PROOF OF AN INVERSE INEQUALITY FOR POLYNOMIALS

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Denote all polynomials of order less than or equal to N by $\mathbb{P}_N$ over the interval $[-1,1]$.

Proposition. For all $v \in \mathbb{P}_N$ with $v(1) = 0$, there exists a constant C independent of N such that

$$\int_{-1}^{1} (1+x) v_x^2 \, dx \le CN^2 \int_{-1}^{1} v^2 \, dx. \tag{0.1}$$

Proof. Note that $v = \sum_{n=0}^{N-1} a_n (L_n - L_{n+1})$ and $v(1) = 0$ since $L_n(1) = L_{n+1} = 1$.

$$\begin{aligned}
\int_{-1}^{1} v^2 \, dx &= \int_{-1}^{1} [\sum_{n=0}^{N-1} a_n (L_n - L_{n+1})]^2 \, dx \\
&= \int_{-1}^{1} [a_0 L_0 + \sum_{n=1}^{N-1} (a_n - a_{n-1}) L_n - a_{N-1} L_N]^2 \, dx \\
&= 2a_0^2 + \sum_{n=0}^{N-1} (a_n - a_{n-1})^2 \frac{2}{2n+1} + a_{N-1}^2 \frac{2}{2N+1}.
\end{aligned}$$

Noting that

$$-(1+x)(L_n - L_{n+1})_x = (n+1)(L_n + L_{n+1}), \tag{0.2}$$

we have

$$\begin{aligned}
\int_{-1}^{1} (1+x) v_x^2 \, dx &= \int_{-1}^{1} (1+x) [\sum_{n=0}^{N-1} a_n (L_n - L_{n+1})_x] [\sum_{n=0}^{N-1} a_n (L_n - L_{n+1})_x] \, dx \\
&= \int_{-1}^{1} [\sum_{n=0}^{N-1} -(n+1) a_n (L_n + L_{n+1})] [\sum_{n=0}^{N-1} a_n (L_n - L_{n+1})_x] \, dx \\
&= \int_{-1}^{1} [\sum_{m,n=0}^{N-1} -(n+1) a_m a_n (L_n + L_{n+1})(L_m - L_{m+1})_x \, dx \\
&= -\sum_{m,n=0}^{N-1} (n+1) a_m a_n \int_{-1}^{1} (L_n + L_{n+1})(L_m - L_{m+1})_x \, dx \\
&= 2 \sum_{n=0}^{N-1} a_n^2 (n+1),
\end{aligned}$$

where we used the fact that

$$\int_{-1}^{1} (L_n + L_{n+1})(L_m - L_{m+1})_x \, dx = -2\delta_{m,n}.$$

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Hence

$$\frac{\int_{-1}^{1} v^2\,dx}{\int_{-1}^{1}(1+x)v_x^2\,dx} \geq \frac{a_{N-1}^2\frac{2}{2N+1}}{\sum_{n=0}^{N-1} a_n^2(n+1)} \quad \geq \quad \frac{a_{N-1}^2\frac{2}{2N+1}}{\sum_{n=0}^{N-1} a_n^2 N} = O(N^{-2}).$$

It remains to verify that (0.2). Recall that

$$(2n+1)xL_n = nL_{n-1} + (n+1)L_{n+1} \tag{0.3}$$

and

$$(2n+1)L_n = (L_{n+1} - L_{n-1})_x. \tag{0.4}$$

Taking derivative with respect to x in (0.3) leads to

$$(2n+1)x(L_n)_x + (2n+1)L_n = n(L_{n-1})_x + (n+1)(L_{n+1})_x. \tag{0.5}$$

Rewrite (0.5) as, by (0.4),

$$\begin{aligned}(2n+1)x(L_n)_x + (2n+1)L_n &= (2n+1)(L_{n-1})_x + (n+1)[(L_{n+1} - L_{n-1})]_x \\ &= (2n+1)(L_{n-1})_x + (2n+1)(n+1)L_n,\end{aligned}$$

and thus simplifying gives

$$x(L_n)_x = (L_{n-1})_x + nL_n. \tag{0.6}$$

Similarly, we have

$$x(L_n)_x = (L_{n+1})_x - (n+1)L_n \tag{0.7}$$

By (0.6) and (0.7), we have (0.2).

□