the error analysis of DG-type methods, indicating the possible presence of more delicate cancellation or superconvergence mechanisms [36].

This paper is organized as follows. In Section 2, we introduce some basic notations, the semi-discrete ESDG method, and some useful lemmas. In Section 3, we derive the error estimate for the ESDG method (2.9), then give the detailed discussion on our assumptions and extend our proof to the so-called ESOFDG scheme. Numerical results are reported in Section 4. We finally give concluding remarks in Section 5. Some technical proofs of lemmas are left in Appendix A.

2. Preliminaries. In this section, we will first give a brief review of the entropy analysis for (1.1) on the PDE level. Then we introduce the quadrature rules and the SBP operators [7], which mimic integration by parts at the discrete level. After that, we can describe the ESDG method. At the end of this section, we will give some useful auxiliary results that will be used.

2.1. Entropy analysis. A strictly convex function $U(\mathbf{u})$ is called an entropy function for (1.1) if there exist entropy fluxes, $\mathbf{F}(\mathbf{u}) = (F_1(\mathbf{u}), F_2(\mathbf{u}), \ldots, F_d(\mathbf{u}))^T$, such that

$$U'(\mathbf{u})\mathbf{f}'_m(\mathbf{u}) = F'_m(\mathbf{u}), \quad m = 1, \ldots, d. \tag{2.1}$$

Here $U'(\mathbf{u})$ and $F'_m(\mathbf{u})$ are understood as row vectors, and $\mathbf{f}'_m(\mathbf{u})$ is the $n \times n$ Jacobian matrix. We call the $(U(\mathbf{u}), \mathbf{F}(\mathbf{u}))$ an entropy pair. The entropy variable and the entropy Hessian are defined by

$$\mathbf{v}(\mathbf{u}) := U'(\mathbf{u})^T, \quad H(\mathbf{u}) := U''(\mathbf{u}).$$

In the error analysis below, we assume uniform convexity on the relevant set of states: there exist constants $C_1, C_2 > 0$ such that

$$C_1 I_n \le H(\mathbf{u}) \le C_2 I_n,$$

where I_n is the $n \times n$ identity matrix. Consequently, $H(\mathbf{u})$ is symmetric positive-definite, and the mapping $\mathbf{u} \mapsto \mathbf{v}(\mathbf{u})$ is invertible. We also define the entropy potential fluxes as follows:

$$\psi_m(\mathbf{v}) := \mathbf{v}^T \mathbf{f}_m(\mathbf{u}(\mathbf{v})) - F_m(\mathbf{u}(\mathbf{v})), \quad m = 1, \dots, d.$$

One can easily verify that $\psi_m'(\mathbf{v}) = \mathbf{f}_m(\mathbf{u}(\mathbf{v}))^T$. In addition, for a unit vector $\mathbf{n} = (n_1, \dots, n_d)^T$, we set

$$\mathbf{f}_{\mathbf{n}}(\mathbf{u}) := \sum_{m=1}^{d} n_m \mathbf{f}_m(\mathbf{u}), \quad F_{\mathbf{n}}(\mathbf{u}) := \sum_{m=1}^{d} n_m F_m(\mathbf{u}), \quad \psi_{\mathbf{n}}(\mathbf{v}) := \sum_{m=1}^{d} n_m \psi_m(\mathbf{v}).$$

In general, the physical relevant solution $\mathbf{u}$ of (1.1) is meant to satisfy the following entropy inequality

$$\partial_t U(\mathbf{u}) + \sum_{m=1}^{d} \partial_{x_m} F_m(\mathbf{u}) \le 0, \tag{2.2}$$

in the distribution sense for all entropy pairs. Formally integrating (2.2) in space with the assumption that $\mathbf{u}$ is compactly supported or periodic, we obtain

$$\frac{d}{dt} \int_\Omega U(\mathbf{u}) dx \le 0,$$

which indicates that the total amount of entropy is non-increasing in time. For scalar conservation laws, any convex function U defines an entropy function with entropy fluxes $F_m(u) = \int^u U'(s) f_m'(s) ds$. For hyperbolic systems, the existence of entropy functions is equivalent to the symmetrizability of the systems [19]. Fortunately, for most systems we are interested in, such as shallow water equations, compressible Euler equations, and MHD equations, we are able to find the entropy functions with physical meaning.

For numerical aspects, we would like to find our semi-discrete scheme to satisfy (2.2). Such property of schemes is referred to as *entropy-stable* (ES). To design ES schemes, the following entropy conservative fluxes and entropy stable fluxes are proposed by Tadmor [32, 33], and defined as follows.

Definition 2.1. For $m = 1, \dots, d$, a two-point numerical flux $\mathbf{f}_{m,S}(\mathbf{u}_L, \mathbf{u}_R)$ is called entropy conservative with respect to U if it is consistent, symmetric, and satisfies

$$(\mathbf{v}_R - \mathbf{v}_L)^T \mathbf{f}_{m,S}(\mathbf{u}_L, \mathbf{u}_R) = \psi_{m,R} - \psi_{m,L},$$

where $\mathbf{v}_{L,R}$ and $\psi_{m,L,R}$ are the entropy variables and entropy potential fluxes at left and right states. Furthermore, given entropy conservative fluxes in all coordinate directions, we define the directional entropy conservative flux by

$$\mathbf{f}_{\mathbf{n},S}(\mathbf{u}_L, \mathbf{u}_R) := \sum_{m=1}^{d} n_m \mathbf{f}_{m,S}(\mathbf{u}_L, \mathbf{u}_R).$$

Definition 2.2. A consistent two-point numerical flux $\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_L, \mathbf{u}_R)$ is called entropy stable with respect to U if it satisfies

$$(\mathbf{v}_R - \mathbf{v}_L)^T \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_L, \mathbf{u}_R) \le \psi_{\mathbf{n},R} - \psi_{\mathbf{n},L}.$$

2.2. Quadrature rules and SBP operators. In this subsection, we will briefly review the quadrature rules and SBP operators in order to rewrite the DG method under the SBP framework. Assume that $\Omega \subset \mathbb{R}^d$ is a bounded polygonal domain. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of conforming, shape-regular, and quasi-uniform simplicial partitions of Ω. We denote $h_K := \operatorname{diam}(K)$, $h := \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{E}_h$ be the set of all interior faces of $\mathcal{T}_h$, and $\mathcal{E}_K$ be the set of all faces of $K \in \mathcal{T}_h$. For each $\gamma \in \mathcal{E}_h$, we fix a unit normal vector $\mathbf{n}_\gamma$. If $\gamma = K^- \cap K^+$, then labels K^- and K^+ are chosen such that $\mathbf{n}_\gamma$ points from K^- to K^+. For a piecewise smooth function w, we denote the two traces of w on γ by

$$w^- := w|_{K^-}, \quad w^+ := w|_{K^+}.$$

When a single element K is fixed and $\gamma \in \mathcal{E}_K$, we also write $w^{\gamma,K} := w|_K$ on γ, namely the trace of w on γ taken from the inner side of K. Let $\widehat{K}$ be a fixed reference simplex and, for each $K \in \mathcal{T}_h$, let

$$F_K(\widehat{\mathbf{x}}) = A_K\widehat{\mathbf{x}} + \mathbf{b}_K =: \mathbf{x} \tag{2.3}$$

be the affine mapping from $\widehat{K}$ onto K. We assume that the volume and surface quadrature rules, as well as the corresponding nodal SBP operators on K, are obtained by affine transformation of fixed reference-element rules and operators. For notational convenience, all definitions below are stated directly on the element K. For each simplex $K \in \mathcal{T}_h$, suppose that there is a quadrature rule of degree at least $2k-1$ on K, associated with quadrature nodes and positive weights $\{(\mathbf{x}_j^K, \omega_j^K)\}_{j=1}^{N_{Q,k}}$, $N_{Q,k} \ge N_{P,k}$ with

$$N_{P,\nu} = \dim \mathcal{P}^\nu(\mathbb{R}^d), \quad 0 \le \nu \le k.$$

For each $\gamma \in \mathcal{E}_h$, suppose that there is a quadrature rule of degree at least $2k$ on γ, associated with quadrature nodes and positive weights $\{(\mathbf{x}_j^\gamma, \tau_j^\gamma)\}_{j=1}^{N_{B,k}}$. Now we take $\{p_l\}_{l=1}^{N_{P,\nu}}$ as the set of basis functions of $\mathcal{P}^\nu(\mathbb{R}^d)$, which means

$$\{p_l\}_{l=1}^{N_{P,0}} \subset \{p_l\}_{l=1}^{N_{P,1}} \subset \cdots \subset \{p_l\}_{l=1}^{N_{P,k}}.$$

Then we define the Vandermonde matrices, whose columns are nodal values:

$$V_\nu^K = [\vec{p}_1^K, \cdots, \vec{p}_{N_{P,\nu}}^K], \quad V_\nu^\gamma = [\vec{p}_1^\gamma, \cdots, \vec{p}_{N_{P,\nu}}^\gamma],$$

where $\vec{p}_l^K = [p_l(\mathbf{x}_1^K), \cdots, p_l(\mathbf{x}_{N_{Q,k}}^K)]^T, \vec{p}_l^\gamma = [p_l(\mathbf{x}_1^\gamma), \cdots, p_l(\mathbf{x}_{N_{B,k}}^\gamma)]^T$. We also define polynomial differential matrices $\hat{D}_m$ such that

$$\frac{\partial}{\partial x_m} p_l(\mathbf{x}) = \sum_{r=1}^{N_{P,k}} \hat{D}_{m,rl} p_r(\mathbf{x}).$$

Then $V_k^K \hat{D}_m$ is the Vandermonde matrix of $\{\partial_{x_m} p_l(\mathbf{x})\}_{l=1}^{N_{P,k}}$ on K. For a scalar function w, define the volume and surface nodal vectors by

$$\vec{w}^K := \left[w(\mathbf{x}_1^K), \dots, w(\mathbf{x}_{N_{Q,k}}^K)\right]^T, \quad \vec{w}^{\gamma,K} := \left[w(\mathbf{x}_1^\gamma), \dots, w(\mathbf{x}_{N_{B,k}}^\gamma)\right]^T.$$

Denote

$$M^K := \mathrm{diag}(\omega_1^K, \dots, \omega_{N_{Q,k}}^K), \quad B^\gamma := \mathrm{diag}(\tau_1^\gamma, \dots, \tau_{N_{B,k}}^\gamma).$$

We then define the continuous and discrete inner products on K and γ that

$$(u, w)_K := \int_K uw dx, \quad (u, w)_{K,\omega} := \sum_{j=1}^{N_{Q,k}} \omega_j^K u(\mathbf{x}_j^K) w(\mathbf{x}_j^K) = (\vec{u}^K)^T M^K \vec{w}^K,$$

$$(u, w)_\gamma := \int_\gamma uw ds, \quad \langle u, w\rangle_{\gamma,\tau} := \sum_{s=1}^{N_{B,k}} \tau_s^\gamma u(\mathbf{x}_s^\gamma) w(\mathbf{x}_s^\gamma) = (\vec{u}^{\gamma,K})^T B^\gamma \vec{w}^{\gamma,K}.$$

In order to obtain the *nodal* SBP property, we recall the definitions of *degree k difference matrix* D_m^K and extrapolation matrices $\{R^{\gamma,K}\}_{\gamma\in\mathcal{E}_K}$, for which the following two conditions hold:

1. Exactness: both D_m^K and $R^{\gamma,K}$ are exact for polynomials of degree k, i.e.,

$$D_m^K V_k^K = V_k^K \hat{D}_m, \quad R^{\gamma,K} V_k^K = V_k^\gamma. \tag{2.4}$$

2. Summation-by-parts: setting $S_m^K := M^K D_m^K$ and $E^{\gamma,K} := (R^{\gamma,K})^T B^\gamma R^{\gamma,K}$, we have

$$S_m^K + (S_m^K)^T = \sum_{\gamma\in\mathcal{E}_K} n_m^{\gamma,K} E^{\gamma,K}. \tag{2.5}$$

We also define L^2 projection matrices under the discrete inner product:

$$P_\nu^K = \big((V_\nu^K)^T M^K V_\nu^K\big)^{-1} (V_\nu^K)^T M^K, \quad 0 \le \nu \le k.$$

In particular, for $\nu = 0$, we have the orthogonality of the operator P_0^K that

$$\vec{c}^T M^K (\vec{w}^K - V_0^K P_0^K \vec{w}^K) = 0, \quad \forall \vec{w}^K \in \mathbb{R}^{N_{Q,k}},$$

where $\vec{c} \in \mathbb{R}^{N_{Q,k}}$ is a constant vector. The existence of SBP difference matrices is ensured by the following theorem [7].

THEOREM 2.3. *Assume that we have extrapolation matrices $R^{\gamma,K}$ satisfying the exactness property. Then the difference matrices, given by the formula*

$$D_m^K = \frac{1}{2}(M^K)^{-1} \sum_{\gamma\in\mathcal{E}_K} n_m^{\gamma,K} \big(R^{\gamma,K} + V_k^\gamma P_k^K\big)^T B^\gamma \big(R^{\gamma,K} - V_k^\gamma P_k^K\big) + V_k^K \widehat{D}_m P_k^K.$$

satisfy (2.4) *and* (2.5).

For the choice of the extrapolation matrices $R^{\gamma,K}$, the readers can refer to [7]. According to the construction of P_0^K and V_0^K, one can easily check that $Q^K := I_{N_{Q,k}} - V_0^K P_0^K$ satisfy

$$Q^K = (Q^K)^2, \quad (M^K Q^K)^T = M^K Q^K. \tag{2.6}$$

For the vector-valued function $\mathbf{w} : \mathbb{R}^d \to \mathbb{R}^n$, we introduce the vector of nodal values

$$\vec{\mathbf{w}}^K := \big[\mathbf{w}(\mathbf{x}_1^K)^T, \dots, \mathbf{w}(\mathbf{x}_{N_{Q,k}}^K)^T\big]^T \in \mathbb{R}^{nN_{Q,k}},$$

$$\vec{\mathbf{w}}^{\gamma,K} := \left[\mathbf{w}^{\gamma,K}(\mathbf{x}_1^{\gamma})^T, \dots, \mathbf{w}^{\gamma,K}(\mathbf{x}_{N_{B,k}}^{\gamma})^T\right]^T \in \mathbb{R}^{nN_{B,k}}.$$

as well as the Kronecker products

$$\begin{aligned}
&\mathbf{M}^K = M^K \otimes I_n, \quad \mathbf{B}^{\gamma} = B^{\gamma} \otimes I_n, \quad \mathbf{D}_m^K = D_m^K \otimes I_n, \quad \mathbf{R}^{\gamma,K} = R^{\gamma,K} \otimes I_n,\\
&\hat{\mathbf{D}}_m = \hat{D}_m \otimes I_n, \quad \mathbf{V}_\nu^K = V_\nu^K \otimes I_n, \quad \mathbf{V}_\nu^{\gamma} = V_\nu^{\gamma} \otimes I_n, \quad \mathbf{Q}^K = Q^K \otimes I_n,\\
&\mathbf{P}_\nu^K := P_\nu^K \otimes I_n, \quad \mathbf{S}_m^K := S_m^K \otimes I_n.
\end{aligned}$$

All the properties before remain valid for vector-valued nodal data under the Kronecker-product convention introduced above. For a volume nodal vector $\vec{\mathbf{w}}^K$, we define the discrete h-norm by

$$\|\vec{\mathbf{w}}^K\|_{h,K}^2 := (\vec{\mathbf{w}}^K)^T \mathbf{M}^K \vec{\mathbf{w}}^K, \quad \|\vec{\mathbf{w}}\|_h^2 := \sum_{K\in\mathcal{T}_h} \|\vec{\mathbf{w}}^K\|_{h,K}^2,$$

where the corresponding global broken nodal vector is

$$\vec{\mathbf{w}} := \left(\vec{\mathbf{w}}^{\,K}\right)_{K\in\mathcal{T}_h} \in \prod_{K\in\mathcal{T}_h} \mathbb{R}^{nN_{Q,k}}.$$

We also define the $\ell^q, 1 \le q \le \infty$ norms by

$$\|\vec{\mathbf{w}}^K\|_{\ell^q,K}^q := \sum_{j=1}^{N_{Q,k}} |\mathbf{w}_j^K|^q, \quad 1 \le q < \infty,$$

and

$$\|\vec{\mathbf{w}}^K\|_{\ell^\infty,K} := \max_{1\le j\le N_{Q,k}} |\mathbf{w}_j^K|,$$

where $\mathbf{w}_j^K = \mathbf{w}(\mathbf{x}_j^K)$ and $|\cdot|$ denotes the Euclidean norm. The corresponding global norms are

$$\|\vec{\mathbf{w}}\|_{\ell^q}^q := \sum_{K\in\mathcal{T}_h} \|\vec{\mathbf{w}}^K\|_{\ell^q,K}^q, \quad \|\vec{\mathbf{w}}\|_{\ell^\infty} := \max_{K\in\mathcal{T}_h} \|\vec{\mathbf{w}}^K\|_{\ell^\infty,K}.$$

Remark 2.4 (Affine transformation of the SBP operators). Under the affine mapping (2.3), and let $\widehat{\gamma} \in \mathcal{E}_{\widehat{K}}$ be the reference face mapped onto $\gamma \in \mathcal{E}_K$, and set

$$J_K := |\det A_K|, \quad J_{\gamma,K} := J_K \left|(A_K^{-1})^T \widehat{\mathbf{n}}_{\widehat{\gamma}}\right|.$$

After a consistent ordering of the volume and face nodes, we obtain

$$\begin{aligned}
&M^K = J_K M^{\widehat{K}}, \quad B^{\gamma} = J_{\gamma,K} B^{\widehat{\gamma}},\\
&D_m^K = \sum_{\alpha=1}^{d} ((A_K^{-1})^T)_{m\alpha} D_\alpha^{\widehat{K}}, \quad R^{\gamma,K} = R^{\widehat{\gamma},\widehat{K}}, \quad Q^K = Q^{\widehat{K}}.
\end{aligned} \tag{2.7}$$

2.3. Entropy stable DG schemes with collocated surface nodes. We now introduce the ESDG scheme that is the focus of this paper. We first recall the classical DG method. Given polynomial degree $k \ge 0$, we define the DG finite element space,

$$\mathbf{W}_h^k = \{\mathbf{w}_h : \mathbf{w}_h|_K = \mathbf{w}_h^K \in \left[\mathcal{P}^k(K)\right]^n, K \in \mathcal{T}_h\}.$$

For (1.1), the classical DG method can be constructed as follows: we seek $\mathbf{u}_h \in \mathbf{W}_h^k$ such that for each $\mathbf{w}_h \in \mathbf{W}_h^k$ and $K \in \mathcal{T}_h$, we have

$$(2.8)\qquad (\partial_t \mathbf{u}_h^K, \mathbf{w}_h^K)_K - \sum_{m=1}^{d} (\mathbf{f}_m(\mathbf{u}_h^K), \partial_m \mathbf{w}_h^K)_K = - \sum_{\gamma \in \mathcal{E}_K} \langle \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_h^{\gamma,K}, \mathbf{u}_h^{\gamma,\tilde{K}}), \mathbf{w}_h^{\gamma,K} \rangle_\gamma,$$

where $\tilde{K}$ denotes the neighboring element sharing the face γ with K, and $\hat{\mathbf{f}}_{\mathbf{n}}$ is the interface numerical flux function. By applying the volume quadrature rule and the surface quadrature rule to (2.8) and replacing the standard difference operator with a flux-differencing term, we obtain the ESDG method introduced in [6],

$$(2.9)\quad \mathbf{M}^K \frac{d\vec{\mathbf{u}}_h^K}{dt} + 2\sum_{m=1}^{d} \left(\mathbf{S}_m^K \circ \mathbf{F}_{m,S}(\vec{\mathbf{u}}_h^K, \vec{\mathbf{u}}_h^K)\right)\vec{\mathbf{1}}^K = \sum_{\gamma \in \mathcal{E}_K} (\mathbf{R}^{\gamma,K})^T \mathbf{B}^\gamma \left(\vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K} - \vec{\hat{\mathbf{f}}}_{\mathbf{n}}^{\gamma,K} \right).$$

Here $\circ$ denotes the Hadamard product and $\vec{\mathbf{1}}^K$ is the vector of ones, and $\mathbf{F}_{m,S}(\cdot,\cdot)$ is the matrix of pairwise combinations of entropy conservative fluxes

$$\mathbf{F}_{m,S}(\vec{\mathbf{u}}_L, \vec{\mathbf{u}}_R) = \begin{bmatrix} \operatorname{diag}(\mathbf{f}_{m,S}(\mathbf{u}_{L,1}, \mathbf{u}_{R,1})) & \cdots & \operatorname{diag}(\mathbf{f}_{m,S}(\mathbf{u}_{L,1}, \mathbf{u}_{R,\mathcal{N}_R})) \\ \vdots & \ddots & \vdots \\ \operatorname{diag}(\mathbf{f}_{m,S}(\mathbf{u}_{L,\mathcal{N}_L}, \mathbf{u}_{R,1})) & \cdots & \operatorname{diag}(\mathbf{f}_{m,S}(\mathbf{u}_{L,\mathcal{N}_L}, \mathbf{u}_{R,\mathcal{N}_R})) \end{bmatrix},$$

for $\vec{\mathbf{u}}_L \in \mathbb{R}^{n\mathcal{N}_L}$ and $\vec{\mathbf{u}}_R \in \mathbb{R}^{n\mathcal{N}_R}$. The $\vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K}$ and $\vec{\hat{\mathbf{f}}}_{\mathbf{n}}^{\gamma,K}$ are defined as

$$\begin{aligned} \vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K} &= \left[(\mathbf{f}_{\mathbf{n}}(\mathbf{u}_{h,1}^{\gamma,K}))^T, \dots, (\mathbf{f}_{\mathbf{n}}(\mathbf{u}_{h,N_{B,k}}^{\gamma,K}))^T \right]^T, \\ \vec{\hat{\mathbf{f}}}_{\mathbf{n}}^{\gamma,K} &= \left[(\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{h,1}^{\gamma,K}, \mathbf{u}_{h,1}^{\gamma,\tilde{K}}))^T, \dots, (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{h,N_{B,k}}^{\gamma,K}, \mathbf{u}_{h,N_{B,k}}^{\gamma,\tilde{K}}))^T \right]^T. \end{aligned}$$

Note that this method requires the collocated surface quadrature nodes, namely, $\{\mathbf{x}_j^\gamma\}_{j=1}^{N_{B,k}} \subset \{\mathbf{x}_j^K\}_{j=1}^{N_{Q,k}}$. Thus $\mathbf{R}^{\gamma,K}$ is simply a restriction onto γ. Under the assumptions that $\mathbf{f}_{m,S}$ is entropy conservative and that $\widehat{\mathbf{f}}_{\mathbf{n}}$ is entropy stable with respect to U, the (2.9) satisfies a semi-discrete entropy inequality [6]. Next, we introduce some auxiliary results that will be used in the error estimates of (2.9). Here and below, for two nonnegative quantities a and b, we write $a \lesssim b$ if there exists a constant $C > 0$, independent of h, such that $a \le Cb$. Moreover, we write $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$ are true. We first give the scaling properties of the quadrature weights and SBP operators.

LEMMA 2.5 (Scaling properties of SBP operators). *Assume that the mesh is shape regular and quasi-uniform; the following estimates hold uniformly for all $K \in \mathcal{T}_h$ and all faces $\gamma \in \mathcal{E}_K$:*

$$\omega_j^K \sim h_K^d, \quad \tau_s^\gamma \sim h_K^{d-1}.$$

Consequently, for any volume nodal vector $\vec{\mathbf{w}}^K$,

$$\|\vec{\mathbf{w}}^K\|_{h,K}^2 \sim h_K^d \|\vec{\mathbf{w}}^K\|_{\ell^2,K}^2.$$

For SBP operators, for any $1 \le q \le \infty$,

$$\|\mathbf{D}_m^K\|_{\ell^q} \lesssim h_K^{-1}, \quad \|\mathbf{M}^K\|_{\ell^q} \lesssim h_K^d, \quad \|\mathbf{R}^{\gamma,K}\|_{\ell^q} \lesssim 1, \quad \|\mathbf{Q}^K\|_{\ell^q} \lesssim 1.$$

Here the operator norms are induced by the corresponding nodal ℓ^q-norms. In addition, since $\mathbf{Q}^K$ is an $\mathbf{M}^K$-orthogonal projection,

$$\|\mathbf{Q}^K \vec{\mathbf{w}}^{\,K}\|_{h,K} \le \|\vec{\mathbf{w}}^{\,K}\|_{h,K}.$$

The proof of Lemma 2.5 is given in the appendix; see Section A.1. We next recall the following inverse estimates [8] for piecewise polynomials.

LEMMA 2.6 (Inverse inequalities). *There exists a constant $C > 0$, independent of h, such that for any $\mathbf{v}_h \in \mathbf{W}_h^k$,*

$$|\mathbf{v}_h|_{H^1} \lesssim h^{-1}\|\mathbf{v}_h\|_{L^2}, \quad \|\mathbf{v}_h\|_{L^\infty} \lesssim h^{-d/2}\|\mathbf{v}_h\|_{L^2}.$$

The next lemma gives the local truncation estimate of the ESDG method, see [6].

LEMMA 2.7 (Local truncation errors of the ESDG method). *Assume that the exact solution $\mathbf{u}$, physical fluxes $\mathbf{f}_m$ and the entropy conservative fluxes $\mathbf{f}_{m,S}$ are sufficiently smooth, and $\hat{\mathbf{f}}_{\mathbf{n}}$ are locally Lipschitz continuous with respect to both arguments. Let*

$$\begin{aligned}\boldsymbol{\tau}_j^K(\mathbf{u}) :=& \frac{d\mathbf{u}_j^K}{dt} + 2\sum_{m=1}^{d}\sum_{\ell=1}^{N_{Q,k}} D_{m,j\ell}^K \mathbf{f}_{m,S}(\mathbf{u}_j^K, \mathbf{u}_\ell^K) \\ &- \sum_{\gamma\in\mathcal{E}_K}\sum_{s=1}^{N_{B,k}} R_{sj}^{\gamma,K}\frac{\tau_s^\gamma}{\omega_j^K}\big(\mathbf{f}_{\mathbf{n}}(\mathbf{u}_s^{\gamma,K}) - \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_s^{\gamma,K}, \mathbf{u}_s^{\gamma,\tilde{K}})\big),\end{aligned}$$

for $j = 1, \ldots, N_{Q,k}$. Then we have

$$\|\vec{\boldsymbol{\tau}}^K(\mathbf{u})\|_{\ell^q,K} \lesssim h^k, \quad 1 \le q \le \infty,$$

where $\vec{\boldsymbol{\tau}}^K(\mathbf{u}) = \big[\boldsymbol{\tau}_1^K(\mathbf{u})^T, \ldots, \boldsymbol{\tau}_{N_{Q,k}}^K(\mathbf{u})^T\big]^T$.

The next lemma, intuitively, measures the difference between the central flux and the entropy conservative flux.

LEMMA 2.8. *Define*

$$\mathbf{f}_{m,C}(\mathbf{a},\mathbf{b}) := \frac{\mathbf{f}_m(\mathbf{a}) + \mathbf{f}_m(\mathbf{b})}{2}, \quad \boldsymbol{\Phi}_m(\mathbf{a},\mathbf{b}) := \mathbf{f}_{m,S}(\mathbf{a},\mathbf{b}) - \mathbf{f}_{m,C}(\mathbf{a},\mathbf{b}), \quad \mathbf{a},\mathbf{b} \in \mathbb{R}^n.$$

Assume that $\mathbf{f}_m \in C^3(\mathbb{R}^n)$, $\mathbf{f}_{m,S} \in C^3(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathbf{f}_{m,S}$ is the symmetric and consistent flux. For any convex compact set $S \subset \mathbb{R}^n$, if $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in S$ and satisfy

$$|\mathbf{a}-\mathbf{b}| + |\mathbf{c}-\mathbf{d}| \lesssim h,$$

then

$$|\boldsymbol{\Phi}_m(\mathbf{a},\mathbf{b}) - \boldsymbol{\Phi}_m(\mathbf{c},\mathbf{d})| \lesssim h\big(|\mathbf{a}-\mathbf{c}| + |\mathbf{b}-\mathbf{d}|\big).$$

The proof of Lemma 2.8 is given in the appendix; see Section A.2.

3. The Main Results. In this section, we establish the error estimate of the ESDG scheme (2.9). Before proceeding to the analysis, we point out a distinction between the ESDG method and the classical DG method. In a classical DG method, the numerical solution is naturally a piecewise polynomial. In contrast, the ESDG scheme is formulated as an evolution equation for nodal vectors, which are not automatically associated with functions in a prescribed polynomial space. To employ standard polynomial approximation and inverse estimates, we assume that each nodal vector admits a polynomial reconstruction.

ASSUMPTION 3.1. *For each element $K \in \mathcal{T}_h$, there exists a finite-dimensional polynomial space $V_h(K)$ satisfying*

$$\mathcal{P}^k(K) \subset V_h(K) \subset \mathcal{P}^r(K),$$

where $r \geq k$ is fixed and independent of h, such that the nodal evaluation map

$$\mathcal{N}_K : V_h(K) \to \mathbb{R}^{N_{Q,k}}, \quad \mathcal{N}_K w := \big(w(\mathbf{x}_1^K), \dots, w(\mathbf{x}_{N_{Q,k}}^K)\big)^T \quad \forall w \in V_h(K),$$

is an isomorphism. It follows

$$\big[\mathcal{P}^k(K)\big]^n \subset \mathbf{V}_h(K) := [V_h(K)]^n \subset \big[\mathcal{P}^r(K)\big]^n.$$

Then every vector of nodal values admits a unique polynomial reconstruction in $\mathbf{V}_h(K)$.

ASSUMPTION 3.2. *Let $\mathbf{u}_h$ denote the piecewise polynomial reconstruction associated with Assumption 3.1. Assume that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} \leq h, \quad 0 \leq t \leq T.$$

3.1. Error estimates for the ESDG scheme.

THEOREM 3.3. *Let U be a sufficiently smooth uniformly convex entropy function. Assume that the exact solution $\mathbf{u}$, the physical fluxes $\mathbf{f}_m$, the entropy conservative fluxes $\mathbf{f}_{m,S}$ are sufficiently smooth, and the entropy stable fluxes $\hat{\mathbf{f}}_{\mathbf{n}}$ are locally Lipschitz continuous with respect to both arguments. Suppose that Assumption 3.1 and Assumption 3.2 holds, and*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim h^k, \quad t = 0.$$

Then the solution of the ESDG scheme (2.9) satisfies

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim h^k, \quad 0 \leq t \leq T.$$

Proof. For each $K \in \mathcal{T}_h$, define

$$\vec{\mathbf{e}}^K := \vec{\mathbf{u}}^K - \vec{\mathbf{u}}_h^K, \quad \vec{\mathbf{e}}_v^K := \vec{\mathbf{v}}^K - \vec{\mathbf{v}}_h^K,$$

where $\vec{\mathbf{v}}^K := [\mathbf{v}(\mathbf{u}_1^K)^T, \cdots, \mathbf{v}(\mathbf{u}_{N_{Q,k}}^K)^T]^T, \vec{\mathbf{v}}_h^K := [\mathbf{v}(\mathbf{u}_{h,1}^K)^T, \cdots, \mathbf{v}(\mathbf{u}_{h,N_{Q,k}}^K)^T]^T$, and $\mathbf{v}(\mathbf{u})$ is the entropy variable. By Taylor's expansion, we have $\vec{\mathbf{e}}_v^K = \mathbf{H}^K \vec{\mathbf{e}}^K$, where

$$\begin{aligned}
\mathbf{H}^K &:= \operatorname{diag}(H_1^K, \cdots, H_{N_{Q,k}}^K) \in \mathbb{R}^{nN_{Q,k} \times nN_{Q,k}}, \\
H_i^K &:= \int_0^1 U''(\mathbf{u}_{h,i}^K + \theta(\mathbf{u}_i^K - \mathbf{u}_{h,i}^K))\, d\theta \in \mathbb{R}^{n \times n}.
\end{aligned}$$

Each H_i^K is symmetric positive definite. Define $\mathbf{f}_{m,C}, \mathbf{\Phi}_m$ as those in Lemma 2.8. With a slight abuse of notation, the corresponding matrices are denoted by $\mathbf{F}_{m,C}(\vec{\mathbf{u}}_L, \vec{\mathbf{u}}_R)$

and $\mathbf{\Phi}_m(\vec{\mathbf{u}}_L, \vec{\mathbf{u}}_R)$ with the similar definition as $\mathbf{F}_{m,S}(\vec{\mathbf{u}}_L, \vec{\mathbf{u}}_R)$. Since $\mathbf{F}_{m,S} = \mathbf{F}_{m,C} + \mathbf{\Phi}_m$, the scheme (2.9) can be rewritten as

$$\begin{aligned} \mathbf{M}^K \frac{d\vec{\mathbf{u}}_h^K}{dt} &+ 2\sum_{m=1}^d \big(\mathbf{S}_m^K \circ \mathbf{F}_{m,C}(\vec{\mathbf{u}}_h^K, \vec{\mathbf{u}}_h^K)\big)\vec{\mathbf{1}}^K + 2\sum_{m=1}^d \big(\mathbf{S}_m^K \circ \mathbf{\Phi}_m(\vec{\mathbf{u}}_h^K, \vec{\mathbf{u}}_h^K)\big)\vec{\mathbf{1}}^K \\ &= \sum_{\gamma\in\mathcal{E}_K} (\mathbf{R}^{\gamma,K})^T \mathbf{B}^\gamma \big(\vec{\mathbf{f}}_{\mathbf{n}}(\vec{\mathbf{u}}_h^{\gamma,K}) - \hat{\vec{\mathbf{f}}}_{\mathbf{n}}(\vec{\mathbf{u}}_h^{\gamma,K}, \vec{\mathbf{u}}_h^{\gamma,\tilde{K}})\big). \end{aligned} \tag{3.1}$$

Using $\mathbf{S}_m^K \vec{\mathbf{1}}^K = \vec{\mathbf{0}}$, we have

$$2\big(\mathbf{S}_m^K \circ \mathbf{F}_{m,C}(\vec{\mathbf{w}}^K, \vec{\mathbf{w}}^K)\big)\vec{\mathbf{1}}^K = \mathbf{S}_m^K \vec{\mathbf{f}}_m(\vec{\mathbf{w}}^K), \quad \forall\, \vec{\mathbf{w}}^K \in \mathbb{R}^{nN_{Q,k}}.$$

By Lemma 2.7, the exact solution satisfies

$$\begin{aligned} \mathbf{M}^K \frac{d\vec{\mathbf{u}}^K}{dt} &+ \sum_{m=1}^d \mathbf{S}_m^K \vec{\mathbf{f}}_m(\vec{\mathbf{u}}^K) + 2\sum_{m=1}^d \big(\mathbf{S}_m^K \circ \mathbf{\Phi}_m(\vec{\mathbf{u}}^K, \vec{\mathbf{u}}^K)\big)\vec{\mathbf{1}}^K \\ &= \sum_{\gamma\in\mathcal{E}_K} (\mathbf{R}^{\gamma,K})^T \mathbf{B}^\gamma \Big(\vec{\mathbf{f}}_{\mathbf{n}}(\vec{\mathbf{u}}^{\gamma,K}) - \hat{\vec{\mathbf{f}}}_{\mathbf{n}}(\vec{\mathbf{u}}^{\gamma,K}, \vec{\mathbf{u}}^{\gamma,\tilde{K}})\Big) + \vec{\mathbf{T}}^K(\mathbf{u}), \end{aligned}$$

where $\vec{\mathbf{T}}^K(\mathbf{u}) = \mathbf{M}^K \vec{\boldsymbol{\tau}}^K(\mathbf{u})$. Subtracting (3.1) from (3.2), we obtain

$$\begin{aligned} \mathbf{M}^K \frac{d\vec{\mathbf{e}}^K}{dt} &+ \sum_{m=1}^d \mathbf{S}_m^K \Delta\vec{\mathbf{f}}_m^K + 2\sum_{m=1}^d \big(\mathbf{S}_m^K \circ \Delta\mathbf{\Phi}_m^K\big)\vec{\mathbf{1}}^K \\ &= \sum_{\gamma\in\mathcal{E}_K} (\mathbf{R}^{\gamma,K})^T \mathbf{B}^\gamma \Big(\Delta\vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K} - \Delta\hat{\vec{\mathbf{f}}}_{\mathbf{n}}^{\gamma}\Big) + \vec{\mathbf{T}}^K(\mathbf{u}). \end{aligned} \tag{3.2}$$

Here,

$$\begin{aligned} \Delta\vec{\mathbf{f}}_m^K &:= \vec{\mathbf{f}}_m(\vec{\mathbf{u}}^K) - \vec{\mathbf{f}}_m(\vec{\mathbf{u}}_h^K), \quad \Delta\mathbf{\Phi}_m^K := \mathbf{\Phi}_m(\vec{\mathbf{u}}^K, \vec{\mathbf{u}}^K) - \mathbf{\Phi}_m(\vec{\mathbf{u}}_h^K, \vec{\mathbf{u}}_h^K), \\ \Delta\vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K} &:= \vec{\mathbf{f}}_{\mathbf{n}}(\vec{\mathbf{u}}^{\gamma,K}) - \vec{\mathbf{f}}_{\mathbf{n}}(\vec{\mathbf{u}}_h^{\gamma,K}), \quad \Delta\hat{\vec{\mathbf{f}}}_{\mathbf{n}}^{\gamma} := \hat{\vec{\mathbf{f}}}_{\mathbf{n}}(\vec{\mathbf{u}}^{\gamma,K}, \vec{\mathbf{u}}^{\gamma,\tilde{K}}) - \hat{\vec{\mathbf{f}}}_{\mathbf{n}}(\vec{\mathbf{u}}_h^{\gamma,K}, \vec{\mathbf{u}}_h^{\gamma,\tilde{K}}). \end{aligned}$$

Left multiplying (3.2) by $(\vec{\mathbf{e}}_v^K)^T$, we obtain

$$\begin{aligned} (\vec{\mathbf{e}}^K)^T \mathbf{H}^K \mathbf{M}^K \frac{d\vec{\mathbf{e}}^K}{dt} = &- \sum_{m=1}^d (\vec{\mathbf{e}}^K)^T \mathbf{H}^K \mathbf{S}_m^K \Delta\vec{\mathbf{f}}_m^K \\ &- 2\sum_{m=1}^d (\vec{\mathbf{e}}^K)^T \mathbf{H}^K \big(\mathbf{S}_m^K \circ \Delta\mathbf{\Phi}_m^K\big)\vec{\mathbf{1}}^K \\ &+ \sum_{\gamma\in\mathcal{E}_K} (\vec{\mathbf{e}}_v^{\gamma,K})^T \mathbf{B}^\gamma \Big(\Delta\vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K} - \Delta\hat{\vec{\mathbf{f}}}_{\mathbf{n}}^{\gamma}\Big) + (\vec{\mathbf{e}}^K)^T \mathbf{H}^K \vec{\mathbf{T}}^K(\mathbf{u}), \end{aligned} \tag{3.3}$$

where $\vec{\mathbf{e}}_v^{\gamma,K} := \mathbf{R}^{\gamma,K} \vec{\mathbf{e}}_v^K$. We then decompose $\mathbf{S}_m^K$ into its symmetric and skew-symmetric parts:

$$\mathbf{S}_m^K = \frac{1}{2}\big(\mathbf{S}_m^K + (\mathbf{S}_m^K)^T\big) + \frac{1}{2}\big(\mathbf{S}_m^K - (\mathbf{S}_m^K)^T\big). \tag{3.4}$$

Let

$$A_m^K := \frac{1}{2}\big(S_m^K - (S_m^K)^T\big), \quad \mathbf{A}_m^K := A_m^K \otimes I_n = \frac{1}{2}\big(\mathbf{S}_m^K - (\mathbf{S}_m^K)^T\big).$$

By (2.5) and (3.4), we have

$$(\vec{\mathbf{e}}^K)^T \mathbf{H}^K \mathbf{M}^K \frac{d\vec{\mathbf{e}}^K}{dt} = \mathcal{S}_1^K + \mathcal{S}_2^K + \mathcal{S}_3^K + \mathcal{S}_4^K, \tag{3.5}$$

where

$$\mathcal{S}_1^K := \sum_{\gamma\in\mathcal{E}_K} (\vec{\mathbf{e}}_v^{\gamma,K})^T \mathbf{B}^\gamma \left(\frac{1}{2}\Delta\vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K} - \Delta\hat{\vec{\mathbf{f}}}_{\mathbf{n}}^\gamma\right), \quad \mathcal{S}_2^K := -\sum_{m=1}^{d} (\vec{\mathbf{e}}^K)^T \mathbf{H}^K \mathbf{A}_m^K \Delta\vec{\mathbf{f}}_m^K,$$

$$\mathcal{S}_3^K := -2\sum_{m=1}^{d} (\vec{\mathbf{e}}^K)^T \mathbf{H}^K \left(\mathbf{S}_m^K \circ \Delta\mathbf{\Phi}_m^K\right)\vec{\mathbf{1}}^K, \quad \mathcal{S}_4^K := (\vec{\mathbf{e}}^K)^T \mathbf{H}^K \vec{\mathbf{T}}^K(\mathbf{u}).$$

Define the entropy-weighted energy by

$$\|\vec{\mathbf{w}}^K\|_{U,K}^2 := (\vec{\mathbf{w}}^K)^T \mathbf{H}^K \mathbf{M}^K \vec{\mathbf{w}}^K, \quad \|\vec{\mathbf{w}}\|_U^2 := \sum_{K\in\mathcal{T}_h} \|\vec{\mathbf{w}}^K\|_{U,K}^2.$$

The uniform convexity and smoothness of U imply $\|\cdot\|_U \sim \|\cdot\|_h$. Then (3.5) becomes

$$\frac{1}{2}\frac{d}{dt}\|\vec{\mathbf{e}}\|_U^2 = \sum_{K\in\mathcal{T}_h} \mathcal{S}_0^K + \sum_{K\in\mathcal{T}_h} \mathcal{S}_1^K + \sum_{K\in\mathcal{T}_h} \mathcal{S}_2^K + \sum_{K\in\mathcal{T}_h} \mathcal{S}_3^K + \sum_{K\in\mathcal{T}_h} \mathcal{S}_4^K, \tag{3.6}$$

where

$$\mathcal{S}_0^K = \frac{1}{2}(\vec{\mathbf{e}}^K)^T \frac{d\mathbf{H}^K}{dt} \mathbf{M}^K \vec{\mathbf{e}}^K.$$

We first estimate $\mathcal{S}_0^K$. Indeed, we have

$$\sum_{K\in\mathcal{T}_h} \mathcal{S}_0^K = \frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{N_{Q,k}} \omega_i^K (\mathbf{e}_i^K)^T \partial_t H_i^K \mathbf{e}_i^K,$$

where

$$\partial_t H_i^K = \int_0^1 U'''(\mathbf{u}_{h,i}^K + \theta(\mathbf{u}_i^K - \mathbf{u}_{h,i}^K))\left[(1-\theta)\partial_t \mathbf{u}_{h,i}^K + \theta\partial_t\mathbf{u}_i^K\right] d\theta \lesssim |\partial_t \mathbf{u}_{h,i}^K| + |\partial_t \mathbf{u}_i^K|$$

According to (2.9), using $\mathbf{D}_m^K \vec{\mathbf{1}}^K = \vec{\mathbf{0}}$ and the consistency of the entropy conservative flux, we obtain

$$\begin{aligned}
\partial_t \mathbf{u}_{h,i}^K = & -2\sum_{m=1}^{d}\sum_{j=1}^{N_{Q,k}} D_{m,ij}^K \mathbf{f}_{m,S}(\mathbf{u}_{h,i}^K, \mathbf{u}_{h,j}^K) \\
& + \sum_{\gamma\in\mathcal{E}_K}\sum_{s=1}^{N_{B,k}} R_{si}^{\gamma,K} \frac{\tau_s^\gamma}{\omega_i^K}\left(\mathbf{f}_{\mathbf{n}}(\mathbf{u}_{h,s}^{\gamma,K}) - \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{h,s}^{\gamma,K}, \mathbf{u}_{h,s}^{\gamma,\tilde{K}})\right) \\
= & -2\sum_{m=1}^{d}\sum_{j=1}^{N_{Q,k}} D_{m,ij}^K \Big(\mathbf{f}_{m,S}(\mathbf{u}_{h,i}^K, \mathbf{u}_{h,j}^K) - \mathbf{f}_{m,S}(\mathbf{u}_{h,i}^K, \mathbf{u}_{h,i}^K)\Big) \\
& + \sum_{\gamma\in\mathcal{E}_K}\sum_{s=1}^{N_{B,k}} R_{si}^{\gamma,K} \frac{\tau_s^\gamma}{\omega_i^K}\left(\mathbf{f}_{\mathbf{n}}(\mathbf{u}_{h,s}^{\gamma,K}) - \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{h,s}^{\gamma,K}, \mathbf{u}_{h,s}^{\gamma,\tilde{K}})\right).
\end{aligned}$$

By Lemma 2.5, the smoothness of $\mathbf{f}_{m,S}$, the local Lipschitz continuity and consistency of $\hat{\mathbf{f}}_{\mathbf{n}}$, the smoothness of the exact solution and Assumption 3.2, we have

$$|\partial_t \mathbf{u}_{h,i}^K| \lesssim h^{-1} \sum_{m=1}^{d} \sum_{j=1}^{N_{Q,k}} |\mathbf{u}_{h,i}^K - \mathbf{u}_{h,j}^K| + h^{-1} \sum_{\gamma\in\mathcal{E}_K} \sum_{s=1}^{N_{B,k}} |\mathbf{u}_{h,s}^{\gamma,K} - \mathbf{u}_{h,s}^{\gamma,\tilde{K}}| \lesssim 1$$

Since the exact solution is smooth, we also have $|\partial_t \mathbf{u}_i^K| \lesssim 1$, and hence $\|\partial_t H_i^K\|_{\ell^2} \lesssim 1$. It follows that

$$\left|\sum_{K\in\mathcal{T}_h} \mathcal{S}_0^K\right| \lesssim \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{N_{Q,k}} \omega_i^K |\mathbf{e}_i^K|^2 = \|\vec{\mathbf{e}}\|_h^2 \lesssim \|\vec{\mathbf{e}}\|_U^2. \tag{3.7}$$

We then estimate $\mathcal{S}_1^K$. At the surface quadrature node $\mathbf{x}_s^\gamma$, we denote $\mathbf{e}_{v,s}^{\gamma,K^\pm} := \mathbf{v}_s^\gamma - \mathbf{v}_{h,s}^{\gamma,K^\pm}$, $\mathbf{e}_s^{\gamma,K^\pm} := \mathbf{u}_s^\gamma - \mathbf{u}_{h,s}^{\gamma,K^\pm}$. Since the outward normal of K^- on γ is $\mathbf{n}_\gamma$, while the outward normal of K^+ on γ is $-\mathbf{n}_\gamma$, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \mathcal{S}_1^K = \sum_{\gamma\in\mathcal{E}_h} \sum_{s=1}^{N_{B,k}} \tau_s^\gamma \Bigg\{ &(\mathbf{e}_{v,s}^{\gamma,K^-})^T \left(\hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^-}, \mathbf{u}_{h,s}^{\gamma,K^+}) - \frac{1}{2}\big(\mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma) + \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^-})\big)\right) \\ &+ (\mathbf{e}_{v,s}^{\gamma,K^+})^T \left(\frac{1}{2}\big(\mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma) + \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^+})\big) - \hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^-}, \mathbf{u}_{h,s}^{\gamma,K^+})\right) \Bigg\}.\end{aligned}$$

We combine the two terms associated with the entropy-stable flux and then rewrite the resulting expression by adding and subtracting the entropy-conservative flux:

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \mathcal{S}_1^K = \sum_{\gamma\in\mathcal{E}_h} \sum_{s=1}^{N_{B,k}} \tau_s^\gamma \Bigg\{ &\big(\mathbf{v}_{h,s}^{\gamma,K^+} - \mathbf{v}_{h,s}^{\gamma,K^-}\big)^T \Big(\hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^-}, \mathbf{u}_{h,s}^{\gamma,K^+}) - \mathbf{f}_{\mathbf{n}_\gamma,S}(\mathbf{u}_{h,s}^{\gamma,K^-}, \mathbf{u}_{h,s}^{\gamma,K^+})\Big) \\ &+ \big(\mathbf{v}_{h,s}^{\gamma,K^+} - \mathbf{v}_{h,s}^{\gamma,K^-}\big)^T \mathbf{f}_{\mathbf{n}_\gamma,S}(\mathbf{u}_{h,s}^{\gamma,K^-}, \mathbf{u}_{h,s}^{\gamma,K^+}) - \frac{1}{2}(\mathbf{e}_{v,s}^{\gamma,K^-})^T\big(\mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma) + \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^-})\big) \\ &+ \frac{1}{2}(\mathbf{e}_{v,s}^{\gamma,K^+})^T\big(\mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma) + \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^+})\big) \Bigg\}.\end{aligned}$$

By the definitions of entropy conservative flux and entropy stable flux, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \mathcal{S}_1^K \le& \sum_{\gamma\in\mathcal{E}_h} \sum_{s=1}^{N_{B,k}} \tau_s^\gamma \Bigg\{ \widetilde{\psi}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^+}) - \widetilde{\psi}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^-}) - \\ &\frac{1}{2}(\mathbf{e}_{v,s}^{\gamma,K^-})^T\big(\mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma) + \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^-})\big) + \frac{1}{2}(\mathbf{e}_{v,s}^{\gamma,K^+})^T\big(\mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma) + \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_{h,s}^{\gamma,K^+})\big) \Bigg\} \\ =& \sum_{\gamma\in\mathcal{E}_h} \sum_{s=1}^{N_{B,k}} \tau_s^\gamma \left(G_{\gamma,s}(\mathbf{e}_s^{\gamma,K^+}) - G_{\gamma,s}(\mathbf{e}_s^{\gamma,K^-})\right) \qquad (3.8) \\ \le& \sum_{\gamma\in\mathcal{E}_h} \sum_{s=1}^{N_{B,k}} \tau_s^\gamma \left(|G_{\gamma,s}(\mathbf{e}_s^{\gamma,K^+})| + |G_{\gamma,s}(\mathbf{e}_s^{\gamma,K^-})|\right).\end{aligned}$$

Here $\widetilde{\psi}_{\mathbf{n}_\gamma}(\mathbf{u}) := \psi_{\mathbf{n}_\gamma}(\mathbf{v}(\mathbf{u}))$, and

$$G_{\gamma,s}(\boldsymbol{\mu}) := \frac{1}{2}\big(\mathbf{v}(\mathbf{u}_s^\gamma) - \mathbf{v}(\mathbf{u}_s^\gamma - \boldsymbol{\mu})\big)^T\big(\mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma) + \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma - \boldsymbol{\mu})\big) + \widetilde{\psi}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma - \boldsymbol{\mu}) - \widetilde{\psi}_{\mathbf{n}_\gamma}(\mathbf{u}_s^\gamma).$$

Denote $\boldsymbol{\eta} := \mathbf{e}_s^{\gamma,K^{\pm}}$. For the simplicity of proof, define

$$g_{\gamma,s}(z) = G_{\gamma,s}(z\boldsymbol{\eta}), \quad \mathbf{v}_{\gamma,s}(z) = \mathbf{v}(\mathbf{u}_s^{\gamma} - z\boldsymbol{\eta}),$$
$$\mathbf{f}_{\gamma,s}(z) = \mathbf{f}_{\mathbf{n}_\gamma}(\mathbf{u}_s^{\gamma} - z\boldsymbol{\eta}), \quad \tilde{\psi}_{\gamma,s}(z) = \tilde{\psi}_{\mathbf{n}_\gamma}(\mathbf{u}_s^{\gamma} - z\boldsymbol{\eta}).$$

The entropy relation gives $\tilde{\psi}'_{\gamma,s}(z) = (\mathbf{v}'_{\gamma,s}(z))^T \mathbf{f}_{\gamma,s}(z)$. Together with the smoothness of the entropy variables and physical fluxes, we have

$$g_{\gamma,s}(0) = g'_{\gamma,s}(0) = g''_{\gamma,s}(0) = 0, \quad |g''_{\gamma,s}(z)| \lesssim z|\boldsymbol{\eta}|^3.$$

Taylor's formula with integral remainder therefore gives

$$|G_{\gamma,s}(\boldsymbol{\eta})| = |g_{\gamma,s}(1)| \le \int_0^1 (1-z)|g''_{\gamma,s}(z)|dz \lesssim |\boldsymbol{\eta}|^3 \int_0^1 (1-z)z dz \lesssim |\boldsymbol{\eta}|^3$$

By Assumption 3.2 and Lemma 2.5, we derive the estimate from (3.8)

$$\sum_{K\in\mathcal{T}_h} \mathcal{S}_1^K \lesssim h \sum_{\gamma\in\mathcal{E}_h} \sum_{s=1}^{N_{B,k}} \tau_s^{\gamma} \left(|\mathbf{e}_s^{\gamma,K^-}|^2 + |\mathbf{e}_s^{\gamma,K^+}|^2 \right) \lesssim \|\vec{\mathbf{e}}\|_h^2 \lesssim \|\vec{\mathbf{e}}\|_U^2. \tag{3.9}$$

We next estimate $\mathcal{S}_2^K$. Taylor's expansion at the exact solution gives

$$\Delta \vec{\mathbf{f}}_m^K = \mathbf{J}_m^K \vec{\mathbf{e}}^K + \vec{\mathbf{r}}_m^K,$$

where

$$\mathbf{J}_m^K := \operatorname{diag}\big(\mathbf{f}'_m(\mathbf{u}_1^K), \dots, \mathbf{f}'_m(\mathbf{u}_{N_{Q,k}}^K)\big),$$

and the Taylor's remainder satisfies

$$|\mathbf{r}_{m,i}^K| \lesssim |\mathbf{e}_i^K|^2, \quad 1 \le i \le N_{Q,k}.$$

For a fixed point $\mathbf{x}_0^K \in K$, we define

$$H_0^K := U''(\mathbf{u}(\mathbf{x}_0^K)), \quad J_{m,0}^K := \mathbf{f}'_m(\mathbf{u}(\mathbf{x}_0^K)),$$
$$\mathbf{H}_0^K := I_{N_{Q,k}} \otimes H_0^K, \quad \mathbf{J}_{m,0}^K := I_{N_{Q,k}} \otimes J_{m,0}^K.$$

Then

$$\begin{aligned}\mathcal{S}_2^K =& -\sum_{m=1}^{d} (\vec{\mathbf{e}}^K)^T (\mathbf{H}^K - \mathbf{H}_0^K) \mathbf{A}_m^K (\mathbf{J}_m^K \vec{\mathbf{e}}^K + \vec{\mathbf{r}}_m^K) \\ &- \sum_{m=1}^{d} (\vec{\mathbf{e}}^K)^T \mathbf{H}_0^K \mathbf{A}_m^K ((\mathbf{J}_m^K - \mathbf{J}_{m,0}^K)\vec{\mathbf{e}}^K + \vec{\mathbf{r}}_m^K) - \sum_{m=1}^{d} (\vec{\mathbf{e}}^K)^T \mathbf{H}_0^K \mathbf{A}_m^K \mathbf{J}_{m,0}^K \vec{\mathbf{e}}^K.\end{aligned}$$

By the property of the Kronecker product, we obtain

$$\mathbf{H}_0^K \mathbf{A}_m^K \mathbf{J}_{m,0}^K = (I_{N_{Q,k}} \otimes H_0^K)(A_m^K \otimes I_n)(I_{N_{Q,k}} \otimes J_{m,0}^K) = A_m^K \otimes (H_0^K J_{m,0}^K).$$

Differentiating (2.1), we obtain

$$H(\mathbf{u})\mathbf{f}'_m(\mathbf{u}) = F''_m(\mathbf{u}) - \sum_{r=1}^{n} v_r(\mathbf{u}) f''_{m,r}(\mathbf{u}),$$

where $f''_{m,r}$ denotes the Hessian matrix of the r-th component of $\mathbf{f}_m$. Since $F''_m(\mathbf{u})$ and $f''_{m,r}(\mathbf{u})$, $r = 1, \ldots, n$ are symmetric matrices, then the $H(\mathbf{u})\mathbf{f}'_m(\mathbf{u})$ is symmetric matrix. Since A_m^K is a skew-symmetric matrix, we have

$$\sum_{m=1}^{d} (\vec{\mathbf{e}}^K)^T \mathbf{H}_0^K \mathbf{A}_m^K \mathbf{J}_{m,0}^K \vec{\mathbf{e}}^K = 0.$$

By a priori Assumption 3.2, the smoothness of the exact solution, and $\|\mathbf{A}_m^K\|_{\ell^2} \lesssim h^{d-1}$, we have

$$\sum_{K \in \mathcal{T}_h} \mathcal{S}_2^K \lesssim \sum_{K \in \mathcal{T}_h} h^d \|\vec{\mathbf{e}}^K\|_{\ell^2,K}^2 \lesssim \|\vec{\mathbf{e}}\|_h^2 \lesssim \|\vec{\mathbf{e}}\|_U^2. \tag{3.10}$$

We then estimate $\mathcal{S}_3^K$. By Assumption 3.2 and Lemma 2.8, we have

$$\|(\Delta \boldsymbol{\Phi}_m^K)_{ij}\|_{\ell^2} \lesssim h\big(|\mathbf{e}_i^K| + |\mathbf{e}_j^K|\big).$$

Thus,

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \mathcal{S}_3^K &\lesssim \sum_{K \in \mathcal{T}_h} \sum_{m=1}^{d} \sum_{i,j=1}^{N_{Q,k}} |\mathbf{e}_i^K| \|H_i^K\|_{\ell^2} |(S_m^K)_{ij}| \|(\Delta \boldsymbol{\Phi}_m^K)_{ij}\|_{\ell^2} \\ &\lesssim \sum_{K \in \mathcal{T}_h} h^d \sum_{j=1}^{N_{Q,k}} |\mathbf{e}_j^K|^2 \lesssim \sum_{K \in \mathcal{T}_h} \|\vec{\mathbf{e}}^K\|_{h,K}^2 \lesssim \|\vec{\mathbf{e}}\|_h^2 \lesssim \|\vec{\mathbf{e}}\|_U^2. \end{aligned} \tag{3.11}$$

Finally, for $\mathcal{S}_4^K$, by Lemmas 2.5 and 2.7, together with the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \mathcal{S}_4^K &\le \sum_{K \in \mathcal{T}_h} \|\vec{\mathbf{e}}^K\|_{\ell^2,K} \|\mathbf{H}^K \vec{\mathbf{T}}^K(\mathbf{u})\|_{\ell^2,K} \\ &\lesssim \left(\sum_{K \in \mathcal{T}_h} h^{-d} \|\vec{\mathbf{e}}^K\|_{h,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} h^{2k+2d} \right)^{1/2} \\ &\lesssim h^k \|\vec{\mathbf{e}}\|_h \lesssim h^k \|\vec{\mathbf{e}}\|_U \lesssim h^{2k} + \|\vec{\mathbf{e}}\|_U^2. \end{aligned} \tag{3.12}$$

Substituting the estimates (3.7), (3.9), (3.10), (3.11), and (3.12) into (3.6) yields

$$\frac{1}{2} \frac{d}{dt} \|\vec{\mathbf{e}}\|_U^2 \lesssim \|\vec{\mathbf{e}}\|_U^2 + h^{2k}.$$

Gronwall's inequality, the initial approximation, and the norm equivalence complete the proof. ∎

3.2. Discussion of the assumptions. We now discuss the two assumptions introduced at the outset of this section. Since the reconstruction space for the system is given by $\mathbf{V}_h(K) = [V_h(K)]^n$, the polynomial reconstruction assumption reduces to the corresponding scalar problem. We therefore first discuss the properties and existence of the scalar reconstruction space $V_h(K)$. We then justify the a priori L^∞ bound for the system via a standard continuity argument.

Polynomial reconstruction. Under Assumption 3.1, each nodal vector is uniquely identified with a polynomial in $V_h(K)$. This correspondence is applied in two main contexts. First, the positivity of the quadrature weights, together with the uniqueness of the polynomial reconstruction from nodal values, implies the uniform equivalence of the discrete norm to the standard $L^2(K)$ norm:

$$C_1\|w_h\|_{L^2(K)}^2 \le \|w_h\|_{h,K}^2 \le C_2\|w_h\|_{L^2(K)}^2, \quad w_h \in V_h(K),$$

where $C_1, C_2 > 0$ are independent of h. Thus, the quadrature-based norm used in our error estimate is equivalent to the usual broken L^2 norm on the reconstructed polynomial space. Second, since $V_h(K) \subset \mathcal{P}^r(K)$, where r is fixed independently of h, the standard inverse estimate yields

$$\|w_h\|_{L^\infty(K)} \le Ch_K^{-d/2}\|w_h\|_{L^2(K)} \le Ch_K^{-d/2}\|w_h\|_{h,K}, \quad w_h \in V_h(K).$$

In particular, this estimate can be applied to the reconstructed numerical solution u_h. We now propose a method to check whether the assumption holds for the given quadrature nodes.

PROPOSITION 3.4 (Polynomial reconstruction from nodal values). *Let V_k^K be the Vandermonde matrix associated with the volume quadrature nodes and $\mathcal{P}^k(K)$, as defined in Subsection 2.2. Then the following statements are equivalent:*

1. *V_k^K has full column rank.*
2. *There exists an integer $r \ge k$ and a polynomial space $V_h(K)$ satisfying*

$$\mathcal{P}^k(K) \subset V_h(K) \subset \mathcal{P}^r(K), \quad \dim V_h(K) = N_{Q,k},$$

such that the nodal evaluation map on the volume quadrature nodes is an isomorphism from $V_h(K)$ onto $\mathbb{R}^{N_{Q,k}}$.

The proof of Proposition 3.4 is given in the appendix; see Section A.3. Next, we provide three examples illustrating the assumption.

Example 3.5 (One-dimensional Gauss-Lobatto nodes). When $d = 1$, the $k+1$ Gauss-Lobatto nodes are distinct. Hence the nodal evaluation map on $\mathcal{P}^k(\widehat{K})$ is an isomorphism onto $\mathbb{R}^{k+1}$, and one may simply take $V_h(\widehat{K}) = \mathcal{P}^k(\widehat{K})$, and hence Assumption 3.1 holds with $r = k$.

For the next two examples, let

$$\widehat{K} := \{(\widehat{x}_1, \widehat{x}_2) : \widehat{x}_1 \ge 0, \quad \widehat{x}_2 \ge 0, \quad \widehat{x}_1 + \widehat{x}_2 \le 1\},$$

and let $\lambda_1, \lambda_2, \lambda_3$ be its barycentric coordinates. We denote by $\mathrm{Per}(a, b, c)$ the set of all distinct permutations of (a, b, c).

Example 3.6 (Triangular quadrature nodes for $k = 1$). When $k = 1$, the six quadrature nodes constructed in [6] have barycentric coordinates

$$\hat{\mathcal{X}}_1 = \mathrm{Per}\,(0, a, b)\,, \quad a = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad b = \frac{1}{2} + \frac{\sqrt{3}}{6}.$$

We order the nodes as

$$\widehat{\mathbf{x}}_1 = (0, a, b),\ \widehat{\mathbf{x}}_2 = (0, b, a),\ \widehat{\mathbf{x}}_3 = (a, 0, b),\ \widehat{\mathbf{x}}_4 = (b, 0, a),\ \widehat{\mathbf{x}}_5 = (a, b, 0),\ \widehat{\mathbf{x}}_6 = (b, a, 0).$$

Following the standard construction in classic finite element methods, we can construct the Lagrange basis as

$$
\begin{aligned}
L_1 &= \frac{1}{2} - \lambda_1 - 3\sqrt{3}\,\lambda_2\lambda_3(\lambda_2-\lambda_3) - 3\lambda_1\lambda_3(\lambda_3-\lambda_1) - 3\lambda_1\lambda_2(\lambda_2-\lambda_1),\\
L_2 &= \frac{1}{2} - \lambda_1 - 3\sqrt{3}\,\lambda_2\lambda_3(\lambda_3-\lambda_2) - 3\lambda_1\lambda_3(\lambda_3-\lambda_1) - 3\lambda_1\lambda_2(\lambda_2-\lambda_1),\\
L_3 &= \frac{1}{2} - \lambda_2 - 3\sqrt{3}\,\lambda_1\lambda_3(\lambda_1-\lambda_3) - 3\lambda_2\lambda_3(\lambda_3-\lambda_2) - 3\lambda_1\lambda_2(\lambda_1-\lambda_2),\\
L_4 &= \frac{1}{2} - \lambda_2 - 3\sqrt{3}\,\lambda_1\lambda_3(\lambda_3-\lambda_1) - 3\lambda_1\lambda_2(\lambda_1-\lambda_2) - 3\lambda_2\lambda_3(\lambda_3-\lambda_2),\\
L_5 &= \frac{1}{2} - \lambda_3 - 3\sqrt{3}\,\lambda_1\lambda_2(\lambda_1-\lambda_2) - 3\lambda_2\lambda_3(\lambda_2-\lambda_3) - 3\lambda_1\lambda_3(\lambda_1-\lambda_3),\\
L_6 &= \frac{1}{2} - \lambda_3 - 3\sqrt{3}\,\lambda_1\lambda_2(\lambda_2-\lambda_1) - 3\lambda_1\lambda_3(\lambda_1-\lambda_3) - 3\lambda_2\lambda_3(\lambda_2-\lambda_3).
\end{aligned}
$$

Clearly $V_h(\widehat{K}) := \operatorname{span}\{L_i\}_{i=1}^6$ satisfies our assumption. It is worth noting that one cannot simply take $V_h(\widehat{K}) = \mathcal{P}^2(\widehat{K})$, even though $\dim \mathcal{P}^2(\widehat{K}) = 6$. Indeed, the nonzero polynomial

$$
q := \frac{1}{6} - \big(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\big)
$$

vanishes at all six nodes in $\widehat{\mathcal{X}}_1$.

Example 3.7 (Triangular quadrature nodes for $k=2$). When $k=2$, the ten quadrature nodes constructed in [6] have barycentric coordinates

$$
\widehat{\mathcal{X}}_2 = \left\{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right\} \cup \operatorname{Per}\left(0,\frac{1}{2},\frac{1}{2}\right) \cup \operatorname{Per}\left(0,\frac{1}{2}-\frac{\sqrt{15}}{10},\frac{1}{2}+\frac{\sqrt{15}}{10}\right).
$$

Since $N_{Q,2} = 10 = \dim \mathcal{P}^3(\widehat{K})$, one may take $V_h(\widehat{K}) := \mathcal{P}^3(\widehat{K})$. It is easy to check

$$
\mathcal{P}^2(\widehat{K}) \subset V_h(\widehat{K}) = \mathcal{P}^3(\widehat{K}),
$$

and the nodal evaluation map is an isomorphism from $V_h(\widehat{K})$ onto $\mathbb{R}^{10}$.

Remark 3.8. For higher-order triangular quadrature nodes (e.g., $k=3,4$), as shown in [6], we can still numerically construct $V_h(\hat{K})$ that satisfies Assumption 3.1. Due to space limitations, we omit the construction details.

A priori assumption. We next justify Assumption 3.2 by a standard continuity argument. Let

$$
T^* := \sup\left\{t \in (0,T] : \|\mathbf{u}(s) - \mathbf{u}_h(s)\|_{L^\infty} \le h, \quad 0 \le s \le t\right\}.
$$

On the interval $[0,T^*]$, Assumption 3.2 holds, and hence Theorem 3.3 gives

$$
\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_h \lesssim h^k, \quad 0 \le t \le T^*.
$$

Assumption 3.1 allows us to use the standard interpolation estimate and the inverse estimate to obtain

$$
\begin{aligned}
\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{L^\infty} &\le \|\mathbf{u}(t) - \mathcal{I}_h\mathbf{u}(t)\|_{L^\infty} + \|\mathcal{I}_h\mathbf{u}(t) - \mathbf{u}_h(t)\|_{L^\infty}\\
&\lesssim h^{k+1} + h^{-d/2}\|\mathcal{I}_h\mathbf{u}(t) - \mathbf{u}_h(t)\|_h \lesssim h^{k+1} + h^{k-d/2},
\end{aligned}
$$

where $\mathcal{I}_h\mathbf{u}|_K \in \mathbf{V}_h(K)$ interpolates u componentwise at $\left\{\mathbf{x}_i^K\right\}_{i=1}^{N_{Q,k}}$ for each $K \in \mathcal{T}_h$. If $k - \frac{d}{2} > 1$, then for sufficiently small h we have

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{L^\infty} \leq \frac{1}{2}h < h, \quad 0 \leq t \leq T^*.$$

By the continuity of $\mathbf{u}_h(t)$ in time, this improved estimate implies $T^* = T$.

3.3. Extension to the ESOFDG scheme. In [26], the authors proposed the ESOFDG scheme to control spurious oscillations by adding a damping term to the original ESDG formulation. In this subsection, we derive an error estimate for the ESOFDG scheme, which reads

$$\begin{aligned}\mathbf{M}^K \frac{d\vec{\mathbf{u}}_h^K}{dt} &+ 2\sum_{m=1}^{d} \left(\mathbf{S}_m^K \circ \mathbf{F}_{m,S}(\vec{\mathbf{u}}_h^K, \vec{\mathbf{u}}_h^K)\right)\vec{\mathbf{1}}^K \\ &= \sum_{\gamma\in\mathcal{E}_K} (\mathbf{R}^{\gamma,K})^T\mathbf{B}^\gamma \left(\vec{\mathbf{f}}_{\mathbf{n}}^{\gamma,K} - \vec{\hat{\mathbf{f}}}_{\mathbf{n}}^{\gamma,K}\right) - \sigma^K(\mathbf{u}_h)\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{u}}_h^K.\end{aligned} \tag{3.13}$$

The damping coefficient $\sigma^K(\mathbf{u}_h)$ is defined as follows:

$$\sigma^K(\mathbf{u}_h) = \max_{1\leq s\leq n}\left(\sum_{l=0}^{1}\frac{h_K^{2l}}{l+1}\sum_{|\alpha|=l}\frac{1}{N_\Gamma}\sum_{\nu\in\partial K}[\![(\mathbf{L}\partial^\alpha\mathbf{u}_h)_s|_\nu]\!]^2\right)^{\frac{1}{2}}, \tag{3.14}$$

where N_Γ stands for the number of faces of ∂K, and $[\![\mathbf{w}|_\nu]\!]$ denotes the jump of a function $\mathbf{w}$ on the vertex $\nu \in \partial K$. The matrix $\mathbf{L}$ comes from the characteristic decomposition such that $\sum_{i=1}^d n_i\mathbf{f}_i'(\overline{\mathbf{u}}_h) = \mathbf{L}^{-1}\mathbf{\Lambda}\mathbf{L}$, where $\mathbf{n} = (n_1, \cdots, n_d)^T$ denotes a unit outward normal. $\overline{\mathbf{u}}_h$ denotes some average of $\mathbf{u}_h$ at ν, [26]. Hence, for each individual jump, the same matrix is applied to the two traces. Moreover, a priori Assumption 3.2 implies that $\mathbf{L}$ is uniformly bounded:

$$\|\mathbf{L}\|_{\ell^2} \leq C_L,$$

where C_L is independent of h, the element, and the face. The following lemma gives the estimate required for the additional damping term.

LEMMA 3.9. *Let $\sigma^K(\mathbf{u}_h)$ be defined in (3.14), we have*

$$\sigma^K(\mathbf{u}_h) \lesssim h_K^{k+1} + h_K^{-d/2}\sum_{\tilde{K}\in\Omega_K}\|\vec{\mathbf{e}}^{\tilde{K}}\|_{h,\tilde{K}}.$$

Under Assumption 3.2, this further implies

$$\sigma^K(\mathbf{u}_h) \lesssim h_K.$$

The proof of Lemma 3.9 is given in the appendix; see Section A.4. Now we can obtain the error estimate of the ESOFDG scheme for hyperbolic systems in the following Theorem.

THEOREM 3.10. *For the symmetrizable system of conservation laws (1.1), assume that the solution $\mathbf{u}$ and the flux function $\mathbf{f}_m$ are sufficiently smooth with bounded derivatives. Let $\mathbf{u}_h$ be the numerical solution of the ESOFDG scheme (3.13) and U be a sufficiently smooth uniformly convex entropy function. Assume that the entropy* *conservative fluxes* $\mathbf{f}_{m,S}$ *are sufficiently smooth, and the entropy stable fluxes* $\hat{\mathbf{f}}_{\mathbf{n}}$ *are locally Lipschitz continuous with respect to both arguments. Suppose that Assumption 3.1 and Assumption 3.2 holds, and*

$$\|\mathbf{u}-\mathbf{u}_h\|_h \lesssim h^k, \quad t=0.$$

Then the solution of the ESOFDG scheme (3.13) *satisfies*

$$\|\mathbf{u}-\mathbf{u}_h\|_h \lesssim h^k, \quad 0\le t\le T.$$

Proof. Notice that

$$\sigma^K(\mathbf{u})=0$$

for the smooth solution $\mathbf{u}$. Repeating the derivation in the proof of Theorem 3.3, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\vec{\mathbf{e}}\|_U^2 = \sum_{K\in\mathcal{T}_h}\mathcal{S}_0^K + \sum_{K\in\mathcal{T}_h}\mathcal{S}_1^K + \sum_{K\in\mathcal{T}_h}\mathcal{S}_2^K + \sum_{K\in\mathcal{T}_h}\mathcal{S}_3^K + \sum_{K\in\mathcal{T}_h}\mathcal{S}_4^K + \sum_{K\in\mathcal{T}_h}\mathcal{S}_{OF}^K,$$

where

$$\sum_{K\in\mathcal{T}_h}\mathcal{S}_{OF}^K := \sum_{K\in\mathcal{T}_h}\sigma^K(\mathbf{u}_h)(\vec{\mathbf{e}}^K)^T\mathbf{H}^K\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{u}}_h^K, \tag{3.15}$$

and $\mathcal{S}_i^K$, $0\le i\le 4$, are the same as those in (3.6). Thus, we only need to estimate $\mathcal{S}_{OF}^K$. We rewrite (3.15) as

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\mathcal{S}_{OF}^K &= -\sum_{K\in\mathcal{T}_h}\sigma^K(\mathbf{u}_h)(\vec{\mathbf{e}}^K)^T\mathbf{H}^K\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{e}}^K + \sum_{K\in\mathcal{T}_h}\sigma^K(\mathbf{u}_h)(\vec{\mathbf{e}}^K)^T\mathbf{H}^K\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{u}}^K\\ &=: \sum_{K\in\mathcal{T}_h}\mathcal{O}_1^K + \sum_{K\in\mathcal{T}_h}\mathcal{O}_2^K.\end{aligned}$$

We first estimate $\mathcal{O}_1^K$. By Lemmas 2.5 and 3.9, together with the uniform boundedness of $\mathbf{H}^K$, we have

$$\begin{aligned}\mathcal{O}_1^K &= -\sigma^K(\mathbf{u}_h)(\vec{\mathbf{e}}^K)^T\mathbf{H}^K\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{e}}^K \le \sigma^K(\mathbf{u}_h)\left|(\vec{\mathbf{e}}^K)^T\mathbf{H}^K\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{e}}^K\right|\\ &\le \sigma^K(\mathbf{u}_h)\|\vec{\mathbf{e}}^K\|_{\ell^2,K}\|\mathbf{H}^K\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{e}}^K\|_{\ell^2,K} \lesssim h^{d+1}\|\vec{\mathbf{e}}^K\|_{\ell^2,K}^2 \lesssim \|\vec{\mathbf{e}}^K\|_{h,K}^2.\end{aligned}$$

Therefore,

$$\sum_{K\in\mathcal{T}_h}\mathcal{O}_1^K \lesssim \|\vec{\mathbf{e}}\|_h^2 \lesssim \|\vec{\mathbf{e}}\|_U^2.$$

For $\mathcal{O}_2^K$, the construction $\mathbf{Q}^K = Q^K\otimes I_n$ and the smoothness of $\mathbf{u}$ imply

$$\begin{aligned}\left|(\mathbf{Q}^K\vec{\mathbf{u}}^K)_j\right| &= \left|\mathbf{u}_j^K - \frac{\sum_{i=1}^{N_{Q,k}}\omega_i^K\mathbf{u}_i^K}{\sum_{i=1}^{N_{Q,k}}\omega_i^K}\right| = \left|\frac{\sum_{i=1}^{N_{Q,k}}\omega_i^K(\mathbf{u}_j^K-\mathbf{u}_i^K)}{\sum_{i=1}^{N_{Q,k}}\omega_i^K}\right|\\ &\le \frac{\sum_{i=1}^{N_{Q,k}}\omega_i^K|\mathbf{u}_j^K-\mathbf{u}_i^K|}{\sum_{i=1}^{N_{Q,k}}\omega_i^K} \lesssim h.\end{aligned}$$

Consequently,

$$\sum_{K\in\mathcal{T}_h}\mathcal{O}_2^K \le \sum_{K\in\mathcal{T}_h}\left|\sigma^K(\mathbf{u}_h)(\vec{\mathbf{e}}^K)^T\mathbf{H}^K\mathbf{M}^K\mathbf{Q}^K\vec{\mathbf{u}}^K\right|$$

$$
\begin{aligned}
&\lesssim \sum_{K\in\mathcal{T}_h} \sigma^K(\mathbf{u}_h) h^d \|\vec{\mathbf{e}}^K\|_{\ell^2,K} \|\mathbf{Q}^K \vec{\mathbf{u}}^K\|_{\ell^2,K} \\
&\lesssim \sum_{K\in\mathcal{T}_h} \sigma^K(\mathbf{u}_h) h^{1+d/2} \|\vec{\mathbf{e}}^K\|_{h,K} \\
&\lesssim \sum_{K\in\mathcal{T}_h} \left(h^{k+2+d/2} \|\vec{\mathbf{e}}^K\|_{h,K} + h \|\vec{\mathbf{e}}^K\|_{h,K} \sum_{\widetilde{K}\in\Omega_K} \|\vec{\mathbf{e}}^{\widetilde{K}}\|_{h,\widetilde{K}} \right) \\
&\lesssim h^{2k+4} + \|\vec{\mathbf{e}}\|_h^2.
\end{aligned}
$$

It follows that

$$
\sum_{K\in\mathcal{T}_h} \mathcal{S}_{OF}^K \lesssim \|\vec{\mathbf{e}}\|_U^2 + h^{2k+4}.
$$

Combining this estimate with the estimates of $\mathcal{S}_i^K$, $0 \le i \le 4$, gives

$$
\frac{1}{2}\frac{d}{dt}\|\vec{\mathbf{e}}\|_U^2 \lesssim \|\vec{\mathbf{e}}\|_U^2 + h^{2k},
$$

Then Gronwall's inequality and the norm equivalence complete the proof. ∎

4. Numerical experiments.

In this section, we show some numerical results to validate our error estimate in Section 3. We use a nonuniform mesh, which is 20% random perturbation of the uniform mesh for one-dimensional tests. Two-dimensional tests are performed on unstructured triangular meshes generated by Gmsh [17]. The initial data are specified at the volume quadrature nodes. For the numerical experiments, we use SBP discretizations associated with polynomial degrees $k = 1, 2, 3$ and perform a comparison between the ESDG and ESOFDG schemes. Temporal discretization is achieved by the classical third-order SSP Runge–Kutta method [20]. We set the time step τ size as

$$
\tau = \frac{\mathrm{CFL}}{\sigma_0 + \lambda_0} h^{\max(1, \frac{k+1}{3})},
$$

for one-dimensional problems, and the condition for two-dimensional problems is similar. Here, $\sigma_0 = \max_K \sigma^K(\mathbf{u}_h)$ defined in (3.14) and we set $\sigma_0 = 0$ for the ESDG method, λ_0 is the maximum of the spectral radius of the Jacobian $\mathbf{f}'(\mathbf{u})$ over each element. In order to mitigate the contribution of temporal discretization errors to the total error, we adopt a fixed CFL number of 0.1 for the numerical experiments described subsequently. For scalar conservation laws, the entropy-stable flux is taken to be the local Lax–Friedrichs flux. In the system case, we solve the compressible Euler equations; the entropy-conservative flux is that recommended by Chandrashekar [5], and the entropy-stable flux is realized via the local Lax–Friedrichs flux, with the extreme wave speeds suitably approximated [6].

4.1. One-dimensional problem.

Example 4.1. We first consider the Burgers equation $u_t + (\frac{u^2}{2})_x = 0$ with periodic boundary condition. The initial condition is $u_0(x) = e^{\cos x}\sin x + (\sin x)^2, x \in (0, 2\pi)$. We adopt the entropy function $U(u) = u^2 + e^u$ and compute the solution at $T = 0.4$.

The results of ESDG and ESOFDG are listed in Table 1. It is reported that the convergence rates are below the optimal order and vary with the polynomial degree k, which is better than our estimate. Such a phenomenon is broadly observed in experiments [6, 7, 26].

TABLE 1

Errors and orders of the ESDG and ESOFDG schemes for the one-dimensional Burgers equation at the final time $T = 0.4$.

	ESDG						ESOFDG					
	$k=1$		$k=2$		$k=3$		$k=1$		$k=2$		$k=3$	
N	$\|u-u_h\|_h$	Order	$\|u-u_h\|_h$	Order	$\|u-u_h\|_h$	Order	$\|u-u_h\|_h$	Order	$\|u-u_h\|_h$	Order	$\|u-u_h\|_h$	Order
16	2.81E−01	–	4.77E−02	–	1.62E−02	–	2.85E−01	–	4.82E−02	–	1.74E−02	–
32	1.01E−01	1.480	1.10E−02	2.120	1.62E−03	3.322	1.01E−01	1.503	1.12E−02	2.106	1.65E−03	3.402
64	3.85E−02	1.386	2.03E−03	2.432	1.74E−04	3.213	3.86E−02	1.382	2.02E−03	2.473	1.76E−04	3.228
128	1.27E−02	1.595	4.05E−04	2.328	1.66E−05	3.390	1.27E−02	1.598	4.04E−04	2.319	1.67E−05	3.401
256	4.00E−03	1.672	6.56E−05	2.625	1.10E−06	3.923	3.99E−03	1.673	6.55E−05	2.624	1.10E−06	3.925
512	1.25E−03	1.678	1.09E−05	2.595	7.88E−08	3.798	1.25E−03	1.678	1.09E−05	2.594	7.88E−08	3.798

4.2. Two-dimensional problem.

Example 4.2. We now consider the vortex evolution problem for two-dimensional Euler equations [6]. The computational domain is $[0, 20]^2$ and the initial condition is given by

$$
\begin{aligned}
&w_1(\mathbf{x}, 0) = 1 - (x_2 - y_2)\phi(r), \quad w_2(\mathbf{x}, 0) = 1 + (x_1 - y_1)\phi(r), \\
&\frac{p(\mathbf{x}, 0)}{\rho(\mathbf{x}, 0)} = 1 - \frac{\gamma - 1}{2\gamma}\phi(r)^2, \quad p(\mathbf{x}, 0) = \rho(\mathbf{x}, 0)^\gamma.
\end{aligned}
$$

Here $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. We take $(y_1, y_2) = (10, 10)$ and $\phi(r) = \frac{5}{2\pi} e^{\frac{1-r^2}{2}}$. The exact solution is $\mathbf{u}(x_1, x_2, t) = \mathbf{u}(x_1 - t, x_2 - t, 0)$. We use the exact solution to prescribe boundary conditions and list the computational results at $T = 0.1$.

As reported in Table 2, the observed convergence rates remain slightly below the optimal threshold $k + 1$. Furthermore, the ESOFDG scheme achieves rates that are almost indistinguishable from those of the ESDG scheme. This verifies that the inclusion of damping terms does not sacrifice accuracy when applied to the Euler systems.

TABLE 2

Errors and orders of the ESDG and ESOFDG schemes for the two-dimensional Euler equation at the final time $T = 0.1$.

	ESDG						ESOFDG					
	$k=1$		$k=2$		$k=3$		$k=1$		$k=2$		$k=3$	
$20/h$	$\|\mathbf{u}-\mathbf{u}_h\|_h$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_h$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_h$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_h$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_h$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_h$	Order
16	3.32E−01	–	8.98E−02	–	2.60E−02	–	3.31E−01	–	9.00E−02	–	2.65E−02	–
32	1.54E−01	1.111	2.30E−02	1.968	2.42E−03	3.422	1.53E−01	1.110	2.30E−02	1.969	2.43E−03	3.446
64	6.45E−02	1.252	4.11E−03	2.482	1.93E−04	3.651	6.45E−02	1.249	4.11E−03	2.484	1.93E−04	3.655
128	2.24E−02	1.524	6.68E−04	2.621	1.49E−05	3.696	2.24E−02	1.523	6.68E−04	2.621	1.49E−05	3.697
256	7.66E−03	1.550	1.10E−04	2.602	1.28E−06	3.538	7.66E−03	1.550	1.10E−04	2.603	1.28E−06	3.537
512	2.46E−03	1.640	1.82E−05	2.599	1.16E−07	3.469	2.46E−03	1.640	1.82E−05	2.599	1.16E−07	3.469

5. Concluding Remarks. A rigorous error analysis is established for the semi-discrete entropy-stable discontinuous Galerkin method applied to hyperbolic conservation laws. Through a finite-difference-type consistency–stability framework, k-th order convergence is proved in a quadrature-based norm equivalent to the broken L^2 norm on the finite-dimensional reconstruction space. The analysis extends to the entropy-stable oscillation-free DG method, where the additional damping terms are shown not to degrade the convergence order. Although numerical experiments indicate observed rates up to half an order above the theoretical bound, these results furnish a rigorous convergence theory for this class of entropy-stable schemes. It remains open whether the k-th order bound is sharp for certain numerical examples, and whether sharper $(k+\frac{1}{2})$-th order estimates can be derived within the variational framework of the finite element method.

A. Some technical proofs of Lemmas. In this appendix, we provide some technical proofs of lemmas in the error analysis.

A.1. Proof of Lemma 2.5. We first prove the estimates for scalar nodal data. Let

$$F_K(\widehat{\mathbf{x}}) = A_K\widehat{\mathbf{x}} + \mathbf{b}_K,$$

be the affine mapping from the reference simplex $\widehat{K}$ onto K. By shape regularity,

$$\|A_K\| \lesssim h_K, \quad \|A_K^{-1}\| \lesssim h_K^{-1}, \quad |\det A_K| \sim h_K^d.$$

Then, by the construction of SBP operators on the physical element (2.7), it follows directly $\omega_j^K \sim h_K^d$, $\tau_s^\gamma \sim h_K^{d-1}$, and

$$\|D_m^K\|_{\ell^q} \lesssim h_K^{-1}, \quad \|M^K\|_{\ell^q} \lesssim h_K^d, \quad \|R^{\gamma,K}\|_{\ell^q} \lesssim 1, \quad \|Q^K\|_{\ell^q} \lesssim 1.$$

Consequently, for every scalar volume nodal vector $\vec{w}^{\,K}$,

$$\|\vec{w}^{\,K}\|_{h,K}^2 = \sum_{j=1}^{N_{Q,k}} \omega_j^K |w_j^K|^2 \sim h_K^d \sum_{j=1}^{N_{Q,k}} |w_j^K|^2 = h_K^d \|\vec{w}^{\,K}\|_{\ell^2,K}^2.$$

Moreover, by the orthogonal relation (2.6), we have

$$\|\vec{w}^{\,K}\|_{h,K}^2 = \|Q^K\vec{w}^{\,K}\|_{h,K}^2 + \|(I-Q^K)\vec{w}^{\,K}\|_{h,K}^2 \geq \|Q^K\vec{w}^{\,K}\|_{h,K}^2.$$

Since the nodal dimensions and n are fixed, the scalar operator estimates therefore imply the corresponding estimates for vector-valued operators.

A.2. Proof of Lemma 2.8. For notational simplicity, we fix m and write $\mathbf{\Phi} = \mathbf{\Phi}_m$. Let $\mathbf{x} = \frac{\mathbf{a}+\mathbf{b}}{2}, \mathbf{y} = \frac{\mathbf{a}-\mathbf{b}}{2}$, and define $\mathbf{G}(\mathbf{x},\mathbf{y}) := \mathbf{\Phi}(\mathbf{x}+\mathbf{y},\mathbf{x}-\mathbf{y})$. Since $\mathbf{f}_{m,S}$ is consistent, we have

$$\mathbf{G}(\mathbf{x},0) = \mathbf{0}, \quad D_{\mathbf{x}}\mathbf{G}(\mathbf{x},0) = \mathbf{0}.$$

And $\mathbf{\Phi}$ is symmetric about $\mathbf{a}$ and $\mathbf{b}$, then

$$D_{\mathbf{y}}\mathbf{G}(\mathbf{x},0) = \mathbf{0}.$$

By the fundamental theorem of calculus,

$$D_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{y}) = \int_0^1 \frac{d}{dt} D_{\mathbf{x}}\mathbf{G}(\mathbf{x},t\mathbf{y})\,dt = \int_0^1 D_{\mathbf{y}}D_{\mathbf{x}}\mathbf{G}(\mathbf{x},t\mathbf{y})[\mathbf{y}]\,dt,$$

where $D_{\mathbf{y}}D_{\mathbf{x}}\mathbf{G}(\mathbf{x},t\mathbf{y})[\mathbf{y}] \in \mathbb{R}^{n\times n}$, the (i,j)-th entry is

$$(D_{\mathbf{y}}D_{\mathbf{x}}\mathbf{G}(\mathbf{x},t\mathbf{y})[\mathbf{y}])_{ij} = \sum_{l=1}^{n} \frac{\partial^2(\mathbf{G})_i}{\partial y_l \partial x_j}(\mathbf{x},t\mathbf{y})y_l.$$

Since $\mathbf{\Phi} \in C^3(\mathbb{R}^n \times \mathbb{R}^n)$, the second derivatives of $\mathbf{G}$ are uniformly bounded on the corresponding compact set S. We obtain

$$\|D_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{y})\| \lesssim |\mathbf{y}|.$$

Similarly, we also have

$$\|D_{\mathbf{y}}\mathbf{G}(\mathbf{x},\mathbf{y})\| \lesssim |\mathbf{y}|.$$

Hence,

$$\|D_{\mathbf{a}}\boldsymbol{\Phi}(\mathbf{a},\mathbf{b})\| + \|D_{\mathbf{b}}\boldsymbol{\Phi}(\mathbf{a},\mathbf{b})\| \lesssim |\mathbf{a}-\mathbf{b}|.$$

Finally, since $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in S$ and satisfy

$$|\mathbf{a}-\mathbf{b}| + |\mathbf{c}-\mathbf{d}| \lesssim h,$$

and

$$\begin{aligned}\boldsymbol{\Phi}(\mathbf{a},\mathbf{b}) - \boldsymbol{\Phi}(\mathbf{c},\mathbf{d}) = \int_0^1 \Big(& D_{\mathbf{a}}\boldsymbol{\Phi}(\mathbf{c}+t(\mathbf{a}-\mathbf{c}), \mathbf{d}+t(\mathbf{b}-\mathbf{d}))[\mathbf{a}-\mathbf{c}] \\ & + D_{\mathbf{b}}\boldsymbol{\Phi}(\mathbf{c}+t(\mathbf{a}-\mathbf{c}), \mathbf{d}+t(\mathbf{b}-\mathbf{d}))[\mathbf{b}-\mathbf{d}]\Big)\, dt,\end{aligned}$$

we have

$$|\boldsymbol{\Phi}(\mathbf{a},\mathbf{b}) - \boldsymbol{\Phi}(\mathbf{c},\mathbf{d})| \lesssim h\left(|\mathbf{a}-\mathbf{c}| + |\mathbf{b}-\mathbf{d}|\right).$$

A.3. Proof of Proposition 3.4. We first prove the necessity. Suppose that such a space $V_h(K)$ exists. Since

$$\mathcal{P}^k(K) \subset V_h(K),$$

and

$$\mathcal{N}_K : V_h(K) \to \mathbb{R}^{N_{Q,k}},$$

is an isomorphism, the restriction

$$\mathcal{N}_K|_{\mathcal{P}^k(K)} : \mathcal{P}^k(K) \to \mathbb{R}^{N_{Q,k}},$$

is injective. Hence, the corresponding Vandermonde matrix V_k^K has full column rank.

We now prove the sufficiency. Assume that V_k^K has full column rank. Then

$$\mathcal{N}_K : \mathcal{P}^k(K) \to U := \mathcal{N}_K(\mathcal{P}^k(K)),$$

is an isomorphism. Choose a complementary subspace W such that $\mathbb{R}^{N_{Q,k}} = U \oplus W$. Since the quadrature nodes are pairwise distinct, there exists a polynomial interpolation operator

$$\mathcal{I}_K : \mathbb{R}^{N_{Q,k}} \to \mathcal{P}^{N_{Q,k}-1}(K),$$

such that $\mathcal{N}_K\mathcal{I}_K = I_{\mathbb{R}^{N_{Q,k}}}$. Indeed, one may define

$$\mathcal{I}_K\vec{a} := \sum_{i=1}^{N_{Q,k}} a_i L_i, \quad \forall \vec{a} = (a_1, \cdots, a_{N_{Q,k}})^T \in \mathbb{R}^{N_{Q,k}},$$

where

$$L_i(\mathbf{x}) := \prod_{j\neq i} \frac{(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}-\mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2}.$$

Denote

$$V_h(K) := \mathcal{P}^k(K) + \mathcal{I}_K(W).$$

We claim that the sum is direct. If $p = \mathcal{I}_K\vec{w}$ for some $p \in \mathcal{P}^k(K)$ and $\vec{w} \in W$, then

$$\mathcal{N}_K p = \mathcal{N}_K\mathcal{I}_K\vec{w} = \vec{w} \in U \cap W,$$

and hence $\vec{w} = 0$ and $p = 0$. Moreover, $\mathcal{N}_K$ maps $\mathcal{P}^k(K)$ isomorphically onto U and $\mathcal{I}_K(W)$ isomorphically onto W. Therefore,

$$\mathcal{N}_K : V_h(K) \to U \oplus W = \mathbb{R}^{N_{Q,k}},$$

is an isomorphism. Finally, by the construction above, we have

$$\mathcal{P}^k(K) \subset V_h(K) \subset \mathcal{P}^r(K), \quad r := \max\{k, N_{Q,k} - 1\}.$$

A.4. Proof of Lemma 3.9. We first point out that, for any vector-valued piecewise polynomial $\mathbf{v}_h$ and each pairwise jump, the assumptions on the characteristic matrix give

$$|[\![(\mathbf{L}\partial^\alpha \mathbf{v}_h)_s|_\nu]\!]| = |(\mathbf{L}[\![\partial^\alpha \mathbf{v}_h|_\nu]\!])_s| \le \|\mathbf{L}\|_{\ell^2} |[\![\partial^\alpha \mathbf{v}_h|_\nu]\!]| \lesssim |[\![\partial^\alpha \mathbf{v}_h|_\nu]\!]| \,.$$

Thus, the characteristic transformation does not affect the order of the jump estimates. Let $\mathcal{I}_h \mathbf{u}$ be the componentwise nodal interpolant of $\mathbf{u}$ in the reconstruction space $\mathbf{V}_h(K)$, and set

$$\mathbf{w}_h := \mathcal{I}_h \mathbf{u}, \qquad \mathbf{z}_h := \mathbf{u}_h - \mathbf{w}_h.$$

It is easy to check that when $\mathbf{L}$ is fixed, the triangle inequality holds

$$\sigma^K(\mathbf{u}_h) \le \sigma^K(\mathbf{z}_h) + \sigma^K(\mathbf{w}_h).$$

We first estimate $\sigma^K(\mathbf{w}_h)$. For $|\alpha| = l$, $l = 0, 1$, the smoothness of the exact solution gives

$$[\![\partial^\alpha \mathbf{u}|_\nu]\!] = \mathbf{0}.$$

Therefore, using the uniform boundedness of $\mathbf{L}$ and the standard componentwise interpolation estimate, we obtain

$$h^l \,|[\![(\mathbf{L}\partial^\alpha \mathbf{w}_h)_s|_\nu]\!]| \lesssim h^l \,|[\![\partial^\alpha(\mathbf{w}_h - \mathbf{u})|_\nu]\!]| \lesssim h^l \sum_{\widetilde{K} \in \Omega_K} \|\partial^\alpha(\mathbf{w}_h - \mathbf{u})\|_{L^\infty(\widetilde{K})} \lesssim h^{k+1}.$$

Since the numbers of faces are fixed, the definition of the damping coefficient gives

$$\sigma^K(\mathbf{w}_h) \lesssim h^{k+1}.$$

For $\sigma^K(\mathbf{z}_h)$, the uniform boundedness of $\mathbf{L}$, the inverse estimate, and the norm equivalence give

$$\begin{aligned} h^l \,|[\![(\mathbf{L}\partial^\alpha \mathbf{z}_h)_s|_\nu]\!]| &\lesssim h^l \,|[\![\partial^\alpha \mathbf{z}_h|_\nu]\!]| \lesssim h^l \sum_{\widetilde{K} \in \Omega_K} \|\partial^\alpha \mathbf{z}_h\|_{L^\infty(\widetilde{K})} \lesssim h^{-d/2} \sum_{\widetilde{K} \in \Omega_K} \|\mathbf{z}_h\|_{L^2(\widetilde{K})} \\ &\lesssim h^{-d/2} \sum_{\widetilde{K} \in \Omega_K} \|\vec{\mathbf{z}}^{\widetilde{K}}\|_{h,\widetilde{K}}. \end{aligned}$$

Moreover, we have $\vec{\mathbf{z}}^{\widetilde{K}} = -\vec{\mathbf{e}}^{\widetilde{K}}$ since $\mathcal{I}_h \mathbf{u}$ interpolates $\mathbf{u}$ at the volume quadrature nodes. Therefore

$$\sigma^K(\mathbf{z}_h) \lesssim h^{-d/2} \sum_{\widetilde{K} \in \Omega_K} \|\vec{\mathbf{e}}^{\widetilde{K}}\|_{h,\widetilde{K}}.$$

Combining the preceding estimates yields

$$\sigma^K(\mathbf{u}_h) \lesssim h^{k+1} + h^{-d/2} \sum_{\widetilde{K} \in \Omega_K} \|\vec{\mathbf{e}}^{\widetilde{K}}\|_{h,\widetilde{K}}.$$

Finally, under Assumption 3.2,

$$\|\vec{\mathbf{e}}^{\widetilde{K}}\|_{h,\widetilde{K}} = \left(\sum_{i=1}^{N_{Q,k}} \omega_i^{\widetilde{K}} |\mathbf{e}_i^{\widetilde{K}}|^2\right)^{1/2} \lesssim h^{d/2}\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(\widetilde{K})} \lesssim h^{1+d/2}.$$

Since the number of elements in Ω_K is uniformly bounded, we conclude that

$$\sigma^K(\mathbf{u}_h) \lesssim h^{k+1} + h \lesssim h.$$

This completes the proof.

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